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**Topology
of Lie Groups, I and II**

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TOPOLOGY OF LIE GROUPS, I and II
by Mamoru Mimura and Hiroshi Toda

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Topology of Lie Groups, I

From linear algebra
to the periodicity of **KO**-groups

In Place of a Proper Preface
Dramatis Persona
Students A, B, C
Professors P, Q

A: I am now learning linear algebra in the foundation course—all about matrices, determinants, and such stuff. As long as it stays in two or three dimensions I am all right; I have geometric images relating it to familiar shapes on the plane or in space. However, when things get to the general dimension, I understand the algebraic formulas, but I find it hard to interpret them “geometrically.”

B: It gets worse in the advanced course with new things every day: manifolds, topology, ... all so abstract. When your head has been stuffed with such things, even mathematics that you learned in your foundation course might turn your brain to custard.

A: Since they put linear algebra in the foundation course, I guess it must be pretty basic and important stuff.

B: Linear algebra is not used just in algebra, but also in geometry and analysis. I think it plays a role more like arithmetic in junior and senior high school.

C: I used to like solving problems in Euclidean geometry. Is there anything like that in advanced math?

B: Algebraic geometry, differential geometry, and topology seem to be of that kind to me, but to tell the truth, I have no intuitive feeling for any of the theories we are taught these days. I am really confused. Oh, but there comes Professor P. Would you like to hear a little sermon?

* * *

Prof. P: Since modern mathematics advances from day to day it cannot be helped if students who have just entered the senior course feel confused. All the lectures given in the senior course are just an introduction to one particular area. If more time were available, it would be best to give more examples and diversions so that students could understand without hurry, but as it is, they must fill in the gaps themselves.

B: Can you suggest a suitable book in your own field?

A: And directly connected with what we are learning in the foundation course?
Prof. P: Well, we have just finished *The topology of Lie groups*. You could get a hold of something like that. Lie groups are what one might call generalizations of the general linear and orthogonal groups. The first volume covers linear groups of matrices.

B: I suppose by topology you mean the same thing I have heard about. I find fundamental groups are more or less all right, but when it comes to homology or homotopy, things become hard to understand. It's hard to form any geometrical images.

Prof. P: We would not be happy spending too much time explaining topological concepts, especially the definition and basic properties of cohomology. Since in this book we use them rather than study them, we want the reader to skip over the proofs of fundamental properties and read through the book taking them for granted. It may not be a bad idea, however, after reading this book and learning topological results about classical groups, to go back to the beginning and examine the definitions and basic properties.

Student C: Is this book written kindly enough for beginners to understand?

Prof. P: No, not at all kindly. In studying mathematics the most important thing is to think for oneself. If the reader is led everywhere by the hand, his own ideas will be smothered. On the other hand, there are as few references as possible, so that those who have learned linear algebra in a foundation course should be able to read it. In the later chapters there are high-level results and ideas that cannot be found in any published books or papers.

Prof. Q: By the way, old boy, we all know Lie groups and Lie algebras are closely related, aren't they? How did you manage to write this book without even mentioning Lie algebras?

Prof. P: We are sorry, but we don't know enough about Lie algebras. Anyway, many eminent scholars have published works on this topic. Besides, we did not want too much pure algebra.

A: What are those letters 'iff' in this book?

Prof. P A iff B means that the statements A and B are equivalent. Either one can be replaced by the other. In a few instances we have used ' $\iff$ ' instead of 'iff'.

Prof. Q: I suppose it was a good idea to prove Bott periodicity also for KO -groups. But isn't it a bit of an anticlimax after the big title of *Topology of Lie groups* to offer only the classical groups or at best the spinor group?

Prof. P: Please do not be so impatient. We intend to write about the general theory of Lie groups and exceptional groups in detail in the second volume. Just wait and do not get so worked up just yet, please.

CHAPTER I

Classical Groups

In §1 of this chapter we state some general properties of topological groups and their homogeneous spaces, and in §2 we define an important class of examples: the classical groups, which we examine more concretely. In §3 we represent the compact, connected, classical groups as unions of maximal tori, by making use of diagonalization of normal matrices and, furthermore, we give concrete forms of their Weyl groups. The arguments in Chapters III and IV will make no use of the general theory of Lie groups relying only on special properties of the classical groups. In this book we will be mainly concerned with compact Lie groups. This is all that is needed if one is interested only in topological properties in classical groups, since, as will be shown in §4, a classical group is homeomorphic to a direct product of a Euclidean space and a compact classical Lie group. Incidentally, the general theory of compact Lie groups will be discussed in Chapter V. In §5 we will state the most fundamental properties of differentiable manifolds and Lie groups and show that a classical group is a special kind of a Lie group.

§1. Topological groups and homogeneous spaces

Definition of a topological group. A topological group is essentially a topological space with a continuous group structure. A more precise statement runs as follows:

DEFINITION. A set G is called a *topological group* if it has the following structures:

(i) G is a *group*, i.e., there is a multiplication $\mu: G \times G \rightarrow G$ together with an identity element $e \in G$ and an inverse $\iota: G \rightarrow G$ satisfying the identities (the group axioms):

$$ge = eg = g, \quad (gg')g'' = g(g'g''), \quad gg^{-1} = g^{-1}g = e,$$

where we write $\mu(g, g') = gg'$ and $\iota(g) = g^{-1}$.

(ii) G is a T_1 -space, i.e., there is given a family $\mathcal{O} = \{O_\lambda\}_{\lambda \in \Lambda}$ of subsets (open sets) O_λ of G that satisfies the axiom of a family of open sets (and hence G is a *topological space*): $\emptyset, G \in \mathcal{O}$; if for each α $O_\alpha \in \mathcal{O}$ then $\bigcup_\alpha O_\alpha \in \mathcal{O}$; if $O_1, O_2 \in \mathcal{O}$ then $O_1 \cap O_2 \in \mathcal{O}$ and G also satisfies the

T_1 -axiom (the separation axiom):

every element g of G is closed (i.e., $G - g \in \mathcal{O}$).

(iii) The functions $\mu: G \times G \rightarrow G$ and $\iota: G \rightarrow G$ are continuous.

As is well known, the product of three elements g , g' , and g'' may be written as $gg'g''$ forgetting the parentheses by the associative law $g(g'g) = (gg')g''$. Equally well known are the definitions such as that of a continuous map between topological spaces X , Y :

$f: X \rightarrow Y$ is *continuous* if the inverse image $f^{-1}(O)$ of an open set O is open. This is equivalent to the condition that the inverse image of a closed set (the complement of an open set) is closed.

A topology is defined on a product space $X \times Y$ by the condition that a set O is *open* in $X \times Y$ iff for any point (x, y) of O , there are open sets O_X, O_Y such that

$$x \in O_X, \quad y \in O_Y, \quad \text{and} \quad O_X \times O_Y \subset O.$$

In the context of topological groups, we will use the usual terminology of groups and topological spaces:

G is *abelian* (topological group) if $gg' = g'g$ for any $g, g' \in G$.

H is a *subgroup* of G if H is a subspace of G and $h, h' \in H$ implies $h^{-1}h' \in H$.

N is a *normal subgroup* of G if N is a subgroup of G and $gN = Ng$ for any $g \in G$.

In general, for subsets A and B of G , we use the following notation:

$$A \cdot B = \{ab | a \in A, b \in B\};$$

$$N(A) = \{g \in G | gA = Ag\} \text{—the } \textit{normalizer} \text{ of the subset } A \text{ in } G;$$

$Z(A) = \{z \in G | za = az \text{ for any } a \in A\}$ —the *centralizer* of the subset A in G ;

$$Z(G) = \{z \in G | gz = zg \text{ for any } g \in G\} \text{—the } \textit{center} \text{ of } G;$$

H is called a *closed (open) subgroup* of G if H is a subgroup of G and a closed (open) subset of G .

As for a homomorphism of topological groups we require continuity; namely,

$f: G \rightarrow G'$ is called a *homomorphism* of topological groups if f is continuous and satisfies $f(gg') = f(g)f(g')$ for any $g, g' \in G$.

Two topological groups G and G' are called *isomorphic* if there are homomorphisms $\varphi: G \rightarrow G'$ and $\psi: G' \rightarrow G$ of topological groups such that

$$\psi \circ \varphi = 1_G \text{ (identity),} \quad \varphi \circ \psi = 1_{G'}.$$

The map φ is an isomorphism of groups and a homeomorphism of topological spaces.

We give some simple examples of topological groups.

1.1. An arbitrary group G is a topological group with discrete topology (that is every subset of G is open. Equivalently, every element of G is both closed and open). We call such a group G a discrete group.

NOTATION. $\mathbf{R}$ denotes the set of real numbers;

$\mathbf{C}$ denotes the set of complex numbers ($\mathbf{C} = \{a + bi | a, b \in \mathbf{R}\}$, $i^2 = -1$);

$\mathbf{H}$ denotes the set of quaternions ($\mathbf{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbf{R}\}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$); $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

As is well known, $\mathbf{R}$ and $\mathbf{C}$ are commutative fields and $\mathbf{H}$ is a noncommutative field. We denote conjugates by

$$\begin{aligned}\bar{a} &= a \quad (a \in \mathbf{R}), \quad \overline{a + bi} = a - bi \quad (a + bi \in \mathbf{C}), \\ \overline{a + bi + cj + dk} &= a - bi - cj - dk \quad (a + bi + cj + dk \in \mathbf{H})\end{aligned}$$

and define the absolute value of $x \in \mathbf{F}$ by

$$|x| = \sqrt{x\bar{x}} = \sqrt{\bar{x}x} \geq 0.$$

(If $x = a + bi + cj + dk$, $|x| = \sqrt{a^2 + b^2 + c^2 + d^2}$.) We make $\mathbf{F}$ a metric space by taking $|x - y|$ as the distance between two points x and y of $\mathbf{F}$. By introducing a topology in $\mathbf{F}$ in the usual way (O is an open set of $\mathbf{F}$ iff for any $x \in O$, there is $\varepsilon > 0$ such that $|x - y| < \varepsilon$ implies $y \in O$), $\mathbf{F}$ becomes a topological space. Moreover, the four arithmetical operations, excluding division by zero, are continuous with respect to this topology. Therefore we have:

1.2. For $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$, $\mathbf{F}$ is an abelian topological group with respect to addition. $\mathbf{F}^* = \mathbf{F} - \{0\}$ is a topological group (abelian for $\mathbf{F} = \mathbf{R}, \mathbf{C}$) with respect to multiplication.

NOTATION. $\mathbf{F}^n = \{(x_1, \dots, x_n) | x_i \in \mathbf{F}\}$ is the n -dimensional vector space over $\mathbf{F}$ ($= \mathbf{R}, \mathbf{C}, \mathbf{H}$), with addition and scalar multiplication given respectively by

$$\begin{aligned}(x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n), \\ x(x_1, \dots, x_n) &= (xx_1, \dots, xx_n), \quad (x_1, \dots, x_n)x = (x_1x, \dots, x_nx).\end{aligned}$$

The norm (absolute value, length) is given by

$$\|(x_1, \dots, x_n)\| = \sqrt{|x_1|^2 + \dots + |x_n|^2}.$$

As above, $\mathbf{F}^n$ is a metric space with distance $\|x - y\|$ between $x, y \in \mathbf{F}^n$. In particular, $\mathbf{R}^n$ is called the n -dimensional Euclidean space.

1.3. $\mathbf{F}^n$ ($\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) is an abelian topological group with respect to the addition. $\mathbf{C}^n$ is isomorphic to $\mathbf{R}^{2n}$, $\mathbf{H}^n$ to $\mathbf{C}^{2n}$ and to $\mathbf{R}^{4n}$ as a topological group.

In fact, the isomorphisms

$$r: \mathbb{C}^n \rightarrow \mathbb{R}^{2n}, \quad c': \mathbb{H}^n \rightarrow \mathbb{C}^{2n}$$

are given respectively by

$$\begin{aligned} r(a_1 + b_1 i, \dots, a_n + b_n i) &= (a_1, \dots, a_n, b_1, \dots, b_n), \quad a_i, b_i \in \mathbb{R}, \\ c'(\alpha_1 + j\beta_1, \dots, \alpha_n + j\beta_n) &= (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n), \quad \alpha_i, \beta_i \in \mathbb{C} \end{aligned} \quad (1.3')$$

1.4. The homomorphism $f: \mathbb{F}^n \rightarrow \mathbb{F}^m$ of vector spaces over $\mathbb{F}$ given by $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ is a homomorphism of abelian topological groups.

Actions of topological groups and homogeneous spaces

DEFINITION. For a topological group G and a topological space X , a continuous map

$$\mu = \mu_X: G \times X \rightarrow X \text{ (or } X \times G \rightarrow X)$$

is called a left (or right) *action* of G on X if it satisfies the following conditions

- (1) $g(g'x) = (gg')x$ (or $(xg)g' = x(gg')$).
- (2) $ex = x$ (or $xe = x$) for the identity $e \in G$,

where we write $\mu(g, x) = gx$ or $\mu(x, g) = xg$. We also call X a G -space.

The action μ is said to be *effective* if $gx = g'x$ (or $xg = xg'$) for every $x \in X$ implies $g = g'$, and *transitive* if for any $x, x' \in X$, there is $g \in G$ such that $gx = x'$ or $xg = x'$.

NOTATION. For a left action, $L_g: X \rightarrow X$ is defined by $L_g(x) = gx$; for a right action, $R_g: X \rightarrow X$ is defined by $R_g(x) = xg$.

The continuity of the action μ implies that L_g (or R_g) is continuous and the conditions (1) and (2) are respectively equivalent to

$$L_{g'} \circ L_g = L_{g'g} \text{ (or } R_{g'} \circ R_g = R_{gg'}), \quad (1')$$

$$L_e = 1_X \text{ (or } R_e = 1_X). \quad (2')$$

Therefore the maps L_g and $L_{g^{-1}}$ (or R_g and $R_{g^{-1}}$) are inverses of each other and so they are homeomorphisms of X onto itself. We call L_g (or R_g) a *left (topological) transformation* (or *right (topological) transformation*) and G a *transformation group* of X .

Left transformations $\mu: G \times X \rightarrow X$ are in one-to-one correspondence with right transformations $\mu': X \times G \rightarrow X$ since

$$\mu(g, x) = \mu'(x, g^{-1}).$$

Hence it is sufficient to only consider actions on one side.

DEFINITION. A topological space X is called the *homogeneous space* of a topological group G if the topological group G acts on the topological space X transitively.

Since for every pair of points x, x' of a homogeneous space X there is a homeomorphism $L_x: X \rightarrow X$ that maps x to x' , any local property of a point of X is independent of the point. This is the origin of the name; homogeneous space. Clearly we have:

1.5. Any subgroup H of a topological group G acts on G and so G is itself a homogeneous space.

Quotient spaces and homogeneous spaces. In general, when an equivalence relation $\sim$ is given on a topological space X we obtain a topological space $X/\sim$, called the *quotient space* of X by the relation $\sim$, by defining a topology on the set of equivalence classes $X/\sim$ in the following way: let $\pi: X \rightarrow X/\sim$ be the natural projection taking an element to its equivalence class, then

A is open (closed) in $X/\sim$ iff $\pi^{-1}(A)$ is open (closed) in X .

The map π is continuous and is called the *projection*. If there is a map $f: X \rightarrow Y$ from a topological space X onto a set Y , one can define $Y = X/\sim$ by means of the relation: $x \sim x'$ iff $f(x) = f(x')$. The topological space Y with the quotient topology is called the *identification space* of X by f . Moreover, if Y is already a topological space and if its topology coincides with that of an identification space, then it is called an *identification topology* and f is said to be an *identification map*.

If a topological group G acts on a topological space X from left (or right), we can define an equivalence relation on X by setting $x \sim x'$ iff $gx = x'$ (equivalently $xg = x'$), for some $g \in G$. We use the notation

$$X/G = X/\sim, \quad \pi: X \rightarrow X/G,$$

for the quotient space and the projection. Each point of X/G is an equivalence class under $\sim$ and we can write

$$Gx = \{gx | g \in G\} \quad (\text{or } xG = \{xg | g \in G\}),$$

which we call an *orbit* of x by G for some representative x .

THEOREM 1.6. Let G be a topological group and H a subgroup of G .

(1) The right (or left) action (multiplication) of H on G induces a projection $\pi: G \rightarrow G/H$ that is continuous and open (i.e., the image of an open set is open). Then G/H is a homogeneous space of G with respect to the left (or right) action of G .

(2) If H is a closed subgroup then G/H is a regular space (i.e., a T_1 -space with the property that for a closed set F and an arbitrary point $p \notin F$ there are open sets U, V satisfying $p \in U, F \subset V$, and $U \cap V = \emptyset$), and so G/H is a Hausdorff space (i.e., for any two points p, q there are open sets U and V such that $p \in U, q \in V$, and $U \cap V = \emptyset$). In particular, a topological group is a regular space.

PROOF. (1) Let O be an open set of G . R_h is a homeomorphism so $h \in H$ implies $R_h(O)$ is open. Thus, $\pi^{-1}\pi(O) = \cup R_h(O)$ is open and hence $\pi(O)$ is open. So, π is an open map. The continuity of π is clear.

Next, we define an action of G on G/H , $\varphi: G \times G/H \rightarrow G/H$, by

$$\varphi(g, g'H) = gg'H.$$

For an open set O of G/H , the commutativity of the diagram below implies that $(1 \times \pi)^{-1}\varphi^{-1}(O) = \mu^{-1}\pi^{-1}(O)$ is open.

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ \downarrow 1 \times \pi & & \downarrow \pi \\ G \times G/H & \xrightarrow{\varphi} & G/H \end{array}$$

Using the fact that the product of open maps is again an open map, we see that $\varphi^{-1}(O) = (1 \times \pi)(1 \times \pi)^{-1}(O)$ is open, and hence φ is continuous. Clearly φ gives a transitive action from the left and so G/H is a homogeneous space of G .

(2) H is closed and L_g is a homeomorphism so $\pi^{-1}(\pi(g)) = gH = L_g(H)$ is closed. Thus, every point $\pi(g)$ of G/H is closed, and it follows that G/H is a T_1 -space.

It remains to show that for a closed subset F of G/H and an arbitrary point $p \in F$ there are open sets U and V satisfying $p \in U$, $F \subset V$, $U \cap V = \emptyset$. Since G acts transitively on G/H , we may suppose that p is the class (H) of the identity e . Since F is closed there is an open set U_0 such that $F \cap U_0 = \emptyset$ and $p \in U_0$. From the continuity of the group action of G it follows that there is an open set W satisfying $e \in W$ and $W^{-1} \cdot W = \{w_1^{-1}w_2 | w_1, w_2 \in W\} \subset \pi^{-1}(U_0)$. The set $W \cdot \pi^{-1}(F) = \cup Wg$ ($g \in \pi^{-1}(F)$) is open. Since π is an open map, both $U = \pi(W)$ and $V = \pi(W \cdot \pi^{-1}(F))$ are open and such that $p \in U$ and $F \subset V$. If we suppose that $y \in U \cap V$, then there are $g_1, g_2 \in W$ and $g \in \pi^{-1}(F)$ satisfying $y = \pi(g_2) = \pi(g_1g)$. Thus we have $h \in H$ such that $g_2h = g_1g$, from which we deduce that $\pi(gh^{-1}) \in F \cap U_0 = \emptyset$ from $gh^{-1} = g_1^{-1}g_2 \in W^{-1} \cdot W \subset \pi^{-1}(U_0)$. Thus we get $U \cap V = \emptyset$.

Since G is a T_1 -space, $\{e\}$ is a closed subgroup and so $G = G/\{e\}$ is a regular space. Q.E.D.

Suppose we are given an action of G on X . For a point x of X , the set

$$G_x = \{g \in G | gx = x\} \text{ (or } G_x = \{g \in G | xg = x\})$$

is a subgroup of G (Note: $\{x\}$ is closed so G_x is closed) and is called the *isotropy group* at x (the *stabilizer* of x). For $x \in X$ we define a map $\mu_x: G \rightarrow X$ by

$$\mu_x(g) = gx \text{ (or } xg).$$

Obviously:

1.7. (1) G acts transitively iff μ_x is surjective.

(2) G acts effectively iff $\bigcap_{x \in X} G_x = \{e\}$.

THEOREM 1.8. Let X be a homogeneous space of G .

(1) μ_x induces a bijection $h_x: G/G_x \rightarrow X$ such that $\mu_x = h_x \circ \pi_x$, where $\pi_x: G \rightarrow G/G_x$ is the projection.

(2) For elements $g \in G$, $x \in X$, $y = g \cdot x$ the diagram below is commutative, where $G_y = gG_xg^{-1}$ and A_g is a homeomorphism given by $A_g(g'G_x) = (gg'g^{-1})G_x$.

$$\begin{array}{ccc} G/G_x & \xrightarrow{h_x} & X \\ \downarrow A_g & & \downarrow L_g \\ G/G_y & \xrightarrow{h_y} & X \end{array}$$

(3) If μ_x (or h_x) is an open map, then h_x is a homeomorphism.

PROOF. (1) Observe that $\pi_x(g_1) = \pi_x(g_2)$ iff $g_1^{-1}g_2 \in G_x$ iff $g_1^{-1}g_2x = x$ iff $\mu_x(g_1) = \mu_x(g_2)$. Thus the equality $h_x(\pi_x(g)) = \mu_x(g)$ determines the injection h_x and so by (1.7)(1), μ_x surjective implies that h_x is surjective. Since U is an open set of X , $\mu_x^{-1}(U) = \pi_x^{-1}(h_x^{-1}(U))$ is open and hence $h_x^{-1}(U)$ is open. Thus h_x is a continuous bijection.

(2) For $g_1 \in G$, $L_g h_x(\pi_x(g_1)) = L_g \mu_x(g_1) = gg_1 \cdot x = gg_1g^{-1} \cdot y = h_y A_g(\pi_y(g_1))$. This implies the above diagram is commutative. Clearly we have $G_y = A_g G_x = gG_xg^{-1}$.

(3) When μ_x is an open map, and if O is open in G/G_x then $\pi_x^{-1}(O)$ is open and hence $h_x(O) = \mu_x(\pi_x^{-1}(O))$ is open. So h_x is an open map, which implies that h_x is a homeomorphism. Q.E.D.

Using this theorem, one can easily prove

1.9. Let $f: G \rightarrow G'$ be a homomorphism of topological groups. Then the kernel $\text{Ker } f = f^{-1}(e)$ of f is a closed, normal subgroup of G and f induces a monomorphism $\bar{f}: G/\text{Ker } f \rightarrow G'$. Moreover, if f is an epimorphism and an open map, $\bar{f}$ is an isomorphism between topological groups.

To check whether $\mu_x: G \rightarrow X$ in Theorem 1.8 is an open map, the next lemma is useful. First we need the notions of compactness, local-compactness, and the second axiom of countability from general topology.

A compact (resp. Lindelöf) set is characterized by the property that an arbitrary open covering has a finite (resp. countable) open subcovering. An important example is a closed bounded set in the Euclidean space $\mathbb{R}^n$. A continuous image and a closed subset of a compact set are also compact. A compact subset of a Hausdorff space is a closed set. A locally compact space is a space in which every point has a compact neighborhood.

A topological space satisfies the second axiom of countability if it has a countable open basis (i.e., an arbitrary open set is a union of sets belonging

to the basis). The main examples are $\mathbf{R}^n$ and its subspaces.

LEMMA 1.10. Suppose that a Hausdorff space X is a homogeneous space of G . If either

(α) G is compact or

(β) G and X are locally compact and G satisfies the second axiom of countability,

then $\mu_x: G \rightarrow X$ is an open map and hence $h_x: G/G_x \rightarrow X$ is a homeomorphism.

PROOF. (α) If G is compact, G/G_x is a compact Hausdorff space by Theorem 1.6. Being continuous and bijective, $h_x: G/G_x \rightarrow X$ is a homeomorphism. By Theorem 1.6, $\pi: G \rightarrow G/G_x$ is an open map, hence $\mu_x = h_x \circ \pi$ is an open map.

(β) Suppose (β) is satisfied. It suffices to show that when O is an open set of G and $g \in O$, then $\mu_x(g)$ is an interior point of $\mu_x(O)$.

We may choose a neighborhood V of e such that $g \cdot V = L_g(V) \subset U$. By the regularity and the local compactness of G and by the continuity of the group action, we can choose a neighborhood U of e such that $\bar{U}$ is compact and $\bar{U}^{-1} \cdot \bar{U} \subset V$. G satisfies the second axiom of countability so G is a Lindelöf space. Since $G = \bigcup_{g \in G} gU$ there is a countable sequence $\{g_n\}$ such that $G = \bigcup_n g_n U$ and hence

$$X = \mu_x(G) = \bigcup_n \mu_x(g_n \bar{U}).$$

If we suppose that for some n , $\mu_x(g_n \bar{U})$ has an interior point $\mu_x(g_n h) = (g_n h)x$, $h \in \bar{U}$, then x is an interior point of $(g_n h)^{-1} \mu_x(g_n \bar{U}) = \mu_x(h^{-1} \bar{U}) \subset \mu_x(\bar{U}^{-1} \cdot \bar{U}) \subset \mu_x(V)$ and hence $\mu_x(g) = gx$ is an interior point of $g\mu_x(V) = \mu_x(gV) \subset \mu_x(O)$.

Thus it suffices to deduce a contradiction from the assumption that $\mu_x(g_n \bar{U})$ does not contain interior points for any n . $\bar{U}$ is compact so $\mu_x(g_n \bar{U})$ is compact and hence $\mu_x(g_n \bar{U})$ is closed. Let W_0 be a nonempty open set of X such that $\bar{W}_0$ is compact. By induction on n we will construct a sequence of nonempty open sets $W_0, W_1, \dots, W_n, \dots$ such that $\bar{W}_n \subset W_{n-1} - \mu_x(g_n \bar{U})$ as follows. Since $\mu_x(g_n \bar{U})$ is a closed set having no interior points, $W_{n-1} - \mu_x(g_n \bar{U})$ is a nonempty open set. As X is a locally-compact, regular space, we can choose a nonempty open set W_n such that $\bar{W}_n \subset W_{n-1} - \mu_x(g_n \bar{U})$. Then $\bar{W}_0 \supset \bar{W}_1 \supset \bar{W}_2 \supset \dots$ is a decreasing sequence of nonempty compact sets and hence $\bigcap_n \bar{W}_n \neq \emptyset$. On the other hand, the equalities $\bar{W}_n \cap \mu_x(g_n \bar{U}) = \emptyset$ and $X = \bigcup_n \mu_x(g_n \bar{U})$ imply $\bigcap_n \bar{W}_n = \emptyset$ and so we get a contradiction. Q.E.D.

The topological group $\mathbf{F}^* = \mathbf{F} - \{0\}$ acts on $\mathbf{F}^{n+1} - \{0\}$ from the left by the scalar product

$$x(x_0, x_1, \dots, x_n) = (xx_0, xx_1, \dots, xx_n).$$

DEFINITION. $P^n(\mathbf{F}) = \mathbf{F}^{n+1} - \{0\}/\mathbf{F}^*$ is called the n -dimensional projective space over $\mathbf{F}$.

A point of $P^n(\mathbf{F})$ is thought of as a one-dimensional subspace passing through $x = (x_0, x_1, \dots, x_n) \neq 0$ and designated by

$$[x_0 : x_1 : \dots : x_n] \in P^n(\mathbf{F}).$$

On the other hand, as

$$S(\mathbf{F}^{n+1}) = \{x \in \mathbf{F}^{n+1} \mid \|x\| = 1\} \quad (\text{the unit sphere})$$

is a closed, bounded set of $\mathbf{F}^{n+1}$, it is a compact, Hausdorff space and homeomorphic with the unit sphere $S^{d(n+1)-1}$ in the Euclidean space $\mathbf{R}^{d(n+1)}$, where

$$d = 1 \quad (\mathbf{F} = \mathbf{R}), \quad d = 2 \quad (\mathbf{F} = \mathbf{C}), \quad d = 4 \quad (\mathbf{F} = \mathbf{H}).$$

The above action restricts to an action of $S(\mathbf{F}^1)$ on $S(\mathbf{F}^{n+1})$. Then $S(\mathbf{F}^{n+1})/S(\mathbf{F}^1)$ is a compact, Hausdorff space and by Lemma 1.10 we obtain the following homeomorphisms:

$$\begin{aligned} P^n(\mathbf{F}) &= \mathbf{F}^{n+1} - \{0\}/\mathbf{F}^* = S(\mathbf{F}^{n+1})/S(\mathbf{F}^1) \\ &= S^{d(n+1)-1}/S^{d-1} \quad (\text{homeomorphic}). \end{aligned} \quad (1.11)$$

A connected topological group. A continuous image of a *connected* topological space (i.e., the space cannot be represented as the union of two disjoint nonempty open subsets) is again connected. A topological space can be represented as the union of *connected components* (i.e., maximal connected subspaces).

THEOREM 1.12. (1) *The connected component G_0 of the identity of a topological group G is a closed, normal subgroup and connected components are in one-to-one correspondence with cosets of G/G_0 .*

(2) *A connected topological group is generated by a neighborhood of the identity.*

(3) *If a normal subgroup H of a connected topological group G has the discrete topology, then $H \subset Z(G)$.*

PROOF. (1) The correspondence $g \mapsto g^{-1}$ is continuous and maps e to e hence $G_0^{-1} \subset G_0$. For any $g_0 \in G_0$ the correspondence $g \mapsto g_0 g$ is continuous and maps e to $g_0 \in G_0$ hence $g_0 G_0 \subset G_0$. So G_0 is a subgroup. Obviously G_0 is a closed set. For any $g_1 \in G$ the correspondence $g \mapsto g_1 g g_1^{-1}$ is continuous and maps e to e hence $g_1 G_0 g_1^{-1} \subset G_0$. Thus G_0 is a normal subgroup. The rest is easy.

(2) Let U be an open set containing the identity and G_1 a subgroup generated by U . $g \in G_1$ implies $g \cdot U \subset G_1$, and G_1 is an open set, since $g \cdot U$ is a neighborhood of g . Conversely, if $g \in \overline{G_1}$ then $g \cdot U \cap G_1 \neq \emptyset$, which implies $g \in G_1$, and hence G_1 is closed. G is connected so $G = G_1$.

(3) If for some $h \in H$, we define $f: G \rightarrow H$ by $f(g) = ghg^{-1}$, then f is continuous and $f(e) = h$. G is connected and H is discrete so $f(G) = \{h\}$ and $ghg^{-1} = h$, which implies $h \in Z(G)$. Q.E.D.

Let G be a topological group and H a subgroup of G . Then by the continuity of the projection $\pi: G \rightarrow G/H$, G is connected, which implies that G/H is connected. In order to prove the converse we need some conditions.

LEMMA 1.13. *Let G be a topological group and H a subgroup of G . If H and G/H are connected, so is G .*

PROOF. It suffices to deduce a contradiction from the assumption that there exist nonempty open sets O_1 and O_2 satisfying $G = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. By Theorem 1.6, π is an open map so $\pi(O_1)$ and $\pi(O_2)$ are open. Since $G/H = \pi(O_1) \cup \pi(O_2)$ is connected, there exists an element $g \in G$ such that $gH \in \pi(O_1) \cap \pi(O_2)$. Let $H_i = g^{-1}(gH \cap O_i)$ ($i = 1, 2$), then the H_i 's are nonempty open sets of H and satisfy $H_1 \cap H_2 = \emptyset$ and $H = H_1 \cup H_2$. This contradicts the connectedness of H . Q.E.D.

DEFINITION. A continuous map $w: I \rightarrow X$ from the closed unit interval $I = [0, 1]$ to a topological space X is called a *path*, $w(0)$ is called the *initial point*, and $w(1)$ the *terminal point*. We also refer to w as a *path connecting* $x_0 = w(0)$ to $x_1 = w(1)$. A topological space X is said to be *arcwise connected*, if there is a path connecting any two points of X . A topological space X is said to be *locally arcwise connected*, if there is an arcwise connected neighborhood in an arbitrary neighborhood of each point of X .

All topological groups and homogeneous spaces considered in this book are locally arcwise connected. As in the case of usual connectivity, the set of all points that can be connected by a path to a point x is called the *arcwise connected component* of the point x . It is obvious from the definition that each arcwise connected component of a locally arcwise connected topological space is open. Hence we have

1.14. *A connected and locally arcwise connected topological space is arcwise connected.*

§2. Classical groups

General linear groups. Let K be a commutative field and V a vector space (a linear space or a K -module) over K . An isomorphic (bijective) linear map from V to itself

$$f: V \cong V \text{ (linear: } f(v + v') = f(v) + f(v'), f(\alpha v) = \alpha f(v), \alpha \in K)$$

is called a (regular) *linear transformation* and the set of all linear transformations of V is called the *general linear transformation group of V* and is denoted by

$$GL(V) = \text{Aut}(V) = \{f: V \cong V \text{ (linear)}\}.$$

Of course, the product in $GL(V)$ is given by the composition of linear maps

$$gf = g \circ f: V \xrightarrow{f} V \xrightarrow{g} V,$$

and the identity is 1_V .

Let V be finite dimensional and $n = \dim_K V$. We denote by

$$M(n, K) = \{(a_{ij})_{i,j=1,\dots,n} | a_{ij} \in K\},$$

the set of square matrices of order n . Then $M(n, K)$ forms a (matrix) ring with respect to the matrix addition and product. We choose a K -basis $B = \{b_1, \dots, b_n\}$ and define a map $\psi_B: GL(V) \rightarrow M(n, K)$ by

$$\psi_B(f) = (a_{ij}), \quad f(b_j) = \sum_i a_{ij} b_i.$$

As is well known, we have:

2.1. ψ_B is a product preserving injection and its image coincides with the group $GL(n, K)$ of all regular matrices in $M(n, K)$. By using the determinant $\det: M(n, K) \rightarrow K$, we have

$$\psi_B: GL(V) \cong GL(n, K) = \{A \in M(n, K) | \det A \neq 0\}.$$

The group $GL(n, K)$ is called the general linear group of K of order n and

$$SL(n, K) = \{A \in GL(n, K) | \det A = 1\}$$

forms a subgroup of $GL(n, K)$ called the special linear group.

From now on we consider the case $K = \mathbf{F} = \mathbf{R}, \mathbf{C}$. $M(n, \mathbf{F})$ is given a metric with respect to the coordinates a_{ij} of a matrix $A = (a_{ij})$ and is homeomorphic with $\mathbf{F}^{n^2}$ ($= \mathbf{R}^{n^2}, \mathbf{C}^{n^2} \cong \mathbf{R}^{2n^2}$). Addition, multiplication, the scalar product, and the determinant are continuous functions on the set of matrices, since all of them can be expressed by means of polynomials in the coordinates a_{ij} . The inverse matrix of a matrix is expressed by means of rational functions with the determinant as their denominator. It follows from the above that:

2.2. For $\mathbf{F} = \mathbf{R}$ and $\mathbf{C}$, $GL(n, \mathbf{F})$ is a topological group with the topology of an open subset of $M(n, \mathbf{F})$, and $SL(n, \mathbf{F})$ is its closed subgroup. $M(n, \mathbf{F})$, $GL(n, \mathbf{F})$, and $SL(n, \mathbf{F})$ are locally compact metric spaces satisfying the second axiom of countability.

Let V be a vector space over $\mathbf{R}$. Then $V \otimes_{\mathbf{R}} \mathbf{C}$ is a vector space over $\mathbf{C}$. We define the complexification $c: GL(V) \rightarrow GL(V \otimes_{\mathbf{R}} \mathbf{C})$ of general linear groups over these vector spaces by

$$c(f) = f \otimes 1_{\mathbf{C}}.$$

In particular, as we have $V \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}^n$ for $V = \mathbf{R}^n$, the complexification is a map

$$c: GL(n, \mathbf{R}) \rightarrow GL(n, \mathbf{C}),$$

and it is obtained as the restriction to $GL(n, \mathbf{R})$ of the extension of the coefficient field from $\mathbf{R}$ to $\mathbf{C}$

$$c: M(n, \mathbf{R}) \rightarrow M(n, \mathbf{C})$$

in the matrix ring. Obviously we have

2.3. $c: M(n, \mathbf{R}) \rightarrow M(n, \mathbf{C})$ is a monomorphism of algebras over $\mathbf{R}$.

$$c(GL(n, \mathbf{R})) = GL(n, \mathbf{C}) \cap \text{Im } c,$$

$$A \in M(n, \mathbf{C}) \text{ belongs to } \text{Im } c \Leftrightarrow A = \overline{A}.$$

Conversely, when V is a vector space over $\mathbf{C}$, we denote by $V_{\mathbf{R}}$ the same space considered as a vector space over $\mathbf{R}$. The map

$$r: GL(V) \rightarrow GL(V_{\mathbf{R}})$$

of general linear groups is a naturally defined monomorphism of topological groups. In particular, when $V = \mathbf{C}^n$, we have $V_{\mathbf{R}} = \mathbf{R}^{2n}$ by (1.3') and hence we obtain the map

$$r: GL(n, \mathbf{C}) \rightarrow GL(2n, \mathbf{R}),$$

which is the restriction of the map

$$r: M(n, \mathbf{C}) \rightarrow M(2n, \mathbf{R})$$

obtained as follows. Express $X \in M(n, \mathbf{C})$ as $X = A + iB$ with $A, B \in M(n, \mathbf{R})$, then we define

$$r(X) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M(2n, \mathbf{R}).$$

In fact, the action of $X \in GL(n, \mathbf{C})$ on $v \in \mathbf{C}^n$ is given by $X \cdot v = {}^t(X^t v)$ and hence for $v = x + iy$ ($x, y \in \mathbf{R}^n$) we have

$${}^t(X \cdot v) = (A + iB) {}^t(x + iy) = (A^t x - B^t y) + i(A^t y + B^t x),$$

which implies the above expression for $r(X)$. The following properties of r can be verified by simple calculations (where $X^* = {}^t \overline{X} = \overline{{}^t X}$, I_n is the unit matrix, and O is the zero matrix):

2.4. (1) The map r is a homomorphism of algebras over $\mathbf{R}$ satisfying

$$r(X^*) = r(X)^* = {}^t(r(X)).$$

(2) $Y \in r(M(n, \mathbf{C}))$ iff $J_n Y J_n^{-1} = Y \in M(2n, \mathbf{R})$, where $J_n = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}$.

(3) $X \in M(n, \mathbf{C})$,

$$\eta_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix} \text{ implies } \eta_n (cr(X)) \eta_n^{-1} = \begin{pmatrix} X & O \\ O & \overline{X} \end{pmatrix}.$$

(4) $r(GL(n, \mathbf{C})) = GL(2n, \mathbf{R}) \cap r(M(n, \mathbf{C}))$.

Representations of topological groups. Suppose that a topological group G acts on a vector space V over $\mathbf{F} (= \mathbf{R}, \mathbf{C})$ by $\mu: G \times V \rightarrow V$. If $L_g: V \rightarrow V$

is a linear map for all $g \in G$, (V, μ) or μ is called a *representation* of G on V and V a *representation space* of the representation μ . Also V is called a G -module in the sense that it is an F -module on which G acts.

We say that (V, μ) is a *real representation* if $F = \mathbf{R}$ and a *complex representation* if $F = \mathbf{C}$.

For a representation (V, μ) of G , we define a map $\varphi: G \rightarrow GL(V)$ by

$$\varphi(g)(v) = L_g(v) = \mu(g, v) = g \cdot v.$$

The map φ is a homomorphism of groups, since $\varphi(gg') = L_{gg'} = L_g \circ L_{g'} = \varphi(g)\varphi(g')$. Here, we make $GL(V)$ a topological group by giving it the topology for which ψ_B of (2.1) is a homeomorphism. (It is shown below that the topology can be introduced independently of the choice of the basis B .) Then the continuity of φ is equivalent to that of the coefficients $\psi_B(\varphi(g)) = (a_{ij}(g))$ where $g\mathbf{b}_j = \sum_i a_{ij}(g)\mathbf{b}_i$. Therefore we have:

2.5. *The above correspondence is a bijection between the set of representations $\mu: G \times V \rightarrow V$ of G on V and the set of homomorphisms $\varphi: G \rightarrow GL(V)$ (or $\psi_B \circ \varphi: G \rightarrow GL(n, F)$).*

(From now on a homomorphism φ will also be called a *representation* of G .)

Let $f: V \rightarrow V$ be an element of $GL(V)$. We say that a representation $\mu: G \times V \rightarrow V$ and the representation $\mu': G \times V \rightarrow V$ defined by $\mu'(g, v) = f(\mu(g, f^{-1}(v)))$ are *equivalent*, since they are essentially the same. When two representations μ and μ' correspond to homomorphisms φ and $\varphi': G \rightarrow GL(V)$, we have that μ and μ' are equivalent iff there is an $f \in GL(V)$ such that $\varphi'(g) = f\varphi(g)f^{-1}$.

In particular, since replacing a basis B by B' in $\psi_B \circ \varphi$ corresponds to an inner automorphism, $\psi_B \circ \varphi$ and $\psi_{B'} \circ \varphi$ are equivalent representations over F^n . This also shows that the topology in $GL(V)$ determined by ψ_B is independent of the choice of B .

DEFINITION. A representation (V, μ) of a topological group G is said to be *reducible*, if there is a vector subspace W of V such that $0 \neq W \neq V$ and $\mu(G \times W) \subset W$. Such a W is called a G -invariant subspace. A representation (V, μ) is called *irreducible* if there is not such W .

Given a G -invariant subspace W , we obtain a representation $(W, \mu|_W)$ by restricting μ to W . We call it the *restriction* of the representation (V, μ) to W .

A representation (V, μ) of G is said to be *completely reducible* if V is a direct sum of G -invariant subspaces W_i and the restriction $(W_i, \mu|_{W_i})$ to each W_i is irreducible.

If we put $GL(V, W) = \{f \in GL(V) | f(W) \subset W\}$, then the homomorphism φ corresponding to a representation (V, μ) having W as a

G -invariant subspace is of the form

$$\varphi: G \rightarrow GL(V, W) \subset GL(V).$$

If we choose a basis $B = \{b_1, \dots, b_n\}$ of V such that $\{b_1, \dots, b_r\}$ is a basis of W , we see that $GL(V, W)$ is isomorphic to

$$GL(n, r, F) = \left\{ \begin{pmatrix} A & C \\ O & B \end{pmatrix} \mid A \in GL(r, F), B \in GL(n-r, F) \right\},$$

under the isomorphism $\psi_B: GL(V) \cong GL(n, F)$. Moreover, if there is a G -invariant subspace W' such that $V = W \oplus W'$ (direct sum), then φ is of the form

$$\varphi: G \rightarrow GL(W) \otimes GL(W') \subset GL(V),$$

where $GL(W) \times GL(W') = \{f \in GL(V) \mid f(W) \subset W, f(W') \subset W'\}$.

In this case we write

$$(V, \mu) = (W, \mu|_W) \oplus (W', \mu|_{W'}) \text{ (direct sum).}$$

We can also define that (V, μ) is *completely reducible* if

$$(V, \mu) = \bigoplus_i (W_i, \mu|_{W_i})$$

with $(W_i, \mu|_{W_i})$ irreducible.

If we choose a new base B such that $\{b_{r+1}, \dots, b_n\}$ is a basis of W' , $GL(W) \times GL(W')$ is seen to be isomorphic to

$$GL(r, F) \times GL(n-r, F) = \left\{ A \oplus B = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in GL(n, r, F) \right\}$$

under ψ_B .

In general, for subgroups H_i ($i = 1, 2$) of $GL(n_i, F)$, we define a subgroup $H_1 \times H_2$ by

$$H_1 \times H_2 = \{A \oplus B \mid A \in H_1, B \in H_2\} \subset GL(n_1 + n_2, F).$$

Unitary groups and orthogonal groups. A vector space V over C is called a *complex inner product space*, if there is an inner product

$$(\cdot, \cdot): V \times V \rightarrow C,$$

satisfying the following conditions:

2.6.

- (1) $(v, v) \geq 0$, $(v, v) = 0$ iff $v = 0$ (*positive definite*);
- (2) $(u, v) = \overline{(v, u)}$, $(u + u', v) = (u, v) + (u', v)$;
- (3) $(\alpha u, v) = \alpha(u, v)$ ($\alpha \in C$).

A vector space V over R is called a *real inner product space* if there is given an inner product $(\cdot, \cdot): V \times V \rightarrow R$ satisfying the same conditions, where condition (2) takes the form $(u, v) = (v, u)$, and $\alpha \in R$ in condition (3).

A subset $\{e_1, \dots, e_r\}$ of V is called an *orthonormal system* if it satisfies $(e_i, e_j) = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

An easy calculation shows that the vectors of an orthonormal system are linearly independent. If these vectors span V , they form a basis of V , which is then called an *orthonormal basis*. If the orthonormal complement $\{v \in V | (v, e_i) = 0\}$ of an orthonormal system $\{e_1, \dots, e_r\}$ is nontrivial, one can choose an element e_{r+1} of the complement with length $\|e_{r+1}\| = \sqrt{(e_{r+1}, e_{r+1})} = 1$ so that $\{e_1, \dots, e_r, e_{r+1}\}$ is again an orthonormal system. It follows that there exists an orthonormal basis if V is finite dimensional.

In particular, in $\mathbf{C}^n$ (or $\mathbf{R}^n$) we define a special inner product

$$(u, v) = u \cdot v^* = \sum_{i=1}^n u_i \bar{v}_i \quad \left(\text{or } \sum_{i=1}^n u_i v_i \right)$$

for vectors $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbf{C}^n$ (or $\mathbf{R}^n$). The length of u is then given by $\|u\| = \sqrt{(u, u)}$.

A linear transformation f of a complex (real) metric space V , preserving the inner product

$$f: V \cong V \text{ (linear), } (f(u), f(v)) = (u, v),$$

is called a *unitary* (or *orthogonal*) *transformation* of $GL(V)$, and the subgroup of $GL(V)$ consisting of such transformations is called a *unitary* (or *orthogonal*) *transformation group* and is denoted by

$$U(V) \subset GL(V).$$

In particular for the case $V = \mathbf{C}^n$ (or $\mathbf{R}^n$) of the inner product given above, we say that

$$U(n) = U(n, \mathbf{C}) = U(\mathbf{C}^n) \text{ is the unitary group of order } n,$$

$$O(n) = U(n, \mathbf{R}) = U(\mathbf{R}^n) \text{ is the orthogonal group of order } n.$$

In matrix terms they can be defined as

$$U(n, \mathbf{F}) = \{A \in M(n, \mathbf{F}) | AA^* = I_n\} \quad (\mathbf{F} = \mathbf{R}, \mathbf{C}).$$

That is, $U(n) = U(n, \mathbf{C})$ is the group of unitary matrices of order n , $O(n) = U(n, \mathbf{R})$ is the group of orthogonal matrices ($A^t A = I_n$) of order n and $U(n, \mathbf{F})$ is a closed subgroup of $GL(n, \mathbf{F})$. Observe that $A \in U(n, \mathbf{F})$ iff $A^{-1} = A^*$ which is equivalent to $A^* A = I_n$ and to $AA^* = I_n$. We denote

$$SU(n, \mathbf{F}) = U(n, \mathbf{F}) \cap SL(n, \mathbf{F}) \quad (\mathbf{F} = \mathbf{R}, \mathbf{C}).$$

In particular we say that

$$SU(n) = SU(n, \mathbf{C}) \text{ is the special unitary group of order } n,$$

$$SO(n) = SU(n, \mathbf{R}) \text{ is the special orthogonal group of order } n \\ \text{or the rotation group of order } n.$$

If we consider 2.3 and 2.4 restricted to orthogonal groups and unitary groups, they take the following forms:

(2.7). (1) $c: O(n) \rightarrow U(n)$ and $c: SO(n) \rightarrow SU(n)$ are monomorphisms of topological groups, and

$A \in U(n)$ (or $SU(n)$) belongs to $\text{Im } c$ iff $\bar{A} = A$.

(2) $r: U(n) \rightarrow SO(2n)$ is a monomorphism, and

$A \in \text{Im } r$ iff $J_n A J_n^{-1} = A$, where $J_n = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}$.

(3) $A \in U(n)$,

$\eta_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}$ implies $\eta_n(c(X))\eta_n^{-1} = A \oplus \bar{A}$.

THEOREM 2.8. The topological groups $U(n)$, $O(n)$, $SU(n)$, and $SO(n)$ are compact.

PROOF. It suffices to show that they are bounded, since they are closed subspaces of $M(n, \mathbb{C}) \cong \mathbb{R}^{2n^2}$ or $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$. The equality $AA^* = I_n$ for $A = (a_{ij}) \in U(n)$ implies $|a_{ij}|^2 = \bar{a}_{ij}a_{ij} \leq \sum_{k=1}^n a_{ik}\bar{a}_{ik} = 1$ and hence $U(n)$ is bounded. The other cases are similar. Q.E.D.

A representation $\mu: G \times V \rightarrow G$ of a topological group G with a complex (or real) metric space V as a representation space is called a *unitary* (or *orthogonal*) *representation*, if it preserves the inner product:

$$(g \cdot u, g \cdot v) = (u, v) \quad (g \in G; u, v \in V).$$

Then the homomorphism $\varphi: G \rightarrow GL(V)$ corresponding to μ takes values in $U(V)$. Similarly to 2.6 we have:

2.9. The set of unitary (or orthogonal) representations of G over V is in one to one correspondence with the set of homomorphisms of topological groups $\varphi: G \rightarrow U(V)$. (The map φ is also called a unitary (or orthogonal) representation.)

An equivalence of two unitary (or orthogonal) representations is given by an automorphism preserving the inner product on V , i.e., in terms of the corresponding homomorphisms φ and $\varphi': G \rightarrow U(V)$, φ and φ' (or μ and μ') are called *equivalent* if there is an element $f \in U(V)$ such that $\varphi' = f\varphi f^{-1}$.

The representation (or action)

$$\mu: U(n, \mathbb{F}) \otimes \mathbb{F}^n \rightarrow \mathbb{F}^n \quad (\mathbb{F} = \mathbb{R}, \mathbb{C})$$

corresponding to the identity map $U(n, \mathbb{F}) \rightarrow U(n, \mathbb{F}) = U(\mathbb{F}^n)$ is given by $\mu(A, v) = {}^t(A^t v) = v^t A$. As the action preserves the inner product and hence the length, it acts on the unit sphere $S(\mathbb{F}^n)$.

THEOREM 2.10. *Let $\mathbf{F} = \mathbf{R}, \mathbf{C}$. The unit sphere $S(\mathbf{F}^n)$ is a homogeneous space of $U(n, \mathbf{F})$ and the isotropy group of $e_1 = (1, 0, \dots, 0)$ is*

$$1 \times U(n-1, \mathbf{F}) = \left\{ \begin{pmatrix} 1 & O \\ O & A \end{pmatrix} \mid A \in U(n-1, \mathbf{F}) \right\}:$$

$$U(n, \mathbf{F})/1 \times U(n-1, \mathbf{F}) = S(\mathbf{F}^n) \cong S^{d^n-1} \text{ (homeomorphism).}$$

If $n > 1$, the same is true with SU in place of U .

PROOF. It suffices to show that the map $\pi: U(n, \mathbf{F}) \rightarrow S(\mathbf{F}^n)$ defined by setting $\pi(A) = {}^t(A'e_1)$ is surjective and that $\pi^{-1}(e_1) = 1 \times U(n-1, \mathbf{F})$. Since the element $v_1 \in S(\mathbf{F}^n)$ satisfies $\|v_1\| = 1$, $\{v_1\}$ forms an orthonormal system. As we have observed before, we can construct an orthonormal basis $\{v_1, v_2, \dots, v_n\}$. If we put $A = ({}^t v_1, {}^t v_2, \dots, {}^t v_n)$, the map π is surjective, since $A^*A = I_n$ and $A'e_1 = {}^t v_1$. Conversely, if we express $A \in U(n, \mathbf{F})$ as $A = ({}^t v_1, {}^t v_2, \dots, {}^t v_n)$, then $\{v_1, \dots, v_n\}$ is an orthonormal basis. Observe that $v_1 = e_1$ if $\pi(A) = {}^t e_1$ and that $a_{i1} = (v_i, e_1) = 0$ if $v_i = (a_{i1}, \dots, a_{in})$ with $i > 1$. Hence $A \in 1 \times U(n-1, \mathbf{F})$. Clearly $\pi(1 \times U(n-1, \mathbf{F})) = {}^t e_1$, which implies $\pi^{-1}({}^t e_1) = 1 \times U(n-1, \mathbf{F})$. If $n > 1$, then one can change v_n so that $\det A = 1$. Q.E.D.

A subspace W of a metric vector space V is also a metric space with the same metric and the orthogonal complement $W^\perp$ of W in V is defined by

$$W^\perp = \{v \in V \mid (v, w) = 0, \forall w \in W\}.$$

THEOREM 2.11. *Let (V, μ) be a unitary (or orthogonal) representation of a topological group G .*

(1) *If W is G -invariant, so is $W^\perp$; and*

$$(V, \mu) = (W, \mu|_W) \oplus (W^\perp, \mu|_{W^\perp}).$$

(2) *If V is of finite dimension then (V, μ) is completely reducible.*

PROOF. Let $g \in G$ and $u \in W^\perp$. We have $g^{-1} \cdot w \in W$ for any $w \in W$ and so $(g \cdot u, w) = (g^{-1} g \cdot u, g^{-1} \cdot w) = (u, g^{-1} \cdot w) = 0$. Thus $g \cdot u \in W^\perp$, which implies (1): $W^\perp$ is G -invariant. Applying (1) repeatedly we obtain (2). Q.E.D.

Incidentally, the group

$$O(n, \mathbf{F}) = \{A \in M(n, \mathbf{F}) \mid A^t A = I_n\} \quad (\mathbf{F} = \mathbf{R}, \mathbf{C})$$

is a closed subgroup of $GL(n, \mathbf{F})$ and $O(n, \mathbf{R}) = O(n) = U(n, \mathbf{R})$. $O(n, \mathbf{C})$ is called a *complex orthogonal group*.

Symplectic groups. So far we have considered the cases $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. The case $\mathbf{F} = \mathbf{H}$, the field of quaternions, can be discussed similarly. However, as $\mathbf{H}$ is a noncommutative field, the map $f: \mathbf{H}^n \rightarrow \mathbf{H}^n$ defined by

$${}^t(f(v)) = A^t v, \quad v \in \mathbf{H}^n, \quad (*)$$

corresponding to a square matrix A of order n in $M(n, \mathbf{H})$ is linear with respect to the right action of $\mathbf{H}$, i.e., it is a homomorphism of the right $\mathbf{H}$ -module $V = \mathbf{H}^n$. We define

$$GL(V) = \{f: V \rightarrow V | f \text{ is a right } \mathbf{H}\text{-module isomorphism}\},$$

and call it the *general linear group* of V . Using a basis $B = \{b_1, \dots, b_n\}$ of V as a right $\mathbf{H}$ -module we define a map $\psi_B: GL(V) \rightarrow M(n, \mathbf{H})$ by $\psi_B(f) = (x_{ij})$, where $f(b_j) = \sum_i b_i x_{ij}$. We then have

2.12. $\psi_B: GL(V) \rightarrow M(n, \mathbf{H})$ is a product preserving monomorphism whose image is

$$GL(n, \mathbf{H}) = \{X \in M(n, \mathbf{H}) | X \text{ is regular} \\ \text{(i.e., there exists } X^{-1} \text{ such that } XX^{-1} = I_n)\}.$$

The group $GL(n, \mathbf{H})$ is called the *general linear group of order n* .

Since the equality $\bar{x}y = y\bar{x}$ holds for any $x, y \in \mathbf{H}$, we see that

$$A, B \in M(n, \mathbf{H}) \text{ implies } (AB)^* = B^* A^*,$$

and hence

$$Sp(n) = U(n, \mathbf{H}) = \{A \in M(n, \mathbf{H}) | AA^* = I_n\}$$

is a subgroup of $GL(n, \mathbf{H})$, which we call the *symplectic group of order n* . If we introduce the topology of $\mathbf{H}^{n^2} \cong \mathbf{R}^{4n^2}$ in $M(n, \mathbf{H})$ with respect to the coordinates and regard $GL(n, \mathbf{H})$ and $Sp(n)$ as its subspaces, they become topological groups.

Ordinarily the symplectic group $Sp(n)$ is defined as a subgroup of $U(2n) = U(2n, \mathbf{C})$. To clarify this we define an isomorphism c' by

$$c': \mathbf{H}^n \rightarrow \mathbf{C}^{2n} \quad c'(a_1 + \mathbf{j}b_1, \dots, a_n + \mathbf{j}b_n) = (a_1, \dots, a_n, b_1, \dots, b_n),$$

regarding $\mathbf{H}^n$ as a right $\mathbf{C}$ -module. Then if $f: \mathbf{H}^n \rightarrow \mathbf{H}^n$ corresponds under $(*)$ to $X \in M(n, \mathbf{H})$, we denote by $c'(X)$ the matrix corresponding to $c'fc'^{-1}: \mathbf{C}^{2n} \rightarrow \mathbf{C}^{2n}$ and thus define

$$c': M(n, \mathbf{H}) \rightarrow M(2n, \mathbf{C}),$$

which is given explicitly as follows. Express $X \in M(n, \mathbf{H})$ as $X = A + \mathbf{j}B$ with $A, B \in M(n, \mathbf{C})$. We have $\mathbf{j}B = \overline{B}\mathbf{j}$ and hence

$$X^t v = (A + \mathbf{j}B)^t (x + \mathbf{j}y) = (A^t x - \overline{B}^t y) + \mathbf{j}(\overline{A}^t y + B^t x),$$

for $v = x + \mathbf{j}y$ with $x, y \in \mathbf{C}^n$.

Therefore we have

$$c'(X) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in M(2n, \mathbf{C}) \quad (X = A + \mathbf{j}B). \quad (2.13)$$

The following properties of c' can be verified by an easy calculation:

(2.14). (1) c' is a monomorphism of algebras over $\mathbf{R}$ satisfying

$$c'(X^*) = c'(X)^*.$$

(2) $Y \in c'(M(n, \mathbf{H}))$ iff $J_n Y J_n^{-1} = \bar{Y} \in M(n, \mathbf{C})$, where $J_n = \begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}$.

(3) $X \in M(n, \mathbf{H})$,

$$\eta_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & -kI_n \end{pmatrix} \text{ implies } \eta_n (qc'(X)) \eta_n^{-1} = \begin{pmatrix} X & O \\ O & X \end{pmatrix}.$$

(4) $c'(GL(n, \mathbf{H})) = GL(2n, \mathbf{C}) \cap c'(M(n, \mathbf{H}))$.

Here the inclusion

$$q: M(m, \mathbf{C}) \rightarrow M(m, \mathbf{H}),$$

obtained by extending the coefficient field from $\mathbf{C}$ to $\mathbf{H}$, is a ring homomorphism and satisfies $q(X^*) = q(X)^*$.

Next, for $\mathbf{F} = \mathbf{R}, \mathbf{C}$ we consider

$$Sp(n, \mathbf{F}) = \{A \in M(2n, \mathbf{F}) \mid {}^t A J_n A = J_n\},$$

which is easily seen to be a closed subgroup of $GL(2n, \mathbf{F})$. In particular, we call $Sp(n, \mathbf{R})$ the *real symplectic group*, and $Sp(n, \mathbf{C})$ the *complex symplectic group*.

Then we have

$$c': Sp(n) \cong Sp(n, \mathbf{C}) \cap U(2n) \text{ (isomorphism)}. \quad (2.15)$$

In fact, as c' is a monomorphism, $X \in Sp(n)$ iff $XX^* = I_n$. This is equivalent to the condition that $c'(X)c'(X)^* = c'(XX^*) = I_{2n}$, which is equivalent to $c'(X) \in U(2n)$. Also, for $A \in U(2n)$, ${}^t A J_n A = J_n$ iff $J_n A J_n^{-1} = ({}^t A)^{-1} = ({}^t A)^* = \bar{A}_n$. This proves (2.15).

2.16. The subgroup $GL(\mathbf{F}^{2n})$ corresponding to $Sp(n, \mathbf{F}) \subset GL(2n, \mathbf{F})$ consists of all isomorphisms $f: \mathbf{F}^{2n} \rightarrow \mathbf{F}^{2n}$ such that

$$f(x_1, \dots, x_{2n}) = (x'_1, \dots, x'_{2n}) \quad \text{and} \quad f(y_1, \dots, y_{2n}) = (y'_1, \dots, y'_{2n})$$

satisfy

$$\sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i) = \sum_{i=1}^n (x'_i y'_{n+i} - x'_{n+i} y'_i).$$

Recall that the inner product in $\mathbf{C}^n$ and $\mathbf{R}^n$ is defined by $(u, v) = u \cdot v^* = \bar{v}^t u$. We define an inner product $(\cdot, \cdot): \mathbf{H}^n \times \mathbf{H}^n \rightarrow \mathbf{H}$ in $\mathbf{H}^n$ by

$$(u, v) = \bar{v}^t u.$$

One can easily show that

2.17. (1) $(v, v) = \|v\|^2 \geq 0$, $(v, v) = 0$ iff $v = 0$.

(2) $(v, u) = (\bar{u}, v)$, $(u + u', v) = (u, v) + (u', v)$.

(3) $(u\alpha, v) = (u, v)\alpha$, $(u, v\alpha) = \bar{\alpha}(u, v)$, $(\alpha u, v) = (u, \bar{\alpha}v)$ ($\alpha \in \mathbf{H}$).

Given an orthonormal system $\{e_1, \dots, e_r\}$ with respect to the inner product of $\mathbf{H}^n$, one can construct an orthonormal basis containing it. Therefore the proofs of Theorems 2.8 and 2.10 can be applied to the case $Sp(n) = U(n, \mathbf{H})$. Thus

THEOREM 2.18. *The topological group $Sp(n)$ is compact, acts effectively on $S(\mathbf{H}^n)$, and satisfies*

$$Sp(n)/1 \times Sp(n-1) \cong S(\mathbf{H}^n) \cong S^{4n-1} \quad (\text{isomorphism}).$$

The topological groups $GL(n, \mathbf{F})$, $U(n, \mathbf{F})$ ($\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$), and $SL(n, \mathbf{F})$, $SU(n, \mathbf{F})$, $O(n, \mathbf{F})$, $Sp(n, \mathbf{F})$ ($\mathbf{F} = \mathbf{R}, \mathbf{C}$), obtained in this section, form part of a family known collectively as the *classical groups*. In this book, by a classical group we mean one of the aforementioned groups.

§3. Diagonalization of normal matrices

Diagonal matrices and normal matrices. Throughout this section $\mathbf{F}$ is one of $\mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

NOTATION. $D(n, \mathbf{F}) = \{A = (a_{ij}) \in M(n, \mathbf{F}) | a_{ij} = 0 \ (i \neq j)\}$ denotes the set of *diagonal matrices*.

DEFINITION. An element $A \in M(n, \mathbf{F})$ is called *normal* if $AA^* = A^*A$.

Since $U(n, \mathbf{F}) = \{A \in M(n, \mathbf{F}) | AA^* = A^*A = I_n\}$, we obtain:

3.1. *Every element (matrix) of $D(n, \mathbf{F})$ and $U(n, \mathbf{F})$ is normal.*

As we always have the equality $(AB)^* = B^*A^*$ for any $A, B \in M(n, \mathbf{F})$, the element $UAU^{-1} = UAU^*$, which is a modification of A by $U \in U(n, \mathbf{F})$, satisfies $(UAU^{-1})^* = UA^*U^{-1}$. It follows easily from this that

3.2. *If $A \in M(n, \mathbf{F})$ is normal and $U \in U(n, \mathbf{F})$ then UAU^{-1} is normal. In particular, for $D \in D(n, \mathbf{F})$, and $U \in U(n, \mathbf{F})$, UDU^{-1} is normal.*

Conversely, a normal matrix $A \in M(n, \mathbf{F})$ is said to be *diagonalizable*, if there is $U \in U(n, \mathbf{F})$ such that $U^{-1}AU \in D(n, \mathbf{F})$. We will consider whether or not a normal matrix is diagonalizable for each case $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

When we express a diagonal matrix $D \in D(n, \mathbf{F})$ as

$$D = \lambda_1 \oplus \dots \oplus \lambda_n = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

the set $\{\lambda_1, \dots, \lambda_n\}$ is called the set of *diagonal elements* of D .

Diagonalization of normal matrices over $\mathbf{C}$

DEFINITION. Let $A \in M(n, \mathbf{C})$. A root $\lambda \in \mathbf{C}$ of the equation of order n $\det(xI_n - A) = 0$ is called a *characteristic value* of A . In the set of

characteristic values $\{\lambda_1, \dots, \lambda_n\}$ we count repeated roots with multiplicity. For a characteristic value λ of A , the subspace

$$V_\lambda = \{v \in \mathbb{C}^n | A^t v = \lambda^t v\},$$

of $\mathbb{C}^n$ is called the *eigenspace* of λ and a nonzero vector of V_λ is called a *characteristic vector* of A corresponding to λ .

There always exists a (characteristic) vector $v \neq 0$ satisfying $(\lambda I_n - A)^t v = 0$, since λ is a root of $\det(xI_n - A) = 0$ and so $V_\lambda \neq 0$.

THEOREM 3.3. *If $A \in M(n, \mathbb{C})$ is normal, then there exists $U \in U(n) = U(n, \mathbb{C})$ such that $U^{-1}AU \in D(n, \mathbb{C})$, and the set of diagonal elements is the set of characteristic values of A . Moreover, given an ordering on the $\{\lambda_1, \dots, \lambda_n\}$ of characteristic roots of A , one can choose U such that $U^{-1}AU = \lambda_1 \oplus \dots \oplus \lambda_n$.*

PROOF. Let λ be a characteristic value of A and consider the eigenspace $V_\lambda \neq 0$. If $V_\lambda = \mathbb{C}^n$, then $A = \lambda I_n$ and the theorem is obvious. Suppose $\dim_{\mathbb{C}} V_\lambda = r < n$ and consider the orthogonal complement $V_\lambda^\perp$ of V_λ . The linear transformation $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ corresponding to A is given by $f(v) = {}^t(A^t v) = v^t A$. Since $v \in V_\lambda$ iff $f(v) = \lambda v \in V_\lambda$, we have

$$f(v\bar{A}) = v\bar{A}^t A = v^t (A\bar{A}^*) = v^t (A^* A) = v^t A\bar{A} = f(v)\bar{A} = \lambda(v\bar{A}).$$

So $v \in V_\lambda$ implies $v\bar{A} \in V_\lambda$. Besides, for $w \in V_\lambda^\perp$ one has

$$f(w, v) = f(w)v^* = w^t A v^* = w(v\bar{A})^* = (w, v\bar{A}) = 0.$$

Thus, $w \in V_\lambda^\perp$ implies $f(w) \in V_\lambda^\perp$.

Now, if we choose an orthonormal basis $\{u_1, \dots, u_n\}$ of $\mathbb{C}^n$ such that $\{u_1, \dots, u_r\}$ is an orthonormal basis of V_λ , then $\{u_{r+1}, \dots, u_n\}$ is an orthonormal basis of $V_\lambda^\perp$ and hence, there is $B = (b_{jk}) \in M(n-r, \mathbb{C})$ such that $A^t u_i = \lambda^t u_i$ ($1 \leq i \leq r$) and $A^t u_j = \sum_{k=r+1}^n b_{jk} {}^t u_k$ ($r < j \leq n$). Therefore, putting $U_0 = ({}^t u_1, \dots, {}^t u_n) \in U(n)$, one has

$$AU_0 = U_0(\lambda I_r \oplus B).$$

It is easy to see that if A is normal then $U_0^{-1}AU_0 = \lambda I_r \oplus B$ is normal and hence B is normal.

By induction on n we can conclude that B is diagonalizable by $U_1 \in U(n-r)$, since the theorem is obvious for $n = 1$. Hence A is diagonalizable by $U = U_0(U_1 \oplus U_1)$.

Since $\det(xI_n - U^{-1}AU) = \det(xI_n - A)$, the characteristic values of A coincide with those of $U^{-1}AU$ and the set of these values with that of the diagonal matrix $U^{-1}AU$.

Let τ be a permutation of n letters $\{1, \dots, n\}$. For a permutation matrix $P_\tau = (\delta_{i\tau(j)})$ we have

$$P_\tau^{-1}(\lambda_1 \oplus \dots \oplus \lambda_n)P_\tau = \lambda_{\tau(1)} \oplus \dots \oplus \lambda_{\tau(n)}.$$

Since $P_\tau \in U(n)$, we can diagonalize a normal matrix so that the diagonal elements are arranged in a given order. Q.E.D.

COROLLARY 3.4. *Under the above assumption, if $\{\lambda_1, \dots, \lambda_s\}$, $s \leq n$, are the distinct characteristic values, then C^n decomposes as a direct sum of mutually orthogonal V_{λ_i} .*

Normal matrices over $\mathbf{R}$ and $\mathbf{H}$. The characteristic values of a matrix A of $M(n, \mathbf{R})$ are defined to be that of A considered as a matrix over $\mathbf{C}$ by the inclusion (complexification) $c: M(n, \mathbf{R}) \rightarrow M(n, \mathbf{C})$. If λ is a characteristic value of $A \in M(n, \mathbf{R})$, then so is its conjugate $\bar{\lambda}$. By 2.4(1) if an element $X \in M(n, \mathbf{C})$ is normal then $r(X) \in M(2n, \mathbf{R})$ is normal.

In particular, $r(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M(2, \mathbf{R})$ is normal. It is not diagonalizable by elements of $O(2)$ if $b \neq 0$.

THEOREM 3.5. *If $A \in M(n, \mathbf{R})$ is normal, then there is $T \in O(n)_\lambda = U(n, \mathbf{R})$ such that $T^{-1}AT = \lambda_1 \oplus \dots \oplus \lambda_{n-2m} \oplus r(\alpha_1) \oplus \dots \oplus r(\alpha_m)$ ($\lambda_i \in \mathbf{R}$, $\alpha_j \in \mathbf{C}$). Here (the set of the characteristic values) $= \{\lambda_i, \alpha_j, \bar{\alpha}_j\}$, and we can choose T such that $T^{-1}AT$ coincides with a direct sum, in any given order, of elements λ_i 's and matrices each taken from a conjugate pair $\{r(\alpha_j), r(\bar{\alpha}_j)\}$.*

PROOF. When A has real characteristic values, one can use an induction argument similar to the proof of Theorem 3.3. So we suppose that a characteristic value α of A is not real. Observe that $\bar{\alpha}$ is then also a characteristic value. If we consider A as a matrix in $M(n, \mathbf{C})$, by Corollary 3.4, $\alpha \neq \bar{\alpha}$ implies that V_α is orthogonal to $V_{\bar{\alpha}}$. Since $u \in V_\alpha$ iff $\bar{u} \in V_{\bar{\alpha}}$, for an orthonormal basis $\{u_i\}$ of V_α one has an orthonormal basis $\{\bar{u}_i\}$ of $V_{\bar{\alpha}}$. Thus one can choose an orthonormal basis $\{u_1, \bar{u}_1, \dots, u_m, \bar{u}_m\}$ of C^n ($n = 2m$) such that

$$A^t u_j = \alpha_j^t u_j, \quad A^t \bar{u}_j = \bar{\alpha}_j^t \bar{u}_j.$$

Putting

$$t_{2j-1} = \frac{1}{\sqrt{2}}(u_j + \bar{u}_j), \quad t_{2j} = \frac{i}{\sqrt{2}}(u_j - \bar{u}_j),$$

we obtain a set $\{t_1, t_2, \dots, t_{2m}\}$ that is an orthonormal basis of C^m satisfying $t_i = \bar{t}_i$, hence $t_i \in \mathbf{R}^n$ and we have an orthonormal basis of $\mathbf{R}^n$, implying $T = ({}^t t_1, \dots, {}^t t_{2m}) \in O(n)$.

It is now easy to verify that $T^{-1}AT = r(\alpha_1) \oplus \dots \oplus r(\alpha_m)$. Thus the first half of the theorem is proved. The proof of the second half is similar to that of Theorem 3.3, observing

$$r(\bar{\alpha}) = {}^t r(\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} r(\alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Q.E.D.

For the case $\mathbf{F} = \mathbf{H}$ one can diagonalize a normal matrix so that the diagonal elements belong to $\mathbf{C}$.

THEOREM 3.6. *For a normal $A \in M(n, H)$, there is an $X \in Sp(n) = U(n, H)$ such that $X^{-1}AX = \lambda_1 \oplus \cdots \oplus \lambda_n$ ($\lambda_i \in \mathbb{C}$), where $\{\lambda_1, \bar{\lambda}_1, \dots, \lambda_n, \bar{\lambda}_n\}$ = (the set of all characteristic values of $c'(A) \in M(2n, \mathbb{C})$). Moreover one can choose X so that $X^{-1}AX$ coincides with a direct sum in any given order of elements, each taken from a conjugate pair $\{\lambda_i, \bar{\lambda}_i\}$ of such characteristic values.*

PROOF. First notice that $c': H^n \rightarrow \mathbb{C}^{2n}$ is an isomorphism of right $\mathbb{C}$ -modules and that the monomorphism $c': M(n, H) \rightarrow M(2n, \mathbb{C})$ is given by $c'(A) \cdot {}^t c'(v) = c'(A \cdot {}^t v)$. By 2.14(1), if A is normal then $c'(A)$ is normal. For any characteristic value λ of $c'(A)$ and any $v \in H^n$,

$$c'(v) \in V_\lambda \quad \text{iff} \quad c'(A \cdot {}^t v) = c'({}^t v)\lambda.$$

This is equivalent to $A \cdot {}^t v = {}^t v\lambda$ and to $A \cdot {}^t(vj) = {}^t(vj)\bar{\lambda}$ and so, putting $V'_\lambda = \{v \in H^n | A \cdot {}^t v = {}^t v\lambda\}$, we obtain

$$c': V'_\lambda \cong V_\lambda, \quad v \in V'_\lambda \quad \text{iff} \quad vj \in V_{\bar{\lambda}}.$$

Next, we choose an orthonormal basis $\{u_1, \dots, u_{2n}\}$ of $\mathbb{C}^{2n}$ that satisfies

$$c'(A) \cdot {}^t u_i = {}^t u_i \lambda_i, \quad c'(A) \cdot {}^t u_{n+i} = {}^t u_{n+i} \bar{\lambda}_i,$$

$$J_n \cdot {}^t u_i = {}^t \bar{u}_{n+i}, \quad (1 \leq i \leq n)$$

for $J_n \in M(2n, \mathbb{C})$ given in 2.14(2), using the fact that

$$J_n \cdot {}^t c'(v) = {}^t c'(vj), \quad v \in H^n.$$

In fact, let us consider each V_λ decomposing $\mathbb{C}^{2n}$ by Corollary 3.4. If λ is not a real number, then V_λ is orthogonal to $V_{\bar{\lambda}}$ and hence an orthonormal basis $\{u_i\}$ of V_λ gives an orthonormal basis $\{u_{n+i}\}$ of $V_{\bar{\lambda}}$ satisfying ${}^t u_{n+i} = \overline{{}^t u_i}$. If λ is a real number, $V'_\lambda = V_{\bar{\lambda}}$ is a right H -module. Take an orthonormal basis $\{v_i\}$ of it and put $u_i = c'(v_i)$ and $u_{n+i} = c'({}^t v_i j)$ ($= (J_n \cdot {}^t u_i)^*$). It follows from the relation below, between $c': H^n \rightarrow \mathbb{C}^{2n}$ and the inner product, that $\{u_i, u_{n+i}\}$ forms an orthonormal basis of V_λ ($\lambda \in \mathbb{R}$). For any $u, v \in H^n$

$$(u, v) = (c'(u), c'(v)) + j\alpha \quad (\alpha \in \mathbb{C}), \quad (c'(u), c'(uj)) = 0.$$

Putting $U = \{{}^t u_1, \dots, {}^t u_{2n}\} \in U(2n, \mathbb{C})$, we have

$$c'(A)U = U(\lambda_1 \oplus \cdots \oplus \lambda_n \oplus \bar{\lambda}_1 \oplus \cdots \oplus \bar{\lambda}_n),$$

$$J_n U = ({}^t \bar{u}_{n+1}, \dots, {}^t \bar{u}_{2n}, -{}^t \bar{u}_1, \dots, -{}^t \bar{u}_n) = ({}^t \bar{u}_1, \dots, {}^t \bar{u}_{2n}) J_n = \bar{U} J_n.$$

Since by (2.15) there is $X \in Sp(n)$ satisfying $c'(X) = U$, we have

$$c'(X^{-1}AX) = U^{-1}c'(A)U = \lambda_1 \oplus \cdots \oplus \lambda_n \oplus \bar{\lambda}_1 \oplus \cdots \oplus \bar{\lambda}_n = c'(\lambda_1 \oplus \cdots \oplus \lambda_n),$$

whence $X^{-1}AX = \lambda_1 \oplus \cdots \oplus \lambda_n$, as c' is a monomorphism. The latter half is easy, once we observe that $j^{-1}\lambda j = \bar{\lambda}$ ($\lambda \in \mathbb{C}$). Q.E.D.

COROLLARY 3.7. *The monomorphism c' induces $Sp(n) \cong Sp(n, \mathbb{C}) \cap SU(2n)$ (isomorphism). In particular $c': Sp(1) \cong SU(2)$ (isomorphism).*

PROOF. In the proof of the theorem, $\det(c'(A)) = \det(U^{-1}c'(A)U)$, $U^{-1}c'(A)U = \lambda_1 \oplus \dots \oplus \lambda_n \oplus \bar{\lambda}_1 \oplus \dots \oplus \bar{\lambda}_n \in U(2n)$, which implies $|\lambda_i| = 1$ and hence $\det(c'(A)) = \prod |\lambda_i|^2 = 1$ so $c'(A) \in SU(2n)$. By (2.15), $c'(Sp(n)) = Sp(n, \mathbb{C}) \cap SU(2n)$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = J_1^{-1} A J_1$ and so ${}^t A J_1 A = J_1$ implies $A \in Sp(1, \mathbb{C})$. Thus $c'(Sp(1)) = Sp(1, \mathbb{C}) \cap SU(2) = SU(2)$. Q.E.D.

Compact, connected classical groups and their Weyl groups. The general theory of compact, connected Lie groups will be considered in Chapter V. Here we will study each of them in concrete terms. Among the compact, connected classical groups listed in Theorems 2.8 and 2.18, only $O(n)$ is not connected since the continuous map $\det: O(n) \rightarrow S(\mathbb{R}) = \{1, -1\}$ is surjective. So we consider the remaining ones.

Each of $G = U(n)$, $SU(n)$, $SO(2n)$, $SO(2n+1)$, $Sp(n)$ has a subgroup T^n called the *standard maximal torus* of G , defined as follows:

$$\begin{aligned} \text{For } G = U(n), T^n &= \{\lambda_1 \oplus \dots \oplus \lambda_n | \lambda_i \in U(1) = S(\mathbb{C}) \subset \mathbb{C}\}; \\ \text{for } G = SU(n+1), T^n &= \left\{ \lambda_1 \oplus \dots \oplus \lambda_{n+1} | \lambda_i \in U(1), \prod \lambda_i = 1 \right\} \quad (3.8) \\ &= (T^{n+1} \cap SU(n+1)). \end{aligned}$$

$$\begin{aligned} \text{For } G = SO(2n), T^n &= \{\mathbf{r}(\lambda_1) \oplus \dots \oplus \mathbf{r}(\lambda_n) | \lambda_i \in U(1)\}; \\ \text{for } G = SO(2n+1), T^n &= \{\mathbf{r}(\lambda_1) \oplus \dots \oplus \mathbf{r}(\lambda_n) \oplus 1 | \lambda_i \in U(1)\} \quad (3.9) \\ &= (T^n \times 1). \end{aligned}$$

$$\text{For } G = Sp(n), T^n = \{\lambda_1 \oplus \dots \oplus \lambda_n | \lambda_i \in \mathfrak{q}(U(1)) \subset \mathbb{H}\}. \quad (3.10)$$

The group $T = U(1)$ is homeomorphic to the circle $S^1 \cong S(\mathbb{C})$ and each T^n above is isomorphic to a direct product of n copies of T . Thus T^n is an n -dimensional torus (in the topological sense). Also T^n is an (arcwise) connected, compact, connected, abelian group.

THEOREM 3.11. *Let $G = U(n)$, $SU(n+1)$, $SO(2n)$, $SO(2n+1)$, or $Sp(n)$, then for each standard maximal torus T^n given above, we have*

$$G = \bigcup_{g \in G} g T^n g^{-1}.$$

PROOF. Any element (matrix) A of G is normal by 3.1. If $G = U(n)$ or $Sp(n)$, then by Theorems 3.4 and 3.6 there is $g \in G$ such that

$$g^{-1} A g = \lambda_1 \oplus \dots \oplus \lambda_n \quad (\lambda_i \in \mathbb{C}).$$

Also $g^{-1} A g \in G$ implies that every λ_i belongs to $U(1)$. Therefore, $g^{-1} A g \in T^n$ and hence $A \in g T^n g^{-1}$, which implies $G = \bigcup g T^n g^{-1}$, since A is arbitrary.

When $G = SU(n+1)$, for each $A \in U(n+1)$, there is $U \in U(n+1)$ such that

$$U^{-1}AU = \lambda_1 \oplus \cdots \oplus \lambda_{n+1},$$

where $\prod \lambda_i = \det(U^{-1}AU) = \det(A) = 1$. Since $|\det U| = 1$, we choose $c \in \mathbb{C}$ such that $\det U = c^{n+1}$ and put $g = c^{-1}U$. Then $g \in SU(n+1)$ and $g^{-1}Ag = \lambda_1 \oplus \cdots \oplus \lambda_{n+1} \in T^n$.

Let $m = 2n, 2n+1$ and consider $A \in SO(m)$. By Theorem 3.5 there is $g \in O(m)$ such that

$$B = g^{-1}Ag = \mathbf{r}(\lambda_1) \oplus \cdots \oplus \mathbf{r}(\lambda_k) \oplus t_1 \oplus \cdots \oplus t_{m-2k} \quad (\lambda_i \in \mathbb{C}, t_j \in \mathbb{R}).$$

Since B belongs to $SO(m)$, $B^t B = I_m$ and $\det B = 1$. $B^t B = I_m$ implies $\mathbf{r}(\lambda_i) \in O(2)$, $t_j \in O(1)$ and hence $\lambda_i \in U(1)$, $t_j = \pm 1$. $\det B = \det(\mathbf{r}(\lambda_i)) = 1$ implies that the number of t_j with $t_j = -1$ is even. Finally, it follows that we can choose g such that $B \in T^n$, since we can change the order of the t_j 's and since $\mathbf{r}(1) = 1 \oplus 1$, $\mathbf{r}(-1) = (-1) \oplus (-1)$ hold. When $\det g = -1$, replacing g by $((-1) \oplus I_{m-1})g \in SO(m)$, we again have $g^{-1}Ag \in T^n$ by the equality $((-1) \oplus 1)^{-1} \mathbf{r}(\lambda_i)((-1) \oplus 1) = \mathbf{r}(\bar{\lambda}_i)$. Q.E.D.

COROLLARY 3.12. *The spaces $U(n)$, $SO(n)$, and $Sp(n)$ are arcwise connected.*

PROOF. Since each subgroup gT^ng^{-1} is homeomorphic to T^n , it is arcwise connected, and since all of them contain the identity element, their union $G = \bigcup gT^ng^{-1}$ is also arcwise connected. Q.E.D.

DEFINITION. The quotient group

$$W(G) = N(T^n)/T^n$$

of the normalizer $N(T^n)$ of T^n in each G of Theorem 3.11 is called the *Weyl group* of G .

The Weyl group $W(G)$ of G acts on the standard maximal torus T^n of G , $\mu: W(G) \times T^n \rightarrow T^n$, as follows:

$$\mu(\varphi, t) = ntn^{-1} \quad (\varphi = nT^n \in W(G), t \in T^n).$$

The action is well defined, independently of the choice of the representative $n \in N(T^n)$ of $\varphi \in W(G)$.

THEOREM 3.13. *The action of $W(G)$ on T^n is effective. Thus one can regard an element of $W(G)$ as a topological transformation of T^n . Viewed in this way for any G , $W(G)$ is a finite group consisting of the following transformations, where we express a point t of T^n as a direct sum of elements (coordinates) λ_i of $U(1)$, $\mathbf{r}(U(1))$, or $\mathbf{q}(U(1))$.*

For $G = U(n)$, $W(G)$ is the group of permutations of the coordinates in T^n

$$\sigma: \lambda_1 \oplus \cdots \oplus \lambda_n \rightarrow \lambda_{\sigma(1)} \oplus \cdots \oplus \lambda_{\sigma(n)}.$$

For $G = SU(n+1)$, $W(G)$ is the group of permutations of the coordinates in T^n

$$\sigma: \lambda_1 \oplus \cdots \oplus \lambda_{n+1} \rightarrow \lambda_{\sigma(1)} \oplus \cdots \oplus \lambda_{\sigma(n+1)}.$$

For $G = SO(2n)$, $W(G)$ is the group of compositions of permutations and of an even number of changes of sign of the coordinates in T^n

$$\sigma: \lambda_1 \oplus \cdots \oplus \lambda_n \rightarrow \lambda_{\sigma(1)}^{\varepsilon(1)} \oplus \cdots \oplus \lambda_{\sigma(n)}^{\varepsilon(n)}, \quad \varepsilon(i) = \pm 1, \quad \prod_{i=1}^n \varepsilon(i) = 1.$$

For $G = SO(2n+1)$, $W(G)$ is the group of compositions of permutations and of changes of sign of the coordinates in T^n

$$\sigma: \lambda_1 \oplus \cdots \oplus \lambda_n \rightarrow \lambda_{\sigma(1)}^{\varepsilon(1)} \oplus \cdots \oplus \lambda_{\sigma(n)}^{\varepsilon(n)}, \quad \varepsilon(i) = \pm 1.$$

For $G = Sp(n)$, $W(G)$ is the same as for $G = SO(2n+1)$.

PROOF. The case $G = U(n)$. Let $A = (a_{ij}) \in N(T^n) \subset U(n)$ be a representative of $\varphi \in W(U(n))$. Then for arbitrary $t \in T^n$ there must exist $t' \in T^n$ such that $At = t'A$. Putting $t = \lambda_1 \oplus \cdots \oplus \lambda_n$, $t' = \lambda'_1 \oplus \cdots \oplus \lambda'_n$, this condition is equivalent to

$$a_{ij}\lambda_j = \lambda'_i a_{ij} \quad (i, j = 1, 2, \dots, n).$$

Since A is regular, for each i there is some j such that $a_{ij} \neq 0$ and then $\lambda_j = \lambda'_i$. Since t is arbitrary, we may choose λ_j so that they are distinct. Then, for each j , there is only one i for which $\lambda_j = \lambda'_i$. So there is an injection from $\{1, 2, \dots, n\}$ to itself, i.e., a permutation σ such that $\lambda'_i = \lambda_{\sigma(i)}$. Thus the action of φ on T^n is a permutation, and conversely any arbitrary permutation σ is induced by $\varphi = \{P_\sigma\}$, where $P_\sigma = (\delta_{i\sigma(i)}) \in N(T^n)$. Therefore it only remains to show that the action is effective.

If the action of $\varphi = \{A\}$ is trivial then $tA = At$ for any $t \in T^n$. By the above argument we have $\lambda_i = \lambda'_i$, and $i \neq j$ implies $a_{ij} = 0$; that is, $\varphi = e$ for $A \in D(n, \mathbb{C}) \cap U(n) = T^n$. Thus the action of $W(U(n))$ is effective.

The case $SU(n+1)$. The above argument applies. The only difference is that we take the class of $(\text{sign } \sigma \oplus I_n)P_\sigma$ as φ inducing the permutation σ .

The case $G = Sp(n)$. In order to make use of the proof of the case $U(n)$, we need the following:

3.14. For $\lambda, \lambda' \in \mathfrak{q}(U(1)) \subset Sp(1)$ and $q \in \mathbb{H}$, one has

$$\begin{aligned} q\lambda = \lambda'q, q \neq 0 \text{ implies } \lambda' = \lambda \text{ or } \lambda' = \bar{\lambda} (= \lambda^{-1}), \\ q\lambda = \lambda q \text{ implies } q \in \mathbb{C} \text{ or } \lambda = \pm 1. \end{aligned}$$

This can either be shown directly by a calculation with quaternions or deduced from the proof of the case $G = SU(2)$, using the monomorphism $c': \mathbb{H} = M(1, \mathbb{H}) \rightarrow M(2, \mathbb{C})$ and $c'(\mathfrak{q}(\alpha)) = \alpha \oplus \bar{\alpha}$.

If we choose t so that the pairs $\{\lambda_j, \bar{\lambda}_j\}$ are all distinct, then using 3.14, the proof of the theorem is similar to the case $G = U(n)$, where we act with j to change the signs of the coordinates: $j\lambda j^{-1} = \bar{\lambda}$.

The cases $G = SO(2n)$ and $SO(2n+1)$. We have the following analogue of 3.14:

3.15. For $\lambda, \lambda' \in \mathfrak{r}(U(1)) \subset SO(2)$ and $B \in M(2, \mathbf{R})$, one has

$$B\lambda = \lambda'B, B \neq 0 \text{ implies } \lambda' = \lambda \text{ or } \lambda' = \lambda^{-1},$$

$$B\lambda = \lambda B \text{ implies } B = tB' \text{ for } t \in \mathbf{R}, B' \in SO(2), \text{ or } \lambda = \pm I_2.$$

This can also be shown either by a direct calculation with matrices of order 2 or by means of the monomorphism $c: M(2, \mathbf{R}) \rightarrow M(2, \mathbf{C})$ and $c(\mathfrak{r}(\alpha)) = \alpha \oplus \bar{\alpha}$. In the case $G = SO(2n)$, we divide $A \in SO(2n) \subset M(2n, \mathbf{R})$ into 2 by 2 blocks and write $A = (A_{ij})$ with $A_{ij} \in M(2, \mathbf{R})$. Similarly to the above, $At = t'A$ iff $A_{ij}\lambda_j = \lambda'_i A_{ij}$. Now, noticing that $P_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, corresponding to a transposition τ , satisfies $P_\tau \lambda P_\tau = \lambda^{-1}$, we see that the action of $W(SO(2n))$ is effective, that the action is similar to that of $W(Sp(n))$ if one is permitted to take A in $O(2n)$, and that one can take $A = (P_\tau^{e(1)} \oplus \dots \oplus P_\tau^{e(n)})P_{\sigma'}$ as a representative of each $\varphi = \{A\}$, where σ' is an even permutation given by $\sigma'(2i-1) = 2\sigma(i) - 1$, $\sigma'(2i) = 2\sigma(i)$, and $e(i) = 0, 1$. Since $\det A = \prod (-1)^{e(i)}$, $A \in SO(2n)$ iff $\prod e(i) = 1$. Thus the case $G = SO(2n)$ is complete.

For the case of $G = SO(2n+1)$, it suffices to replace A, t, t' , etc., by $A \oplus 1, t \oplus 1, t' \oplus 1 \in SO(2n+1)$ respectively. Noticing that $t \oplus 1$ is of the form $t \oplus 1 = \lambda_1 \oplus \dots \oplus \lambda_n \oplus \mathfrak{r}(1)$, every element of $W(SO(2n+1))$ can be expressed as $\{(P_\tau^{e(1)} \oplus \dots \oplus P_\tau^{e(n)} \oplus \prod e(i))(P_\sigma \oplus 1)\}$ and we see that the theorem is also true for this case. Q.E.D.

Let $\Sigma_n = \{\sigma: \text{permutation of } n \text{ letters}\}$ be the symmetric group, $\Lambda_n = \{\sigma: \sigma \text{ is an even permutation of } n \text{ letters}\}$ the alternating group and H_n the subgroup of Σ_{2n} given by the following ($H_n = \Sigma_n \wr \Sigma_2$: the wreath product):

$$\sigma \in H_n \text{ iff there is } \rho \in \Sigma_n \text{ such that } \sigma\{2i-1, 2i\} = \{2\rho(i)-1, 2\rho(i)\}.$$

COROLLARY 3.16. $W(U(n)) \cong \Sigma_n$, $W(SU(n+1)) \cong \Sigma_{n+1}$, $W(Sp(n)) \cong H_n$,

$$W(SO(2n)) \cong H_n \cap \Lambda_{2n}, \quad W(SO(2n+1)) \cong H_n \quad (\text{isomorphisms}).$$

§4. The exponential map of matrices and the polar decomposition of classical groups

The exponential map of matrices. For $A = (a_{ij}) \in M(n, \mathbf{F})$ ($\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) we consider the series of matrices

$$I_n + A + A^2/2! + \dots + A^k/k! + \dots$$

If $\text{Max } |a_{ij}| \leq r$, then $|a_{ij}^{(k)}| < (nr)^k$ for $A^k = (a_{ij}^{(k)})$ and hence, the series consisting of the absolute value of each element in the above series is bounded by $\Sigma (nr)^k/k! = e^{nr}$. As a consequence we obtain 4.1.

4.1. The map $\exp: M(n, \mathbf{F}) \rightarrow M(n, \mathbf{F})$ defined by $\exp A = \sum_{k=0}^{\infty} A^k/k!$ is an analytic map, regarding $M(n, \mathbf{F}) = \mathbf{R}^{dn^2}$ (or $M(n, \mathbf{C}) = \mathbf{C}^{n^2}$).

The map $\exp$ is called the *exponential map* of matrices and $\exp A$ the *exponential matrix* of A .

The following properties are easily verified:

4.2. (1) $AB = BA$ implies $\exp(A + B) = \exp A \cdot \exp B$. In particular, $\exp(-A) = (\exp A)^{-1}$;

(2) $\exp(M(n, \mathbf{F})) \subset GL(n, \mathbf{F})$;

(3) For $P \in GL(n, \mathbf{F})$, $\exp(PAP^{-1}) = P(\exp A)P^{-1}$;

(4) $\exp(A^*) = (\exp A)^*$;

(5) For $\mathbf{F} = \mathbf{R}, \mathbf{C}$,

$$\exp({}^t A) = {}^t(\exp A), \quad \exp(\overline{A}) = \overline{\exp A}, \quad \det(\exp A) = e^{\text{tr}(A)}.$$

If the set of characteristic values of A is $\{\lambda_1, \dots, \lambda_n\}$, then the set of characteristic values of $\exp A$ is $\{e^{\lambda_1}, \dots, e^{\lambda_n}\}$.

THEOREM 4.3. The exponential map $\exp$ is a homeomorphism of a neighborhood of the zero matrix O to a neighborhood of the identity element I_n (i.e., $\exp$ is a local homeomorphism at O).

PROOF. Under the identification $M(n, \mathbf{F}) = \mathbf{R}^{dn^2}$, the Jacobian matrix of $\exp$ at O is I_{dn^2} . The theorem follows from the implicit function theorem. Q.E.D.

COROLLARY 4.4. For each classical group, $\exp: M(n, \mathbf{F}) \rightarrow GL(n, \mathbf{F})$ induces a local homeomorphism of the following subspaces (over $\mathbf{R}$):

$$\{A \in M(n, \mathbf{F}) | A^* = -A\} \rightarrow U(n, \mathbf{F}),$$

$$\{A \in M(n, \mathbf{F}) | \text{tr}(A) = 0\} \rightarrow SL(n, \mathbf{F}) \quad (\mathbf{F} = \mathbf{R}, \mathbf{C}),$$

$$\{A \in M(n, \mathbf{F}) | A^* = -A, \text{tr}(A) = 0\} \rightarrow SU(n, \mathbf{F}) \quad (\mathbf{F} = \mathbf{R}, \mathbf{C}),$$

$$\{A \in M(n, \mathbf{F}) | {}^t A = -A\} \rightarrow O(n, \mathbf{F}) \quad (\mathbf{F} = \mathbf{R}, \mathbf{C}),$$

$$\{A \in M(2n, \mathbf{F}) | J_n A J^{-1} = -{}^t A\} \rightarrow Sp(n, \mathbf{F}) \quad (\mathbf{F} = \mathbf{R}, \mathbf{C}).$$

PROOF. By 4.2 each condition in the above corresponds to the condition defining a classical group. Q.E.D.

THEOREM 4.5. The maps in Corollary 4.4 are surjective for $U(n) = U(n, \mathbf{C})$, $Sp(n) = U(n, \mathbf{H})$, $SO(n) = SU(n, \mathbf{R})$, $SU(n) = SU(n, \mathbf{C})$.

PROOF. The map $\exp: L \rightarrow T^n$ is surjective for $T^n \subset U(n)$ and $L = \{a_1 i \oplus \dots \oplus a_n i | a_i \in \mathbf{R}\}$. Since $gLg^{-1} \subset \{A \in M(n, \mathbf{C}) | A^* = -A\}$ for $G \in U(n)$, the map in Corollary 4.4 is surjective to $U(n)$ by 4.2(3) and Theorem 3.11. The rest is similar. Q.E.D.

Hermitian matrices

DEFINITION. We call a matrix in $H(n, \mathbf{F}) = \{A \in M(n, \mathbf{F}) | A^* = A\}$ a *Hermitian matrix*.

The following are obvious:

4.6. (1) If $A \in H(n, \mathbf{F})$ then A is normal.

(2) If $A \in H(n, \mathbf{F})$, and $U \in U(n, \mathbf{F})$ then $UAU^{-1} \in H(n, \mathbf{F})$.

(3) $H(n, \mathbf{F}) \cap D(n, \mathbf{F}) = D(n, \mathbf{R})$.

LEMMA 4.7. For any $A \in H(n, \mathbf{F})$ there exists $U \in U(n, \mathbf{F})$ such that $UAU^{-1} \in D(n, \mathbf{R})$ and the diagonal elements (which are real numbers) of UAU^{-1} are determined uniquely up to order.

PROOF. The cases $\mathbf{F} = \mathbf{C}, \mathbf{H}$ are clear from Theorems 3.3 and 3.6 and 4.6(3). Also for the case $\mathbf{F} = \mathbf{R}$, one has $T^{-1}AT \in H(n, \mathbf{F})$ in Theorem 3.5 and hence $r(\lambda_i)^* = f(\lambda_i)$, which implies $\bar{\lambda}_i = \lambda_i$ and so $\lambda_i \in \mathbf{R}$, hence $T^{-1}AT \in D(n, \mathbf{R})$. Q.E.D.

DEFINITION. The diagonal elements of UAU^{-1} in Theorem 4.7 are called the *characteristic values* of the Hermitian matrix A . A Hermitian matrix is said to be *positive definite* if all its characteristic values are positive.

NOTATION. $H^+(n, \mathbf{F}) = \{A \in H(n, \mathbf{F}) | A \text{ is positive definite}\}$.

LEMMA 4.8. $\exp: H(n, \mathbf{F})$ is an injection of $H(n, \mathbf{F})$ into $H^+(n, \mathbf{F})$.

PROOF. If, for a given $A \in H(n, \mathbf{F})$, we have $U \in U(n, \mathbf{F})$ such that $UAU^{-1} = \lambda_1 \oplus \cdots \oplus \lambda_n$, $\lambda_i \in \mathbf{R}$, then $U(\exp A)U^{-1} = \exp(UAU^{-1}) = e^{\lambda_1} \oplus \cdots \oplus e^{\lambda_n}$, $e^{\lambda_i} > 0$, and hence $\exp A \in H^+(n, \mathbf{F})$. Making use of a permutation matrix we choose U so that

$$UAU^{-1} = a_1 I_{n_1} \oplus \cdots \oplus a_r I_{n_r}, \quad a_1 < a_2 < \cdots < a_r, \quad n_1 + \cdots + n_r = n.$$

Now suppose $\exp A = \exp B$, $B \in H(n, \mathbf{F})$, we choose U_0 so that

$$U_0 B U_0^{-1} = b_1 I_{m_1} \oplus \cdots \oplus b_s I_{m_s}, \quad b_1 < b_2 < \cdots < b_s, \quad m_1 + \cdots + m_s = n.$$

The characteristic values of $\exp A$ are $\{n_i \text{ copies of } e^{a_i}\}$ with $e^{a_1} < \cdots < e^{a_r}$. Since the analogous result holds for $\exp B$, we see that $r = s$, $a_i = b_i$, $n_i = m_i$ ($i = 1, \dots, r$). Thus

$$UAU^{-1} = U_0 B U_0^{-1} = a_1 I_{n_1} \oplus \cdots \oplus a_r I_{n_r},$$

$$U_0 U^{-1} (e^{a_1} I_{n_1} \oplus \cdots \oplus e^{a_r} I_{n_r}) = U_0 \exp A U^{-1} = (e^{a_1} I_{n_1} \oplus \cdots \oplus e^{a_r} I_{n_r}) U_0 U^{-1}.$$

Comparing each element of the second equality, we see that there are $U_i \in U(n, \mathbf{F})$ such that $U_0 U^{-1} = U_1 \oplus \cdots \oplus U_r$. Then, as $U_0 U^{-1}$ commutes with $a_1 I_{n_1} \oplus \cdots \oplus a_r I_{n_r} = UAU^{-1}$, we have

$$U_0 B U_0^{-1} = UAU^{-1} = (U_0 U^{-1})(UAU^{-1})(U_0 U^{-1})^{-1} = U_0 A U_0^{-1}$$

hence $B = A$.

Thus we see that $\exp$ is injective on $H(n, \mathbf{F})$. Q.E.D.

LEMMA 4.9. *The map $\exp: H(n, \mathbf{F}) \rightarrow H^+(n, \mathbf{F})$ is a homeomorphism.*

PROOF. First, for

$$D^+(n, \mathbf{R}) = H^+(n, \mathbf{F}) \cap D(n, \mathbf{R}) = \{\nu_1 \oplus \cdots \oplus \nu_n | \nu_i > 0\}$$

we define $\log: D^+(n, \mathbf{R}) \rightarrow D(n, \mathbf{R})$ by

$$\log(\nu_1 \oplus \cdots \oplus \nu_n) = \log \nu_1 \oplus \cdots \oplus \log \nu_n.$$

The map $\log$ is a homeomorphism and is the inverse of $\exp: D(n, \mathbf{R}) \rightarrow D^+(n, \mathbf{R})$.

For any $A \in H^+(n, \mathbf{F})$, there is U such that $UAU^{-1} \in D^+(n, \mathbf{R})$. If we put $B = \log(UAU^{-1})$, then $\exp(U^{-1}BU) = U^{-1}(\exp B)U = U^{-1}(UAU^{-1})U = A$, whence $\exp$ is surjective. Because $\exp$ is a continuous bijection by Lemma 4.8, we can prove that it is a homeomorphism by showing that it is a closed map. That is, it suffices to show that if $\lim \exp A_i = B \in H^+(n, \mathbf{F})$ for an arbitrary sequence $A_1, A_2, \dots$ of a closed set X of $H(n, \mathbf{F})$, then $B \in \exp(X)$.

Now, for each A_i we choose $U_i \in U(n, \mathbf{F})$ such that $U_i A_i U_i^{-1} = D_i \in D(n, \mathbf{R})$. Since $U(n, \mathbf{F})$ is compact (by Theorems 2.8 and 2.18), there is a convergent subsequence of $\{U_i\}$. So by rechoosing, we may suppose from the beginning that $U = \lim U_i \in U(n, \mathbf{F})$. Since we have

$$\lim \exp D_i = \lim \exp(U_i A_i U_i^{-1}) = \lim U_i (\exp A_i) U_i^{-1} = UB U^{-1} \in D^+(n, \mathbf{R})$$

by putting $D = \log(UBU^{-1}) \in D(n, \mathbf{R})$, we obtain

$$D = \log(\lim \exp D_i) = \lim \log \exp D_i = \lim D_i$$

by the continuity of $\log$.

Put $A = U^{-1}DU$, then the closedness of X implies

$$A = U^{-1}(\lim D_i)U = \lim U^{-1}D_i U = \lim U_i^{-1}D_i U_i = \lim A_i \in X,$$

$$B = \lim \exp A_i = \exp \lim A_i = \exp A \in \exp(X). \quad \text{Q.E.D.}$$

COROLLARY 4.10. *Both $H(n, \mathbf{F})$ and $H^+(n, \mathbf{F})$ are homeomorphic to $\mathbf{R}^l$ where $l = dn(n-1)/2 + n$.*

PROOF. $H(n, \mathbf{F})$ is an $\mathbf{R}$ -subspace of $\mathbf{R}^{dn^2}$ and its matrix $A = (a_{ij})$ is determined by $a_{ii} \in \mathbf{R}$ and $a_{ij} \in \mathbf{F} (i < j)$. Hence $\dim_{\mathbf{R}} H(n, \mathbf{F}) = dn(n-1)/2 + n$. Q.E.D.

The polar decomposition of noncompact classical groups

THEOREM 4.11. *The map $f: U(n, \mathbf{F}) \times H(n, \mathbf{F}) \rightarrow GL(n, \mathbf{F})$ defined by setting $f(U, H) = U \cdot \exp H$ is a homeomorphism, and hence*

$$GL(n, \mathbf{F}) \cong U(n, \mathbf{F}) \times \mathbf{R}^{dn(n-1)/2+n} \quad (\text{isomorphism}).$$

PROOF. By Lemma 4.9 and Corollary 4.10 it suffices to show that the map $g: U(n, \mathbf{F}) \times H^+(n, \mathbf{F}) \rightarrow GL(n, \mathbf{F})$ defined by $g(U, H) = UH$ is a homeomorphism. Using the homeomorphism $\exp$ in Lemma 4.9 we define a map

$$\sqrt{\cdot}: H^+(n, \mathbf{F}) \rightarrow H^+(n, \mathbf{F}) \quad \text{by} \quad \sqrt{B} = \exp(\tfrac{1}{2} \exp^{-1} B).$$

Then $\sqrt{\cdot}$ is a homeomorphism satisfying $(\sqrt{B})^2 = B$.

Let us construct the inverse map of g . A matrix $A \in GL(n, \mathbf{F})$ satisfies $(A^*A)^* = A^*A^{**} = A^*A \in H(n, \mathbf{F})$. Choosing $P \in U(n, \mathbf{F})$ such that $P(A^*A)P^{-1} = D \in D(n, \mathbf{R})$, we set $B = PAP^{-1} = (b_{ij})$ and then $D = B^*B$ with diagonal elements $\lambda_i = \sum \bar{b}_{ji}b_{ji} = \sum |b_{ji}|^2 \geq 0$, whence $A^*A \in H^+(n, \mathbf{F})$. Moreover, if we put $H = \sqrt{A^*A}$ and $U = AH^{-1}$, then $H = H^*$, and $HH^{-1} = I_n$, hence $(H^{-1})^* = (H^{-1})^*H = I_n$ and $(H^{-1})^* = H^{-1}$. Thus

$$UU^* = AH^{-1}(AH^{-1})^* = AH^{-1}H^{-1}A^* = A(H^2)^{-1}A^* = A(A^*A)^{-1}A^* = I_n.$$

Therefore $U \in U(n, \mathbf{F})$ and the continuous map $h: GL(n, \mathbf{F}) \rightarrow U(n, \mathbf{F}) \times H^+(n, \mathbf{F})$ is defined by

$$h(A) = (AH^{-1}, H),$$

where $H = \sqrt{A^*A}$. Obviously $gh(A) = (AH^{-1})H = A$. Conversely, $hg(U, H) = h(UH) = (UHH_0^{-1}, H_0)$ and one has

$$H_0 = \sqrt{(UH)^*UH} = \sqrt{H^*U^*UH} = \sqrt{H^2} = H, \quad UHH_0^{-1} = U.$$

Thus g and h are continuous maps, inverse to each other, and so g is a homeomorphism. Q.E.D.

As for the noncompact classical groups other than $GL(n, \mathbf{F})$ the following holds:

COROLLARY 4.12. Let $\mathbf{F} = \mathbf{R}, \mathbf{C}$. The map f in Theorem 4.11 gives the following homeomorphisms:

- (1) $SU(n, \mathbf{F}) \times \mathbf{R}^{dn(n-1)/2+n} \cong SL(n, \mathbf{F})$;
- (2) $U(n) \times \mathbf{R}^{n(n+1)} \cong Sp(n, \mathbf{R})$, $Sp(n) \times \mathbf{R}^{n(2n+1)} \cong Sp(n, \mathbf{C})$;
- (3) $O(n) \times \mathbf{R}^{n(n-1)/2} \cong O(n, \mathbf{C})$.

PROOF. It can be verified that the map f of the theorem induces the following homeomorphisms:

$$\begin{aligned} SU(n, \mathbf{F}) \times \{A \in H(n, \mathbf{F}) \mid \text{tr}(A) = 0\} &\rightarrow SL(n, \mathbf{F}); \\ \mathbf{r}(U(n)) \times \{A \in H(2n, \mathbf{R}) \mid {}^t A + J_n A J_n^{-1} = 0\} &\rightarrow Sp(n, \mathbf{R}); \\ \mathbf{c}'(Sp(n)) \times \{A \in H(2n, \mathbf{C}) \mid {}^t A + J_n A J_n^{-1} = 0\} &\rightarrow Sp(n, \mathbf{C}); \\ \mathbf{c}(O(n)) \times \{A \in H(n, \mathbf{C}) \mid A + \bar{A} = 0\} &\rightarrow O(n, \mathbf{C}). \end{aligned}$$

Each of these sets $\{A \in \dots\}$ is a linear space over $\mathbf{R}$ with dimension given above. Q.E.D.

In general, if a topological group G contains a (maximal) compact subgroup H and a subspace homeomorphic to $\mathbf{R}^m$ such that the multiplication of G gives a homeomorphism

$$H \times \mathbf{R}^m \cong G,$$

we refer to the above homeomorphism as a *polar decomposition* of G . It is known that in general a connected Lie group has a polar decomposition (*Iwasawa decomposition*).

Homotopy invariants, topological properties such as (generalized) cohomology groups, and homotopy groups of a topological group G having the polar decomposition $G \cong H \times \mathbf{R}^m$ can be identified with those of the compact subgroup H . This is the reason why in the present book we will study the topological properties of compact topological groups and in particular, the compact classical groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ in its first half.

§5. Differentiable manifolds and Lie groups

Differentiable manifolds

DEFINITION. An m -dimensional topological manifold M is a Hausdorff space, each point of which has a neighborhood homeomorphic to the m -dimensional Euclidean space $\mathbf{R}^m$.

Then there is an open covering $M = \bigcup_{\alpha} U_{\alpha}$ of M and a homeomorphism $\psi_{\alpha}: V_{\alpha} \cong U_{\alpha}$ of an open set V_{α} of $\mathbf{R}^m$ with U_{α} . If $U_{\alpha} \cap U_{\beta} \neq \emptyset$ then the map

$$f_{\beta\alpha} = \psi_{\beta}^{-1} \psi_{\alpha}: V_{\alpha} \cap \psi_{\alpha}^{-1}(U_{\alpha} \cap U_{\beta}) \rightarrow V_{\beta} \cap \psi_{\beta}^{-1}(U_{\alpha} \cap U_{\beta})$$

is a homeomorphism. We call $\{U_{\alpha}, \psi_{\alpha}\}$ a (system of) *coordinate neighborhoods* or simply an *atlas* of a topological manifold M , and $f_{\beta\alpha}$ a *coordinate transformation*.

Obviously, a topological manifold is locally compact. Normally we assume that topological manifolds are paracompact (cf. Chapter II, §2).

5.1. If a homogeneous space, in particular a topological group, has a point with a neighborhood homeomorphic to an open set of $\mathbf{R}^m$, then it is an m -dimensional manifold.

Let V be an open set of $\mathbf{R}^m$, $f: V \rightarrow \mathbf{R}^n$ a function (a map) and $\mathbf{x} \in V$. If there is an $f'(\mathbf{x}) \in \text{Hom}_{\mathbf{R}}(\mathbf{R}^m, \mathbf{R}^n)$ such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + f'(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|),$$

for sufficiently small $\mathbf{h} \in \mathbf{R}^n$ with length $\|\mathbf{h}\|$ satisfying $\mathbf{x} + \mathbf{h} \in V$, then

we call $f'(x)$ the *differential* of f at x , where $o(t) \in \mathbf{R}^n$ ($t \in \mathbf{R}$) denotes a vector such that $\|o(t)/t\| \rightarrow 0$ when $t \rightarrow 0$. If $x = (x_1, \dots, x_m)$, $h = (h_1, \dots, h_m)$, $f(x) = (f_1(x), \dots, f_n(x))$, and $f'(x)h = (\sum a_{ij} h_j, \dots, \sum a_{in} h_j)$ then $a_{ij} = \partial f_i(x)/\partial x_j$.

Identifying $\text{Hom}_{\mathbf{R}}(\mathbf{R}^m, \mathbf{R}^n)$ with $\mathbf{R}^{mn}$ we have $f': V \rightarrow \mathbf{R}^{mn}$, so we can consider the second differential $f'' = (f')' \in \text{Hom}_{\mathbf{R}}(\mathbf{R}^m, \mathbf{R}^{mn})$. Repeating this process we obtain the successive differentials $f', f'', \dots, f^{(r)}, \dots$. If there exist $f', f'', \dots, f^{(r)}$ that are continuous, then f is called a C^r -function. If f is a C^r -function for all natural numbers r , then f is called a C^∞ -function or a *differentiable function*.

Further, if f can be expanded as a uniformly absolutely-convergent infinite series in a neighborhood of each point of V , then f is called a C^ω -function or a *real analytic function*.

We shall use the symbols ∞ and ω assuming that $0 < 1 < 2 < \dots < r < \dots < \infty < \omega$.

DEFINITION. An atlas $\{U_\alpha, \psi_\alpha\}$ of a topological manifold M is said to be of C^r -class ($r = 0, 1, 2, \dots, \infty, \omega$) if the coordinate transformations $f_{\beta\alpha}$ are all C^r -functions.

Of course, when $s \leq r$, C^r -class implies C^s -class.

DEFINITION. A map $f: M \rightarrow M'$ between topological manifolds M and M' with atlases of C^r -class $\{U_\alpha, \psi_\alpha\}$ and $\{V_\beta, \varphi_\beta\}$ respectively is called a C^s -map ($s \leq r$) if whenever $f(U_\alpha) \cap V_\beta \neq \emptyset$, $\varphi_\beta^{-1} f \psi_\alpha: \psi_\alpha^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \rightarrow \varphi_\beta^{-1}(V_\beta \cap f(U_\alpha))$ is a C^s -function.

Further, f as above is called a C^s -homeomorphism, if it is a homeomorphism and the inverse f^{-1} is a C^s -map.

Two atlases of C^r -class on a topological manifold M are called C^s -equivalent if the identity map $1_M: M \rightarrow M$ is a C^s -homeomorphism with respect to the two atlases.

C^s -equivalence is clearly an equivalence relation.

DEFINITION. A C^r -manifold M is by definition the C^r -equivalence class of a topological manifold M with an atlas of C^r -class.

The C^r -equivalence class of an atlas is called a (C^r) -differentiable structure on M .

Since we devote ourselves exclusively to the study of C^∞ -manifolds, we will call them *differentiable manifolds*. A C^∞ -homeomorphism will be called a *diffeomorphism*.

For example, an open set U of $\mathbf{R}^m$ is a differentiable manifold with an atlas $\{U, 1_U\}$. In particular, the general linear group $GL(n, \mathbf{F})$ is a dn^2 -dimensional differentiable manifold.

Given two differentiable manifolds M and N with atlases $\{U_\alpha, \psi_\alpha\}$ and $\{U'_\beta, \psi'_\beta\}$ respectively, the *product manifold* $M \times N$ is defined by taking $\{U_\alpha \times U'_\beta, \psi_\alpha \times \psi'_\beta\}$ as its atlas.

5.2. (1) The projections $\pi_1: M \times N \rightarrow M$ and $\pi_2: M \times N \rightarrow N$ are C^∞ -maps.

(2) $f: L \rightarrow M \times N$ is a C^∞ -map iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are C^∞ -maps.

The tangent bundle of a differentiable manifold. Given an atlas $\{U_\alpha, \psi_\alpha\}$ on an m -dimensional differentiable manifold M and a family of coordinate transformations $f_{\beta\alpha} = \psi_\beta^{-1} \psi_\alpha$, we introduce a relation $\sim$ on the set of points $(x_\alpha, y_\alpha) \in U_\alpha \times \mathbf{R}^m$ of the topological sum $\coprod_\alpha U_\alpha \times \mathbf{R}^m$ as follows:

$$(x_\alpha, y_\alpha) \sim (x_\beta, y_\beta) \quad \text{iff} \quad x_\alpha = x_\beta \in U_\alpha \cap U_\beta, \quad y_\beta = f'_{\beta\alpha}(y_\alpha), \\ (x_\alpha = \psi_\alpha(v)).$$

The quotient space

$$TM = \coprod_\alpha U_\alpha \times \mathbf{R}^m / \sim$$

is a $2m$ -dimensional differentiable manifold with an atlas $\{U_\alpha \times \mathbf{R}^m, \psi_\alpha \times 1\}$. The differential structure is not changed, even if we replace the atlas with one that is C^∞ -equivalent to it. The map

$$p: TM \rightarrow M \quad \text{defined by} \quad p(x_\alpha, y_\alpha) = x_\alpha$$

is a C^∞ -map and is called the *projection*. The inverse image

$$T_x M = p^{-1}(x),$$

of a point x of M by p is homeomorphic to $\mathbf{R}^m$ by a natural C^∞ -map $x \times \mathbf{R}^m \rightarrow TM$ and is a vector space over $\mathbf{R}$ called the *tangent vector space* at x . A vector in $T_x M$ is called a *tangent vector* of M at x . Summing up,

$$\tau(M) = (TM, p, M, \mathbf{R}^m)$$

is called the *tangent bundle* of M and TM the *total space* or the *tangent bundle space*.

For example, $TU = U \times \mathbf{R}^m$ for an open set U of $\mathbf{R}^m$. In particular,

$$T(GL(n, \mathbf{F})) = GL(n, \mathbf{F}) \times M(n, \mathbf{F}).$$

The differential $df: TM \rightarrow TN$ of a C^∞ -map $f: M \rightarrow N$ between differentiable manifolds M and N is defined by

$$df(x, y) = (f(x), (\varphi_\beta^{-1} f \psi_\alpha)'(y)y), \quad \psi_\alpha(v) = x, v \in U_\alpha.$$

The definition is independent of the choice of α and β and df is a C^∞ -map. The map df satisfies $p \circ df = f \circ p$ and

$$df_x = df|_{T_x M}: T_x M (\cong \mathbf{R}^m) \rightarrow T_{f(x)} N (\cong \mathbf{R}^n)$$

is a linear map.

5.3. (1) $d1_M = 1_{TM}$, $d(g \circ f) = dg \circ df$.

(2) If f is a diffeomorphism then df is a diffeomorphism, and df_x is an isomorphism.

DEFINITION. A differentiable manifold M is called a (*regular*) *submanifold* of a differentiable manifold N if M is a subspace of N , the inclusion $i: M \subset N$ is a C^∞ -map, and di_x is a monomorphism for any $x \in M$.

THEOREM 5.4. *Let $M \subset N$. M is a submanifold of N iff there is an atlas $\{U_\alpha, \psi_\alpha\}$ of N , where $\psi_\alpha: (\mathbb{R}^n \supset) V_\alpha \cong U_\alpha$, such that*

$$\{M \cap U_\beta, \psi_\beta|(\mathbb{R}^m \times 0) \cap V_\beta\},$$

for all β satisfying $M \cap V_\beta \neq \emptyset$, is an atlas of M .

PROOF. (Sufficiency) This is obvious.

(Necessity) First, we consider an atlas $\{U'_\alpha, \phi'_\alpha\}$ on N . Put $U_\alpha = U'_\alpha - (M \cap U'_\alpha)$ and let ψ_α be the restriction of ϕ'_α for each α . Secondly, if for each point $x = \phi'_\alpha(v) \in U'_\alpha$ of M with $v \in V'_\alpha$ we can construct a C^∞ -homeomorphism h of a neighborhood V of the origin of $\mathbb{R}^n$ onto a neighborhood of v such that $h(0) = v$ and $\phi'_\alpha \circ h$ maps $(\mathbb{R}^m \times 0) \cap V$ homeomorphically onto a neighborhood of x in M , we can obtain the required atlas on N by adding $\psi_\beta = \phi'_\beta \circ h$ and $U_\beta = \psi_\beta(V)$ to the above $\{U_\alpha, \psi_\alpha\}$ so that $\{U_\beta\}$ covers M .

Finally, in order to construct such a map h , it is sufficient to show the following:

5.4'. *If f is a C^∞ -map of a neighborhood of 0 in $\mathbb{R}^m$ to a neighborhood of 0 in $\mathbb{R}^n$ such that $f(0) = 0$ and $(df)_0: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a monomorphism, there exists a C^∞ -homeomorphism between sufficiently small neighborhoods of 0 in $\mathbb{R}^n$ such that its restriction to $\mathbb{R}^m \times 0$ coincides with f .*

In fact, let $\pi: \mathbb{R}^n \rightarrow df_0(\mathbb{R}^m)$ be the (orthogonal) projection and $\mathbb{R}^{n-m}$ the complement of $df_0(\mathbb{R}^m)$ and consider $(\pi \circ f) \times 1: U \times \mathbb{R}^{n-m} \rightarrow df_0(\mathbb{R}^m) \times \mathbb{R}^{n-m}$. Since its Jacobian at the origin is not 0, it is a C^∞ -homeomorphism in a neighborhood of the origin. We may take its inverse as h . Q.E.D.

Lie groups and one-parameter subgroups

DEFINITION. G is called a *Lie group* if G is a topological group and a differentiable manifold such that the multiplication $\mu: G \times G \rightarrow G$ and the inverse element $\iota: G \rightarrow G$ are C^∞ -maps.

A map $f: G \rightarrow G'$ is called a *homomorphism of Lie groups* if f is a homomorphism of topological groups and a C^∞ -map.

$f: G \rightarrow G'$ is called an *isomorphism of Lie groups* if f is an isomorphism of topological groups and a diffeomorphism.

5.5. $\mathbb{R}^n$ is an abelian Lie group under vector addition. $GL(n, \mathbb{F})$ is a Lie group.

In fact, multiplication of matrices and sending a matrix to its inverse are C^∞ -maps, since they can be expressed by rational functions of elements.

DEFINITION. Let G be a Lie group. We call a homomorphism $s: \mathbb{R} \rightarrow G$ of Lie groups a *one-parameter subgroup* of G and $ds(0, 1) \in T_e G$ the *initial vector* of s , where $(0, 1)$ is the unit vector of $T_0 \mathbb{R} = 0 \times \mathbb{R}$.

The image of s is a subgroup of G but not necessarily a closed subgroup.

Below we will see that a one-parameter subgroup is determined by its initial vector. To this end we quote the following lemma from the theory of differential equations.

LEMMA 5.6. Let U be a neighborhood of 0 in $\mathbb{R}^m$, V a neighborhood of y_0 in $\mathbb{R}^n$, and $h: V \times U \rightarrow \mathbb{R}^m$ a C^∞ -map. Given the differential equation

$$df(t)/dt = h(y, f(t)), \quad f(0) = 0,$$

where $y \in V$ and $f: \mathbb{R} \rightarrow \mathbb{R}^m$, there exist $\varepsilon > 0$ and a neighborhood $V' \subset V$ of y_0 such that for each $y \in V'$ there is a unique solution f_y of the above differential equation for $t \in (-\varepsilon, \varepsilon)$ such that the map $f: (-\varepsilon, \varepsilon) \times V' \rightarrow \mathbb{R}^m$ defined by $f(t, y) = f_y(t)$ is a C^∞ -map.

DEFINITION. A (tangent) vector field v on a differentiable manifold M is a C^∞ -map $v: M \rightarrow TM$ satisfying $p \circ v = 1_M$.

A vector field v maps each point x of M to a tangent vector $v(x)$ at the point and the map thus defined is a C^∞ -map.

LEMMA 5.7. Let $W \subset \mathbb{R}$ be a neighborhood of the closed interval $[a, b]$, v a vector field on a differentiable manifold M , and $f, g: W \rightarrow M$ C^∞ -maps such that $df, dg: TW = W \times \mathbb{R} \rightarrow TM$ satisfy $df(t, 1) = v(f(t))$, $dg(t, 1) = v(g(t))$ for $[a, b]$. If $f(t_0) = g(t_0)$ for some $t_0 \in [a, b]$, then $f(t) = g(t)$ for all $t \in [a, b]$.

PROOF. Suppose $f(t_1) = g(t_1)$ for $t_1 \in [a, b]$. We choose a coordinate neighborhood $\{U_\alpha, \psi_\alpha\}$ containing $f(t_1)$ such that $\psi_\alpha(0) = f(t_1)$ and replace TM by $V_\alpha \times \mathbb{R}^m$ ($\psi_\alpha: V_\alpha \cong U_\alpha$). We may suppose $t_1 = 0$. Then if we denote $v(x) = (x, h(x))$, $x \in V_\alpha$, by the assumption of the lemma we have

$$df(t)/dt = h(f(t)), \quad dg(t)/dt = h(g(t)).$$

Applying Lemma 5.6 ($n = 0$), one sees that there exists $\varepsilon > 0$ such that $f = g$ on $(t_1 - \varepsilon, t_1 + \varepsilon)$. Let $c = \sup\{d \mid f(t) = g(t), t \in [a, b], d \leq b\}$. Then $c = b$ by the above. Similarly for the infimum. Q.E.D.

Since the left translation $L_x: G \rightarrow G$ ($L_x(g) = xg$) by an element x of a Lie group G is a C^∞ -map, it induces a C^∞ -homeomorphism $dL_x: TG \rightarrow TG$ of tangent bundle spaces.

DEFINITION. A vector field v over a Lie group is called *left invariant* if the diagram below is commutative for every $x \in G$.

$$\begin{array}{ccc} TG & \xrightarrow{dL_x} & TG \\ \uparrow v & & \uparrow v \\ G & \xrightarrow{L_x} & G \end{array}$$

LEMMA 5.8. (1) If we put $v(x) = dL_x(v_0)$ for $v_0 \in T_e G$, then $v: G \rightarrow TG$ is a left invariant vector field of G by this correspondence.

(2) Let v be a left invariant vector field on G and $s: \mathbf{R} \rightarrow G$ a one-parameter subgroup of G . If $v(e) \in T_e G$ coincides with the initial vector $ds(0, 1)$, then $ds(t, 1) = v(s(t))$ for all $t \in \mathbf{R}$.

PROOF. (1) The first half is clear from 5.3(1). The inverse correspondence is given by $v \mapsto v_0 = v(e)$.

(2) Since the diagram below commutes for $t \in \mathbf{R}$, a similar diagram is obtained for the tangent vector spaces.

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{s} & G \\ \downarrow L_t & & \downarrow L_{s(t)} \\ \mathbf{R} & \xrightarrow{s} & G \end{array}$$

By (1) $v(x) = dL_x(ds(0, 1))$. In particular,

$$v(s(t)) = dL_{s(t)}(ds(0, 1)) = ds dL_t(0, 1) = ds(t, 1).$$

Q.E.D.

THEOREM 5.9. In a Lie group G the correspondence assigning to a one-parameter subgroup $s: \mathbf{R} \rightarrow G$ the initial vector $ds(0, 1) \in T_e G$ gives a bijection between the set $\{s\}$ and $T_e G$.

PROOF. If s and s' have the same initial vector v_0 , then by Lemma 5.8 $ds(t, 1) = v(s(t))$ and $ds'(t, 1) = v(s'(t))$ for a left-invariant vector field v satisfying $v_0 = v(e)$. Since $s'(0) = s(0) = e$, we have $s = s'$ by Lemma 5.7. Thus the correspondence is injective.

When an initial vector v_0 , and so, a left-invariant vector field v , is given, we consider a differential equation $ds(t, 1) = v(s(t))$. The equation is solvable for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ with the initial condition $ds(0, 1) = v(e) = v_0$. Further one has

$$\begin{aligned} v(s(a)s(t)) &= dL_{s(a)s(t)} ds(0, 1) = dL_{s(a)} dL_{s(t)} ds(0, 1) = (dL_{s(a)} ds) dL_t(0, 1) \\ &= (ds dL_a)(t, 1) = ds(a+t, 1) = v(s(a+t)). \end{aligned}$$

Therefore by applying Lemma 5.7 to $f(t) = s(a)s(t)$ and $g(t) = s(a+t)$ in the range $|a| < \varepsilon/2$ and $|t| < \varepsilon/2$, one obtains $s(a)s(t) = s(a+t)$.

For general $t \in \mathbf{R}$ choosing a natural number n satisfying $|t/n| < \varepsilon/2$, we put $s(t) = (s(t/n))^n$. It is easy to see that $s(t)$ is well defined independently of the choice of n , that $s(a+t) = s(a)s(t)$ holds, and that s is a C^∞ -map. Thus s is a one-parameter subgroup with the initial vector $ds(0, 1) = v_0$. Hence the correspondence is surjective. Q.E.D.

The exponential map in a Lie group and closed subgroups

DEFINITION. For a one parameter subgroup s with the initial vector $v_0 \in T_e G$ we call

$$\exp: T_e G \rightarrow G \quad \text{defined by} \quad \exp(v_0) = s(1)$$

the exponential map.

For the case $G = GL(n, \mathbb{F})$, we have $T_e G = M(n, \mathbb{F})$. Putting $s(t) = \exp(tA)$ for $A \in M(n, \mathbb{F})$, we have

$$s(t+t') = \exp(tA + t'A) = \exp(tA)\exp(t'A) = s(t)s(t'),$$

since tA commutes with $t'A$. In other words, s is a one-parameter subgroup. Thus the exponential map defined above is seen to be a generalization of the exponential map of matrices.

THEOREM 5.10. *The exponential map $\exp$ is a diffeomorphism of a neighborhood of $\mathbf{0} \in T_e G$ onto a neighborhood of $e \in G$.*

PROOF. We restrict TG to a coordinate neighborhood U_α containing e and consider $U_\alpha \times \mathbb{R}^m$. The left-invariant vector field v_y corresponding to a vector $y = v_0$ in $T_e G \cong \mathbb{R}^m = \exp \mathbb{R}^m$ is given by $v_y(x) = (x, h(y, x))$, where $h(y, x) = dL_x(y)$. Then $s(t) = \exp(ty)$ is a solution of the differential equation $(d/dt)s(t) = h(y, s(t))$. Since h is a C^∞ -map, $\exp(ty)$ is a C^∞ -map in a neighborhood of $\mathbf{0}$ by Lemma 5.6. The differential $d(\exp)_0: T(T_e G) = T_e G \rightarrow T_e G$ of $\exp$ is obviously the identity map. By using the implicit function theorem, $\exp$ is seen to be a diffeomorphism in a neighborhood of $\mathbf{0}$. Q.E.D.

COROLLARY 5.11. *If $T_e G$ is a direct sum of $\mathbb{R}$ -vector subspaces $V_1, \dots, V_k$, then the map $f: T_e G \rightarrow G$ defined by $f(v_1 \oplus \dots \oplus v_k) = \exp v_1 \dots \exp v_k$ ($v_i \in V_i$) is a diffeomorphism of a neighborhood of $\mathbf{0}$ to that of e .*

PROOF. f is differentiable and df_0 is the identity map hence f is locally a diffeomorphism. Q.E.D.

LEMMA 5.12. *Let H be a closed subgroup of a Lie group G and $W = \{v \in T_e G | \exp(tv) \in H, \forall t \in \mathbb{R}\}$. Then W is a vector subspace of $T_e G$ and $\exp|_W: W \rightarrow H$ is a diffeomorphism of a neighborhood of $\mathbf{0}$ with a neighborhood of e .*

PROOF. First, we show the following:

$$\mathbf{0} \neq v_n \in T_e G, \quad \exp v_n \in H, \quad \lim_n v_n = \mathbf{0}, \quad \lim_n \frac{v_n}{\|v_n\|} = v \text{ implies } v \in W. \quad (*)$$

Since $\lim_n \|v_n\| = 0$, $\|v_n\| \neq 0$, there is a sequence of integers $\{a_n\}$ such that $\lim_n a_n \|v_n\| = t$. Then $\exp(tv) = \lim_n \exp(a_n v_n) = \lim_n (\exp v_n)^{a_n}$, where $(\exp v_n)^{a_n} \in H$, and since H is closed, $\exp(tv) \in H$ and so $v \in W$.

Next we show that

$$\begin{aligned} \text{if we put } \mu(x, y) &= \exp^{-1}(\exp x \cdot \exp y) \quad (x, y \in V, \\ \exp x \cdot \exp y &\in U), \text{ then } \mu(x, y) = x + y + o(\|x\| + \|y\|), \end{aligned} \quad (**)$$

where V and U are neighborhoods of $\mathbf{0}$ and e respectively such that $\exp: V \cong U$ (homeomorphism).

Since μ is a C^∞ -map, there are $z \in T_e G$ and $a, b \in \mathbb{R}$ such that

$$\mu(x, y) = z + ax + by + o(\|x\| + \|y\|).$$

$\mu(0, y) = y$ implies $z = 0$, $b = 1$. $\mu(x, 0) = x$ implies $a = 1$.

We prove that W is a vector subspace. Obviously, $w \in W$ and $t \in \mathbb{R}$ imply $tw \in W$. Let $w_1, w_2 \in W$ such that $w_1 + w_2 \neq 0$. If we put $h(t) = \mu(tw_1, tw_2)$, then $h(t) = t(w_1 + w_2) + o(t)$ by (**). Hence, if we put $v_n = h(1/n)$ for sufficiently large n , $w = w_1 + w_2$, $v = w/\|w\|$, we can see that the condition (*) is satisfied and thus $w_1 + w_2 \in W$. Thus we have proved that W is a vector subspace.

Next we decompose $T_e G = W^\perp \oplus W$ and put $f(w', w) = \exp w'$. $\exp w$ ($w' \in W^\perp$, $w \in W$). Then $f: T_e G \rightarrow G$ is a diffeomorphism of a neighborhood V' of 0 onto a neighborhood U' of e by Corollary 5.11. It follows that f is a homeomorphism of $V' \cap f^{-1}(U' \cap H)$ onto $U' \cap H$.

Obviously $V' \cap W \subset U' \cap f^{-1}(U' \cap H)$. If there is a neighborhood of 0 in $V' \cap f^{-1}(U' \cap H)$ that is contained in $V' \cap W$, then the latter half of the lemma follows. So it is sufficient to deduce a contradiction, by supposing otherwise. It follows from the third assumption that there exists a sequence of points $\{(w'_n, w_n)\}$ such that $f(w'_n, w_n) \in H$, $\lim_n f(w'_n, w_n) = e$ and $w'_n \neq 0$. We have $\exp w_n, f(w'_n, w_n) \in H$ so $\exp w'_n \in H$, and as the unit sphere $S(W^\perp)$ of $W^\perp$ is compact, we can replace w'_n by a subsequence with $\lim_n w'_n/\|w'_n\| = w' \in S(W^\perp)$. Using (*) $w' \in W$ implies $w' \in W^\perp \cap W = \{0\}$, and hence $\|w'\| = 1$ and $w' = 0$, which is a contradiction. Q.E.D.

THEOREM 5.13. *A closed subgroup H of a Lie group is itself a Lie group with the differentiable structure of a submanifold of G and $T_e H = \{v \in T_e G \mid \exp(tv) \in H, \forall t \in \mathbb{R}\}$.*

PROOF. Choose a neighborhood U of e in G , then $\{L_g U, L_g \circ \exp\}_{g \in G}$ is an atlas on G . If we take W as in Lemma 5.12 and define coordinate neighborhoods of H using $V \cap W$ and $\exp$ on it, H becomes a differentiable manifold. Also it is clear that H is a submanifold of G . Then the group operation in H is C^∞ -map, and $T_e H = W$. Q.E.D.

COROLLARY 5.14. *Closed subgroups of $GL(n, \mathbb{F})$ and in particular, the classical groups are Lie groups.*

DEFINITION. A Lie group isomorphic to a closed subgroup of $GL(n, \mathbb{F})$ is called a *linear Lie group*.

Homomorphisms of Lie groups and homogeneous spaces

LEMMA 5.15. *A continuous homomorphism $f: \mathbb{R} \rightarrow G$ of the additive group $\mathbb{R}$ of real numbers to a Lie group G is a one-parameter subgroup, that is, differentiability follows from continuity.*

PROOF. We consider $T_e G = \mathbb{R}^m$ and suppose that $\exp$ is a homeomorphism of a 2ε -neighborhood $V_{2\varepsilon}$ of $\mathbf{0}$ onto a neighborhood U_2 of e in G . Let V_ε be the ε -neighborhood of $\mathbf{0}$ and $U_1 = \exp V_\varepsilon$, then it is easily seen that for any element g of U_1 there is uniquely an element $h \in U_1$ satisfying $h^2 = g$.

We choose $a > 0$ so that $f([-a, a]) \subset U_\varepsilon$ and put $f(a) = \exp v$, $v \in V_\varepsilon$. Then by the above we have $f(a/2)^2 = f(a) = \exp v = \exp(v/2)^2$, which implies $f(a/2) = \exp(v/2)$. Repeating it, $f(a/2^k) = \exp(v/2^k)$, hence $f(ra) = \exp(rv)$ for a rational number $r = n/2^k \in [-1, 1]$. Such r 's are dense in $[-1, 1]$ and so $f(ta) = \exp(tv)$ for $t \in [-1, 1]$. Hence $f(ta) = \exp(tv)$ for any $t \in \mathbb{R}$; that is, f is a one-parameter subgroup with initial vector v . Q.E.D.

THEOREM 5.16. Let G and G' be Lie groups. If $f: G \rightarrow G'$ is a homomorphism of topological groups, then f is a homomorphism of Lie groups.

PROOF. It suffices to show that f is a C^∞ -map in a neighborhood of the identity element. Let us define $f_i: \mathbb{R} \rightarrow G'$ by $f_i(t) = f(\exp tv_i)$ for an $\mathbb{R}$ -basis $\{v_1, \dots, v_m\}$ of $T_e G$. Then the f_i 's are continuous homomorphisms. By Lemma 5.15 there are $v'_i \in T_e G'$ such that $f_i(t) = \exp(tv'_i)$.

Next we define $h: \mathbb{R}^m \rightarrow G$ and $h': \mathbb{R}^m \rightarrow G'$ by $h(t_1, \dots, t_m) = \exp t_1 v_1 \cdots \exp t_m v_m$ and $h'(t_1, \dots, t_m) = \exp t_1 v'_1 \cdots \exp t_m v'_m$ respectively. Then they are C^∞ -maps. Further, h is a diffeomorphism in a neighborhood of $\mathbf{0}$ by Corollary 5.11. Since f is a homomorphism, we have

$$f(h(t_1, \dots, t_m)) = f_1(t_1) \cdots f_m(t_m) = h'(t_1, \dots, t_m).$$

Therefore $f = h' \circ h^{-1}$ is a C^∞ -map in a neighborhood of $\mathbf{0}$. Q.E.D.

THEOREM 5.17. If G is a Lie group and H a closed subgroup, then the homogeneous space G/H is a differentiable manifold with a differentiable structure satisfying the following properties:

- (1) the projection $\pi: G \rightarrow G/H$ is a C^∞ -map;
- (2) the action of $G: G \times G/H \rightarrow G/H$ is a C^∞ -map;
- (3) for each point of G/H , there exist its neighborhood U and a C^∞ -map $s: U \rightarrow G$ such that $\pi \circ s = 1_U$. (We say that the projection has a local cross-section.)

PROOF. If we take the open neighborhoods V' and V of $\mathbf{0}$ small enough in the definition of $f: T_e G = W^\perp \oplus W \rightarrow G$, $(f(w', w) = \exp w' \exp w)$, then by the proof of Lemma 5.12 f is a diffeomorphism of $V' \times V$ onto a neighborhood $U' \subset G$ of e such that $f(\mathbf{0} \times V) = U' \cap H$.

Moreover, if we take V'_0 and V_0 so small that $U'^{-1}_0 \cdot U'_0 \subset U'$ where $U'_0 = f(V'_0 \times V_0)$, then for $(w', w), (w'_1, w_1) \in V'_0 \times V_0$ we have that $\pi f(w', w) = \pi f(w'_1, w_1)$, which implies $\exp w' \exp w = \exp w'_1 \exp w_1 \cdot h$

($h \in H$). This implies $\exp w'_1 \cdot h \in U' \cap H$, which $\exp w'_1 \cdot h = \exp w_2$ ($w_2 \in V$). Therefore, $f(w', w) = f(w'_1, w_2)$ and hence $w' = w'_1$.

Thus the following map, defined independently of $w \in V_0$, is bijective:

$$\psi: V'_0 \xrightarrow{\times w} V'_0 \times V_0 \xrightarrow{f} U'_0 \xrightarrow{\pi} \pi(U'_0).$$

Since π is an open map, $U_0 = \pi(U'_0)$ is an open neighborhood of $\pi(e)$. One can see that ψ is a homeomorphism by using the local compactness of V'_0 .

We take $\{L_g U_0, L_g \circ \psi\}_{g \in G}$ as a system of neighborhoods of G/H . Since the coordinate transformation of $L_g U_0$ to $L_{xg} U_0$ is given by L_x ($x \in U'$), it is restriction of the following composition:

$$V'_0 \xrightarrow{\times 0} V'_0 \times V_0 \xrightarrow{f} U'_0 \xrightarrow{L_x} U'_0 \xrightarrow{f^{-1}} V'_0 \times V_0 \xrightarrow{\text{proj}} V'_0.$$

Therefore coordinate transformations are C^∞ -maps and G/H is a differentiable manifold. Properties (1), (2), and (3) are easily verified; in the case (3) we may consider

$$s: L_g U_0 \xrightarrow{L_g^{-1}} U_0 \xrightarrow{\psi^{-1}} V'_0 \xrightarrow{\times 0} V'_0 \times V_0 \xrightarrow{f} U'_0 \xrightarrow{L_g} L_g U'_0 \subset G,$$

with $U = L_g U_0$. Q.E.D.

COROLLARY 5.18. *If G is a Lie group and H and K are closed subgroups such that $H \supset K$, then the projection $\pi: G/K \rightarrow G/H$ is a C^∞ -map and has a local cross-section.*

PROOF. This is seen by decomposing $W = T_e H$ further as $W = W' \oplus T_e K$. A local cross-section is obtained from that of $G \rightarrow G/H$. Q.E.D.

The existence of a local cross-section will be crucial in the theory of fibre bundles in the next chapter.

CHAPTER II

Covering Spaces and Fibre Bundles

The main object of this chapter is to introduce various facts from algebraic topology. In §1 we consider fundamental groups and also (universal) covering spaces and covering transformation groups associated with them. In §2 we state some elementary properties of fibre bundles. For details we refer, for example, to Steenrod [13]. In §3 we introduce the definition of homotopy groups and state their properties and the notion of a fibration, which is a generalization of that of a fibre bundle. In order to define spinor groups we introduce in §4 the notion of a local isomorphism of topological groups, in particular we study the properties of covering groups of classical groups. Then in §5 after considering the relations between Lie groups and Lie algebras, we describe the relations between classical groups and spinor groups of low rank: $\text{Spin}(n)$ for $n = 3, 4, 5, 6$. In §6 we introduce the notion of n -universal bundles. After demonstrating the existence of classifying spaces, we prove the classification theorem, which states that the set of isomorphism classes of fibre bundles over a space X corresponds bijectively to the set of homotopy classes of maps from X to the classifying space.

§1. Fundamental groups and covering spaces

Homotopy classes and fundamental groups. For subspaces $A_1, \dots, A_n$ and $B_1, \dots, B_n$ of topological spaces X and Y respectively, and a (continuous) map $f: X \rightarrow Y$ satisfying $f(A_i) \subset B_i$ we write

$$f: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n).$$

DEFINITION. Two maps $f, g: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n)$ are called *homotopic* if there exists a map $H: (X \times I, A_1 \times I, \dots, A_n \times I) \rightarrow (Y, B_1, \dots, B_n)$ such that

$$H(x, 0) = f(x), \quad H(x, 1) = g(x) \quad (x \in X).$$

q

We call H a *homotopy* between f and g , and write

$$f \simeq g: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n),$$

if f and g are homotopic.

The following is easy to verify:

1.1. (1) The homotopy relation $\simeq$ is an equivalence relation.

(2) The homotopy relation is preserved under composition:

$$\begin{aligned} \text{if } f &\simeq g: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n) \\ \text{and } f' &\simeq g': (Y, B_1, \dots, B_n) \rightarrow (Z, C_1, \dots, C_n), \\ \text{then } f' \circ f &\simeq g' \circ g: (X, A_1, \dots, A_n) \rightarrow (Z, C_1, \dots, C_n). \end{aligned}$$

DEFINITION. We denote by $[f]$ the set of maps homotopic to $f: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n)$ and call $[f]$ the homotopy class of f . The systems of spaces $(X, A_1, \dots, A_n)$ and $(Y, B_1, \dots, B_n)$ are called homotopically equivalent if there are maps

$$f: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n)$$

and

$$g: (Y, B_1, \dots, B_n) \rightarrow (X, A_1, \dots, A_n)$$

such that

$$g \circ f \simeq 1: (X, A_1, \dots, A_n) \rightarrow (X, A_1, \dots, A_n)$$

and

$$f \circ g \simeq 1: (Y, B_1, \dots, B_n) \rightarrow (Y, B_1, \dots, B_n).$$

In this case we also say that they have the same homotopy type. In particular, a space is said to be contractible if it has the same homotopy type as a point.

Spaces (or systems of spaces) of the same homotopy type have sets of homotopy classes in bijective correspondence with each other and are regarded as equivalent when considering fundamental groups, homotopy groups, (co)-homology groups, K -groups, and so on as we will see below.

For a given homotopy $H: X \times I \rightarrow Y$, the map

$$h_t: X \rightarrow Y \quad (0 \leq t \leq 1)$$

defined by $h_t(x) = H(x, t)$ is called a homotopy, and then $h_0 \simeq h_1: X \rightarrow Y$.

Now we consider homotopy classes of paths.

NOTATION.

$$\begin{aligned} \pi_1(X, x_0, x_1) &= \{w: (I, 0, 1) \rightarrow (X, x_0, x_1)\} / \simeq \\ &= \{\text{homotopy classes } [w] \text{ of paths } w \text{ with the initial point } \\ &\quad x_0 \text{ and the terminal point } x_1\} \end{aligned}$$

$$\begin{aligned} \pi_1(X, x_0) &= \pi_1(X, x_0, x_0) = \{w: (I, \partial I) \rightarrow (X, x_0)\} / \simeq \\ &= \{\text{homotopy classes } [w] \text{ of closed paths } w \text{ in } X \text{ with the} \\ &\quad \text{base point } x_0\}. \end{aligned}$$

DEFINITION. For two paths $w, w': I \rightarrow X$ satisfying $w(1) = w'(0)$ we define a product of paths:

$$w * w': I \rightarrow X \quad \text{by} \quad (w * w')(t) = \begin{cases} w(2t) & (0 \leq t \leq 1/2), \\ w(2t-1) & (1/2 \leq t \leq 1). \end{cases}$$

The inverse $\bar{w}: I \rightarrow X$ of a path $w: I \rightarrow X$ is defined by

$$\bar{w}(t) = w(1 - t).$$

The constant path at $x \in X$ is the map

$$0_x: I \rightarrow X \text{ defined by } 0_x(t) = x.$$

1.2. The following holds whenever the products are defined:

(1) $w_0 \simeq w_1$ and $w'_0 \simeq w'_1$ imply $w_0 * w'_0 \simeq w_1 * w'_1$, and hence one can define a product between homotopy classes of paths by $[w][w'] = [w * w']$.

(2) $([w][w'])([w'']) = [w]([w']([w'']))$ (which we denote by $[w][w']([w''])$).

(3) $[w][\bar{w}] = [0_x]$, $[\bar{w}][w] = [0_y]$ ($w(0) = x$, $w(1) = y$).

(4) $[0_x][w] = [w] = [w][0_y]$ ($w(0) = x$, $w(1) = y$).

(5) $w \simeq w': (I, 0, 1) \rightarrow (X, x_0, x_1)$ and $f \simeq g: (X, x_0, x_1) \rightarrow (Y, y_0, y_1)$ imply $f \circ w \simeq g \circ w': (I, 0, 1) \rightarrow (Y, y_0, y_1)$ and hence a morphism

$$f_*: \pi_1(X, x_0, x_1) \rightarrow \pi_1(Y, y_0, y_1)$$

is defined by $f_*[w] = [f \circ w]$ and $f \simeq g$ implies $f_* = g_*$, and $1_{X_*} = 1$.

(6) $f_*([w][w']) = f_*[w]f_*[w']$, $(f' \circ f)_* = f'_* \circ f_*$.

1.3. (1) The set $\pi_1(X, x_0)$ forms a group under the product of homotopy classes of closed paths.

(2) For a map $f: (X, x_0) \rightarrow (Y, y_0)$, the induced map $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is a homomorphism and

$$f \simeq g \text{ implies } f_* = g_*, \quad (f' \circ f)_* = f'_* \circ f_*, \quad 1_{X_*} = 1.$$

(3) If there exists a path connecting x_0 and x_1 , then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ (isomorphism).

DEFINITION. $\pi_1(X, x_0)$ is called the fundamental group of X at x_0 .

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0) \text{ (isomorphism)}. \quad (1.4)$$

The isomorphism is given by the correspondence $\alpha \mapsto (p_{X*}\alpha, p_{Y*}\alpha)$, where p_X and p_Y are the projections to X and Y respectively. The inverse morphism is given by $(\alpha, \beta) \mapsto (i_{X*}\alpha)(i_{Y*}\beta)$, where $i_X = (\cdot, y_0)$ and $i_Y = (x_0, \cdot)$.

Covering spaces

REMINDER. X is called *locally arcwise connected* if for an arbitrary neighborhood U of each point x of X , there is an arcwise connected neighborhood of x contained in U .

DEFINITION. X is called *1-connected* if X is arcwise connected (i.e., $\pi_0(X) = 0$) and *simply connected* (i.e., $\pi_1(X) = 0$).

DEFINITION. $(\tilde{X}, p, X)$ is called a *covering space* if the following conditions are satisfied.

(1) The spaces $\tilde{X}$ and X are arcwise connected as well as locally arcwise connected,

(2) The map $p: \tilde{X} \rightarrow X$ is a continuous surjection and for any point x of X there is an arcwise connected neighborhood U of x such that each arcwise connected component U_α of $p^{-1}(U)$ is open and that $p|_{U_\alpha}: U_\alpha \cong U$ (homeomorphism). (We call such a U a *fundamental neighborhood*).

We say that X is the *base space* of the covering, p is the *projection* or *covering map*, $\tilde{X}$ is the *total space* or the *covering space* of X , and $p^{-1}(x)$ is the *fibre* over x of X .

It follows from the definition that $p^{-1}(x)$ has the discrete topology and that for a fundamental neighborhood U of x , we have

$$U \times p^{-1}(x) \cong p^{-1}(U) \quad (\text{homeomorphism}).$$

DEFINITION. Two covering spaces $(\tilde{X}, p, X)$ and $(\tilde{Y}, q, Y)$ are called *equivalent* if there are homeomorphisms $\tilde{h}$ and h such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{h}} & \tilde{Y} \\ \downarrow p & & \downarrow q \\ X & \xrightarrow{h} & Y \end{array}$$

is commutative.

We often identify equivalent covering spaces.

DEFINITION. For a covering space $(\tilde{X}, p, X)$ and a map $f: Y \rightarrow X$, a map $\tilde{f}: Y \rightarrow \tilde{X}$ satisfying $p \circ \tilde{f} = f$ is called a *lifting* of f (relative to p).

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow \tilde{f} & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

LEMMA 1.5. Let $(\tilde{X}, p, X)$ be a covering space. For a path $w: I \rightarrow X$ and a point $\tilde{x}$ of $\tilde{X}$ satisfying $p(\tilde{x}) = w(0)$, there exists a unique lifting $\tilde{w}$ of w such that $\tilde{w}(0) = \tilde{x}$. Moreover, if $w \simeq w_1$, then for their liftings $\tilde{w}$ and $\tilde{w}_1$, $w(0) = \tilde{w}_1(0)$ implies $\tilde{w}(1) = \tilde{w}_1(1)$ and $\tilde{w} \simeq \tilde{w}_1$.

PROOF. The lemma is obvious if $w(I)$ is contained in a fundamental neighborhood. In general, $w(I)$ can be covered by a finite number of fundamental neighborhoods since it is compact. From this the first half follows easily. The latter half: given a homotopy $H: I \times I \rightarrow X$ of $w \simeq w_1$ divide I so finely into a finite number of intervals I_i that $H(I_i \times I_j)$ is contained in one fundamental neighborhood. From this we can see that there is a lifting $\tilde{H}: I \times I \rightarrow \tilde{X}$ of H satisfying $\tilde{H}(0 \times I) = \tilde{w}(0) = \tilde{w}_1(0)$. We have that $\tilde{H}(1 \times I)$ is a constant path at $\tilde{x}_1$, since $\tilde{H}(1 \times I) \subset p^{-1}(x_1)$ and $x_1 = w(1) = w_1(1)$ and since $p^{-1}(x_1)$ has discrete topology. Then

$\tilde{w}(1) = \tilde{w}_1(1) = \tilde{x}_1$ and $\tilde{w} \simeq \tilde{w}_1$, since $\tilde{H}|I \times 0 = \tilde{w}$ and $H|I \times 1 = \tilde{w}_1$ by the first half. Q.E.D.

THEOREM 1.6. *Let $(\tilde{X}, p, X)$ be a covering space.*

(1) *For any two points x_0, x_1 of X there is a bijection*

$$p^{-1}(x_0) \cong p^{-1}(x_1).$$

(2) *The map $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is a monomorphism.*

(3) *There is a bijection*

$$\psi: \pi_1(X, x_0) / \text{Im } p_* = \{(\text{Im } p_*)\alpha \mid \alpha \in \pi_1\} \cong p^{-1}(x_0).$$

PROOF. (1) and (2) are obvious by Lemma 1.5.

(3) We fix $\tilde{x}_0 \in p^{-1}(x_0)$. For $[w] \in \pi_1(X, x_0)$ we consider a lift $\tilde{w}$ of w satisfying $\tilde{w}(0) = \tilde{x}_0$ and put $\tilde{\psi}[w] = \tilde{w}(1)$, which determines a map $\tilde{\psi}: \pi_1(X, x_0) \rightarrow p^{-1}(x_0)$ by Lemma 1.5. The map $\tilde{\psi}$ is surjective, since there is a path connecting $\tilde{x}_0$ and an arbitrary point of $p^{-1}(x_0)$. For $[v] \in \pi_1(\tilde{X}, \tilde{x}_0)$, one clearly has $\tilde{\psi}(p_*[v][w]) = \tilde{\psi}[w]$. Conversely, if $\tilde{\psi}[w] = \tilde{\psi}[w_1]$ holds, then $v = w * \bar{w}_1$ is a closed path and satisfies $[w][w_1]^{-1} = p_*[v]$. Thus we obtain the bijection ψ . Q.E.D.

DEFINITION. The number of elements (or the cardinality) of $p^{-1}(x_0)$ is called the *covering degree*.

The following fundamental lemma gives a criterion for the existence of a lifting.

LEMMA 1.7. *Let $(\tilde{X}, p, X)$ be a covering space, Y arcwise connected, and $f: (Y, y_0) \rightarrow (X, x_0)$.*

(1) *Let $\tilde{f}$ and $\tilde{g}: Y \rightarrow \tilde{X}$ be liftings of f . Then $\tilde{f}(y_0) = \tilde{g}(y_0)$ implies $\tilde{f} = \tilde{g}$.*

(2) *If Y is locally arcwise connected and $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}))$, where $\tilde{x} \in p^{-1}(x_0)$ then there exists a unique lifting $\tilde{f}: Y \rightarrow \tilde{X}$ of f such that $\tilde{f}(y_0) = \tilde{x}$. In particular, the assumption is satisfied for any $\tilde{x} \in p^{-1}(x_0)$ when Y is locally arcwise connected and 1-connected.*

PROOF. (1) Let y be an arbitrary point of Y . If we choose a path w connecting y_0 and y , then $\tilde{f} \circ w$ and $\tilde{g} \circ w$ are liftings of $f \circ w$ and satisfy $(\tilde{f} \circ w)(0) = \tilde{f}(y_0) = \tilde{g}(y_0) = (\tilde{g} \circ w)(0)$. Lemma 1.5 implies $\tilde{f}(y) = (\tilde{f} \circ w)(1) = (\tilde{g} \circ w)(1) = \tilde{g}(y)$.

(2) For the elements y and w above, let $\tilde{v}: I \rightarrow \tilde{X}$ be a lifting of $f \circ w$ satisfying $\tilde{v}(0) = \tilde{x}_0$ and put $\tilde{f}(y) = \tilde{v}(1)$. Similarly to the proof of the previous theorem, one can see that a map $\tilde{f}: Y \rightarrow \tilde{X}$ is obtained, since the condition $\text{Im } f_* \subset \text{Im } p_*$ implies that $\tilde{f}(y)$ is independent of the choice of $\tilde{v}$. Clearly $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$ and so it suffices to prove the continuity

of $\tilde{f}$. We take a neighborhood $\tilde{U}$ of $\tilde{f}(y)$ to be one of the arcwise connected components of the inverse image $p^{-1}(U)$ of a fundamental neighborhood U . By the assumption and the continuity of f there is an arcwise connected neighborhood V such that $f(V) \subset U$ and hence $\tilde{f}(V) \subset p^{-1}(U)$. $\tilde{f}(V)$ is arcwise connected and $\tilde{f}(V) \cap \tilde{U} \ni \tilde{f}(y)$ so $\tilde{f}(V) \subset \tilde{U}$. Thus $\tilde{f}$ is continuous. Q.E.D.

Covering transformation groups

DEFINITION. A homeomorphism $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ is called a *covering transformation* of a covering $(\tilde{X}, p, X)$ if $p \circ \tilde{f} = p$.

The set of covering transformations with the composition as multiplication forms a group called the *covering transformation group* of $(\tilde{X}, p, X)$.

LEMMA 1.8. (1) If a map $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ satisfies $p \circ \tilde{f} = p$, then $\tilde{f}$ is a covering transformation.

(2) For two points $\tilde{x}_0$ and $\tilde{x}_1$ of $p^{-1}(x_0)$ there is a covering transformation $\tilde{f}$ such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$ iff for a path w connecting $\tilde{x}_0$ and $\tilde{x}_1$, the element $[p \circ w]$ belongs to the normalizer

$$N(H) = \{\alpha \in \pi_1(X, x_0) | \alpha H = H \alpha\} \quad \text{of} \quad H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

PROOF. Suppose there is a lifting $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ of $p: \tilde{X} \rightarrow X$ such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$. For any element $[a] \in \pi_1(\tilde{X}, \tilde{x}_0)$ we have $p_*[a] = p_*\tilde{f}_*[a]$ since $p = p \circ \tilde{f}$ holds. As $[\bar{w}](\tilde{f}_*[a])[w] \in \pi_1(\tilde{X}, \tilde{x}_0)$, by putting $\alpha = [p \circ w] \in \pi_1(X, x_0)$ we have $\alpha^{-1}(p_*[a])\alpha = p_*([\bar{w}]\tilde{f}_*[a][w]) \in H$. The element $[a]$ is arbitrary so $\alpha^{-1}H\alpha \subset H$ and hence $\alpha \in N(H)$.

Conversely, if we put $\alpha = [p \circ w] \in N(H)$ and $\alpha^{-1} = [p \circ \bar{w}] \in N(H)$ then for any $[a] \in \pi_1(\tilde{X}, \tilde{x}_0)$, there is an element $[b] \in \pi_1(\tilde{X}, \tilde{x}_0)$ such that $\alpha(p_*[a])\alpha^{-1} = p_*[b]$. Therefore $p_*[a] = \alpha^{-1}(p_*[b])\alpha = p_*([\bar{w}][b][w]) \in p_*\pi_1(\tilde{X}, \tilde{x}_1)$ and we have $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_1))$. By applying Lemma 1.7(2) to the case $f = p$ and $\tilde{x} = \tilde{x}_1$, we obtain a lifting $\tilde{f}$ of p satisfying $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$.

Now, if $p = p \circ \tilde{f}$ holds, that is, if $\tilde{f}$ is a lifting of p , then by the first half of the above argument a path w connecting $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{x}_1 = f(\tilde{x}_0)$ satisfies $[p \circ w] \in N(H)$. Hence $[p \circ \bar{w}] = [p \circ w]^{-1} \in N(H)$. Since $\bar{w}$ is a path connecting $\tilde{x}_1$ and $\tilde{x}_0$, there is a lifting $\tilde{g}$ of p satisfying $\tilde{g}(\tilde{x}_1) = \tilde{x}_0$ by the latter half of the above argument. Both $\tilde{g} \circ \tilde{f}$ and $1_{\tilde{X}}$ are liftings of p and satisfy $(\tilde{g} \circ \tilde{f})(\tilde{x}_0) = \tilde{x}_0 = 1_{\tilde{X}}(\tilde{x}_0)$, and so we have $\tilde{g} \circ \tilde{f} = 1_{\tilde{X}}$. Similarly $\tilde{f} \circ \tilde{g} = 1_{\tilde{X}}$. Therefore $\tilde{f}$ is a homeomorphism; namely, a covering transformation and thus (1) is proved. Admitting (1), the above argument gives a proof of (2). Q.E.D.

THEOREM 1.9. The covering transformation group of $(\tilde{X}, p, X)$ is isomorphic to the quotient group $N(H)/H$ by taking a covering transformation $\tilde{f}$ to the coset of $[p \circ w]$ of Lemma 1.8(2) in $N(H)/H$.

PROOF. We consider the image $\psi(N(H)/H) \subset p^{-1}(x_0)$ of $N(H)/H$ by a bijection ψ in Theorem 1.6(3), where ψ maps a coset of $[p \circ w]$ to $w(1)$. Lemma 1.8(2) shows that the map which corresponds to a covering transformation $\tilde{f}$ to $\tilde{f}(\tilde{x}_0) = \tilde{x}_1 = w(1)$ is a surjection to $\psi(N(H)/H)$. Lemma 1.8(1) shows that it is an injection. So it remains to show that the map is a homomorphism.

For two covering transformations $\tilde{f}$ and $\tilde{g}$, let w_1 and w_2 be paths connecting $\tilde{x}_0$ to $\tilde{f}(\tilde{x}_0)$ and to $\tilde{g}(\tilde{x}_0)$ respectively. Then $w_1 * (\tilde{f} \circ w_2)$ is a path connecting $\tilde{x}_0$ and $(\tilde{f} \circ \tilde{g})(\tilde{x}_0)$. Since $[p \circ (w_1 * (\tilde{f} \circ w_2))] = [p \circ w_1][p \circ \tilde{f} \circ w_2] = [p \circ w_1][p \circ w_2]$, the element in $N(H)/H$ corresponding to $\tilde{f} \circ \tilde{g}$ is a product of the elements corresponding to $\tilde{f}$ and $\tilde{g}$. Thus we see that our map is a homomorphism. Q.E.D.

DEFINITION. A covering space is said to be *regular* if, for arbitrary points $\tilde{x}_1, \tilde{x}_2$ of $p^{-1}(x_0)$, there is a covering transformation $\tilde{f}$ such that $\tilde{f}(\tilde{x}_1) = \tilde{x}_2$.

THEOREM 1.10. (1) A covering space $(\tilde{X}, p, X)$ is regular iff $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

(2) For a regular covering space $(\tilde{X}, p, X)$, the quotient group $G = \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ acts on $\tilde{X}$ without fixed points (i.e., $g\tilde{x} = \tilde{x}$ implies $g = e$) and, for the projection $\pi: \tilde{X} \rightarrow \tilde{X}/G$, the covering spaces $(\tilde{X}, p, X)$ and $(\tilde{X}, \pi, \tilde{X}/G)$ are equivalent.

PROOF. (1) is obvious by Lemma 1.8(2). (2) is also clear. Q.E.D.

THEOREM 1.11. Suppose that a group G with discrete topology acts on an arcwise connected, locally arcwise connected, topological space $\tilde{X}$ and that each point x of $\tilde{X}$ has a neighborhood V satisfying the condition:

$$V \cap L_g(V) \neq \emptyset \text{ implies } g = e.$$

Then, for the projection $p: \tilde{X} \rightarrow \tilde{X}/G$, $(\tilde{X}, p, \tilde{X}/G)$ is a regular covering space and its covering transformation group is isomorphic with G .

PROOF. One can suppose that V is arcwise connected. Then one may choose $p(V)$ as a fundamental neighborhood containing $p(x)$ and hence one sees easily that $(\tilde{X}, p, \tilde{X}/G)$ is a covering space. It is obvious that the covering space is regular and has a covering transformation group isomorphic to G . Q.E.D.

Universal covering spaces

DEFINITION. A covering space $(\tilde{X}, p, X)$ is said to be *universal*, if $\tilde{X}$ is 1-connected.

THEOREM 1.12. Let $(\tilde{X}, p, X)$ be a universal covering space and $G = \pi_1(X, x_0)$. Then

(1) The covering $(\tilde{X}, p, X)$ is regular. So G acts on $\tilde{X}$ without fixed

points and $(\tilde{X}, p, X)$ is equivalent to $(\tilde{X}, \pi, \tilde{X}/G)$;

(2) For any covering space $(\tilde{X}', p', X)$, there exists a universal covering space $(\tilde{X}, p'', \tilde{X}')$ such that $p = p' \circ p''$. In fact, $\tilde{X}' \cong \tilde{X}/H$ with $H = p'_*(\pi_1(\tilde{X}', \tilde{x}'_0))$;

(3) A universal covering space over X is unique up to equivalence.

PROOF. (1) is obvious from the definition of regularity and Theorem 1.10.

(2) By applying Lemma 1.7(2) to a covering space $(\tilde{X}', p', X)$ and a map $p: \tilde{X} \rightarrow X$, we can obtain a map $p'': \tilde{X} \rightarrow \tilde{X}'$ such that $p = p' \circ p''$. Let $p''(\tilde{x}_0) = \tilde{x}'_0$. For any $\tilde{x}' \in \tilde{X}'$ we consider a path $\tilde{w}'$ connecting $\tilde{x}'_0$ and $\tilde{x}'$ and take a lifting $\tilde{w}$ of $p' \circ \tilde{w}'$ with respect to $(\tilde{X}, p, X)$ satisfying $\tilde{w}(0) = \tilde{x}_0$. By Lemma 1.7(1) $p'' \circ \tilde{w} = \tilde{w}'$, which implies $p''(\tilde{w}(1)) = \tilde{x}'$. Hence p'' is surjective. We take a fundamental neighborhood U common to $(\tilde{X}', p', X)$ and $(\tilde{X}, p, X)$ and denote by U'_β every component of $p'^{-1}(U)$. Then it is easily verified that $(\tilde{X}, p'', \tilde{X}')$ is a covering space with a fundamental neighborhood U'_β . The covering transformation $\tilde{f}$ in $(\tilde{X}, p'', \tilde{X}')$ is the one in $(\tilde{X}, p, X)$. This interpretation corresponds to the monomorphism $p'_*: \pi_1(\tilde{X}') \rightarrow \pi_1(X)$ under the isomorphism of Theorem 1.9. Thus $\tilde{X}' \cong \tilde{X}/\text{Im } p'_* = \tilde{X}/H$.

(3) follows directly from (2). Q.E.D.

This theorem is meaningless if there does not exist a universal covering space $\tilde{X}$ for X . For example, the space

$$X = \mathbf{R}^2 - \{(\frac{1}{n}, 0) | n = 1, 2, 3, \dots\}$$

is arcwise connected as well as locally arcwise connected, but one can see from the following theorem that there is no universal covering space with base space X .

DEFINITION. A space X is called *locally semisimply connected* if for any neighborhood of each point x of X , there is a neighborhood U of x such that $i_*: \pi_1(U, x) \rightarrow \pi_1(X, x)$ for the inclusion $i: U \rightarrow X$.

THEOREM 1.13. Let X be an arcwise connected and locally arcwise connected topological space. Then there exists a universal covering space $(\tilde{X}, p, X)$ with base space X if and only if X is locally semisimply connected.

PROOF. If there exists a universal covering space $(\tilde{X}, p, X)$, we can see from the diagram that for a fundamental neighborhood U , X is locally semisimply connected.

$$\begin{array}{ccc} \pi_1(U_\alpha) & \longrightarrow & \pi_1(\tilde{X}) = 0 \\ \cong \downarrow (p|_{U_\alpha})_* & & \downarrow p_* \\ \pi_1(U) & \xrightarrow{i_*} & \pi_1(X) \end{array}$$

Conversely, we will construct a universal covering space $(\tilde{X}, p, X)$ when X is locally semisimply connected as follows.

Let x_0 be a fixed point of X . Put $\tilde{X} = \{[u] \text{ the homotopy class of a path } u: (I, 0, 1) \rightarrow (X, x_0, x_1) \text{ with base point } x_0\}$ and define a map $p: \tilde{X} \rightarrow X$ by $p[u] = u(1)$.

Introduce a topology in $\tilde{X}$ by specifying that V is a neighborhood of $[u]$ iff there is a neighborhood $U \subset X$ of $p[u] = u(1)$ such that $[u * u'] \in V$ for any path $u': I \rightarrow U$ with initial point $u(1)$. It is then clear that p is a continuous surjection.

We choose an arcwise connected neighborhood U of $x \in X$ such that $i_* = 0: \pi_1(U, x) \rightarrow \pi_1(X, x)$. For an arbitrary point $[u]$ of $p^{-1}(U)$ we take a path v in U connecting $p[u]$ and x and put $\tilde{x}_\alpha = [u * v] \in p^{-1}(x)$. It follows from the equality $i_* = 0$ that $\tilde{x}_\alpha$ is determined independently of the choice of v . Let U_α be the set of such $[u]$, then $p^{-1}(U) = \bigcup_\alpha U_\alpha$, $U_\alpha \cap U_\beta = \emptyset$ for $\alpha \neq \beta$ and $p|_{U_\alpha}: U_\alpha \rightarrow U$ is bijective. Moreover, as the neighborhoods U_α correspond to those in U under p , one can see that $p|_{U_\alpha}$ is a homeomorphism and that every U_α is open in $p^{-1}(U)$. Thus we have proved that $(\tilde{X}, p, X)$ is a covering space.

For any $[w] \in \pi_1(X, x_0)$, a lifting $\tilde{w}: (I, 0) \rightarrow (\tilde{X}, [0])$ of $w: (I, \partial I) \rightarrow (X, x_0)$ is defined by $\tilde{w}(s) = [w_s]$, where $w_s(t) = w(st)$. Putting $\tilde{\psi}[w] = \tilde{w}(1) \in p^{-1}(\tilde{x}_0)$, the map $\tilde{\psi}: \pi_1(X, x_0) \rightarrow p^{-1}(x_0)$ is bijective, since $p^{-1}(x_0) = \{[w]|w: (I, 0, 1) \rightarrow (X, x_0, x_0)\} = \pi_1(X, x_0)$. As the bijection ψ in Theorem 1.6(3) is induced by $\tilde{\psi}$, we have $\text{Im } p_* = \text{Ker } \psi = 0$. So $\psi_1(\tilde{X}) = 0$, since $p_*: \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is a monomorphism; hence, $(\tilde{X}, p, X)$ is a universal covering space. Q.E.D.

COROLLARY 1.14. *If a topological manifold (in particular, a differentiable manifold, a Lie group, or a homogeneous space) is arcwise connected, then there exists a universal covering space with it as base space.*

One can regard a covering space as a special case of a fibre bundle defined in the following section.

§2. Fundamental properties of fibre bundles

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The definition of fibre bundles. Below we state the basic facts about fibre bundles. For details we refer to Steenrod [13].

DEFINITION. (E, p, B, F) is a *product bundle* or a *trivial fibration* if there is a homeomorphism $\phi: B \times F \rightarrow E$ such that $p\phi(b, y) = b$. $\xi = (E, p, B, F)$ is called a *locally trivial fibration* if $p: E \rightarrow B$ is a map and for each point there is an open neighborhood U of the point such that

$$\xi|U = (p^{-1}(U), p|_{p^{-1}(U)}, U, F)$$

is a product bundle.

In this case E is called the *total space*, p the *projection*, B the *base space*, F the *fibre*, $F_b = p^{-1}(b)$, where $b \in B$ the *fibre over b* , U a (*coordinate neighborhood*) and the homeomorphism $\phi_U: U \times F \rightarrow p^{-1}(U)$ a *coordinate function*.

DEFINITION. A map $f: \xi' \rightarrow \xi$ between locally trivial fibrations, $\xi' = (E', p', B', F)$ and $\xi = (E, p, B, F)$ with the same fibre is called a *fibre map* if $f: E' \rightarrow E$ is a map by which each fibre $F_{b'}$ over $b' \in B'$ is mapped to a fibre F_b over $b \in B$.

Then the map $\bar{f}: B' \rightarrow B$ defined by $b = \bar{f}(b')$ satisfies $p \circ f = \bar{f} \circ p'$.

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{\bar{f}} & B \end{array}$$

For a locally trivial fibration $\xi = (E, p, B, F)$ and a map $g: B' \rightarrow B$ we put

$$\begin{aligned} g^{\#}E &= \{(x, b') | p(x) = g(b')\} \subset E \times B', \\ \bar{g}(x, b') &= x, \quad p'(x, b') = b'. \end{aligned}$$

One can see that $g^{\#}\xi = (g^{\#}E, p', B', F)$ is a locally trivial fibration and that $\bar{g}: g^{\#}E \rightarrow E$ is a fibre map satisfying $\bar{g} = g$.

We call $g^{\#}\xi$ the *locally trivial fibration induced from ξ by g* .

Suppose that a topological group G acts on the fibre F from the left.

DEFINITION. $(E, p, B, F, G, \{U_{\alpha}, \phi_{\alpha}\})$ is called a *coordinate bundle* if (E, p, B, F) is a locally trivial fibration with $\{U_{\alpha}, \phi_{\alpha}\}$ as the family of coordinate neighborhoods and coordinate functions, and a (continuous) map $g_{\beta\alpha}: U_{\alpha} \cap U_{\beta} \rightarrow G$ determined by

$$\phi_{\alpha}(b, y) = \phi'_{\beta}(b, g_{\beta\alpha}(b) \cdot y), \quad \text{where } b \in U_{\alpha} \cap U_{\beta}.$$

We call $\{g_{\beta\alpha}(b)\}$ *coordinate transformations* and G the *structure group*.

DEFINITION. A map $f: \xi \rightarrow \xi'$ between two coordinate bundles $\xi = (E, p, B, F, G, \{U_{\alpha}, \phi_{\alpha}\})$ and $\xi' = (E', p', B', F, G, \{U'_{\beta}, \phi'_{\beta}\})$ is called a *bundle map* if f is a fibre map between locally trivial fibrations and the map $\bar{g}_{\beta\alpha}: U_{\alpha} \cap \bar{f}^{-1}(U'_{\beta}) \rightarrow G$ determined by $f\phi_{\alpha}(b, y) = \phi'_{\beta}(\bar{f}(b), \bar{g}_{\beta\alpha}(b) \cdot y)$ is continuous, where $b \in U_{\alpha} \cap \bar{f}^{-1}(U'_{\beta})$.

DEFINITION. Two coordinate bundles $\xi = (E, p, B, F, G, \{U_{\alpha}, \phi_{\alpha}\})$ and $\xi' = (E, p, B, F, G, \{U'_{\beta}, \phi'_{\beta}\})$ differing only in their coordinate systems are called *equivalent in the strict sense* if the identity map $1_E: E \rightarrow E$ is a bundle map $1_E: \xi \rightarrow \xi'$.

This is easily seen to be an equivalence relation.

DEFINITION. An equivalence class of coordinate bundles by the equivalence in the strict sense is called a *fibre bundle* and denoted by $\xi = (E, p, B, F, G)$.

When we discuss fibre bundles, we use coordinate neighborhoods, coordinate functions, and coordinate transformations of a suitable coordinate bundle that is representative of the fibre bundle. Terms referring to coordinate bundles such as total space, projection, base space, fibre, structure group, fibre map, and bundle map are also used in discussing fibre bundles.

Given a coordinate bundle $\xi = (E, p, B, F, G, \{U_\alpha, \phi_\alpha\})$, if one considers a map

$$\coprod_{\alpha} \phi_{\alpha}: \coprod_{\alpha} U_{\alpha} \times F \rightarrow E,$$

of the topological sum of $U_{\alpha} \times F$ to the total space E , then E is seen to be constructed as follows

$$E = \coprod_{\alpha} U_{\alpha} \times F / \sim; \quad (b, f)(\in U_{\alpha} \times F) \sim (b, g_{\beta\alpha}(b) \cdot f)(\in U_{\beta} \times F).$$

From this we obtain the following theorem.

THEOREM 2.1. (1) *Coordinate bundles having B , F , and G as base space, fibre, and structure group respectively are in one-to-one correspondence with families $(U, g) = \{U_{\alpha}, g_{\beta\alpha}\}$ of open coverings $\{U_{\alpha}\}$ of B and maps $g_{\beta\alpha}: U_{\alpha} \cap U_{\beta} \rightarrow G$ satisfying the relation*

$$g_{\gamma\beta}(b)g_{\beta\alpha}(b) = g_{\gamma\alpha}(b), \quad b \in U_{\alpha} \cap U_{\beta} \cap U_{\gamma}.$$

(2) *Fibre bundles with B , F , G as above are in one to one correspondence with limits $\varinjlim (U, g)_{\lambda}$ with respect to refinement and restriction of maps.*

The tangent bundle TM of a differentiable manifold M is constructed as above with $g_{\beta\alpha}(b) \in GL(n, \mathbf{R})$. Therefore the tangent bundle of M is a fibre bundle:

$$\tau_M = (TM, p, M, \mathbf{R}^n, GL(n, \mathbf{R})).$$

DEFINITION. A fibre bundle with fibre $\mathbf{F}^n$ and structure group G that is a subgroup of $GL(n, \mathbf{F})$ is called an n -dimensional $\mathbf{F}$ -vector bundle.

THEOREM 2.2. (1) *Every fibre $\mathbf{F}_b^n = p^{-1}(b)$ of an n -dimensional $\mathbf{F}$ -vector bundle $\xi = (E, p, B, \mathbf{F}^n, GL(n, \mathbf{F}))$ has the structure of a natural right $\mathbf{F}$ -module (a vector space when $\mathbf{F} = \mathbf{R}, \mathbf{C}$), and the right action of $\mathbf{F}: E \times \mathbf{F} \rightarrow E$ and addition $\{(\mathbf{u}, \mathbf{v}) | p(\mathbf{u}) = p(\mathbf{v})\} \rightarrow E$ are continuous.*

(2) *Conversely, if a locally trivial fibration $(E, p, B, \mathbf{F}^n)$ has continuous addition and the right action of $\mathbf{F}$, then it has the structure of an n -dimensional $\mathbf{F}$ -vector bundle.*

Simple properties of fibre bundles

THEOREM 2.3. *If in a bundle map $f: \xi \rightarrow \xi'$, $B = B'$, and $\bar{f} = 1_B$, then $f: E \rightarrow E'$ is a homeomorphism and the inverse map $f^{-1}: E' \rightarrow E$ gives a bundle map $f^{-1}: \xi' \rightarrow \xi$.*

DEFINITION. Two fibre bundles ξ and ξ' with the same B, F, G are called *equivalent* (Notation. $\xi \equiv \xi'$) if there is a bundle map $f: \xi \rightarrow \xi'$ such that $\bar{f} = 1_B$.

THEOREM 2.4. *The locally trivial fibration induced from a fibre bundle $\xi' = (E', p', B', F, G)$ by a map $g: B \rightarrow B'$ is a fibre bundle $g^*\xi' = (g^*E', p, B, F, G)$ and $\tilde{g}: g^*\xi' \rightarrow \xi'$ is a bundle map. Conversely, for a bundle map $f: \xi \rightarrow \xi'$, we have $\xi \equiv \bar{f}^*\xi'$.*

THEOREM 2.5. *Let $\{g_{\beta\alpha}\}$ and $\{g'_{\beta\alpha}\}$ be the coordinate transformations of two fibre bundles $\xi = (E, p, B, F, G)$ and $\xi' = (E', p', B', F, G)$ with a common system of coordinate neighborhoods $\{U_\alpha\}$. Then $\xi \equiv \xi'$ iff there is a family $\{\lambda_\alpha\}$ of maps $\lambda_\alpha: U_\alpha \rightarrow G$ satisfying the following identities:*

$$g'_{\beta\alpha}(b) = \lambda_\beta(b)^{-1} g_{\beta\alpha}(b) \lambda_\alpha(b), \quad b \in U_\alpha \cap U_\beta.$$

These theorems can be verified directly from the definitions.

DEFINITION. A topological space is called *paracompact* if for an arbitrary open covering $\{U_\alpha\}$ of X , $X = \bigcup_\alpha U_\alpha$, there exists a closed covering $\{V_\beta\}$ that is a refinement of $\{U_\alpha\}$ (i.e., each V_β is contained in a certain U_α) and *locally finite* (i.e., each point of X has a neighborhood W such that the number of β such that $W \cap V_\beta \neq \emptyset$ is finite).

We quote the following results from the theory of topological spaces:

LEMMA 2.6. (1) *A paracompact Hausdorff space is normal.*

(2) *A locally compact Hausdorff space satisfying the second axiom of countability is a paracompact space.*

(3) *For an open covering $\{U_\alpha\}$ of a paracompact Hausdorff space, there are its locally finite refinement $\{V_\beta\}$ and an open covering $\{W_\beta\}$ satisfying $V_\beta \subset \bar{W}_\beta$. Moreover, there are continuous functions e_β such that $e_\beta(X - W_\beta) = 0$ and $\sum_\beta e_\beta = 1$ (a partition of unity).*

DEFINITION. A map $s: B_0 \rightarrow E$ is called a *cross-section* on $B_0 \subset B$ of a map $p: E \rightarrow B$ if $p \circ s = 1_{B_0}$.

A cross-section $s: B \rightarrow E$ on B is simply called a *cross-section* of p .

A cross-section of a fibre bundle (E, p, B, F, G) is defined to be a cross-section of the projection p .

LEMMA 2.7. *In a fibre bundle (E, p, B, F, G) , if B is a paracompact Hausdorff space and F is homeomorphic with $\mathbb{R}^n$, then a cross-section on a closed set B_0 of B is extendable over B .*

To prove the lemma, one considers a refinement $\{V_\beta\}$ in Lemma 2.6(3) for a system of coordinate neighborhoods $\{U_\alpha\}$, well orders the index set

$\{\beta\}$, and extends successively the cross-section over each $\overline{V}_\beta$ by transfinite induction. At this point, for $p^{-1}(\overline{V}_\beta) \cong \overline{V}_\beta \times \mathbf{R}^n \rightarrow \overline{V}_\beta$, in order to extend a given cross-section on $\overline{W} = \bigcup_{\gamma < \beta} (\overline{V}_\gamma \cap \overline{V}_\beta)$ over $\overline{V}_\beta$, it suffices to show that an arbitrary continuous map $\overline{W} \rightarrow \mathbf{R}^n$ is extendable over $\overline{V}_\beta$. This is assured by the Urysohn lemma, since B is a normal space.

Principal bundle

DEFINITION. A fibre bundle $\xi = (E, p, B, F, G)$ with $G = F$ is called a G -bundle or a *principal G -bundle*, denoted by $\xi = (E, p, B, G)$, if the action of G on itself is given by the multiplication of the topological group.

THEOREM 2.8. *A principal bundle has a cross-section iff the principal bundle is equivalent to the trivial principal bundle.*

DEFINITION. A cross-section $s: U \rightarrow E$ of a map $p: E \rightarrow B$ at a point $b \in B$ is called a *local cross-section* if U is a neighborhood of b and s is a cross-section of $p|_{p^{-1}(U)}: p^{-1}(U) \rightarrow U$.

Obviously, the projection of a locally trivial fibration admits a local cross-section (at every point of the base space).

THEOREM 2.9. *In a G -bundle $\xi = (E, p, B, G)$ a right action $E \times G \rightarrow E$ of G on E satisfying $p(x \cdot g) = p(x)$ can be defined by*

$$\phi_\alpha(b, g) \cdot g' = \phi_\alpha(b, gg'), \quad b \in U_\alpha.$$

Conversely, suppose given a right action $E \times G \rightarrow E$ of a topological group G on a topological space E . If the projection $p: E \rightarrow B = E/G$ admits a (local) cross-section $s_\alpha: U_\alpha \rightarrow E$ on a neighborhood U_α of each point and if $\phi_\alpha: U_\alpha \times G \rightarrow p^{-1}(U_\alpha)$ determined by $\phi_\alpha(b, g) = s_\alpha(b) \cdot g$ is a homeomorphism, then $\xi = (E, p, B, G)$ is a G -bundle.

A locally trivial fibration with the property of the latter half of the theorem can be taken as the definition of a principal bundle. In order to define a fibre bundle, starting with this definition, one can make use of the construction below.

When a topological group G acts on X from the right and on Y from the left, we denote by

$$X \times_G Y = X \times Y / \sim$$

the space obtained as the quotient by the relation

$$(x, y) \sim (x \cdot g, g^{-1} \cdot y), \quad g \in G, x \in X, y \in Y$$

of the product space $X \times Y$. If maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ commute with the action above, they induce a map

$$f \times_G g: X \times_G Y \rightarrow X' \times_G Y'.$$

The following theorem can be easily proved by referring to the definition.

THEOREM 2.10. *For a principal bundle $\xi = (E, p, B, G)$ and F on which G acts from the left, $\xi \times_G F = (E \times_G F, p', B, F, G)$ is a fibre bundle with fibre F where $p'(x, y) = p(x)$. Conversely, if the action of G on F is effective, there exists a G -bundle ξ for any fibre bundle $\xi' = (E', p', B, F, G)$ such that $\xi' \equiv \xi \times_G F$, and the assignment $\xi' \mapsto \xi \times_G F$ gives a one to one correspondence between principal bundles and fibre bundles. There is also a one to one correspondence $f \mapsto f \times_G 1_F$ between bundle maps.*

DEFINITION. If $\xi' \equiv \xi \times_G F$, then ξ' is called the *associated fibre bundle* of the principal bundle ξ and ξ the *associated principal bundle* of the fibre bundle ξ' .

Let E be a topological group, G a closed subgroup, and e the identity element. If the projection

$$p: E \rightarrow E/G$$

has a local cross-section $s: U \rightarrow E$ at $p(e)$, then for each point x of E , a map $s_x: U_x \rightarrow E$ given by $s_x(b) = x \cdot s(x^{-1} \cdot b)$ for $b \in U_x = x \cdot U$ is a local cross-section at $p(x)$ and satisfies the condition of the latter half of Theorem 2.10. Then for any subgroup H of G we have $E \times_G (G/H) = E/H$ and, hence we obtain the following:

THEOREM 2.11. *If $p: E \rightarrow E/G$ in the above has a local cross-section at $p(e)$, then $\xi = (E, p, E/G, G)$ is a G -bundle and $\xi' = (E/H, p', E/G, G/H, G)$ is the associated fibre bundle of ξ with fibre G/H .*

Here the action of the structure group G on the fiber G/H is not in general effective.

For the case where the action of G on F in a fibre bundle (E, p, B, F, G) is not effective, the quotient group G/G_0 by a normal closed subgroup $G_0 = \{g \in G | L_g = 1_F\}$ acts effectively on F and thus a fibre bundle $(E, p, B, F, G/G_0)$ is naturally obtained from the above fibre bundle. In Theorem 2.11 G_0 is a maximal normal subgroup contained in H .

It follows from Corollary 5.18 of Chapter I that:

COROLLARY 2.12. *Let G be a Lie group, H and K subgroups, with $H \supset K$. Then $(G, p, G/H, H)$ is an H -bundle and $(G/K, p, G/H, H/K, H)$ is a fibre bundle.*

For example, $(U(n, \mathbf{F}), p, S(\mathbf{F}^n), U(n-1, \mathbf{F}))$ is a principal bundle. (See Theorem 2.10 of Chapter I.)

Homotopy of bundles For a fibre bundle $\xi = (E, p, B, F, G)$ with coordinate neighborhoods $\{U_\alpha, \phi_\alpha\}$ and the unit interval $I = [0, 1]$, we denote by

$$\xi \times I = (E \times I, p \times 1_I, B \times I, F, G),$$

the fibre bundle with coordinate system given by

$$\{U_\alpha \times I, \Phi_\alpha\}, \quad \text{where } \Phi_\alpha((b, t), y) = (\phi_\alpha(b, y), t) \text{ for } t \in I.$$

2.13. (1) $\pi^*\xi \equiv \xi \times I$ for the projection $\pi: B \times I \rightarrow B$.

(2) $i_t^*(\xi \times I) \equiv \xi$ for the inclusion $i_t: B \rightarrow B \times I$ given by $i_t(b) = (b, t)$.

DEFINITION. We call a bundle map $H: \xi \times I \rightarrow \xi'$ and $h_t = H(\cdot, t): \xi \rightarrow \xi'$ a homotopy of bundles. We then say that h_0 and h_1 are homotopic and write $h_0 \simeq h_1: \xi \rightarrow \xi'$.

The maps $\bar{H}: B \times I \rightarrow B'$ and $\bar{h}_t: B \rightarrow B'$ are ordinary homotopies.

THEOREM 2.14 (The covering homotopy theorem in bundles). *Let ξ and ξ' be fibre bundles with the same structure group G and the fibre F and suppose that the base space of ξ is a paracompact Hausdorff space. If a bundle map $h: \xi \rightarrow \xi'$ and a homotopy $\bar{H}: B \times I \rightarrow B'$ induce $\bar{h} = \bar{h}_0$, then there exists a homotopy of bundles $H: \xi \times I \rightarrow \xi'$ inducing $\bar{H}$ such that $h = h_0$. Here, if $\bar{h}_t = \bar{h}_a$ for $t \in [a, b]$, we can take H so that $h_t = h_a$ for $t \in [a, b]$.*

This theorem is of fundamental importance in the theory of fibre bundles. The theorem is easily proved if ξ is a trivial bundle, and is obvious locally. The paracompactness of the base space is used to extend the local property over the whole.

COROLLARY 2.15. *Let B be a paracompact Hausdorff space.*

(1) $\xi' \equiv i_0^*\xi' \times I$ for a fibre bundle ξ' over $B \times I$.

(2) For a fibre bundle ξ over B' , $f \simeq g: B \rightarrow B'$ iff $f^*\xi \equiv g^*\xi$.

(3) If B is contractible (i.e., $1_B \simeq 0: B \rightarrow B$), then ξ is equivalent to a trivial bundle.

If we restrict the action of the structure group G on the fibre F to a closed subgroup H of G , then H acts on F . One can then naturally regard a fibre bundle $\xi = (E, p, B, F, H)$ with structure group H as a fibre bundle $\xi_G = (E, p, B, F, G)$ with structure group G . Conversely, if there is a fibre bundle ξ with structure group H as above for a fibre bundle $\xi' = (E, p, B, F, G)$ with structure group G such that there is an equivalence $\xi' \equiv \xi_G$ as fibre bundles with structure group G , we say that the structure group G is reducible to H and that ξ is a fibre bundle obtained by reducing the structure group G of ξ' to H .

THEOREM 2.16. *Suppose that the base space B is a paracompact Hausdorff space, the projection $G \rightarrow G/H$ has a local cross-section and $G/H \cong \mathbb{R}^n$ (a homeomorphism). Then the following hold:*

(1) *The structure group of a fibre bundle $\xi = (E, p, B, F, G)$ over B is reducible to H .*

(2) *Suppose that G acts on F effectively. If two fibre bundles obtained by reducing the structure group G to H are equivalent as fibre bundles with structure group G , then they are also equivalent as fibre bundles with structure group H .*

For a proof of the theorem we consider the associated G -bundle ξ_G and its associated fibre bundle $\xi_G/H = \xi_G \times_G (G/H)$. (1) is reduced to showing that ξ_G/H has a section, and this is proved by Lemma 2.7.

(2) The structure group of a G -bundle η_G over $B \times I$ inducing $\xi_G \equiv \xi'_G$ is reducible to H on $B \times \{0, 1\}$ and η_G/H has a section on $B \times \{0, 1\}$. It is proved similarly to Lemma 2.7 that the section is extendable over $B \times I$ and hence the structure group of η_G is reducible to H , from which (2) is obtained.

Applying Theorem 4.11 of Chapter I to this theorem we obtain:

COROLLARY 2.17. *Suppose that the base space B is a paracompact Hausdorff space. There is a one to one correspondence by reduction between equivalence classes of n -dimensional $\mathbf{F}$ -vector bundles with structure group $GL(n, \mathbf{F})$ and those with $U(n, \mathbf{F})$.*

§3. Homotopy groups and fibrations

The fundamental properties of homotopy groups. Homotopy groups are defined as generalizations of the fundamental group to higher dimensions.

NOTATION.

$$\begin{aligned} & [X, A_1, \dots, A_n; Y, B_1, \dots, B_n] \\ &= \{h: (X, A_1, \dots, A_n) \rightarrow (Y, B_1, \dots, B_n)\} / \simeq \\ &= \{\text{the set of homotopy classes } [h] \text{ of the above } h\}. \\ I^n &= \{(t_1, \dots, t_n) \in \mathbf{R}^n | 0 \leq t_i \leq 1, i = 1, \dots, n\}: \text{ the} \\ & \quad \text{unit } n\text{-cube.} \\ \partial I^n &= \{(t_1, \dots, t_n) \in I^n | \prod_{i=1}^n t_i(1-t_i) = 0\}: \text{ the boundary of } I^n. \\ J^{n-1} &= \{(t_1, \dots, t_n) \in I^n | t_n \prod_{i=1}^{n-1} t_i(1-t_i) = 0\} = \partial I^n - I^{n-1} \times 0. \\ \pi_n(X, A, x_0) &= [I^n, \partial I^n, J^{n-1}; X, A, x_0] \quad (x_0 \in A \subset X). \\ \pi_n(X, x_0) &= [I^n, \partial I^n; X, x_0] \quad (= \pi_n(X, x_0, x_0)). \end{aligned}$$

For two maps $h_1, h_2: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ we define addition

$$h_1 + h_2: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$$

by

$$(h_1 + h_2)(t_1, \dots, t_n) = \begin{cases} h_1(2t_1, t_2, \dots, t_n) & (t_1 \leq \frac{1}{2}), \\ h_2(2t_1 - 1, t_2, \dots, t_n) & (t_1 \geq \frac{1}{2}), \end{cases}$$

where the definition makes sense only if $n > 1$, or $n = 1$ and $A = x_0$. It is well known that the addition is defined also on homotopy classes and $\pi_n(X, A, x_0)$ forms a group under this addition. We call it the n th (n -dimensional) homotopy group of (X, A, x_0) . The case $n = 1$ and $A = x_0$ is the fundamental group.

We list the fundamental properties of homotopy groups.

3.1. $f: (X, A, x_0) \rightarrow (Y, B, y_0)$ induces a morphism

$$f_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0) \quad \text{where } f_*[h] = [f \circ h],$$

with the properties: $f \simeq f'$ implies $f_* = f'_*$, $(g \circ f)_* = g_* \circ f_*$, and $(1_X)_* = 1$. f_* is a homomorphism: $f_*(\alpha + \beta) = f_*(\alpha) + f_*(\beta)$ whenever addition is defined.

3.2. If $n > 2$, or $n = 2$ and $A = x_0$, $\pi_n(X, A, x_0)$ is an abelian group.

The class of the constant map $h_0: I^n \rightarrow x_0$ is denoted by $0 \in \pi_n(X, A, x_0)$. When $\pi_n(X, A, x_0)$ is a group, 0 is the identity element.

DEFINITION. A sequence of maps between sets with the distinguished elements

$$0: \cdots \rightarrow G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \rightarrow \cdots$$

is called *exact* if $\text{Ker } f_n (= f_n^{-1}(0)) = \text{Im } f_{n+1} (= f_{n+1}(G_{n+1}))$ for each n .

3.3. The sequence

$$\begin{aligned} \cdots \rightarrow \pi_{n+1}(X, A, x_0) &\xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \\ &\xrightarrow{j_*} \pi_n(X, A, x_0) \rightarrow \cdots \rightarrow \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A) \xrightarrow{i_*} \pi_0(X) \end{aligned}$$

is exact, where $i: (A, x_0) \rightarrow (X, x_0)$ and $j: (X, x_0, x_0) \rightarrow (X, A, x_0)$ are the inclusions and $\partial: \pi_{n+1}(X, A, x_0) \rightarrow \pi_n(A, x_0)$ is given by

$$\partial[h] = [h|I^n \times 0],$$

which is a homomorphism for $n \geq 1$. Further, ∂ is natural (that is, the diagram below is commutative).

$$\begin{array}{ccc} \pi_{n+1}(X, A, x_0) & \xrightarrow{\partial} & \pi_n(A, x_0) \\ \downarrow f_* & & \downarrow f_* \\ \pi_{n+1}(Y, B, y_0) & \xrightarrow{\partial} & \pi_n(B, y_0). \end{array}$$

3.3'. The sequence

$$\begin{aligned} \cdots \rightarrow \pi_{n+1}(X, A, x_0) &\xrightarrow{\partial} \pi_n(A, B, x_0) \xrightarrow{i_*} \\ &\pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, x_0) \rightarrow \cdots \rightarrow \pi_1(X, A, x_0) \end{aligned}$$

is exact, where i and j are the inclusions and the definition of ∂ is similar to the above.

3.4. For a map $h: (I^n, \partial I^n) \rightarrow (X, x_1)$ and a path $w: (I, 0, 1) \rightarrow (X, x_0, x_1)$, there is a homotopy $h_t: I^n \rightarrow X$ such that $h_1 = h$ and $h_t(\partial I^n) = w(t)$. The class $[w]$ induces an isomorphism

$$[w]_*: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0) \quad \text{where } [w]_*[h] = [h_0],$$

which is independent of the choice of representatives w and h , and satisfies

$$[w * w']_* = [w]_* \circ [w']_* \quad \text{and} \quad [0]_* = 1.$$

In particular, $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$.

It follows from this that if X is arcwise connected, $\pi_n(X, x_0)$ is independent of the choice of the base point x_0 up to isomorphism. We sometimes denote it simply by $\pi_n(X)$, omitting x_0 . The same holds for $\pi_n(X, A, x_0)$ if A is arcwise connected, and we denote it by $\pi_n(X, A)$.

3.4'. For a map $h: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_1)$ and a path $w: (I, 0, 1) \rightarrow (A, x_0, x_1)$, there exists a homotopy $h_t: (I^n, \partial I^n) \rightarrow (X, A)$ such that $h_1 = h$ and $h_t(J^{n-1}) = w(t)$, and an isomorphism

$$[w]_*: \pi_n(X, A, x_1) \cong \pi_n(X, A, x_0) \quad \text{for } [w]_*[h] = [h_0]$$

satisfying $[w * w']_* = [w]_* \circ [w']_*$ and $[0]_* = 1$. $\pi_1(A, x_0)$ acts on $\pi_n(X, A, x_0)$.

DEFINITION. X is called n -simple if $\pi_0(X) = 0$ and the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial.

(X, A) is called n -simple if $\pi_0(A) = \pi_0(X) = 0$ and the action of $\pi_1(A)$ on $\pi_n(X, A)$ is trivial.

3.5. The identification map $\psi: I^n \rightarrow I^n / \partial I^n \cong S^n$ induces a bijection $\psi^*: [S^n, *, X, x_0] \cong \pi_n(X, x_0)$ by $\psi^*[h] = [h \circ \psi]$. If X is n -simple then $\psi^*: [S^n, X] \cong \pi_n(X, x_0)$.

3.5'. The identification map $\psi: I^n \rightarrow I^n / J^{n-1} = E^n$ induces a bijection $\psi^*: [E^n, S^{n-1}, *, X, A, x_0] \cong \pi_n(X, A, x_0)$ by $\psi^*[h] = [h \circ \psi]$. If (X, A) is n -simple then $\psi^*: [E^n, S^{n-1}; X, A] \cong \pi_n(X, A, x_0)$.

Fibrations and path spaces We define a fibration by requiring the covering homotopy property to be satisfied only for I^n in the definition of a fibre bundle. To obtain the required properties involving homotopy groups and (co)homology groups, it suffices to consider fibrations as follows:

DEFINITION. (E, p, B) is called a *fibration* (in the sense of Serre) if $p: E \rightarrow B$ is a continuous surjection and has the following *covering homotopy property* for I^n ($n = 0, 1, 2, \dots$): if $f: I^n \rightarrow E$ and $G: I^n \times I \rightarrow B$ satisfies

$$(p \circ f)(x) = G(x, 0), \quad \text{for } x \in I^n,$$

then there exists $F: I^n \times I \rightarrow E$ satisfying

$$G = p \circ F, \quad F(x, 0) = f(x) \quad \text{for } x \in I^n.$$

$$\begin{array}{ccc} I^n & \xrightarrow{f} & E \\ \times 0 \downarrow & \nearrow F & \downarrow p \\ I^n \times I & \xrightarrow{G} & B \end{array}$$

Notions and notation similar to those in §2 are also used in this case.

In the above, E is called the *total space*, p the *projection*, B the *base space*, and $F_b = p^{-1}(b)$ the *fibre over b* .

A *fibre map* $(f, \bar{f}): (E', p', B') \rightarrow (E, p, B)$ between two fibrations is defined to be a pair of maps $f: E' \rightarrow E$ and $\bar{f}: B' \rightarrow B$ satisfying $p \circ f = \bar{f} \circ p'$. Conversely, suppose given a fibre space (E, p, B) and a map $g: B' \rightarrow B$. If we put $g^*E = \{(e, b') \in E \times B' | p(e) = g(b')\}$ and define $f(e, b') = e$ and $p'(e, b') = b'$, then (g^*E, p', B') is a fibration and (f, g) is a fibre map. They are called the *fibration* and the *fibre map induced by g* respectively.

Among other things, it follows easily from the definition that for given fibrations (E, p, B) and (E', p', B') , the product $(E \times E', p \times p', B \times B')$ and the composition $(E, p' \circ p, B')$ when $E' = B$ are also fibrations.

THEOREM 3.6. *For a fibre bundle $\xi = (E, p, B, F, G)$, (E, p, B) is a fibration. In particular, a covering space $(\tilde{X}, p, X)$ is a fibration.*

PROOF. Suppose given $f: I^n \rightarrow E$ and $G: I^n \times I \rightarrow B$ satisfying $p \circ f = G|I^n \times 0$. Consider a fibre bundle $f^*\xi = (E', p', I^n, F, G)$ induced by f , where $E' = \{(e, x) \in E \times I^n | p(e) = f(x)\}$ and $p'(e, x) = x$. By Theorem 2.14 the bundle map $\tilde{f}: f^*\xi \rightarrow \xi$ is extended to $\tilde{F}: f^*\xi \times I \rightarrow \xi$ so that $\tilde{F} = G$. If we define a cross-section $s: I^n \rightarrow E'$ by $s(x) = (f(x), x)$ and put $F = \tilde{F} \circ (s \times 1_I): I^n \times I \rightarrow E' \times I \rightarrow E$, then $p \circ F = G$ and $F|I^n \times 0 = f$. Thus (E, p, B) satisfies the covering homotopy property for I^n .

It is obvious by Lemma 1.7 that a covering space is a fibration, although it may be considered as a special case of a fibre bundle. Q.E.D.

REMARK 3.6'. In the case of a fibre bundle, the covering homotopy property holds for any paracompact Hausdorff space, which is more general than I^n .

DEFINITION. The space $\text{Map}(Y, X) = \{h: Y \rightarrow X \text{ (continuous)}\}$ with the *compact-open topology* defined below is called a *mapping space*: for a compact set C of Y and an open set U of X , the set $W(C, U) = \{h \in \text{Map}(Y, X) | h(C) \subset U\}$ is open and the open sets of $\text{Map}(Y, X)$ are generated by such $\{W(C, U)\}$.

Continuous maps $f: X \rightarrow X'$ and $g: Y' \rightarrow Y$ induce the following continuous maps $f_\#$ and $g^\#$:

$$\begin{aligned} f_\#: \text{Map}(Y, X) &\rightarrow \text{Map}(Y, X') & f_\#(h) &= f \circ h, \\ g^\#: \text{Map}(Y, X) &\rightarrow \text{Map}(Y', X) & g^\#(h) &= h \circ g. \end{aligned}$$

Also, by putting $\phi f(z)(y) = f(z, y)$ for $f: Z \times Y \rightarrow X$ one has

3.7. $f: X \times X \rightarrow X$ is continuous iff $\phi f: Z \rightarrow \text{Map}(Y, X)$ is continuous.

3.8. $p: \text{Map}(Y, X) \times Y \rightarrow X$ (where $p(h, y) = h(y)$) is continuous.

Subsets of $\text{Map}(Y, X)$ are given the subspace topology.

NOTATION. $\Omega(X, A, B) = \{w: (I, 0, 1) \rightarrow (X, A, B)\} \subset \text{Map}(I, X)$ is called a *path space*. We have the following subspaces.

$$\begin{aligned}\Omega(X, A) &= \Omega(X, A, x_0), & \Omega X &= \Omega(X, x_0) = \Omega(X, x_0, x_0), \\ P(X, A) &= \Omega(X, A, X), & PX &= P(X, x_0) = \Omega(X, x_0, X).\end{aligned}$$

THEOREM 3.9. If X is arcwise connected, then $(\Omega(X, A, B), p, A \times B)$, $(\Omega(X, A, B), p_0, A)$ and $(\Omega(X, A, B), p_1, B)$ are fibrations, where $p(w) = (w(0), w(1))$, $p_0(w) = w(0)$, and $p_1(w) = w(1)$.

PROOF. The maps p , p_0 , and p_1 are continuous by 3.8. It follows from the arcwise connectedness of X that they are surjective. Let $Z = I^n$. Suppose that $f: Z \rightarrow \Omega(X, A, B)$ and $H: Z \times I \rightarrow A \times B$ satisfy $H(z, 0) = p(f(z))$. We put $H(z, t) = (H_0(z, t), H_1(z, t))$, then $f(z)(0) = H_0(z, 0)$ and $f(z)(1) = H_1(z, 0)$.

We define $\tilde{H}: Z \times I \rightarrow \Omega(X, A, B)$ as follows:

$$\tilde{H}(z, t)(s) = \begin{cases} H_0(z, t - 4s) & (0 \leq s \leq t/4), \\ f(z) \left(\frac{4s - t}{4 - 2t} \right) & ((4 - t)/4 \leq s \leq 1), \\ H_1(z, 4s + t - 4) & (t/4 \leq s \leq (4 - t)/4). \end{cases}$$

The continuity of $\tilde{H}$ follows from 3.7. Obviously $\tilde{H}$ satisfies $p \circ \tilde{H} = H$ and $\tilde{H}|Z \times 0 = f$ and hence, the covering homotopy property for $Z = I^n$. The other two cases are similar. Q.E.D.

REMARK 3.9'. The path space fibration has the covering homotopy property for arbitrary topological space.

THEOREM 3.10. Let $e_0 \in F_0 = p^{-1}(b_0)$ in a fibration (E, p, B) . Then

$$p_*: \pi_n(E, F_0, e_0) \cong \pi_n(B, b_0) \quad (\text{bijection}).$$

PROOF. We begin with the following lemma.

3.10'. Let (K, L) be a pair homeomorphic to (I^m, J^{m-1}) . If $f: L \rightarrow E$ and $G: K \rightarrow B$ satisfy $p \circ f = G|L$, then there exists $F: K \rightarrow E$ such that $F|L = f$ and $G = p \circ F$.

Without loss of generality one may suppose

$$(K, L) = (I^{m-1} \times I, I^{m-1} \times 0).$$

Then 3.10' is nothing but the covering homotopy property for I^{m-1} .

For arbitrary $G: (I^n, \partial I^n) \rightarrow (B, b_0)$, we define $f: J^{n-1} \rightarrow E$ by $f(J^{n-1}) = e_0$. Then clearly $p \circ f = G|J^{n-1}$. By 3.10' there is $F: I^n \rightarrow E$ such that $p \circ F = G$ and $F(J^{n-1}) = f(J^{n-1}) = e_0$. $p \circ F(\partial I^n) = G(\partial I^n) = b_0$ implies $F(\partial I^n) \subset F_0$; that is, p_* is an epimorphism, since $[F] \in \pi_n(E, F_0, e_0)$ satisfies $p_*[F] = [G]$.

Suppose $p_*[f_0] = p_*[f_1]$ for $f_0, f_1: (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F_0, e_0)$. Let $G: I^{n+1} = I^n \times I \rightarrow B$ be a homotopy to show $p \circ f_0 \simeq p \circ f_1$, satisfying $G(\partial I^n \times I) = b_0$. If we define $f: I^n \times \partial I \cup J^{n-1} \times I$ by $f|I^n \times 0 = f_0$, $f|I^n \times 1 = f_1$, and $f(J^{n-1} \times I) = e_0$, then $p \circ f = G|I^n \times \partial I \cup J^{n-1} \times I$. Since $I^n \times \partial I \cup J^{n-1} \times I = \partial(I^n \times I) - I^{n-1} \times 0 \times I$, $(I^{n+1}, I^n \times \partial I \cup J^{n-1} \times I)$ is homeomorphic with (I^{n+1}, J^n) . By 3.10' there is $F: I^{n+1} \rightarrow E$ such that $p \circ F = G$ and $F|I^n \times \partial I \cup J^{n-1} \times I = f$. Since $p \circ F(\partial I^n \times I) = G(\partial I^n \times I) = b_0$ implies $F(\partial I^n \times I) \subset F_0$, F shows that $f_0 \simeq f_1: (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F_0, e_0)$; that is, $[f_0] = [f_1]$. Thus p_* is a monomorphism. Q.E.D.

COROLLARY 3.11. *The sequence*

$$\begin{aligned} \cdots \rightarrow \pi_n(F_0) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\Delta} \pi_{n-1}(F) \rightarrow \cdots \\ \rightarrow \pi_0(F_0) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B) \end{aligned}$$

is exact, where $\Delta = \partial \circ p_*^{-1}: \pi_n(B) \cong \pi_n(E, F) \rightarrow \pi_{n-1}(F_0)$.

PROOF. In the exact sequence of 3.3 for $(X, A) = (E, F_0)$ one makes use of the bijection of Theorem 3.10, while the last part can be verified directly from the covering homotopy property. Q.E.D.

COROLLARY 3.12. *For a covering space $(\tilde{X}, p, X)$, $p_*: \pi_n(\tilde{X}) \cong \pi_n(X)$ if $n > 1$.*

PROOF. That $p^{-1}(x_0)$ has discrete topology implies $\pi_n(p^{-1}(x_0)) = 0$ for $n > 0$. By Corollary 3.11, $0 \rightarrow \pi_n(\tilde{X}) \rightarrow \pi_n(X) \rightarrow 0$ is exact for $n > 1$. Hence p_* is an isomorphism. Q.E.D.

n-connectedness

DEFINITION. X is called *n*-connected if $\pi_i(X, x_0) = 0$ for $0 \leq i \leq n$. (X, A) is called *n*-connected if $\pi_0(A) = \pi_0(X) = 0$ and $\pi_i(X, A, x_0) = 0$ for $1 \leq i \leq n$.

The definition is independent of the choice of the base point x_0 by 3.4 and 3.4'.

LEMMA 3.13. (1) *If $\pi_0(A) = \pi_0(X) = 0$ for $A \subset X$, then (X, A) is *n*-connected iff $i_*: \pi_i(A) \rightarrow \pi_i(X)$ is an isomorphism for $i < n$ and an epimorphism for $i \leq n$.*

(2) *If (E, p, B) is a fibration and $F = p^{-1}(b_0)$, then B is *n*-connected iff $i_*: \pi_i(F) \rightarrow \pi_i(E)$ is an isomorphism for $i < n$ and an epimorphism for $i \leq n$.*

PROOF. The statements (1) and (2) are easily obtained by 3.3 and Corollary 3.11. Q.E.D.

THEOREM 3.14. *The *n*-dimensional sphere S^n is $(n-1)$ -connected.*

PROOF. Consider an arbitrary map $f: (I^k, \partial I^k) \rightarrow (S^n, e_1)$. Let U be a 4ε -neighborhood of $-e_1$ for sufficiently small $\varepsilon > 0$. We ε -approximate f by a differentiable (or simplicial) map $g: \|f(x) - g(x)\| < \varepsilon$. We denote by $\rho(x)$ the distance between $f(x)$ and $-e_1$ for $x \in f^{-1}(U)$ and define a homotopy $f: (I^k, \partial I^k) \rightarrow (S^n, e_1)$ as follows: if $\rho(x) \geq 3\varepsilon$, $f_t(x) = f(x)$; if $3\varepsilon \geq \rho(x)$, putting $\lambda(x) = \text{Max}(1, 3, -\rho(x)/\varepsilon)$, $f_t(x)$ is the point dividing the arc of a great circle (a geodesic) connecting $f(x)$ and $g(x)$ in the ratio $t\lambda(x) : 1 - t\lambda(x)$. We have $f_0 = f$ and f_1 coincides with g on $f^{-1}(V)$ for an ε -neighborhood V of $-e_1$. If $k < n$, then the points that are not in the image of g are dense, and so there is $v \in S^n$ such that $f_1(I^k) \subset S^n - v$ is homeomorphic with $\mathbb{R}^n$, hence we have $f_1 \simeq 0$. Thus for $k \leq n-1$, $f \simeq f_1 \simeq 0$; in other words, S^n is $(n-1)$ -connected. Q.E.D.

THEOREM 3.15. $U(n+m, \mathbb{F})/1 \times U(m, \mathbb{F})$ and $GL(n+m, \mathbb{F})/1 \times GL(m, \mathbb{F})$ are both $(d(m+1)-2)$ -connected, where $d=1$ ($\mathbb{F}=\mathbb{R}$), $d=2$ ($\mathbb{F}=\mathbb{C}$), and $d=4$ ($\mathbb{F}=\mathbb{H}$). When $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, $SU(n+m, \mathbb{F})/1 \times SU(m, \mathbb{F})$ and $SL(n+m, \mathbb{F})/1 \times SL(m, \mathbb{F})$ are also $(d(m+1)-2)$ -connected.

PROOF. From Theorems 2.10 and 2.18 of Chapter I, one has

$$U(m+1, \mathbb{F})/1 \times U(m, \mathbb{F}) \cong S(\mathbb{F}^{m+1}) \cong S^{d(m+1)-1} \quad (\text{homeomorphisms}). \quad (3.16)$$

The latter is $(d(m+1)-2)$ -connected by Theorem 3.14. All the homomorphisms

$$\begin{aligned} \pi_i(U(m, \mathbb{F})) &\rightarrow \pi_i(U(m+1, \mathbb{F})) \\ &\rightarrow \pi_i(U(m+2, \mathbb{F})) \rightarrow \cdots \rightarrow \pi_i(U(m+n, \mathbb{F})) \end{aligned}$$

induced by the inclusions are epimorphisms for $i \leq d(m+1)-2$ and monomorphisms for $i < d(m+1)-2$ by Lemma 3.13(2). Again applying Lemma 3.13(2) to the result obtained by composing these homomorphisms, one sees that $U(n+m, \mathbb{F})/1 \times U(m, \mathbb{F})$ is $(d(m+1)-2)$ -connected. The inclusion

$$U(n, \mathbb{F}) \rightarrow U(n, \mathbb{F}) \times 0 \rightarrow U(n, \mathbb{F}) \times \mathbb{R}^{dn(n-1)/2+n} \cong GL(n, \mathbb{F})$$

is a homotopy equivalence by Theorem 4.11 of Chapter I. It follows that we may replace $U(n+i, \mathbb{F})$ by $GL(n+i, \mathbb{F})$ in the above arguments and that $GL(n+m, \mathbb{F})/1 \times GL(m, \mathbb{F})$ is $(d(m+1)-2)$ -connected. Since

$$\begin{aligned} SU(n+m, \mathbb{F})/1 \times SU(m, \mathbb{F}) &\cong U(n+m, \mathbb{F})/1 \times U(m, \mathbb{F}), \\ SL(n+m, \mathbb{F})/1 \times SL(m, \mathbb{F}) &\cong GL(n+m, \mathbb{F})/1 \times GL(m, \mathbb{F}), \end{aligned}$$

for $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, the latter half of the theorem follows. Q.E.D.

COROLLARY 3.17. For $i < n-1$, $\pi_i(SO(n)) \cong \pi_i(SO(n+1))$.

For $i < 2n$, $\pi_i(U(n)) \cong \pi_i(U(n+1))$, and $\pi_i(SU(n)) \cong \pi_i(SU(n+1))$.

For $i < 4n+2$, $\pi_i(Sp(n)) \cong \pi_i(Sp(n+1))$.

§4. Covering groups

Covering spaces of H -spaces

DEFINITION. X is called an H -space with a multiplication μ and a unit x_0 , if $\mu: (X \times X, (x_0, x_0)) \rightarrow (X, x_0)$ is a map satisfying

$$\mu \circ i_1 \simeq 1_X \simeq \mu \circ i_2: (X, x_0) \rightarrow (X, x_0),$$

where $i_1, i_2: X \rightarrow X \times X$ are given by $i_1(x) = (x, x_0)$, $i_2(x) = (x_0, x)$.

In particular, a topological group is an H -space.

For an H -space X the product $f \cdot g$ of maps $f, g: Y \rightarrow X$ is defined by

$$(f \cdot g)(y) = \mu(f(y), g(y)), \quad y \in Y$$

using the multiplication μ . For $B \subset Y$, if $f(B) = g(B) = x_0$, then $(f \cdot g)(B) = x_0$.

THEOREM 4.1. If X is an H -space, then

$$[f \cdot g] = [f] + [g] \in \pi_n(X, x_0),$$

for $f, g: (I^n, \partial I^n) \rightarrow (X, x_0)$ and in particular the fundamental group $\pi_1(X, x_0)$ is an abelian group.

PROOF. It follows from the definition of an H -space that $0 \cdot f \simeq f \cdot 0 \simeq f$ for the constant map 0 at x_0 . Also $f \simeq g$ and $f' \simeq g'$ imply $f \cdot g \simeq f' \cdot g'$. Note that these also hold for $f + g$. Further, the equality

$$(f + g) \cdot (f' + g') = (f \cdot f') + (g \cdot g')$$

holds. Therefore the theorem follows from the following homotopies:

$$\begin{aligned} f + g &\simeq (f \cdot 0) + (0 \cdot g) = (f + 0) \cdot (0 + g) \simeq f \cdot g, \\ g + f &\simeq (0 \cdot g) + (f \cdot 0) = (0 + f) \cdot (g + 0) \simeq f \cdot g. \end{aligned}$$

Q.E.D.

A covering space of an H -space admits naturally the structure of an H -space.

THEOREM 4.2. Suppose that the base space of a covering space $(\tilde{X}, p, X)$ is an H -space with multiplication μ and unit x_0 . Let $F_0 = p^{-1}(x_0)$ be the fibre over x_0 .

(1) For each point $\tilde{x}_0$ of F_0 there exists a unique multiplication $\tilde{\mu}$ with a unit $\tilde{x}_0$ such that $\tilde{X}$ is an H -space and such that the diagram below commutes.

$$\begin{array}{ccc} \tilde{X} \times \tilde{X} & \xrightarrow{\tilde{\mu}} & \tilde{X} \\ \downarrow p \times p & & \downarrow p \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

(2) With respect to $\tilde{\mu}$, F_0 is an abelian group and $\psi: \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong F_0$ is an isomorphism of groups, and so F_0 is isomorphic to the covering transformation group.

PROOF. (1) Applying Lemma 1.7(2) to $f = \mu \circ (p \times p): (\tilde{X} \times \tilde{X}, (\tilde{x}_0, \tilde{x}_0)) \rightarrow (X, x_0)$. Since $\tilde{X}$ is arcwise connected and locally arcwise connected, so is $\tilde{X} \times \tilde{X}$. Combining (1.4) and Theorem 4.1 we obtain

$$f_*(\pi_1(\tilde{X} \times \tilde{X}, (\tilde{x}_0, \tilde{x}_0))) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

By Lemma 1.7 there exists a unique $\tilde{\mu}$ such that $\tilde{\mu}(\tilde{x}_0, \tilde{x}_0) = \tilde{x}_0$ and the diagram in (1) is commutative. To show that $\tilde{\mu}$ satisfies the condition of a multiplication of an H -space, for example, to construct a homotopy $\tilde{H}$ showing that $\tilde{\mu} \circ \tilde{i}_1 \simeq 1_{\tilde{X}}$, one can apply an argument, similar to the one above, to the homotopy H showing that $\mu \circ i_1 \simeq 1_X$ in the diagram.

$$\begin{array}{ccc} \tilde{X} \times I & \xrightarrow{\tilde{H}} & \tilde{X} \\ \downarrow p \times 1 & & \downarrow p \\ X \times I & \xrightarrow{H} & X \end{array}$$

(2) Since $\pi_1(X)$ is an abelian group by Theorem 4.1, $(\tilde{X}, p, X)$ is regular by Theorem 1.10. It follows from Theorem 1.9 that $\pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is isomorphic to the covering transformation group. There is a bijection $\psi: \pi_1(X, x_0)/\text{Im } p_* \cong F_0$. The map ψ assigns to $[w] \in \pi_1(X, x_0)$ an element $\tilde{x}_1 = \tilde{w}(1) \in F_0$, where $\tilde{w}$ is a lift of w satisfying $\tilde{w}(0) = \tilde{x}_0$. For $[w_2] \in \pi_1(X, x_0)$ we similarly consider $\tilde{w}_2$ and $\tilde{x}_2 = \tilde{w}_2(1) \in F_0$. Now we define

$$\tilde{w}_3: I \rightarrow \tilde{X} \text{ by } \tilde{w}_3(t) = \tilde{\mu}(\tilde{w}_2(t), \tilde{x}_1), \quad t \in I,$$

where $\tilde{w}_3(0) = \tilde{\mu}(\tilde{x}_0, \tilde{x}_1) = (\tilde{\mu} \circ \tilde{i}_1)(\tilde{x}_1)$. If one takes the homotopy H in the proof of (1), $H|_{x_0 \times I}$ is a constant map, and $\tilde{H}|_{\tilde{x}_1 \times I}$ is its lift and a constant map by Lemma 1.7(1). Therefore $\tilde{\mu} \circ \tilde{i}_1(\tilde{x}_1) = \tilde{x}_1$, which implies $\tilde{w}_3(0) = \tilde{x}_1 = \tilde{w}(1)$ and hence one can define a path $\tilde{w} * \tilde{w}_3$ connecting $\tilde{x}_0$ and $\tilde{\mu}(\tilde{x}_2, \tilde{x}_1)$. By the commutativity of the diagram in (1), one obtains $p \circ \tilde{w}_3 = \mu \circ i_2 \circ (p \circ \tilde{w}_2) \simeq p \circ \tilde{w}_2$, and hence

$$[p \circ (\tilde{w} * \tilde{w}_3)] = [p \circ \tilde{w} * p \circ \tilde{w}_2] = [w][w_2] = [w_2][w] \quad (\pi_1(X): \text{abelian}).$$

This shows $\psi([w_2][w]) = \tilde{\mu}(\tilde{x}_2, \tilde{x}_1)$, i.e., ψ is an isomorphism. Q.E.D.

THEOREM 4.3. If the multiplication μ in Theorem 4.2 satisfies the axioms of group multiplication, so does $\tilde{\mu}$. That is,

(1) If μ is associative ($\mu \circ (\mu \times 1_X) = \mu \circ (1_X \times \mu)$) then $\tilde{\mu}$ is associative.

(2) If $\mu \circ i_1 = 1_X = \mu \circ i_2$ then $\tilde{\mu} \circ \tilde{i}_1 = 1_{\tilde{X}} = \tilde{\mu} \circ \tilde{i}_2$.

(3) If μ has an inverse $\iota: X \rightarrow X$ ($\mu(\iota(x), x) = x_0 = \mu(x, \iota(x))$, $\forall x \in X$), then $\tilde{\mu}$ has also an inverse $\tilde{\iota}: \tilde{X} \rightarrow \tilde{X}$.

PROOF. $\tilde{\mu} \circ (\tilde{\mu} \times 1_{\tilde{X}})$ and $\tilde{\mu} \circ (1_{\tilde{X}} \times \tilde{\mu}): \tilde{X} \times \tilde{X} \times \tilde{X} \rightarrow \tilde{X}$ are both lifts of $\mu \circ (\mu \times 1_X) \circ (p \times p \times p): X \times X \times X \rightarrow X$ and coincides at $(\tilde{x}_0, \tilde{x}_0, \tilde{x}_0)$ and so $\tilde{\mu} \circ (\tilde{\mu} \times 1_{\tilde{X}}) = \tilde{\mu} \circ (1_{\tilde{X}} \times \tilde{\mu})$ by Lemma 1.7(1); similarly for (2). As for (3) we take a lift $\tilde{\iota}$ of $\iota \circ p: \tilde{X} \rightarrow X \rightarrow X$ satisfying $\tilde{\iota}(\tilde{x}_0) = \tilde{x}_0$. The map ι satisfies $\mu \circ (\iota \times 1) \circ d = 0 = \mu \circ (1 \times \iota) \circ d$, where $d: X \rightarrow X \times X$ is the diagonal map. One can show that $\tilde{\mu} \circ (\tilde{\iota} \times 1) \circ \tilde{d} = 0 = \tilde{\mu} \circ (1 \times \tilde{\iota}) \circ \tilde{d}$ similarly as in (1) and obtain (3). Q.E.D.

Universal covering groups. It follows from Theorems 4.2 and 4.3 that:

4.4. If X is a topological group, then for a covering space $(\tilde{X}, p, X)$, one can introduce uniquely the structure of a topological group on $\tilde{X}$ such that $p: \tilde{X} \rightarrow X$ is a homomorphism and an arbitrary point $\tilde{e}$ of the fibre over the unit e of X is the unit.

DEFINITION. A (universal) covering space $(\tilde{G}, p, G)$ consisting of topological groups $\tilde{G}$ and G and a homomorphism $p: \tilde{G} \rightarrow G$ is called a (universal) covering group. Also $\tilde{G}$ is called a (universal) covering group.

LEMMA 4.5. Let $\tilde{G}$ be an arcwise connected, locally arcwise connected topological group and $p: \tilde{G} \rightarrow G$ an epimorphism of topological groups. $(\tilde{G}, p, G)$ is a covering group iff $\text{Ker } p$ has discrete topology in $\tilde{G}$.

PROOF. The necessity is obvious. If $\text{Ker } p$ has the discrete topology in $\tilde{G}$, there is a neighborhood U of $\tilde{e}$ such that $UU^{-1} \cap \text{Ker } p = \{\tilde{e}\}$. The neighborhood $V = \tilde{g}U$ of $\tilde{g} \in \tilde{G}$ satisfies the condition of Theorem 1.11 and hence $(\tilde{G}, p, G = \tilde{G}/\text{Ker } p)$ is a covering space. Q.E.D.

THEOREM 4.6. For a covering group $(\tilde{G}, p, G)$, $\text{Ker } p$ is contained in the center of $\tilde{G}$, that is, $\tilde{x}\tilde{g} = \tilde{g}\tilde{x}$ for any $\tilde{x} \in \text{Ker } p$ and $\tilde{g} \in \tilde{G}$.

PROOF. For $\tilde{x} \in \text{Ker } p$, $L_{\tilde{x}}: \tilde{G} \rightarrow \tilde{G}$ and $R_{\tilde{x}}: \tilde{G} \rightarrow \tilde{G}$ are both covering transformations ($p \circ L_{\tilde{x}} = p = p \circ R_{\tilde{x}}$) and satisfy $L_{\tilde{x}}(\tilde{e}) = \tilde{x} = R_{\tilde{x}}(\tilde{e})$, hence $L_{\tilde{x}} = R_{\tilde{x}}$ by Lemma 1.7(1). So $\tilde{x}\tilde{g} = \tilde{g}\tilde{x}$ for all $\tilde{g} \in \tilde{G}$. Q.E.D.

COROLLARY 4.7. If a normal subgroup H of an arcwise connected, locally arcwise connected topological group G has discrete topology as subspace of G , then H is contained in $Z(G)$ (cf. Theorem 1.12 of Chapter I).

All the results on the universal covering group obtained so far are summed up in the following:

THEOREM 4.8. (1) An arcwise connected, locally arcwise connected, locally semisimply connected topological group G , in particular a connected Lie group G , has a universal covering group $(\tilde{G}, p, G)$.

(2) If a topological group G is the base space of a universal covering group $(\tilde{G}, p, G)$, then the structure of a topological group can be uniquely introduced in the total space $\tilde{G}$ so that p is a homomorphism and $\text{Ker } p$ is a discrete subgroup contained in the center satisfying $\psi: \pi_1(G) \cong \text{Ker } p$ (isomorphism).

(3) For a universal covering group $(\tilde{G}, p, G)$, any covering group with base space G is equivalent to a covering group $(\tilde{G}/H, p', G)$ obtained by a suitable subgroup H of $\text{Ker } p$.

The covering groups of classical groups

LEMMA 4.9. The universal covering group over $U(1) \cong SO(2) \cong S^1$ is $\mathbf{R}$ and $\pi_1(S^1) \cong \mathbf{Z}$ (the infinite cyclic group).

PROOF. Since the subgroup $\mathbf{Z} = \{\text{integers}\}$ of the topological group $\mathbf{R}$ has the discrete topology, $(\mathbf{R}, p, \mathbf{R}/\mathbf{Z})$ is a covering group by Lemma 3.5. Here $\mathbf{R}/\mathbf{Z}$ is isomorphic with S^1 and $\mathbf{R}$ is contractible and hence $\mathbf{R}$ is the universal covering group of S^1 and $\psi: \pi_1(S^1) \cong \mathbf{Z}$. Q.E.D.

Now we will study the centers of compact, connected classical groups.

THEOREM 4.10. (1) The center of $U(n)$ is $\{cI_n | c \in \mathbf{C}, |c| = 1\} \cong U(1) \cong \mathbf{R}/\mathbf{Z}$.

(2) The center of $SU(n+1)$ is $\{cI_{n+1} | c \in \mathbf{C}, c^{n+1} = 1\} \cong \mathbf{Z}_{n+1}$ (a cyclic group of order $n+1$).

(3) The center of $Sp(n)$ is $\{I_n, -I_n\} \cong \mathbf{Z}_2$.

(4) The center of $SO(2n+1)$ is $\{I_{2n+1}\}$ and that of $SO(2n)$ ($n > 1$) is $\{I_{2n}, -I_{2n}\} \cong \mathbf{Z}_2$.

PROOF. We denote one of these groups by G and its standard maximal torus by T^n . Then by Theorem 3.11 of Chapter I any element x (of the center) of G can be expressed as $x = g t g^{-1}$ where $g \in G$ and $t \in T^n$. Since x is an element of the center, $xg = gx$ and $x = t \in T^n$. Moreover, since the action of $W(G) = N(T^n)/T^n$ on T^n is an inner automorphism, x is invariant under the action. If we consider the action of $W(G)$ in Theorem 3.13 of Chapter I, the form of x is determined.

In the case $G = U(n)$; $x = \lambda_1 \oplus \cdots \oplus \lambda_n$ is invariant under permutations, which implies $x = cI_n$. Since $x = cI_n \in U(n)$, we have $|c| = 1$. Obviously, cI_n is an element of the center. The case $SU(n+1)$ is similar but only with the condition that $\det(cI_{n+1}) = 1$ iff $c^{n+1} = 1$.

In the case $G = Sp(n)$; first note $x = cI_n$ and then $c = \bar{c}$ implies $c = \pm 1$.

In the case $G = SO(2n+1)$; $W(G)$ is the same as in $G = Sp(n)$ and so $x = \pm I_{2n} \oplus I_1$. Since $x = -I_{2n} \oplus I_1$ does not commute with the element $I_{2n-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the center of $SO(2n+1)$ consists of I_{2n+1} .

In the case $G = SO(2n)$; $x = r(cI_n)$ and $x = x^{-1}$ imply $x = \pm I_{2n}$ for $n > 1$. ($SO(2) \cong U(1)$ is abelian and is its own center). Q.E.D.

The quotient group of one of the classical groups above by the center is called a *projective classical group* and is denoted by

$$PU(n) = U(n)/\{cI_n\} = SU(n)/\{cI_n | c^n = 1\},$$

$$PSp(n) = Sp(n)/\{I_n, -I_n\},$$

$$PSO(2n) = SO(2n)/\{I_{2n}, -I_{2n}\} \quad (PSO(2n+1) = SO(2n+1)).$$

LEMMA 4.11. *$PSp(1)$ is isomorphic with $SO(3)$. That is, there is a universal covering group $(Sp(1), \rho, SO(3))$ such that $\text{Ker } \rho \cong \mathbb{Z}_2$.*

PROOF. When we identify the set $\mathbf{H}$ of quaternions with $\mathbf{R}^4$ with a basis $\{1, i, j, k\}$, the absolute value of an element of $\mathbf{H}$ corresponds to the length of the vector in $\mathbf{R}^4$. For an element x of $Sp(1) = \{x \in \mathbf{H} | |x| = 1\}$ we put $\rho(x)(q) = xqx^{-1}$ where $q \in \mathbf{H} = \mathbf{R}^4$. The map $\rho(x): \mathbf{R}^4 \rightarrow \mathbf{R}^4$ is a linear map preserving length and so it is an orthogonal transformation. Since $\rho(x)(1) = x1x^{-1} = 1$, $\rho(x)$ gives rise to an orthogonal transformation of the complement $\mathbf{R}^3$ with a basis $\{i, j, k\}$ orthogonal to 1 in $\mathbf{R}^4$. Obviously $\rho(xy) = \rho(x)\rho(y)$ and hence a homomorphism $\rho: Sp(1) \rightarrow SO(3)$ is obtained. We have $\rho(Sp(1)) \subset SO(3)$, as $Sp(1)$ is connected.

By direct calculation one can see that $\rho(\cos \theta + i \sin \theta)$ is a rotation by angle 2θ about the axis $i\mathbf{R}$ and that $\rho(\cos \theta + j \sin \theta)$ is a rotation by angle 2θ about the axis $j\mathbf{R}$. As is well known, all rotations in the 3-dimensional Euclidean space can be expressed as a composition of rotations of these two kinds. This implies that $\rho: Sp(1) \rightarrow SO(3)$ is surjective. We have

$$x \in \text{Ker } \rho \quad \text{iff} \quad xqx^{-1} = \rho(x)(q) = q \quad (\forall q \in \mathbf{R}^3)$$

thus, for $x \in \text{Ker } \rho$ we have $xi = ix$, $xj = jx$ so $x = \pm 1$. Therefore $\text{Ker } \rho = \{1, -1\}$ coincides with the center of $Sp(1)$ and so $PSp(1)$ is isomorphic with $SO(3)$. The latter half of the theorem follows from Theorem 3.14 and the fact that $Sp(1) \cong S^3$. Q.E.D.

THEOREM 4.12. (1) $SU(n)$ and $Sp(n)$ are 1-connected.

(2) $\pi_1(U(n)) \cong \mathbb{Z}$, $\pi_1(SO(2)) \cong \mathbb{Z}$.

(3) $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$.

PROOF. (1) $Sp(1) = S(\mathbf{H}) \cong S^3$ is 1-connected by Theorem 3.14. Obviously $SU(1) = \{I_1\}$ is 1-connected. It follows from Lemma 4.9 that $\pi_1(U(1)) \cong \pi_1(SO(2)) \cong \mathbb{Z}$. Also it follows from Lemma 4.11 and Theorem 4.8(2) that $\pi_1(SO(3)) \cong \text{Ker } \rho \cong \mathbb{Z}_2$.

Thus, by induction on n , the theorem is easily proved using Corollary 3.17. Q.E.D.

The groups $SU(n)$ and $Sp(n)$ are their own universal covering groups, and the base space of any covering group that has one of them as total space is its quotient group by a subgroup of the center and hence it is determined in Theorem 4.10. As for the universal covering space of $U(n)$, the following results hold:

4.13. (1) For $p: SU(n) \times \mathbf{R} \rightarrow U(n)$ defined by $p(A, t) = e^{2\pi i t} A$, $(SU(n) \times \mathbf{R}, p, U(n))$ is a universal covering group and $\text{Ker } p$ is the infinite cyclic group generated by $(e^{2\pi i/n} I_n, -1/n)$.

(2) For $p: SU(n) \times U(1) \rightarrow U(n)$ defined by $p(A, c) = cA$, $(SU(n) \times \mathbf{R}, p, U(n))$ is a covering group and $\text{Ker } p$ is a cyclic group of order n .

DEFINITION. The universal covering group of $SO(n)$ for $n \geq 3$ is called the n th spinor group, denoted by $\text{Spin}(n)$.

THEOREM 4.4. (1) The center of $\text{Spin}(2n+1)$ is isomorphic to $\mathbb{Z}_2$.

(2) The center of $\text{Spin}(4n)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and that of $\text{Spin}(4n+2)$ is isomorphic to $\mathbb{Z}_4$.

PROOF. (1) The kernel of $p: \text{Spin}(2n+1) \rightarrow SO(2n+1)$ is contained in the center. On the other hand, as the center of $SO(2n+1)$ consists of the identity element only, $\text{Ker } p \cong \mathbb{Z}_2$ coincides with the center of $\text{Spin}(2n+1)$.

(2) Since $\text{Spin}(2m)$ is the universal covering space of $PO(2m) = SO(2m)/\{\pm I_{2m}\}$ of the covering degree 4, the center of $\text{Spin}(2m)$ contains a subgroup of order 4. On the other hand, the center of $\text{Spin}(2m)$ is mapped to the center of $SO(2m)$ by $p: \text{Spin}(2m) \rightarrow SO(2m)$ with $\text{Ker } p \cong \mathbb{Z}_2$ and hence the center of $\text{Spin}(2m)$ is a group of order 4, that is, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$. So the center is isomorphic either to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ according to whether the inverse image $p^{-1}(-I_{2m})$ of $-I_{2m}$ contains an element of order 4 or only elements of order 2.

If we define $w: I \rightarrow SO(2m)$ by $w(t) = (e^{\pi i t}) \oplus \cdots \oplus r(e^{\pi i t})$, then the homotopy class of the path w may be regarded as an element of $\text{Spin}(2m)$ (cf. the proof of Theorem 1.13). Since $p[w] = w(1) = -I_{2m}$, it suffices to study $[w]^2$, where $[w]^2$ is represented by a path $u: I \rightarrow SO(2m)$ defined by $u(t) = r(e^{2\pi i t}) \oplus \cdots \oplus r(e^{2\pi i t})$. If we denote by u_j the path obtained by replacing all the terms except the j th one with I_2 in the definition of $u(t)$, then $u = u_1 \cdots u_m$ and besides each closed path $[u_j]$ is easily seen to be a generator of $\pi_1(SO(2m))$. In other words, denoting by α the generator of $\text{Ker } p$, one has $[w]^2 = [u] = [u_1] \cdots [u_m] = \alpha^m$. Therefore $[w]$ is of order 2 if m is even, and of order 4 if m is odd. Thus (2) is proved. Q.E.D.

It follows from this theorem that the covering groups obtained as a quotient space of the spinor group other than itself are just $SO(2n+1)$ for $\text{Spin}(2n+1)$, and $SO(4n+2)$ and $PO(4n+2)$ for $\text{Spin}(4n+2)$. There are, however, two more possibilities for $\text{Spin}(4n)$ besides $SO(4n)$ and $PO(4n)$.

THEOREM 4.15. If $\text{Ker } p$, A , and A' are the distinct subgroups of order 2 in the center of $\text{Spin}(4n)$, then $\text{Spin}(4n)/A$ is isomorphic with $\text{Spin}(4n)/A'$.

PROOF. It suffices to find an automorphism of $\text{Spin}(4n)$ mapping A to A' . In fact, a conjugation of $SO(4n)$ by the element $(-1) \oplus I_{4n-1} \in O(4n)$ induces a transformation of the paths in $SO(4n)$ and hence of $\text{Spin}(4n)$. Then the element w in the proof of Theorem 4.14 is mapped to w' defined by $w'(t) = r(e^{\pi i t}) \oplus r(e^{\pi i t}) \oplus \cdots \oplus r(e^{\pi i t})$ and $w * w' \simeq u_2 \cdots u_{2n}$ so $[w'] = [w] \cdot \alpha \neq [w]$. This shows the existence of an automorphism of $\text{Spin}(4n)$ which maps A generated by $[w]$ to A' generated by $[w']$. Q.E.D.

DEFINITION. The Lie group $\text{Spin}(4n)/A$ determined by the theorem is called a semispinor group, denoted by $Ss(4n)$. It is known that $Ss(4n)$

is not isomorphic to $SO(4n)$ for $n \neq 2$ and that $Ss(8)$ is isomorphic to $SO(8)$.

Local isomorphisms of topological groups

LEMMA 4.16. *Let G and G' be topological groups, U a neighborhood of the identity in G , and $f: U \rightarrow G'$ a map such that $x, y, xy \in U$ implies $f(x)f(y) = f(xy)$. If G is 1-connected, then f is uniquely extendable to a homomorphism $\bar{f}: G \rightarrow G'$ of topological groups.*

PROOF. Take an element x of G and consider a path $w: I \rightarrow G$ connecting the identity e to x . Then divide I into subintervals $[t_i, t_{i+1}]$, $0 = t_0 < t_1 < \dots < t_n = 1$, satisfying the condition that $w(s)^{-1}w(t) \in U$ for $s, t \in [t_i, t_{i+1}]$. Choose a neighborhood V of e so that $V^{-1}V \subset U$ and cover I with $\{w^{-1}(L_g V), g \in G\}$. Since I is compact, one can choose a subdivision so fine that each $[t_i, t_{i+1}]$ is contained in some $w^{-1}(L_g V)$. Then, for $s, t \in [t_i, t_{i+1}]$, $w(s) = gv_1$, $w(t) = gv_2$ for $v_1, v_2 \in V$ implies $w(s)^{-1}w(t) = v_1^{-1}v_2 \in V^{-1}V \subset U$. Thus there exists a subdivision as above. For such a subdivision we define

$$\bar{f}_w(x) = f(w(t_1))f(w(t_1)^{-1}w(t_2)) \cdots f(w(t_{n-1})^{-1}w(t_n)).$$

Now we consider the subdivision with a subdivision point $t' \in [t_i, t_{i+1}]$ added. Since $w(t_i)^{-1}$, $w(t')$, $w(t')^{-1}w(t_{i+1}) \in U$, one has

$$f(w(t_i)^{-1}w(t'))f(w(t')^{-1}w(t_{i+1})) = f(w(t_i)^{-1}w(t_{i+1}))$$

by the assumption of the theorem. This shows that $\bar{f}_w(x)$ is invariant under refinements of the subdivision. For given two subdivisions, we can consider a refinement common to these and hence we can see that $\bar{f}_w(x)$ is determined independently of the choice of the subdivision.

For another element $y \in G$, $\bar{f}_u(y)$ is defined by a path u connecting e and y . Then by defining a path v connecting x and xy by $v(t) = \bar{f}_w(x)u(t)$, we have

$$\bar{f}_w(x)\bar{f}_u(y) = \bar{f}_{w \cdot v}(xy).$$

Next, we will show that $\bar{f}_w(x)$ is determined independently of the choice of a path w . Since G is 1-connected, $w \simeq w'$ holds for any other path w' connecting e and x . Let $W: I \times I \rightarrow G$ be a homotopy to show $w \simeq w'$. Similarly as before, we divide $I \times I$ into sufficiently small squares with side $1/n$ so that $t, t' \in [j/n, (j+1)/n]$, $s, s' \in [i/n, (i+1)/n]$ imply $W(s, t)^{-1}W(s', t') \in U$. Using the assumption of the theorem one can show that $\bar{f}_{w_{i+1}}(x) = \bar{f}_{w_i}(x)$, where $w_i(t) = W(t, i/n)$. Therefore $\bar{f}_w(x) = \bar{f}_{w_0}(x) = \bar{f}_{w_n}(x) = \bar{f}_{w'}(x)$, that is, $\bar{f}_w(x)$ is determined independently of w . If we denote it by $\bar{f}(x)$, we have by the above

$$\bar{f}(x)\bar{f}(y) = \bar{f}(xy).$$

The continuity of $\bar{f}$ is derived from that of f , the continuity of the multiplication and the inverse of the topological group. Since G is generated by U , the map $\bar{f}$ is an extension of f . Q.E.D.

COROLLARY 4.17. *If G and G' are Lie groups and f is differentiable, then $\bar{f}$ is differentiable and is a homomorphism of Lie groups.*

DEFINITION. Topological groups G and G' are called *locally isomorphic* if there are neighborhoods U and U' of the identities of G and G' respectively and a homeomorphism $f: U \cong U'$ such that $x, y, xy \in U$ implies $f(x)f(y) = f(xy)$.

Lie groups G and G' are called *locally isomorphic* if f in the above is a diffeomorphism.

4.18. *Local isomorphism is an equivalence relation. Also, in a covering space $(\tilde{G}, p, G)$, $\tilde{G}$ and G are locally isomorphic.*

THEOREM 4.19. *If topological groups (or Lie groups) G and G' are 1-connected, then G and G' are locally isomorphic iff G and G' are isomorphic.*

PROOF. Let $f: U \cong U'$ be a given local isomorphism and consider $f: U \rightarrow G'$ and $f^{-1}: U' \rightarrow G$. Then they can be extended to homomorphisms $\bar{f}: G \rightarrow G'$ and $\bar{f}^{-1}: G' \rightarrow G$. Both $\bar{f}^{-1} \circ \bar{f}: G \rightarrow G$ and 1_G are extensions of the inclusion $U \rightarrow G$. By the uniqueness of the extension, one has $\bar{f}^{-1} \circ \bar{f} = 1_G$. Similarly, $\bar{f} \circ \bar{f}^{-1} = 1_{G'}$. Thus G and G' are isomorphic.

The converse is clear. Q.E.D.

COROLLARY 4.20. *Let G and G' be connected Lie groups and $\tilde{G}$ and $\tilde{G}'$ their universal covering groups. Then G and G' are locally isomorphic iff $\tilde{G}$ and $\tilde{G}'$ are isomorphic.*

§5. Lie groups and Lie algebras

The definition of Lie algebra. Let $\mathbf{k} = \mathbf{R}$ or $\mathbf{k} = \mathbf{C}$

DEFINITION. $\mathfrak{g}$ is called a *Lie algebra* over $\mathbf{k}$ (a *real Lie algebra* or a *complex Lie algebra*) if $\mathfrak{g}$ is a vector space over $\mathbf{k}$ having a product

$$[\ , \]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g},$$

which satisfies the following properties:

- (1) (skew symmetry) $[Y, X] = -[X, Y]$, whence $[X, X] = 0$
- (2) (linearity) $[aX + a'X', Y] = a[X, Y] + a'[X', Y]$, ($a, a' \in \mathbf{k}$)
- (3) (the Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Unless otherwise specified, all Lie algebras are to be finite dimensional.

NOTATION. For vector subspaces $\mathfrak{a}, \mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ over $\mathbf{k}$,

$$[\mathfrak{a}, \mathfrak{b}] = \{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}.$$

DEFINITION. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over $\mathbf{k}$.

$\mathfrak{h}$ is called a *Lie subalgebra* of $\mathfrak{g}$ if $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

$\mathfrak{a}$ is called an *ideal* of $\mathfrak{g}$ if $\mathfrak{a}$ is a Lie subalgebra such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$.

$\mathfrak{g}$ is called an *abelian Lie algebra* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

$\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ is called a *homomorphism* of Lie algebras if φ is a linear map such that $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$ for $X, Y \in \mathfrak{g}$.

$\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ is called an *isomorphism* of Lie algebras if φ is a bijective homomorphism of Lie algebras.

$\mathfrak{g}$ and $\mathfrak{g}'$ are called *isomorphic* if there is an isomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ of Lie algebras.

The following can be easily verified.

5.1. If $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ is a homomorphism of Lie algebras then $\text{Im } \varphi = \varphi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{g}'$ and $\text{Ker } \varphi = \varphi^{-1}(0)$ is an ideal of $\mathfrak{g}$.

5.1'. If $\mathfrak{a}$ is an ideal of a Lie algebra $\mathfrak{g}$, then on the quotient vector space $\mathfrak{g}/\mathfrak{a}$ one can uniquely introduce the structure of a Lie algebra such that the projection $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$ is a homomorphism of Lie algebras. ($\mathfrak{g}/\mathfrak{a}$ is called the quotient Lie algebra). In 5.1,

$$\mathfrak{g}/\text{Ker } \varphi \cong \text{Im } \varphi \quad (\text{isomorphism}).$$

5.2. Let V be a vector space of finite dimension over $\mathbf{k}$. $\text{Hom}_{\mathbf{k}}(V, V)$ is a Lie algebra over $\mathbf{k}$, if we define $[f, g] = f \circ g - g \circ f$ for $f, g: V \rightarrow V$.

We denote it by $\mathfrak{gl}(V)$.

If we identify $\text{Hom}_{\mathbf{k}}(V, V)$ with $M(n, \mathbf{k})$, the set of square matrices fixing a basis of V , then $M(n, \mathbf{k})$ is a Lie algebra with $[A, B] = AB - BA$.

We denote it by $\mathfrak{gl}(n, \mathbf{k})$.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over $\mathbf{k}$. The direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ of vector spaces with product defined by

$$[(X, X'), (Y, Y')] = ([X, Y], [X', Y']), \quad X, Y \in \mathfrak{g}, \quad X', Y' \in \mathfrak{g}'$$

is called the *direct sum* of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, and is denoted by $\mathfrak{g} \oplus \mathfrak{g}'$.

5.3. In the direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, $\mathfrak{g}$ and $\mathfrak{g}'$ are ideals and $(\mathfrak{g} \oplus \mathfrak{g}')/\mathfrak{g} \cong \mathfrak{g}'$, $(\mathfrak{g} \oplus \mathfrak{g}')/\mathfrak{g}' \cong \mathfrak{g}$, $\mathfrak{g} \oplus \mathfrak{g}' \cong \mathfrak{g}' \oplus \mathfrak{g}$, $(\mathfrak{g} \oplus \mathfrak{g}') \oplus \mathfrak{g}'' \cong \mathfrak{g} \oplus (\mathfrak{g}' \oplus \mathfrak{g}'')$ (isomorphisms). Conversely, if there are ideals $\mathfrak{a}$ and $\mathfrak{a}'$ such that $\mathfrak{a} + \mathfrak{a}' = \mathfrak{g}$ and $\mathfrak{a} \cap \mathfrak{a}' = 0$, then $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}'$.

DEFINITION. A Lie algebra $\mathfrak{g}$ is called *simple* if $\mathfrak{g}$ is not abelian and $\mathfrak{g}$ does not have ideals except 0 and $\mathfrak{g}$.

A Lie algebra is called *semisimple* if $\mathfrak{g}$ is a direct sum of simple Lie algebras.

We omit the details of the theory of Lie algebras, since there are many good books concerned with it, and we will just quote the results whenever they are needed.

The Lie algebra of tangent vector fields. In §5 of Chapter I a tangent vector field v of a differentiable manifold M was defined as a cross-section

$$v: M \rightarrow TM \text{ (} C^\infty\text{-map) such that } p \circ v = 1_M.$$

Since vector addition and multiplication by a scalar in the tangent vector space TM are C^∞ -maps, addition and multiplication by a scalar of tangent vector fields v and v' are defined by

$$(v + v')(x) = v(x) + v'(x), \quad (av)(x) = av(x), \quad a \in \mathbb{R}$$

and the set of all vector fields forms a real vector space (of infinite dimension).

Now we look at the tangent vector fields from a different point of view.

$C(M) = \{f | C^\infty\text{-map } f: M \rightarrow \mathbb{R}\}$ is an algebra of infinite dimension over $\mathbb{R}$ with

$$(f + g)(x) = f(x) + g(x), \quad (af)(x) = af(x), \quad (fg)(x) = f(x)g(x).$$

DEFINITION. $X: C(M) \rightarrow C(M)$ is called a *differential operator* on the differential manifold M if X is a linear map such that

$$X(fg) = X(f)g + fX(g).$$

NOTATION. $\mathcal{X}(M) = \{\text{differential operators on } M\}$.

5.4. $\mathcal{X}(M)$ is a Lie algebra (of infinite dimension) with

$$(X + Y)(f) = X(f) + Y(f), \quad (aX)(f) = aX(f), \\ [X, Y](f) = X(Y(f)) - Y(X(f)).$$

Further, $\mathcal{X}(M)$ is a left $C(M)$ -module with $(fX)(g) = f(X(g))$.

A C^∞ -map $f: M \rightarrow \mathbb{R}$, which is an element of $C(M)$, induces a bundle map $df: TM \rightarrow T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ of tangent vector spaces. We define

$$X_v(f) = \pi_2 \circ df \circ v: M \rightarrow TM \rightarrow T\mathbb{R} \rightarrow \mathbb{R}$$

for a tangent vector field $v: M \rightarrow TM$ of M and for the map f above, where $\pi_2: T\mathbb{R} = \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection onto the second factor. We may consider that $X_v: C(M) \rightarrow C(M)$ is defined by

$$(df \circ v)(x) = (f(x), X_v(f)(x)), \quad x \in M.$$

THEOREM 5.5. The assignment $v \mapsto X_v$ is an isomorphism of the real vector space of all tangent vector fields on M onto the real vector space $\mathcal{X}(M)$ of differential operators. (It follows that one can define a tangent vector field on M by a differential operator $X \in \mathcal{X}(M)$.)

PROOF. $C(M)$ acts on tangent vector fields by $(fv)(x) = f(x)v(x)$. Since the differential manifold M is a paracompact Hausdorff space, we can choose a partition of unity $\{e_\alpha\}$ associated with a coordinate neighborhood system so that $e_\alpha \in C(M)$. Since we have $v = \sum_\alpha e_\alpha v$ for a tangent vector field v and $X = \sum_\alpha e_\alpha X$ for $X \in C(M)$, we see that it suffices to

consider the theorem for each coordinate neighborhood; that is, it suffices to prove the theorem for the case $M = \mathbf{R}^m$.

A tangent vector field $v: \mathbf{R}^m \rightarrow T\mathbf{R}^m = \mathbf{R}^m \times \mathbf{R}^m$ on $\mathbf{R}^m$ is expressed as

$$v(x) = (x, (v_1(x), \dots, v_m(x))), \quad v_1, \dots, v_m \in C(\mathbf{R}^m).$$

Also, $df: T\mathbf{R}^m \rightarrow T\mathbf{R} = \mathbf{R} \times \mathbf{R}$ for $f \in C(\mathbf{R}^m)$ is given by

$$df(x, (a_1, \dots, a_m)) = \left(f(x), a_1 \frac{\partial f}{\partial x_1}(x) + \dots + a_m \frac{\partial f}{\partial x_m}(x) \right)$$

and hence X_v is given by

$$X_v f(x) = \sum_{i=1}^m v_i(x) \frac{\partial f}{\partial x_i}(x).$$

That X_v is a differential operator is clear from this equality. Moreover it implies the theorem, since $\mathcal{Z}(\mathbf{R}^m)$ has a basis $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}$ as $C(\mathbf{R}^m)$ -module (the proof of this is omitted). Q.E.D.

5.6. Let $h: M \rightarrow M'$ be a C^∞ -map, and $v: M \rightarrow TM$ and $v': M' \rightarrow TM'$ tangent vector fields. If one assigns $v \mapsto X \in \mathcal{Z}(M)$ and $v' \mapsto X' \in \mathcal{Z}(M')$ by Theorem 5.5, then

$$dh \circ v = v' \circ h \text{ iff } X'(f') \circ h = X(f' \circ h) \quad (\forall f' \in C(M')).$$

If v and v' are related as above, v' is determined by v and h and so one can express it as

$$v' = h_* v, \quad X' = h_* X.$$

The following equalities hold for $X, Y \in \mathcal{Z}(M)$ for which such h_* is defined.

$$\begin{aligned} h_*(aX + bY) &= ah_*X + bh_*Y \quad (a, b \in \mathbf{R}), \\ h_*[X, Y] &= [h_*X, h_*Y]. \end{aligned} \tag{5.7}$$

In particular,

5.7'. If h is a diffeomorphism then $h_*: \mathcal{Z}(M) \cong \mathcal{Z}(M')$ (isomorphism of Lie algebras).

Lie groups and Lie algebras Let G be a Lie group. For a tangent vector field $v: G \rightarrow TG$ on G and the corresponding differential operator $X \in \mathcal{Z}(G)$, we say that v is a *left-invariant vector field* if $L_g v = v$ ($\forall g \in G$), or equivalently

$$L_{g*} X = X \quad (\forall g \in G).$$

Therefore by (5.7)

$$\mathfrak{g} = \{X \in \mathcal{Z}(G) | L_{g*} X = X \quad (\forall g \in G)\}$$

is a Lie subalgebra of $\mathcal{Z}(G)$.

On the other hand, in §5 of Chapter I we were given the following one to one correspondences:

$$\{\text{left invariant vector fields } \mathbf{v}\} \leftrightarrow \{\text{tangent vector } X = \mathbf{v}(e) \in T_e G \text{ at } e\} \\ \leftrightarrow \{\text{one parameter subgroup } s: \mathbf{R} \rightarrow G, s(t) = \exp(tX)\}.$$

Thus by the correspondence in Theorem 5.5 we have

5.8. *There is an isomorphism $T_e G \cong \mathfrak{g}$ of vector spaces.*

By this isomorphism $T_e G \cong \mathfrak{g}$ we regard $T_e G = \mathfrak{g}$ so that $T_e G$ has the structure of a Lie algebra.

DEFINITION. $\mathfrak{g} = T_e G$ is called the *Lie algebra* of a Lie group G , denoted by $L(G) = \mathfrak{g} = T_e G$.

Let $h: G \rightarrow G'$ be a homomorphism of Lie groups and consider $dh: TG \rightarrow TG'$.

If $\mathbf{v}: G \rightarrow TG$ is a left-invariant vector field, then a left-invariant vector field $\mathbf{v}'$ on G' is given by $\mathbf{v}'(g') = dL_{g'}(dh(\mathbf{v}(e)))$, where $\mathbf{v}'$ is determined by $dh(\mathbf{v}(e)) \in T_e(G')$. If $g' = h(g)$, then $\mathbf{v}'(h(g)) = dL_{h(g)}dh(\mathbf{v}(e)) = dh dL_g(\mathbf{v}(e)) = dh(\mathbf{v}(g))$. Namely, $\mathbf{v}$ and $\mathbf{v}'$ satisfy the relation 5.6 and $\mathbf{v}' = h_* \mathbf{v}$.

It follows from (5.6) and (5.7) that:

5.9. *If $h: G \rightarrow G'$ is a homomorphism of Lie groups, $h_*: L(G) \rightarrow L(G')$ is a homomorphism of Lie algebras. In particular, if H is a Lie subgroup of a Lie group G , $L(H)$ is a Lie subalgebra of $L(G)$.*

We denote the inner automorphism of a Lie group G given by conjugation with an element g by

$$A_g: G \rightarrow G \quad \text{where} \quad A_g(x) = gxg^{-1}, \quad x \in G$$

and we denote the automorphism of Lie algebra $\mathfrak{g} = L(G)$ induced by A_g by

$$Ad(g) = (A_g)_*: \mathfrak{g} \rightarrow \mathfrak{g}.$$

Obviously

$$A_{gg'} = A_g \circ A_{g'} \quad \text{implies} \quad Ad(gg') = Ad(g) \circ Ad(g').$$

Let $\text{Aut}(\mathfrak{g})$ be the group of all the automorphisms of the Lie algebra $\mathfrak{g}$. Then $\text{Aut}(\mathfrak{g})$ is a Lie group, since it is a closed subgroup of $GL(\mathfrak{g}) \cong GL(\dim G, \mathbf{R})$.

THEOREM 5.10. *$Ad: G \rightarrow GL(\mathfrak{g})$ is a homomorphism of Lie groups; that is, Ad is a representation of a Lie group G by the Lie algebra $\mathfrak{g} = L(G)$.*

DEFINITION. We call Ad in the above the *adjoint representation* of a Lie group G .

DEFINITION. A Lie algebra homomorphism from a Lie algebra $\mathfrak{g}$ to the Lie algebra of matrices is called a *representation* of $\mathfrak{g}$.

DEFINITION. If we consider the differential of the adjoint representation $Ad: G \rightarrow GL(\mathfrak{g})$ of G where $\mathfrak{g} = L(G)$, we obtain a homomorphism $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) = M(n, \mathbb{R})$ of Lie algebras with $n = \dim G$. We denote it by

$$ad: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) = M(n, \mathbb{R})$$

and call it the *adjoint representation* of $\mathfrak{g}$.

That is, it is seen (cf. (5.15)) that

$$ad([X, Y]) = [adX, adY], \quad Ad(\exp X) = \exp ad(X),$$

and further that

$$ad(X)Y = [X, Y].$$

THEOREM 5.11 (Ado). *Let $\mathbf{k}$ be a field of characteristic 0. Then an arbitrary Lie algebra of finite dimension over $\mathbf{k}$ has a faithful (i.e., monomorphic) representation of finite dimension.*

If a group has a faithful representation, then it is isomorphic to a subgroup of the group of matrices, and hence as a corollary of the above theorem we have:

COROLLARY 5.11'. *Let $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$. For a Lie group g of finite dimension over $\mathbf{k}$, there exist an integer $n \geq 1$ and a (connected) Lie subgroup G of $GL(n, \mathbf{k})$ such that the Lie algebra of G is isomorphic to $\mathfrak{g}$.*

The Lie algebra $\mathfrak{g} = L(G)$ of a Lie group G is determined only by the behavior in neighborhoods of the identity of G . Therefore the Lie algebra $L(G)$ of G coincides with the Lie algebra $L(G_0)$ of the component G_0 of the identity of G , and so, if G' and G are locally isomorphic (in particular, if G' is a covering group of G), their Lie algebras are isomorphic. A famous result of Lie theory asserts that there is such a one to one correspondence between Lie algebras and the local isomorphism classes of Lie groups. Here we only quote the results without proof (cf. Murakami [6] and Matsushima [11, 11']).

THEOREM 5.12. *If G is a Lie group and $\mathfrak{g} = L(G)$ is its Lie algebra, then there is a one to one correspondence between connected Lie subgroups H of G and Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ (cf. 5.9). Moreover,*

$$H \text{ is normal iff } \mathfrak{h} \text{ is an ideal,} \quad H \text{ is abelian iff } \mathfrak{h} \text{ is abelian.}$$

THEOREM 5.13. *If G is a 1-connected Lie group and G' is a Lie group, then the correspondence that assigns the induced homomorphism $f_*: L(G) \rightarrow L(G')$ of Lie algebras to a homomorphism $f: G \rightarrow G'$ of Lie groups is bijective.*

COROLLARY 5.13'. *If $f_*: L(G) \cong L(G')$ in the above theorem, then (G, f, G') is a universal covering group.*

REMARK. Consider the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$ where $\mathfrak{h}$ is an ideal of $\mathfrak{g} = L(G)$ corresponding to a connected normal subgroup H of G . Then there is a Lie group G' by Theorem 5.11' such that $\mathfrak{g}/\mathfrak{h} = L(G')$. If G is 1-connected, then there is a homomorphism $f: G \rightarrow G'$ such that $H = \text{Ker } f$.

It follows that:

5.14. *If a connected Lie subgroup of a 1-connected Lie group is normal, then it is a closed subgroup.*

The product $[\cdot, \cdot]$ in a Lie algebra can be expressed as follows using the exponential map $\exp: L(G) = T_e G \rightarrow G$:

5.15. *For $X, Y \in L(G)$,*

$$[X, Y] = \frac{d}{dt}(Ad(\exp tX)Y)_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t}(Ad(\exp tX)Y - Y).$$

THEOREM 5.16. *The Lie algebra of the Lie group $GL(n, \mathbf{F})$ is $M(n, \mathbf{F})$ with the product given by $[A, B] = AB - BA$. So the Lie algebra of a closed subgroup G of $GL(n, \mathbf{F})$ (a classical group) is the Lie algebra $\mathfrak{g} = \{A \in M(n, \mathbf{F}) | \exp(tA) \in G, \forall t \in \mathbf{R}\}$ with the product given above.*

Spinor groups of low dimension

NOTATION. E_{ij} = (the square matrix with (i, j) -component = 1 and the other components = 0) $\in M(n, \mathbf{F})$

$$L(U(n, \mathbf{F})) = \{A \in M(n, \mathbf{F}) | {}^t A = -\bar{A}\},$$

$$L(SU(n, \mathbf{F})) = \{A \in M(n, \mathbf{F}) | {}^t A = -\bar{A}, \text{tr } A = 0\}.$$

Hence the following result is easily obtained.

THEOREM 5.17. *The algebras of classical groups have the following bases over $\mathbf{R}$:*

$$L(O(n)) = L(SO(n)) : \{E_{ij} - E_{ji}; 1 \leq i < j \leq n\};$$

$$L(U(n)) : \{E_{ij} - E_{ji}; 1 \leq i < j \leq n\}$$

$$\cup \{i(E_{ij} + E_{ji}); 1 \leq i \leq j \leq n\};$$

$$L(SU(n)) : \{E_{ij} - E_{ji}, i(E_{ij} + E_{ji}); 1 \leq i < j \leq n\}$$

$$\cup \{i(E_{ii} - E_{i+1, i+1}); 1 \leq i < n\};$$

$$L(Sp(n)) : \{E_{ij} - E_{ji}; 1 \leq i < j \leq n\}$$

$$\cup \{\alpha(E_{ij} + E_{ji}); 1 \leq i \leq j \leq n, \alpha = i, j, k\}.$$

The product $[A, B] = AB - BA$ in these Lie algebras can be calculated by means of the following identities:

$$E_{ij}E_{jk} = E_{ik}, \quad E_{ij}E_{lk} = 0 \quad (j \neq l). \quad (5.17')$$

For example, if one puts

$$R_{ij} = E_{ij} - E_{ji}, \quad (R_{ji} = -R_{ij}, \quad R_{ii} = 0)$$

in $L(SO(n))$, then the product in $L(SO(n))$ is given by

$$[R_{ij}, R_{jk}] = R_{ik} \quad (i \neq j, j \neq k),$$

$$[R_{ij}, R_{kl}] = 0 \quad (i, j, k, l \text{ are distinct from each other}),$$

(where $\exp tR_{ij}$ is a rotation by angle t in the (i, j) -plane).

On the other hand, since $L(Sp(1))$ has a basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the products are given by

$$[\mathbf{i}, \mathbf{j}] = 2\mathbf{k}, \quad [\mathbf{j}, \mathbf{k}] = 2\mathbf{i}, \quad [\mathbf{i}, \mathbf{k}] = -2\mathbf{j}$$

one obtains, by the correspondence $2R_{12} \mapsto \mathbf{i}$, $2R_{13} \mapsto \mathbf{j}$, $2R_{23} \mapsto -\mathbf{k}$:

5.18. $L(SO(3)) \cong L(Sp(1))$. Hence there is a (double) universal covering $p: Sp(1) \rightarrow SO(3)$ so that $\text{Spin}(3) \cong Sp(1)$ (cf. Lemma 4.11).

If we define a morphism $\varphi_4: L(SO(4)) \rightarrow L(Sp(1)) \oplus L(Sp(1))$ by

$$\varphi_4(2R_{12}) = (\mathbf{i}, \mathbf{i}), \quad \varphi_4(2R_{13}) = (\mathbf{j}, \mathbf{j}), \quad \varphi_4(2R_{23}) = (-\mathbf{k}, -\mathbf{k}),$$

$$\varphi_4(2R_{14}) = (-\mathbf{k}, \mathbf{k}), \quad \varphi_4(2R_{24}) = (-\mathbf{j}, \mathbf{j}), \quad \varphi_4(2R_{34}) = (\mathbf{i}, -\mathbf{i}),$$

it can easily be verified that φ_4 is an isomorphism of Lie algebras. Thus

5.19. $L(SO(4)) \cong L(Sp(1) \times Sp(1))$, $\text{Spin}(4) \cong Sp(1) \times Sp(1)$, and there is a double covering $p: Sp(1) \times Sp(1) \rightarrow SO(4)$ such that the diagram below is commutative, where d is the diagonal map and i is the inclusion map defined by $i(A) = A \oplus 1$:

$$Sp(1) \xrightarrow{d} Sp(1) \times Sp(1)$$

$$\downarrow p \qquad \qquad \downarrow p$$

$$Sp(3) \xrightarrow{i} SO(4)$$

$$Ss(4) \cong (Sp(1)/\mathbf{Z}_2) \times Sp(1) \cong SO(3) \times Sp(1). \quad (5.19')$$

We regard $Sp(1) \times Sp(1) \subset Sp(2)$ by the correspondence $(A, B) \mapsto A \oplus B$.

THEOREM 5.20. There is a universal covering $p: Sp(2) \rightarrow SO(5)$ such that the following diagram commutes.

$$Sp(1) \times Sp(1) \longrightarrow Sp(2)$$

$$\downarrow p \qquad \qquad \downarrow p$$

$$SO(4) \xrightarrow{i} SO(5)$$

Hence $\text{Spin}(5) \cong Sp(2)$.

PROOF. We define $\varphi_5: L(SO(5)) \rightarrow L(Sp(2))$, which is an extension of the above φ_4 to the remaining basis elements R_{i5} ($i = 1, 2, 3, 4$), as follows:

$$\varphi_5(2R_{15}) = \mathbf{k}\tilde{\mathbf{R}}, \quad \varphi_5(2R_{25}) = \mathbf{j}\tilde{\mathbf{R}}, \quad \varphi_5(2R_{35}) = -\mathbf{i}\tilde{\mathbf{R}},$$

$$\varphi_5(2R_{45}) = E_{21} - E_{12}, \quad \text{where } \tilde{\mathbf{R}} = E_{12} + E_{21}.$$

Then φ_5 is an isomorphism of Lie algebras, since it is easily verified to be a homomorphism of Lie algebras. The theorem is then easily obtained from this. Q.E.D.

We consider the inclusion

$$c' : Sp(2) \subset SU(4), \quad L(Sp(2)) \subset L(SU(4))$$

given by $c' : M(2, \mathbf{H}) \rightarrow M(4, \mathbf{C})$ where $c'(A + jB) = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$.

THEOREM 5.21. *There is a universal covering $p : SU(4) \rightarrow SO(6)$ such that the following diagram commutes.*

$$\begin{array}{ccc} Sp(2) & \xrightarrow{c'} & SU(4) \\ \downarrow p & & \downarrow p \\ SO(5) & \xrightarrow{i} & SO(6). \end{array}$$

Hence $\text{Spin}(6) \cong SU(4)$.

PROOF. If we choose a basis of $L(SU(4))$ consisting of

$$\begin{aligned} X_{ij} &= E_{ij} - E_{ji}, & Y_{ij} &= i(E_{ij} + E_{ji}) & (1 \leq i < j \leq 4), \\ T_1 &= i(E_{11} - E_{22}), & T_2 &= i(E_{22} - E_{33}), & T_3 &= i(E_{33} - E_{44}), \end{aligned}$$

then $c' \circ \varphi_5 : L(SO(5)) \rightarrow L(SU(4))$ is given by the following correspondence:

$$\begin{aligned} 2R_{12} &\mapsto T_1 - 2T_2 + T_3, & 2R_{13} &\mapsto -X_{13} - X_{24}, \\ 2R_{23} &\mapsto Y_{13} - Y_{24}, & 2R_{14} &\mapsto Y_{13} - Y_{24}, \\ 2R_{24} &\mapsto X_{13} - X_{24}, & 2R_{34} &\mapsto T_1 - T_3, \\ 2R_{15} &\mapsto -Y_{14} - Y_{23}, & 2R_{25} &\mapsto -X_{14} - X_{23}, \\ 2R_{35} &\mapsto -Y_{12} + Y_{34}, & 2R_{45} &\mapsto -X_{12} - X_{34}. \end{aligned}$$

Here if one further defines a map on the remaining elements of the basis by

$$\begin{aligned} 2R_{16} &\mapsto -X_{14} + X_{23}, & 2R_{26} &\mapsto Y_{14} - Y_{23}, & 2R_{36} &\mapsto -X_{12} + X_{34}, \\ 2R_{46} &\mapsto Y_{12} + Y_{34}, & 2R_{56} &\mapsto T_1 + T_3, \end{aligned}$$

then we obtain an isomorphism $L(SO(6)) \cong L(SU(4))$, and the theorem is proved. Q.E.D.

§6. Classifying spaces

CW-complexes

NOTATION. $D^n = \{(t_1, \dots, t_n) \in \mathbf{R}^n \mid \sum t_i^2 \leq 1\}$ denotes the *unit disk*.

The boundary of D^n is the *unit sphere*

$$S^{n-1} = \{(t_1, \dots, t_n) \in \mathbf{R}^n \mid \sum t_i^2 = 1\} \quad (\partial D^n = S^{n-1}).$$

DEFINITION. A subset e^n of a topological space X is called an *n-cell* of X if there is a (continuous) map $\chi : D^n \rightarrow \bar{e}^n$ (the closure of e^n) $\subset X$

which gives a homeomorphism of $D^n - S^{n-1}$ onto $e^n = \bar{e}^n - \partial e^n$ where $\chi(S^{n-1}) = \partial e^n$. We call χ the *characteristic map* and ∂e^n the *boundary* of e^n .

DEFINITION. K is called a *cell complex* with the *cellular decomposition* $\{e_\alpha\}$ if

- (1) K is a Hausdorff space such that $K = \bigcup_\alpha e_\alpha$, where $e_\alpha \cap e_\beta = \emptyset$ for $\alpha \neq \beta$.
- (2) e_α is an n_α -cell ($n_\alpha = \dim e_\alpha$ is called the *dimension* of e_α).
- (3) $\partial e_\alpha \subset K^{n-1} = \bigcup_{\dim e_\beta \leq n-1} e_\beta$ (which is called the $(n-1)$ -skeleton), where $n = \dim e_\alpha$.

A cell complex is called *finite* (countable) if the number of cells is finite (countable).

L is called a *subcomplex* of K if L is a subspace of K and is a cell complex with a cellular decomposition that consists of all the cells of K contained in L . Then we call (K, L) a *CW-pair*.

A (finite) polyhedron is a finite cell complex. A finite cell complex is a compact Hausdorff space whose topology is uniquely determined. If a CW-complex is not finite, then its topology is not necessarily unique.

DEFINITION. A topological space X is said to have the *weak topology* with respect to a system of subspaces $\{X_\alpha\}$ if X has the following property with respect to $\{X_\alpha\}$: F is a closed set of X iff $F \cap X_\alpha$ is a closed set of X_α for any α .

DEFINITION. K is called a *CW-complex* if K is a cell complex such that the cellular decomposition $\{e_\alpha\}$ satisfies

- (1) (Closure finiteness) the number of β such that $\bar{e}_\alpha \cap e_\beta \neq \emptyset$ is finite for each α ,
- (2) (Weak topology) K has the weak topology with respect to $\{\bar{e}_\alpha\}$.

We quote several properties of CW-complexes without proof.

6.1. (1) A subcomplex of a CW-complex is a closed subspace and a CW-complex.

(2) A map $f: K \rightarrow X$ of a CW-complex is continuous iff $f|_{\bar{e}_\alpha}$ is continuous for any α .

(3) A CW-complex is locally contractible (i.e., each point has a contractible neighborhood). In particular, it is locally arcwise connected and locally simply connected, and each component has a universal covering space.

(4) A CW-complex is paracompact and hence is a normal space.

6.2. (1) If K and L are connected and each cell of $K - L$ is of dimension $> n$ in a CW-pair (K, L) , then (K, L) is n -connected. In particular, if $K^n = e_0$ (a point), K is n -connected.

(2) Suppose that each cell of $K - L$ is of dimension $\leq n$ in a CW-pair (K, L) . If X is $(n-1)$ -connected, any $f: L \rightarrow X$ is extendable over K .

Moreover if (X, A) is n -connected, then for any given $f: (K, L) \rightarrow (X, A)$, there is a homotopy $f_t: (K, L) \rightarrow (X, A)$ such that $f_0 = f$, $f_1(K) \subset A$, and $f_t|L = f|L$.

(3) A CW-pair (K, L) has the homotopy extension property, that is, if an arbitrary map $f: K \rightarrow X$ and an arbitrary homotopy $g_t: L \rightarrow X$ satisfy $f|L = g_0$, then there exists a homotopy $f_t: K \rightarrow X$ such that $f_0 = f$ and $f_t|L = g_t$.

DEFINITION. $f: X \rightarrow Y$ is called a weak homotopy equivalence if f is a bijection of arcwise components (i.e., $f_*: \pi_0(X) \cong \pi_0(Y)$) and $f_*: \pi_i(X_\alpha) \cong \pi_i(Y_\alpha)$ ($\forall i$) for each pair of corresponding arcwise components X_α and Y_α .

Then we say that X and Y are weakly homotopy equivalent. (This is not necessarily an equivalence relation.)

It is obvious that a homotopy equivalence between X and Y is a weak homotopy equivalence. The converse is true if X and Y are CW-complexes.

6.3. (1) (Theorem of J. H. C. Whitehead). Let X and Y be CW-complexes. If $f: X \rightarrow Y$ is a weak homotopy equivalence, then there exists a $g: Y \rightarrow X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, and so X and Y are homotopy equivalent.

(2) For an arbitrary topological space X , there exist a CW-complex K and a weak homotopy equivalence $f: K \rightarrow X$.

A compact differentiable manifold M is triangulable, that is, there is a polyhedron K and a homeomorphism $\varphi: K \cong M$ such that φ is a C^∞ -map on each simplex. Moreover, if M is a submanifold of another compact differentiable manifold N , then there exist a polyhedron L of which K is a subcomplex and a homeomorphism $L \cong N$ which is an extension of φ . For example, using Morse theory L can be obtained from K by a number of handle body constructions. Thus one obtains the following lemma.

LEMMA 6.4. Let $M_1 \subset \dots \subset M_i \subset M_{i+1} \subset \dots$ be a sequence of compact differentiable manifolds with M_i a submanifold of M_{i+1} . The union $\bigcup M_i$ with the weak topology with respect to M_i is a CW-complex.

The n -universal bundle

DEFINITION. A G -bundle $\gamma = (E_\gamma, p_\gamma, B_\gamma, G)$ is called n -universal if for any G -bundle $\xi = (E, p, K, G)$, where K is a CW-complex of $\dim K \leq n$, and for any bundle map $h_L: i^* \xi \rightarrow \gamma$, where $i: L \rightarrow K$ is the inclusion of a subcomplex L , there exists a bundle map $h: \xi \rightarrow \gamma$ such that $h_L = h|_{i^* E}$.

LEMMA 6.5. A G -bundle $\gamma = (E_\gamma, p_\gamma, B_\gamma, G)$ is n -universal iff E_γ is $(n-1)$ -connected.

PROOF. Let γ be an n -universal G -bundle, $f: S^m \rightarrow E_\gamma$ ($m < n$), $\xi = (D^{m+1} \times G, p, D^{m+1}, G)$ a trivial G -bundle and $i: S^m \rightarrow D^{m+1}$ the inclusion

and consider $i^\# \xi = (S^m \times G, p', S^m, G)$. If we put $h_L(x, g) = f(x) \cdot g$ using the action of G on E_γ in Theorem 2.9, we obtain a bundle map $h_L: i^\# \xi \rightarrow \gamma$. By the definition of the n -universality there exists an extension $h: D^{m+1} \times G \rightarrow E_\gamma$ of h_L . If we put $\bar{f}(x) = h(x, e)$ for $x \in D^{m+1}$, then $\bar{f}: D^{m+1} \rightarrow E_\gamma$ is an extension of f . Hence E_γ is $(n-1)$ -connected.

Conversely, supposing that E_γ is $(n-1)$ -connected, we will show that there is an extension $h: \xi \rightarrow \gamma$ of a bundle map $h_L: i^\# \xi \rightarrow \gamma$ to $K = L \cup e_\alpha$ with $\dim e_\alpha \leq n$. We consider the bundle map $\tilde{\chi}_\alpha: \chi_\alpha^\# \xi \rightarrow \xi$ such that $\tilde{\chi}_\alpha|S^m: (\chi_\alpha|S^m)^\# \xi \rightarrow i^\# \xi$ is induced by the characteristic map $\chi_\alpha: (D^{m+1}, S^m) \rightarrow (K, L)$ of e_α . Since D^{m+1} is contractible, we apply Corollary 1.15(2) to the homotopy between the identity map and the constant map, hence $\chi_\alpha^\# \xi = (D^{m+1} \times G, p_1, D^{m+1}, G)$ is a trivial G -bundle. Then, as $h_L \circ (\tilde{\chi}_\alpha|S^m): S^m \times G \rightarrow E_\gamma$ commutes with the action of G , we extend the map $S^m \times e \rightarrow E_\gamma$ to $D^{m+1} \times e$ by the $n+1$ -connectedness of E_γ and then we further extend it to a bundle map $\tilde{h}: D^{m+1} \times G \rightarrow E_\gamma$ by the action of G . The extension h of h_L is obtained by putting $h = \tilde{h} \circ \tilde{\chi}_\alpha^{-1}$ on each point of $p^{-1}(e_\alpha)$. For the general case of extension from $L \cup K^m$ to $L \cup K^{m+1}$, if one proceeds similarly to the above on each cell e_α of $\dim e_\alpha = m+1$, then the continuity of the extension will be assured by the weak topology of K . Q.E.D.

THEOREM 6.6 (The classification theorem). *Let $\gamma = (E_\gamma, p_\gamma, B_\gamma, G)$ be the n -universal G -bundle, and K a CW-complex with $\dim K < n$. For an arbitrary G -bundle $\xi = (E, p, B, G)$, there exists a map $f: K \rightarrow B_\gamma$ such that $\xi \equiv f^\# \gamma$ and $f \simeq g: K \rightarrow B_\gamma$ iff $f^\# \gamma \equiv g^\# \gamma$.*

PROOF. The existence of f is clear from the definition of the n -universal G -bundle and Theorem 2.4. By Corollary 2.15(2), $f \simeq g$ implies $f^\# \gamma \equiv g^\# \gamma$. Conversely, if $\xi = f^\# \gamma \equiv g^\# \gamma$, one can define a bundle map $h_L: i^\# (\xi \times I) \rightarrow \gamma$ using $\tilde{f}$ and $\tilde{g}$ from the restriction of $\xi \times I$ on $L = K \times 0 \cup K \times 1$. There is an extension $h: \xi \times I \rightarrow \gamma$ of h_L , since $\dim(K \times I) \leq n$. Then we have $\bar{h}: K \times I \rightarrow B_\gamma$ which shows $f \simeq g$. Q.E.D.

DEFINITION. If G acts on F effectively, then the associated fibre bundle $\gamma \times_G F$ of the n -universal G -bundle γ is called the n -universal fibre bundle.

COROLLARY 6.7. *If G acts on F effectively, then the analogous result to Theorem 6.6 holds for the n -universal fibre bundle with structure group G and fibre F .*

LEMMA 6.8. *If G is a closed subgroup (e.g., a classical group) of $U(n, \mathbf{F})$, then*

$$(U(n+m, \mathbf{F})/1 \times U(m, \mathbf{F}), p, U(n+m, \mathbf{F})/G \times U(m, \mathbf{F}), G)$$

is the $(d(m+1)-2)$ -universal G -bundle.

PROOF. The proof is clear from Theorem 3.15 and Lemma 6.5 Q.E.D. Similarly, the following holds for the linear Lie group:

6.9. If G is a closed subgroup (i.e., a linear Lie group) of $GL(n, \mathbb{F})$, then $(GL(n+m, \mathbb{F})/1 \times GL(m, \mathbb{F}), p, GL(n+m, \mathbb{F})/G \times GL(m, \mathbb{F}), G)$ is the $(d(m+1)-2)$ -universal G -bundle.

Classifying spaces

DEFINITION. We will denote a G -bundle whose total space is ∞ -connected by $\gamma = (EG, p, BG, G)$ and call it the *universal G -bundle*. The base space BG of the universal G -bundle is called the *classifying space*. If G acts on F effectively, the associated fibre bundle $\gamma \times_G F = (E, p, BG, F, G)$ of the universal G -bundle γ is called the *universal fibre bundle* with fibre F and structure group G .

Corollary 6.7 for this case can be rephrased as follows:

THEOREM 6.10 (The classification theorem). *Let K be a CW-complex. Suppose that G acts on F effectively. Then homotopy classes of (continuous) maps $f: K \rightarrow BG$ are in one to one correspondence with equivalence classes of fibre bundles with base space K , fibre F , and structure group G , by assigning $f \mapsto f^*(\gamma \times_G F)$. In general, one may suppose that K is a paracompact Hausdorff space homotopy equivalent to a CW-complex.*

COROLLARY 6.11. *If the classifying space BG of G can be chosen to be (homotopy equivalent to) a CW-complex, it is unique up to homotopy equivalence.*

Let

$$G(1) \subset G(2) \subset \cdots \subset G(k) \subset G(k+1) \subset \cdots$$

be a sequence of inclusions of Lie groups such that each $G(k)$ is a closed subgroup of $G(k+1)$. We make $G(\infty) = \bigcup_k G(k)$ a topological space by introducing the weak topology with respect to $\{G(k)\}$. $G(\infty)$ is easily seen to be a topological group with the multiplication of $G(k)$.

Let $H(k)$ be a subgroup of $G(k)$ for each k such that

$$H(k) = G(k) \cap H(k+1), \quad k = 1, 2, 3, \dots \quad (6.12)$$

Then one can consider

$$G(k)/H(k) \subset G(k+1)/H(k+1). \quad (6.13)$$

LEMMA 6.14. *The natural inclusion makes $H(\infty)$ a subgroup of $G(\infty)$. If $\bigcup_k G(k)/H(k)$ is given the weak topology, then it is naturally homeomorphic to $G(\infty)/H(\infty)$. Also, the projection $p: G(\infty) \rightarrow G(\infty)/H(\infty)$ has a local cross-section.*

PROOF. One has $H(\infty) \subset G(\infty)$ and $G(\infty)/H(\infty) = \bigcup_k G(k)/H(k)$ as sets. As for the topologies we need only pay attention to a neighborhood of

the identity, since it suffices to consider the question locally. Let $T_e G(k) = V(k)$, $T_e(H(k)) = W(k)$, and $V(k) = W(k)^\perp \oplus W(k)$. If one considers the local homeomorphism $(w', w) \mapsto \exp w' \cdot \exp w$ in the proof of Lemma 5.12 of Chapter I, one can see that it suffices to prove the lemma for the case when $G(k)$ and $H(k)$ are vector spaces $V(k)$ and $W(k)$ over $\mathbb{R}$. One can take $\{\bigcup_k \prod_{i=1}^n [-\varepsilon_i, \varepsilon_i]; \varepsilon_i > 0\}$ as a fundamental system of neighborhoods of 0 in $V(\infty) = \bigcup_k V(k)$. (There is no condition on ε_i as we use the weak topology.) Then one can easily see that there are homeomorphisms

$$V(\infty) \cong (V(\infty)/W(\infty)) \times W(\infty), \\ V(\infty)/W(\infty) = \bigcup_k V(k)/W(k) \quad (\text{weak topology}).$$

Thus the lemma holds also in this case. Q.E.D.

COROLLARY 6.15. *In Lemma 6.14, if there is further a sequence of closed subgroups $\{K(n)\}$ satisfying $K(n) = H(n) \cap K(n+1)$, then*

$$(G(\infty)/K(\infty), p, G(\infty)/H(\infty), H(\infty)/K(\infty), H(\infty))$$

is a fibre bundle.

PROOF. Apply Lemma 6.14 to Theorem 2.11. Q.E.D.

THEOREM 6.16. *If G is a linear Lie group, then there exists a universal G -bundle. If G is a closed subgroup of some $U(n, \mathbb{F})$, there exists a universal G -bundle (EG, p, BG, G) such that BG is a CW-complex.*

PROOF. In the case $G \subset U(n, \mathbb{F})$, if one puts

$$G(k) = U(n+k, \mathbb{F}), \quad H(k) = G \times U(k, \mathbb{F}), \quad K(k) = 1 \times U(k, \mathbb{F})$$

then (6.12) is satisfied for any two of them and one obtains the fibre bundle in Corollary 6.15. Here

$$H(\infty)/K(\infty) = \bigcup_k H(k)/K(k) \cong G, \quad H(\infty) \cong G \times K(\infty),$$

and the action of $H(\infty)$ on $H(\infty)/K(\infty)$ is trivial on $K(\infty)$ and so $H(\infty)$ is reducible to G . That is, one obtains a G -bundle

$$(G(\infty)/K(\infty), p, G(\infty)/H(\infty), G).$$

Since $G(k)/H(k)$ is a compact differentiable manifold, $G(\infty)/H(\infty) = \bigcup_k G(k)/H(k)$ is a CW-complex by Lemma 6.4.

If one can show that $G(\infty)/K(\infty)$ is ∞ -connected, it will follow that the G -bundle above is a universal G -bundle. For any integer $i \geq 0$ and any map $f: S^i \rightarrow G(\infty)/K(\infty)$, the image of f , being compact, is contained in some $G(k)/H(k)$. If we take k sufficiently large, then $f \simeq 0$ by Theorem 3.15 and the ∞ -connectedness follows.

If G is a linear Lie group, we may assume $G \subset GL(n, \mathbb{F})$. The proof is similar except for the part concerning being a CW-complex. Q.E.D.

REMARK 6.17. (1) It will be shown in §2 of Chapter V that a compact Lie group is isomorphic to a subgroup of an orthogonal group of sufficiently large dimension. Therefore a compact Lie group has a classifying space which is a CW-complex.

(2) The remark (1) is unnecessary for compact classical groups, since they are closed subgroups of $U(n, \mathbb{F})$. Among the Lie groups considered through Chapter IV, the only group that is not clearly a closed subgroup of $U(n, \mathbb{F})$ is $\text{Spin}(n)$. In this case the group can be seen to be a subgroup of an orthogonal group, either by the remark (1) or by considering the spin representation defined using Clifford modules.

(3) In general for a topological group G , by introducing a suitable topology in the $(k+1)$ -join $G^{(k+1)} = G * \dots * G$ and letting G act on it naturally, we obtain a k -universal G -bundle $(G^{(k+1)}, p, G^{(k+1)}/G, G)$ and a classifying space $\lim_k G^{(k+1)}/G$ of G , by taking the direct limit. (Milnor)

Let G be a Lie group and H a closed subgroup of it. Then one can see that there is a sequence

$$H \xrightarrow{1} G \xrightarrow{\pi} G/H \xrightarrow{j} BH \xrightarrow{p} BG$$

such that each sequence consisting of consecutive 3 spaces is (equivalent to) a fibration. For the first such sequence this is clear because of the H -bundle $(G, \pi, G/H, H)$.

For the last one, because H acts on EG in the universal G -bundle (EG, p, BG, G) , $(EG, p', EG/H, H)$ is a universal H -bundle, that is, one may take $EG/H = BH$. Then, if we denote by ρ (sometimes by $\rho = \rho(H, G)$) the projection $BH = EG/H \rightarrow BG = EG/G$, we have

6.18. $(BH, \rho, BG, G/H, G)$ is a fibre bundle.

Next, by letting G act on $EG \times (G/H)$ by

$$(x, g'H)g = (xg, g^{-1}g'H), \quad g, g' \in G,$$

we obtain a G -bundle. Since the base space $(EG \times (G/H))/G$ is obtained by imposing the relation:

$$x \sim x' \text{ iff there exists } g \in G \text{ such that } (x, H)g = (x', H),$$

or equivalently there exists $h \in H$ such that $xh = x'$ in EG ;

the base space coincides with $BH = EG/H$.

The projection $p_2: EG \times (G/H) \rightarrow G/H$ is obviously a weak homotopy equivalence, in particular, it is a homotopy equivalence if EG is a CW-complex. Summing up we have 6.19.

6.19. There is a G -bundle $(EG \times (G/H), p, BH, G)$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 H & \xrightarrow{i'} & EG & \xrightarrow{p'} & BH \\
 \downarrow i & & \downarrow i_1 & & \parallel \\
 G & \xrightarrow{i_0} & EG \times (G/H) & \xrightarrow{p} & BH \\
 \downarrow l & & \downarrow p_2 & \nearrow i_2 & \uparrow j \\
 G & \xrightarrow{\pi} & G/H & \xleftarrow{id} & G/H
 \end{array}$$

$$\begin{aligned}
 i_0(g) &= (x_0 g, g^{-1}H), & l(g) &= g^{-1}, \\
 i_1(x) &= (x, H), & i_2(gH) &= (x_0, gH), \\
 p_2(x, gH) &= gH, & j(gH) &= x_0 H,
 \end{aligned}$$

where i_2 and p_2 are weak homotopy equivalences.

The product $(EG \times EG', p \times p', BG \times BG', G \times G')$ of a universal G -bundle (EG, p, BG, G) and a universal G' -bundle (EG', p', BG', G') is a universal $G \times G'$ -bundle. In symbolic form

$$BG \times BG' = B(G \times G'). \quad (6.20)$$

CHAPTER III

Cohomology of Classical Groups and Their Homogeneous Spaces

In this chapter we determine cohomology rings of classical groups and their homogeneous spaces by making use of the Serre spectral sequence. For the sake of convenience in constructing the Serre spectral sequence, we start in §1 with the definition of the cubical chain complex and introduce homology and cohomology with local coefficients as well as cohomology operations. In §2 we construct the Serre spectral sequence of a fibration but we will only sketch an outline here and refer the reader to Serre's original paper [17] for details. We will consider the Gysin sequence in §3 as a special case of the Serre spectral sequence and determine the cohomology rings of the classical groups, the Stiefel manifolds, and classifying spaces as applications. In §4 we consider various conditions which insure that the Serre spectral sequence is trivial and moreover, that determine the action of the Weyl group on the cohomology of a maximal torus. Characteristic classes such as the Chern classes, the Stiefel-Whitney classes, and the Pontrjagin classes are defined in §5. Among symmetric spaces, the cohomology rings of symmetric spaces of classical type, obtained as homogeneous spaces of classical groups, are almost determined in §6.

§1. The cohomology ring of topological spaces

The cubical chain complex

RECOLLECTION. $I^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n | 0 \leq t_i \leq 1\}$ denotes the *unit n -cube*.

NOTATION. $[n] = \{1, 2, \dots, n\}$.

For a monotone increasing function $\alpha: [m] \rightarrow [n]$ ($1 \leq \alpha(1) < \alpha(2) < \dots < \alpha(m) \leq n$), we define $\partial_\alpha^\varepsilon: I^m \rightarrow I^n$ ($\varepsilon = 0, 1$) by

$$\partial_\alpha^\varepsilon(s_1, \dots, s_m) = (t_1, \dots, t_n), \text{ where } t_i = \begin{cases} s_j & (i = \alpha(j)), \\ \varepsilon & (i \notin \text{Im } \alpha). \end{cases}$$

In particular, $\partial_k^\varepsilon: I^{n-1} \rightarrow I^n$ ($1 \leq k \leq n$) is defined by

$$\partial_k^\varepsilon = \partial_\alpha^\varepsilon, \text{ where } \alpha(j) = \begin{cases} j & (j < k), \\ j+1 & (j \geq k); \end{cases}$$

$\pi_k: I^n \rightarrow I^{n-1}$ ($1 \leq k < n$) is defined by

$$\pi_k(t_1, \dots, t_n) = (t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_n).$$

DEFINITION. A singular n -cube u in a topological space X is a continuous map $u: I^n \rightarrow X$.

The (k, ε) -face of a singular n -cube u is the map $\partial_k^\varepsilon u = u \circ \partial_k^\varepsilon: I^{n-1} \rightarrow I^n \rightarrow X$.

A singular n -cube u is called *degenerate* ($n > 0$) if there are v and k such that $u = v \circ \pi_k: I^n \rightarrow I^{n-1} \rightarrow X$, i.e., there is some k such that u is independent of t_k .

NOTATION. $Q_n(X)$ = (the free abelian group generated by singular n -cubes).

$D_n(X)$ = (the subgroup of $Q_n(X)$ generated by degenerate singular n -cubes).

$$C_n(X) = Q_n(X)/D_n(X).$$

DEFINITION. A homomorphism $d: C_n(X) \rightarrow C_{n-1}(X)$ is defined by setting $du = \sum_{k=1}^n (-1)^k (\partial_k^0 u - \partial_k^1 u)$, where we use $d(D_n(X)) \subset D_{n-1}(X)$. Further $d \circ d = 0$ is satisfied and so $(C_n(X), d)$ is a chain complex. We call it the *cubical chain complex*: $C_*(X) = \sum_n C_n(X)$, $(C_n(X) = 0, n < 0)$.

The cohomology ring of a topological space is usually constructed by means of the chain complex generated by singular n -simplices $\Delta^n \rightarrow X$, but it follows from the principle of acyclic models due to Eilenberg-Zilber that one can obtain an equivalent cohomology ring by using cubical chain complexes as above. The latter is more convenient to describe the cohomology, in particular the spectral sequence of a fibration, and so we will construct cohomology rings from cubical chain complexes.

For a (continuous) map $f: X \rightarrow Y$ the map

$$f_\#: C_n(X) \rightarrow C_n(Y) \text{ defined by } f_\#(u) = f \circ u: I^n \rightarrow X \rightarrow Y$$

satisfies $f_\# \circ d = d \circ f_\#$ and hence $f_\#: C_*(X) \rightarrow C_*(Y)$ is the chain map. In this way we obtain a functor of the category of topological spaces to that of chain complexes.

Suppose that $f, g: X \rightarrow Y$ are homotopic $f \simeq g$, that is, there is a map $H: X \times I \rightarrow Y$, satisfying

$$H(x, 0) = f(x), \quad H(x, 1) = g(x) \quad (I = I^1 = [0, 1]).$$

If for $u: I^n \rightarrow X$ we define $Du: I^{n+1} \rightarrow Y$ by $Du(t_1, t_2, \dots, t_{n+1}) = H(u(t_2, \dots, t_{n+1}), t_1)$, then $dD + Dd = g_\# - f_\#$; namely, $f_\#$ and $g_\#$ are chain homotopic (Notation. $f_\# \simeq g_\#$):

$$f \simeq g: X \rightarrow Y \text{ implies } f_\# \simeq g_\#: C_*(X) \rightarrow C_*(Y).$$

In particular one can see that $C_*(I^p) \rightarrow C_*(\text{a point})$ is a chain equivalence, since I^p is contractible. As $C_n(\text{a point}) = 0$ ($n \neq 0$) and $C_0(\text{a point}) \cong \mathbb{Z}$,

we have that

$$\cdots \rightarrow C_n(I^p) \xrightarrow{d} C_{n-1}(I^p) \rightarrow \cdots \rightarrow C_0(I^p) \rightarrow \mathbb{Z} \rightarrow 0$$

is exact, i.e., $C_*(I^p)$ is acyclic.

The cross product $u \times v: I^{p+q} \rightarrow X \times Y$ of singular cubes $u: I^p \rightarrow X$ and $v: I^q \rightarrow Y$ in two topological spaces is defined by

$$(u \times v)(t_1, \dots, t_{p+q}) = (u(t_1, \dots, t_p), v(t_{p+1}, \dots, t_{p+q})).$$

If one of u and v is degenerate, so is $u \times v$, and the boundary formula

$$d(u \times v) = du \times v + (-1)^p u \times dv$$

holds, and hence the cross product of chain complexes

$$\times: C_p(X) \otimes C_q(Y) \rightarrow C_{p+q}(X \times Y)$$

is obtained. By the principle of acyclic models one obtains

$$\times: C_*(X) \otimes C_*(Y) \simeq C_*(X \times Y) \quad (\text{chain equivalence}).$$

The chain equivalence in the opposite direction

$$\square: C_n(X \times Y) \rightarrow \sum_{p+q=n} C_p(X) \otimes C_q(Y)$$

is given by

$$\square(u) = \sum \varepsilon(\alpha) \partial_\alpha^0 u_X \otimes \partial_\beta^1 u_Y, \quad u(t) = (u_X(t), u_Y(t)),$$

where $\alpha: [p] \rightarrow [n]$, $\beta: [q] \rightarrow [n]$, $[n] = \alpha[p] \cup \beta[q]$, $\varepsilon(\alpha) = (-1)^{\nu(\alpha)}$, $\nu(\alpha) = \sum_{i=1}^p (\alpha(i) - i)$.

The cross product is natural and associative:

$$(f \times g)_*(u \times v) = f_* u \times g_* v, \quad (u \times v) \times w = u \times (v \times w).$$

Further, for the switching map $T: X \times Y \rightarrow Y \times X$ defined by $T(x, y) = (y, x)$ and $\tau: C_p(X) \otimes C_q(Y) \rightarrow C_q(Y) \otimes C_p(X)$ defined by $\tau(\alpha \otimes \beta) = (-1)^{pq} \beta \otimes \alpha$, one has

$$T_* \circ \times \simeq \times \circ \tau \quad (\text{chain homotopic}).$$

NOTATION. $C_n(X, k)$ = (the subgroup of $C_n(X)$ generated by $u: I^n \rightarrow X$ satisfying the following condition). A point x_0 is fixed; for $n \leq k$, $u(I^n) = x_0$; for $n > k$, $C_n(X, k)$ is defined inductively by the condition $u \in C_n(X, k)$ iff $\partial_i^e u \in C_{n-1}(X, k)$ ($1 \leq i \leq n$; $e = 0, 1$).

By using the definition of k -connectedness (Chapter II, §3) to construct a homotopy to deform $u: I^n \rightarrow X$ into such u , one can prove the following:

1.1. If X is k -connected then the inclusion $C_n(X, k) \rightarrow C_n(X)$ is a chain equivalence, that is, there is a chain map $\eta: C_n(X) \rightarrow C_n(X, k)$ such that $\eta \circ i = 1$ and $i \circ \eta \simeq 1$.

The homology group. Let (X, A) be a pair of topological spaces ($A \subset X$) and G an abelian group (called a coefficient group).

NOTATION. $C_n(X, A; G) = C_n(X)/C_n(A) \otimes G$ (a chain group);

$$\partial = d \otimes 1 : C_n(X, A; G) \rightarrow C_{n-1}(X, A; G);$$

$$Z_n(X, A; G) = \text{Ker}(\partial : C_n(X, A; G) \rightarrow C_{n-1}(X, A; G))$$

(the group of cycles);

$$B_n(X, A; G) = \text{Im}(\partial : C_{n+1}(X, A; G) \rightarrow C_n(X, A; G))$$

(the group of boundaries);

$$f_{\#} = f_{\#} \otimes 1 : C_n(X, A; G) \rightarrow C_n(Y, B; G) \quad (f : (X, A) \rightarrow (Y, B)).$$

$d \circ d = 0$ implies $\partial \circ \partial = 0$ so $B_n(X, A; G) \subset Z_n(X, A; G)$. $f_{\#}$ is a chain map (i.e., $f_{\#} \circ \partial = \partial \circ f_{\#}$) so $f_{\#}(B_n(X, A; G)) \subset B_n(Y, B; G)$, $f_{\#}(Z_n(X, A; G)) \subset Z_n(Y, B; G)$.

Using these we obtain the following:

DEFINITION. $H_n(X, A; G) = Z_n(X, A; G)/B_n(X, A; G)$ is called the n th (n -dimensional) homology group of the pair (X, A) with coefficients in G . $H_n(X; G) = H_n(X, \emptyset; G)$ is called the homology group of X .

$$H_*(X, A; G) = \sum_n H_n(X, A; G), \quad H_*(X; G) = H_*(X, \emptyset; G).$$

$f_* = (f_{\#})_* : H_n(X, A; G) \rightarrow H_n(Y, B; G)$ denotes the homomorphism induced by $f : (X, A) \rightarrow (Y, B)$.

The following seven axioms (A1) ~ (A7) due to Eilenberg-Steenrod are satisfied:

$$(A1) \quad 1_* = 1 : H_n(X, A; G) \rightarrow H_n(X, A; G),$$

$$(A2) \quad (g \circ f)_* = g_* \circ f_* \quad (f : (X, A) \rightarrow (Y, B), \quad g : (Y, B) \rightarrow (Z, C)).$$

For a triple (X, A, B) , $B \subset A \subset X$, the sequence

$$0 \rightarrow C_n(A, B; G) \xrightarrow{i_*} C_n(X, B; G) \xrightarrow{j_*} C_n(X, A; G) \rightarrow 0$$

is an exact sequence of chain complexes and the boundary homomorphism

$$\partial_* : H_n(X, A; G) \rightarrow H_{n-1}(A, B; G)$$

is defined by $\partial_*\{z\} = \{i_{\#}^{-1}\partial j_{\#}^{-1}(z)\}$, $z \in Z_n(X, A; G)$. An elementary consideration of homological algebra shows the following two axioms:

$$(A3) \quad \partial_* \circ f_* = (f|A)_* \circ \partial_*;$$

$$(A4) \quad (\text{the exactness axiom}) \quad \cdots \xrightarrow{\partial_*} H_n(A, B; G) \xrightarrow{i_*} H_n(X, B; G) \xrightarrow{j_*} H_n(X, A; G) \xrightarrow{\partial_*} H_{n-1}(A, B; G) \xrightarrow{i_*} \cdots \text{ is exact.}$$

$$(A5) \quad (\text{the homotopy axiom}) \quad f \simeq g : (X, A) \rightarrow (Y, B) \text{ implies } f_* = g_* : H_n(X, A; G) \rightarrow H_n(Y, B; G).$$

This follows since $f \simeq g$ implies $f_{\#} \simeq g_{\#}$ (chain homotopic).

(A6) (the excision axiom) If $\bar{U} \subset \text{Int } A$ then

$$i_* : H_n(X - U, A - U; G) \cong H_n(X, A; G).$$

This is proved by partitioning I^n .

(A7) (the *dimension axiom*) For $n \neq 0$, $H_n(\text{a point}; G) = 0$ and $H_0(\text{a point}; G)$ is isomorphic to the coefficient group G .

Further, the following axiom also holds.

(A8) (the *additivity axiom*) $H_n(\Pi_\alpha X_\alpha; G) \cong \sum_\alpha H_n(X_\alpha; G)$.

THEOREM 1.2 (Eilenberg–Steenrod). *These axioms characterize the homology group with coefficients in G of a CW-complex (or a finite CW-complex when omitting (A8)).*

In fact, let

$$u_\alpha: (I^n, \partial I^n) \cong (D^n, S^{n-1}) \rightarrow (e_\alpha, \partial e_\alpha) \subset (K^n, K^{n-1})$$

be the characteristic map of an n -cell e_α of a CW-complex K . Then $C_n(K) = H_n(K^n, K^{n-1})$ is a free module with a basis consisting of the classes of u_α and $C_*(K) = \sum_n C_n(K)$ together with $\partial_*: H_n(K^n, K^{n-1}) \rightarrow H_{n-1}(K^{n-1}, K^{n-2})$ forms a chain complex, and we have a natural isomorphism:

$$H_n(C_*(K) \otimes G) \cong H_n(K; G). \quad (1.2')$$

Using (1.2') one can easily compute the homology groups of a simple space. For example,

$$\begin{aligned} H_i(S^n) &= 0 \quad (i \neq 0, n), \\ H_n(S^n) &\cong H_{n+1}(D^{n+1}, S^n) \cong \mathbf{Z} \quad (n \geq 1). \end{aligned} \quad (1.3)$$

Also from (1.1)

1.4. *If X is k -connected then $H_i(X; G) = 0$ ($0 < i \leq k$), and $H_0(X; G) \cong G$.*

Let R be a commutative ring with 1. If the coefficient group G is an R -module, so is $H_n(X; G)$. In particular, if R is a principal ring (a principal ideal domain), the following *universal coefficient theorem* holds:

$$\begin{aligned} H_n(X, A; G) &\cong H_n(X, A; R) \otimes_R G \\ &\oplus \text{Tor}^R(H_{n-1}(X, A; R), G) \text{ (direct sum)}. \end{aligned} \quad (1.5)$$

Also, since the cross product $\times: C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times Y)$ is a chain equivalence, we have the following *Künneth formula*:

$$\begin{aligned} H_n(X \times Y, A \times Y; R) &\cong \sum_{p+q=n} H_p(X, A; R) \otimes_R H_q(Y; R) \\ &\oplus \sum_{p+q=n} \text{Tor}^R(H_{p-1}(X, A; R), H_q(Y; R)) \quad (R: \text{a principal ring}). \end{aligned} \quad (1.6)$$

Here, the first term corresponds to the cross product (a homomorphism) of the homology group

$$\times: H_p(X, A; R) \otimes H_q(Y; R) \rightarrow H_{p+q}(X \times Y, A \times Y; R)$$

induced by $\times: C_p(X) \otimes C_q(Y) \rightarrow C_{p+q}(X \times Y)$. The cross product is natural, associative, and commutative.

When X is an H -space (in particular, when X is a topological group) the composition of the cross product and the homomorphism μ_* induced by the multiplication $\mu: X \times X \rightarrow X$ maps $\alpha \otimes \beta \mapsto \mu_*(\alpha \times \beta)$, which gives rise to a product

$$H_p(X; R) \otimes H_q(X; R) \rightarrow H_{p+q}(X; R)$$

and hence $H_*(X; R) = \sum H_n(X; R)$ forms a ring with 1. We call it the *Pontrjagin ring* or the *homology ring* of X .

The cohomology ring

NOTATION. $C^n(X, A; G) = \text{Hom}(C_n(X)/C_n(A), G)$ (the cochain group).

$$\delta = d^*: C^{n-1}(X, A; G) \rightarrow C^n(X, A; G) \text{ (i.e., } (\delta h)(u) = h(\partial u) \text{)}.$$

$$Z^n(X, A; G) = \text{Ker}(\delta: C^n(X, A; G) \rightarrow C^{n+1}(X, A; G))$$

(the group of cocycles).

$$B^n(X, A; G) = \text{Im}(\delta: C^{n-1}(X, A; G) \rightarrow C^n(X, A; G))$$

(the group of coboundaries).

$$f^\# = (f_\#)^*: C^n(Y, B; G) \rightarrow C^n(X, A; G) \text{ (} f: (X, A) \rightarrow (Y, B),$$

(f^\# h)(u) = h(f_\#(u)) \text{)}.

DEFINITION. $H^n(X, A; G) = Z^n(X, A; G)/B^n(X, A; G)$ (the n th cohomology group).

$f^* = (f^\#)^*: H^n(Y, B; G) \rightarrow H^n(X, A; G)$ (the homomorphism induced by f).

The coboundary homomorphism

$$\delta^*: H^n(A, B; G) \rightarrow H^{n+1}(X, A; G)$$

is also defined similarly to the case of the homology group and the axioms (A1) ~ (A8) hold literally with the direction of the arrows reversed and with f_* and ∂_* , etc. replaced by f^* and δ^* , etc. The universal coefficient theorem takes the following form:

$$H^n(X, A; G) \cong \text{Hom}_R(H_n(X, A; R), G) \oplus \text{Ext}_R(H_{n-1}(X, A; R), G) \text{ (a direct sum).} \quad (1.7)$$

If $H_{n-1}(X, A; R)$ is R -free (e.g., R is a field), then

$$H^n(X, A; R) \cong \text{Hom}_R(H_n(X, A; R), R) = (\text{the dual of } H_n(X, A; R)).$$

Further, if $H_n(X, A; R)$ is a finitely-generated R -free module, then one has an isomorphism

$$H^n(X, A; R) \cong H_n(X, A; R). \quad (1.8)$$

For $c \in C_n(X, A; R)$ and $f \in C^n(X, A; G) = \text{Hom}_R(C_n(X, A; R), G)$,

$$\langle f, c \rangle = f(c) \in G$$

is called the *Kronecker product*. It satisfies $\langle \delta f, c \rangle = \langle f, \partial c \rangle$ and hence induces the *Kronecker product* between cohomology classes and homology classes

$$\langle \cdot, \cdot \rangle: H^n(X, A; G) \otimes_R H_n(X, A; R) \rightarrow G.$$

(1.8) is equivalent to the following orthogonality conditions:

$$\alpha = 0 \text{ iff } \langle \alpha, x \rangle = 0 \text{ for every } x \quad (G = R), \quad (1.8')$$

$$x = 0 \text{ iff } \langle \alpha, x \rangle = 0 \text{ for every } \alpha \quad (G = R). \quad (1.8'')$$

The (cup) product of $f \in C^p(X; R)$ and $g \in C^q(X; R)$ is defined by

$$(f \cdot g)(u) = \sum \varepsilon(\alpha) f(\partial_\alpha^0 u) g(\partial_\beta^1 u) \in R \quad (u: I^{p+q} \rightarrow X).$$

The cup product is natural and satisfies

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h, \quad \delta(f \cdot g) = \delta f \cdot g + (-1)^p f \cdot \delta g,$$

and hence induces a (cup) product between cohomology classes

$$H^p(X; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X; R).$$

Then $H^*(X; R) = \sum H^n(X; R)$ is an R -algebra with respect to the product and the unit $1 \in H^0(X; R)$ is the cohomology class of $1 \in C^0(X; R)$ given by $1(u) = 1 \in R$. It follows from the homotopy commutativity of the cross product that

$$\alpha\beta = (-1)^{pq} \beta\alpha, \quad \alpha \in H^p(X; R), \quad \beta \in H^q(X; R). \quad (1.9)$$

An element of $C^p(X, A; R)$ can be considered as $f \in C^p(X; R)$ satisfying $f(u) = 0$ for $u(I^p) \subset A$. It follows that the products

$$\cdot: H^p(X, A; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X, A; R)$$

$$\cdot: H^p(X; R) \otimes H^q(X, A; R) \rightarrow H^{p+q}(X, A; R)$$

$$\cdot: H^p(X, A; R) \otimes H^q(X, A; R) \rightarrow H^{p+q}(X, A; R)$$

are defined and satisfy the properties above. They are interrelated via the homomorphism

$$j^*: H^*(X, A; R) \rightarrow H^*(X; R).$$

The relation with $\delta^*: H^*(A; R) \rightarrow H^{*+1}(X, A; R)$ is given by

$$\delta^*(\alpha \cdot i^* \beta) = \delta^* \alpha \cdot \beta \quad \text{for } \alpha \in H^p(A; R), \quad \beta \in H^q(X; R),$$

where $i^*: H^*(X; R) \rightarrow H^*(A; R)$.

Let $p_X: X \times Y \rightarrow X$ and $p_Y: X \times Y \rightarrow Y$ be the projections. Then the cross product

$$\times: H^p(X, A; R) \otimes H^q(Y; R) \rightarrow H^{p+q}(X \times Y, A \times Y; R)$$

is defined by $\alpha \times \beta = p_X^* \alpha \cdot p_Y^* \beta$. One can also define the cross product directly, using the chain equivalence $\square$. Conversely, when the cross product is already given, the (cup) product can be given by $\alpha \cdot \beta = d^*(\alpha \times \beta)$, using the diagonal map $d: X \rightarrow X \times X$.

By virtue of the Künneth formula (1.6) and the duality isomorphism, the cross product gives rise to the following isomorphism (the *Künneth formula*).

1.10.

$$\sum_{p+q=n} H^p(X, A; R) \otimes_R H^q(Y; R) \cong H^n(X \times Y, A \times Y; R),$$

provided $H^p(X, A; R)$ for every p (or $H^q(Y; R)$ for every q) is a finitely generated R -free module.

The Bockstein homomorphism. For a given exact sequence

$$0 \rightarrow H \xrightarrow{f} G \xrightarrow{g} K \rightarrow 0$$

of abelian groups, there is an exact sequence of chain complexes:

$$0 \rightarrow C_n(X; H) \xrightarrow{1 \otimes f} C_n(X; G) \xrightarrow{1 \otimes g} C_n(X; K) \rightarrow 0$$

from which we obtain the *Bockstein exact sequence*:

$$\begin{aligned} \dots \rightarrow H_n(X; H) \xrightarrow{f_*} H_n(X; G) \xrightarrow{g_*} H_n(X; K) \xrightarrow{\beta} \\ H_{n-1}(X; H) \xrightarrow{f_*} \dots \end{aligned} \quad (1.11)$$

We say that f_* and g_* are the *homomorphisms induced by f and g* and β is the *Bockstein homomorphism*. A similar result holds with X replaced by a pair (X, A) . The cohomology case takes the following form:

$$\begin{aligned} \dots \rightarrow H^n(X; H) \xrightarrow{f_*} H^n(X; G) \xrightarrow{g_*} H^n(X; K) \xrightarrow{\beta} \\ H^{n+1}(X; G) \xrightarrow{f_*} \dots \end{aligned} \quad (1.11')$$

These homomorphisms are natural, that is, they commute with the homomorphism induced by a map. Also the relations with the homomorphisms ∂_* and δ^* are given by

$$\partial_* \beta = -\beta \partial_*, \quad \delta^* \beta = -\beta \delta^*. \quad (1.12)$$

The relation with the cross product is given by

$$\beta(x \times y) = \beta x \times y + (-1)^n x \times \beta y \quad (1.13)$$

both for homology and cohomology cases ($n = \deg x$ i.e., $x \in H_n$ or $x \in H^n$).

The relation with the Pontrjagin product or the (cup) product is given by

$$\beta(x \cdot y) = \beta x \cdot y + (-1)^n x \cdot \beta y \quad (n = \deg x) \quad (1.13')$$

For the case $H = G = \mathbb{Z}$ and $K = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$,

$$\rho = g_*: H^n(X; \mathbb{Z}) \rightarrow H^n(X; \mathbb{Z}_p)$$

is called the *mod p reduction* and both $\beta: H^n(X; \mathbb{Z}_p) \rightarrow H^{n+1}(X; \mathbb{Z})$ and $\beta_p = p \circ \beta: H^n(X; \mathbb{Z}_p) \rightarrow H^{n+1}(X; \mathbb{Z})$ are called the *mod p Bockstein homomorphism*. Since β_p satisfies $\beta_p \circ \beta_p = 0$, one can consider the cohomology group $H_{\beta_p}^n(H^*(X; \mathbb{Z}_p))$ with respect to β_p .

We usually denote $H_n(X) = H_n(X; \mathbb{Z})$ and $H^n(X) = H^n(X; \mathbb{Z})$ omitting the coefficient ring $\mathbb{Z}$.

DEFINITION. A topological space X is said to be of *finite type* if $H_n(X)$ is finitely generated for any n . The *Poincaré series* (or the *Poincaré polynomial*) over the coefficient field $\mathbf{k}$ of a topological space X of finite type is defined by the following formula:

$$P_{\mathbf{k}}(X, t) = \sum_n \dim_{\mathbf{k}} H_n(X; \mathbf{k}) \cdot t^n = \sum_n \dim_{\mathbf{k}} H^n(X; \mathbf{k}) \cdot t^n.$$

1.14. (1) $P_{\mathbf{k}}(X \amalg Y, t) = P_{\mathbf{k}}(X, t) + P_{\mathbf{k}}(Y, t)$, where $\amalg$ denotes the topological sum.

(2) $P_{\mathbf{k}}(X \times Y, t) = P_{\mathbf{k}}(X, t) \cdot P_{\mathbf{k}}(Y, t)$, where $\times$ is the direct product.

LEMMA 1.15. Let X be a topological space of finite type, with $P_Q(X, t) = \sum b_n t^n$ and $P_{\mathbb{Z}_p}(X, t) = \sum a_n t^n$.

(1) $a_0 = b_0$, $a_n \geq b_n$ ($n = 1, 2, \dots$).

(2) If the p -primary component of $H_n(X)$ is a direct sum of c_n copies of a cyclic group (of order a power of p), then we have

$$a_n = b_n + c_n + c_{n-1}.$$

In particular,

$$\begin{aligned} a_n = b_n \quad (n = 1, 2, \dots) & \text{ iff } H_*(X) \text{ is } p\text{-torsion free} \\ & \text{ iff } H^*(X) \text{ is } p\text{-torsion free.} \end{aligned}$$

(3) If we denote by c'_n the number of cyclic groups in (2) of order $> p$, then

$$b_n + c'_n + c'_{n-1} = \dim_{\mathbb{Z}_p} H_{\beta_p}^n(H^*(X; \mathbb{Z}_p)).$$

In particular, $\dim_{\mathbb{Z}_p} H_{\beta_p}^n(H^*(X; \mathbb{Z}_p)) = \dim_Q H^n(X; \mathbb{Q})$ implies that the p -primary component of $H_n(X)$ is a direct sum of copies of $\mathbb{Z}_p$.

PROOF. By the fundamental theorem of the theory of (finitely-generated) abelian groups, we decompose the free part F_n of $H_n(X)$ into a direct sum of b_n copies of $\mathbb{Z}$ and the p -primary component of $H_n(X)$ into a direct sum of c_n copies of $\mathbb{Z}_{p^r}$, and apply the universal coefficient theorem. Then we get $\dim H^n(X; \mathbb{Q}) = b_n$ and $\dim H^n(X; \mathbb{Z}_p) = b_n + c_n + c_{n-1}$, where $H_0(X) = F_0$ and $b_0 =$ (the number of arcwise connected components of X). From this we get (1) and (2). As for (3), if we consider $A = \text{Hom}(\mathbb{Z}_{p^r}, \mathbb{Z}_p) \subset H^{n-1}(X; \mathbb{Z}_p)$ and $B = \text{Ext}(\mathbb{Z}_{p^r}, \mathbb{Z}_p) \subset H^n(X; \mathbb{Z}_p)$ corresponding to the direct summand $\mathbb{Z}_{p^r}$ in $H_{n-1}(X)$, β_p maps $A \cong \mathbb{Z}_p$ to $B \cong \mathbb{Z}_p$ and is of

the form $\beta_p = p^{r-1}: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$. If $r = 1$, then β_p is an isomorphism and A and B kill each other in H_β^n . If $r > 1$ then $\beta_p(A) = 0$ and A and B survive in H_β^n . (3) is obtained from these observations. Q.E.D.

Cohomology operations

DEFINITION. A morphism

$$\Phi: H^n(X, A; G) \rightarrow H^{n+q}(X, A; H),$$

which is natural ($f^*\Phi = \Phi f^*$) is called a *cohomology operation of type* $(n, G, n+q, H)$ and q its *degree*. If Φ is given for each n and if it commutes with the coboundary homomorphism ($\delta^*\Phi = \Phi\delta^*$), then the family $\{\Phi\}$ is called a *stable cohomology operation of degree q of type (G, H)* .

The homomorphism φ_* induced by a homomorphism $\varphi: G \rightarrow H$ of coefficient groups is a stable cohomology operation of degree 0. The Bockstein homomorphism β is a cohomology operation of degree 1 and the family $\{(-1)^n\beta\}$ is a stable cohomology operation of degree 1.

If we put $\Phi(x) = x^k$, using the product in the cohomology ring $H^*(X, A; R)$, then the map

$$\Phi: H^n(X, A; R) \rightarrow H^{kn}(X, A; R)$$

is a cohomology operation. If $k = p$ (a prime) and $R = \mathbb{Z}_p$, the map Φ is extendable to a stable operation, which is called the *Steenrod operation*.

For the case $p = 2$, there exists the (*Steenrod*) *squaring operation*

$$Sq^r: H^n(X, A; \mathbb{Z}_2) \rightarrow H^{n+r}(X, A; \mathbb{Z}_2).$$

1.16 (THE FUNDAMENTAL PROPERTIES OF Sq).

- (1) Sq^r is a stable operation satisfying $Sq^r(x) = x^2$ ($x \in H^r$), and this property characterizes Sq^r ($r > 0$).
- (2) $Sq^0 = \text{id}$, $Sq^r = 0$ ($r < 0$), $Sq^r(x) = 0$ ($x \in H^n$, $n < r$).
- (3) $Sq^1 = \beta_2$.
- (4) (the Cartan formula) $Sq^r(xy) = \sum_{s+t=r} Sq^s(x)Sq^t(y)$.
- (5) (the Adem relations)

$$Sq^a Sq^b = \sum_t \binom{b-t-1}{a-2t} Sq^{a+b-t} Sq^t \quad (a < 2b).$$

For the case p an odd prime, there exists the (*Steenrod*) *reduced power operation*

$$\varphi^r: H^n(X, A; \mathbb{Z}_p) \rightarrow H^{n+2r(p-1)}(X, A; \mathbb{Z}_p)$$

1.17 (THE FUNDAMENTAL PROPERTIES OF φ).

- (1) φ^r is a stable cohomology operation satisfying $\varphi^r(x) = x^p$ ($x \in H^{2r}$), and this property characterizes φ^r ($r > 0$).
- (2) $\varphi^0 = \text{id}$, $\varphi^r = 0$ ($r < 0$), $\varphi^r(x) = 0$ ($x \in H^n$, $n < 2r$).

(3) (the Cartan formula) $\wp^r(xy) = \sum_{s+t=r} \wp^s(x)\wp^t(y)$.

(4) (the Adem relations)

$$\wp^a \wp^b = \sum_s (-1)^{a+s} \binom{(b-s)(p-1)-1}{a-sp} \wp^{a+b-s} \wp^s \quad (a < pb),$$

$$\begin{aligned} \wp^a \beta_p \wp^b &= \sum_s (-1)^{a+s} \binom{(b-s)(p-1)}{a-ps} \beta_p \wp^{a+b-s} \wp^s \\ &\quad + \sum_s (-1)^{a+s-1} \binom{(b-s)(p-1)-1}{a-sp-1} \wp^{a+b-s} \beta_p \wp^s \quad (a \leq pb). \end{aligned}$$

(Co)Homology with local coefficients

DEFINITION. $\underline{G} = \{G_x\}_{x \in X}$ is called a *local coefficient group* (or *ring*) over X if G_x is an abelian group (or a ring) and for each path $w: I \rightarrow X$ in X there is given a homomorphism $w_*: G_{w(1)} \rightarrow G_{w(0)}$ of groups (or rings) satisfying:

(a) $0_* = 1$, where 0 is a constant path.

(b) $w_0 \simeq w_1$, a homotopy of paths, implies $w_{0*} = w_{1*}$.

(c) For a product of paths, $(w * w')_* = w_* \circ w'_*$.

For the inverse path $\bar{w}$ of w one has $w * \bar{w} \simeq 0$, which implies $w_* \circ \bar{w}_* = 1$. Similarly, $\bar{w}_* \circ w_* = 1$ and hence w_* is always an isomorphism.

DEFINITION. A local coefficient group (or ring) is called *trivial* if there is a group (ring) G and an isomorphism $\lambda_x: G \rightarrow G_x$ such that $w_* \lambda_{w(1)} = \lambda_{w(0)}$.

When a local coefficient $\underline{G} = \{G_x\}$ is given over a topological space X , we suppose that the coefficient g_u of a nondegenerate singular cube $u: I^n \rightarrow X$ is in $G_{u(0)}$ ($0 = (0, \dots, 0) \in I^n$). Introducing addition in

$$C_n(X; \underline{G}) = \left\{ \sum_u g_u \cdot u \text{ (a finite sum)}, g_u \in G_{u(0)} \right\}$$

by $\sum g_u \cdot u + \sum g'_u \cdot u = \sum (g_u + g'_u) \cdot u$ and a boundary homomorphism ∂ by

$$\partial \left(\sum_u g_u \cdot u \right) = \sum_u \sum_{k=1}^n (-1)^k (w_{k*} g_u \cdot \partial_k^1 u - g_u \cdot \partial_k^0 u),$$

where $w_k(s) = u(0, \dots, 0, \overset{(k)}{s}, 0, \dots, 0)$, we define the chain complex $\{C_n(X; \underline{G}), \partial\}$ of X with local coefficients in $\underline{G}$ and so we define the *homology group of X with local coefficients* as the homology group

$$H_n(X; \underline{G}) = H_n(C_*(X; \underline{G})).$$

The cochain operator of X with local coefficients in $\underline{G}$ is defined as the set of functions assigning a value $f(u)$ in $G_{u(0)}$ to u :

$$C^n(X; \underline{G}) = \{f | f(u) \in G_{u(0)}\}$$

with addition $(f + f')(u) = f(u) + f'(u)$ and a boundary homomorphism

$$\delta f(u) = \sum_{k=1}^n (-1)^k \{w_{k*} f(\partial_k^1 u) - f(\partial_k^0 u)\},$$

and so the cohomology group of X with local coefficients

$$H^n(X; \underline{G}) = H^n(C^*(X; \underline{G}))$$

can be defined as its cohomology group. In the above the path w_k is introduced to give an isomorphism between $G_{u(0)}$ and $G_{\partial_k^1 u(0)}$ so that ∂ and δ are well defined. w_k is a path connecting $u(0)$ and $\partial_k^1 u(0)$ and in the image under u of a path in I^n . Since any two paths in I^n are homotopic, any map satisfying the condition will do as w_k . This fact is also used in proving $\partial\delta = 0$. For the same reason we do not need to specify the path w_β in the following definition of the (cup) product. When $\underline{R} = \{R_x\}$ is a system of local coefficient rings over X , we define the cup product of $f \in C^p(X; \underline{R})$ and $g \in C^q(X; \underline{R})$ by

$$(f \cdot g)(u) = \sum_{\alpha} \varepsilon(\alpha) f(\partial_{\alpha}^0 u) w_{\beta*} g(\partial_{\beta}^1 u),$$

similarly to the usual product. Then the equalities $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ and $\delta(f \cdot g) = (\delta f) \cdot g + (-1)^p f \cdot (\delta g)$ hold, and the (cup) product

$$H^p(X; \underline{R}) \oplus H^q(X; \underline{R}) \rightarrow H^{p+q}(X; \underline{R})$$

is defined so $H^*(X; \underline{R}) = \sum H^n(X; \underline{R})$ is a ring. In addition, $H^*(X; \underline{R})$ possesses similar properties to those of ordinary (co)homology groups but we omit listing them.

LEMMA 1.18. (1) If X is 1-connected then any local coefficient group over X is trivial.

(2) If a system of local coefficient groups over X is trivial, the homology and the cohomology groups (rings) with these local coefficients can be identified with those with ordinary coefficients.

(3) Let X be 0-connected. A system of local coefficient groups $\underline{G} = \{G_x\}$ over X is trivial iff the inclusion $i_x: x \rightarrow X$ induces an isomorphism $i_x^*: H^0(X; \underline{G}) \cong H^0(x; G_x) = G_x$ for every point x of X iff the same holds for a point of X .

PROOF. (1) We put $G = G_x$ and $\lambda_x = w_x$ for a path connecting x and x_0 . Since X is 1-connected, there exists w and any two such are homotopic, and hence $\lambda_x: G \cong G_x$ is determined.

(2) When we replace each G_x by $G = G_x$ using the isomorphism λ_x , w_{k*} in the definition of δ is replaced by the identity map. So we may take $C_n(X; \underline{G}) = C_n(X; G)$ replacing $\sum g_u \cdot u$ by $\sum u \otimes g_u$. Similarly for the cohomology case.

(3) If $\underline{G}$ is trivial, one can see by (2) that i_{x*} is given by λ_x taking $H^0(X; \underline{G}) = H^0(X; G) = G$. Conversely, consider $H^0(X; \underline{G}) = Z^0(X; \underline{G}) \subset C^0(X; \underline{G})$. The condition $\delta f(w) = 0$, where $w: I \rightarrow X$, that $f \in C^0(X; \underline{G})$ is contained in $Z^0(X; \underline{G})$ is $f(\partial_1^0 w) = w_* f(\partial_1^1 w)$, that is, $i_{w(0)}^* = w_* i_{w(1)}^*$. So $\underline{G}$ is trivial. If the condition of triviality $w_* \lambda_{w(1)} = \lambda_{w(0)}$ holds for any closed path, then it also holds for any general path. Q.E.D.

§2. Cohomology of Fibrations (the Serre Spectral Sequence)

Fibrations and relative fibrations. Recall the definition of a fibration (in the sense of Serre) given in Chapter II, §3 that (E, p, B) is a *fibration* if $p: E \rightarrow B$ is a surjection with the covering homotopy property for I^n .

By Theorems 3.6 and 3.9 in Chapter II one has:

2.1. (E, p, B) is a *fibration relative to each of the following projections* p ; *a fibre bundle* (E, p, B, F, G) , *a covering space* $(E = \tilde{X}, p, B = X)$, *and the evaluations* $p = p_1: E = \Omega(X, A, B) \rightarrow B$, $p = p_0: E = \Omega(X, B, A) \rightarrow B$ *of paths.*

Also, a sequence

$$F \xrightarrow{i} E \xrightarrow{p} B$$

obtained by adding a fibre F together with the inclusion into E is called a *fibration (or fibering)*, in the sense that it induces an exact sequence

$$\cdots \rightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\Delta} \pi_{n-1}(F) \rightarrow \cdots \quad (2.1')$$

by Corollary 3.11 of Chapter II.

There are two ways to relativise a fibration.

If we put $E_0 = p^{-1}(B_0)$ for a subspace B_0 of the base space B , the sequence $F \xrightarrow{i} E_0 \rightarrow B_0$ ($b \in B_0$) is also a fibration. In this sense a sequence

$$F \xrightarrow{i} (E, E_0) \xrightarrow{p} (B, B_0)$$

is called a *relative fibration*. Then Theorem 3.10 of Chapter II is easily generalized to give a theorem of the following form:

$$p_*: \pi_n(E, E_0) \cong \pi_n(B, B_0) \quad (\text{bijection}). \quad (2.2)$$

Note that the case $B_0 = (\text{a point})$ is the original theorem:

$$p_*: \pi_n(E, F) \cong \pi_n(B) \quad (\text{bijection}). \quad (2.2')$$

If the total space is ∞ -connected, $\partial: \pi_{n+1}(E, F) \rightarrow \pi_n(B)$ is an isomorphism for $n \geq 1$ by the homotopy exact sequence of the pair (E, F) . Then

$$S = p_* \circ \partial^{-1}: \pi_n(F) \rightarrow \pi_{n+1}(B) \quad (\text{isomorphism})$$

is called the *homotopy suspension*. Particularly important instances are the following two cases.

2.3. (1) For a topological group and its classifying space, . . .

$$S: \pi_n(G) \cong \pi_{n+1}(BG).$$

(2) For a topological space X and its loop space ΩX ,

$$S: \pi_n(\Omega X) \cong \pi_{n+1}(X).$$

In the above, in case (1) EG is ∞ -connected in a fibration (EG, p, BG) (and so it is the universal G -bundle); in case (2) it follows from the following lemma that $\Omega(X, X, x_0)$ is contractible in the path fibration

$$(\Omega(X, X, x_0), p_0, X).$$

DEFINITION. A is a deformation retract of X if $A \subset X$ and there is a homotopy $r_t: X \rightarrow X$ such that $r_0 = 1_X$, $r_1(X) \subset A$, and $r_t|_A = 1_A$.

The mapping cylinder Z_f of $f: X \rightarrow Y$ is defined by

$$Z_f = (X \times I) \amalg Y / \sim, \quad (x, 0) \sim f(x).$$

As is easily seen, Y is a deformation retract of Z_f . Also, the inclusion $i: A \rightarrow X$ is a homotopy equivalence for a deformation retract A of X .

LEMMA 2.4. A is a deformation retract of $\Omega(X, X, A)$, where each point of A is identified with the constant path at the point.

PROOF. Put $r_t(w)(s) = w(t + s - st)$ for $w \in \Omega(X, X, A)$. Q.E.D.

Relativization in the fibre direction is defined as follows.

DEFINITION. $(F, F_0) \xrightarrow{i} (E, E_0) \xrightarrow{p} B$ is called a relative fibration if $F \xrightarrow{i} E \xrightarrow{p} B$ and $F_0 \xrightarrow{i_0} E_0 \xrightarrow{p_0} B$ are fibrations ($i_0 = i|_{F_0}$, $p_0 = p|_{E_0}$).

$((E, E_0), p, B, (F, F_0))$ is called a relative fibre bundle if (E, p, B, F, G) and (E_0, p_0, B, F_0, G) are fibre bundles ($p_0 = p|_{E_0}$), where the action of G on F_0 is the restriction of the action on F .

A relative fibre bundle may be viewed as a relative fibration.

LEMMA 2.5. In a fibre bundle (E, p, B, F, G) , let $\hat{E}$ be the mapping cone of $p: E \rightarrow B$, $\hat{p}: \hat{E} \rightarrow B$ an extension of p defined by $\hat{p}(x, t) = p(x)$ for $x \in E$, and $\hat{F}$ the mapping cone of the constant map $0: F \rightarrow (a \text{ point})$. Then $((\hat{E}, E), \hat{p}, B, (\hat{F}, F), G)$ is a relative fibre bundle.

PROOF. The action of G is extended over $\hat{F}$ by $g \cdot (f, t) = (g \cdot f, t)$ and the coordinate function ϕ_a is extended over $U_a \times \hat{F} \cong \hat{p}^{-1}(U_a)$ by

$$\hat{\phi}_a(b, (f, t)) = (\phi_a(b, f), t).$$

Then $(\hat{E}, \hat{p}, B, \hat{F}, G)$ is a fibre bundle. Q.E.D.

For example, a disk bundle $((\hat{E}, E), p, B, (D^{n+1}, S^n), G)$ is obtained from a sphere bundle (E, p, B, S^n, G) .

Also one may consider that the associated disk bundle $\xi(D^{2n})$ of a vector bundle $\xi = (E, p, B, \mathbb{F}^n, G)$ where $G \subset U(n, \mathbb{F})$ is obtained from the

associated sphere bundle $\xi(S^{d^{n-1}})$ by the construction in Lemma 2.5. The total space $(E(D^n), E(S^{d^{n-1}}))$ thus constructed is homotopy equivalent to $(E, E - s_0(B))$, where $s_0: B \rightarrow E$ is the zero cross section.

The local coefficients of (co)homology associated with a fibration. We will show that in a fibration (E, p, B) the (co)homology groups of fibres F_b over each point b of the base space B form a system of local coefficients over B .

LEMMA 2.6. (1) For any path $w: I \rightarrow B$ in B there is a map $S = S_w: Q_n(F_{w(0)}) \rightarrow Q_n(F_{w(1)})$ satisfying the following conditions: (where for $u: I^n \rightarrow F_b$ we have $Su: I^{n+1} \rightarrow E$)

- (i) $(p \circ Su)(t_1, \dots, t_{n+1}) = w(t_1)$;
- (ii) $\partial_1^0(Su) = u$, $\partial_{i+1}^e(Su) = S(\partial_i^e u)$ ($1 \leq i \leq n$);
- (iii) $S(u' \circ \pi_k) = Su' \circ \pi_{k+1}$ ($1 \leq k \leq n$, $u': I^{n-1} \rightarrow F_b$).

(2) If we put $w_*u = \partial_1^1(S_w u)$ for S_w in (1), then $w \simeq w'$ (a homotopy of paths) implies $w_* \simeq w'_*: C_n(F_{w(0)}) \rightarrow C_n(F_{w(1)})$.

PROOF. (1) The proof is by induction on n . When $n = 0$; we regard $w_t = w(t): I^0 \rightarrow B$ as a homotopy and $u: I^0 \rightarrow F_{w(0)}$ implies $p \circ u = w_0$, and hence by the covering homotopy property for I^0 , there is $u_t: I^0 \rightarrow E$ satisfying $u_0 = u$ and $p \circ u_t = w_t$. We put $Su(t) = u_t$. If u is degenerate then Su is determined by (iii), and must satisfy (i) and (ii). If u is nondegenerate then $Su_0 = Su|0 \times I^n \cup I \times \partial I^n$ is already given by (ii) and $p \circ Su$ is prescribed by (i). One can show the existence of Su , by using the following Lemma 2.7.

(2) Let $W: I^2 \rightarrow B$ be a homotopy $w \simeq w': W(0, t_2) = w(t_2)$, $W(1, t_2) = w'(t_2)$, $W(t_1, 0) = w(0)$, $W(t_1, 1) = w(1)$. If one can construct $Du: I^{n+2} \rightarrow E$ so as to satisfy

- (i)' $(p \circ Du)(t_1, t_2, \dots, t_{n+2}) = W(t_1, t_2)$;
- (ii)' $\partial_1^0(Du) = S_w u$, $\partial_1^1(Du) = S_{w'} u$, $\partial_2^0(Du) = u \circ \pi_1$, $\partial_{i+1}^e(Du) = D(\partial_i^e u)$;
- (iii)' $D(u' \circ \pi_k) = Du' \circ \pi_{k-2}$;

then for $D'u = \partial_2^1(Du)$, one has $dD' + D'd = \partial_1^1 S_{w'} u - \partial_1^1 S_w u$, which implies $w_* \simeq w'_*$. The construction of Du is similar to that of Su and one needs the following lemma to extend Du from $\partial I \times I^{n+1} \cup I \times 0 \times I^n \cup I^2 \times \partial I^n$ over I^{n+2} . Q.E.D.

LEMMA 2.7. Let L be nonempty, contractible and a union of some of the faces $\partial_i^e I^n$ of I^n . If maps $f: L \rightarrow E$ and $h: I^n \rightarrow B$ satisfy $p \circ f = h|L$, then there exists an extension $\tilde{f}: I^n \rightarrow E$ of f (i.e., $\tilde{f}|L = f$) such that $p \circ \tilde{f} = h$.

PROOF. One can show by induction on the number of faces constituting L that there is a homeomorphism $\varphi: I^n \rightarrow I^n$ satisfying $\varphi(I^{n-1} \times 0) = L$. Using this φ , one can suppose that $L = I^{n-1} \times 0 = I^{n-1}$. If we put $g_i(t_1, \dots, t_{n-1}) = h(t_1, \dots, t_{n-1}, t)$, then $p \circ f = g_0$ and hence by the covering homotopy property for I^{n-1} we obtain $f_i: I^{n-1} \rightarrow E$ such that $f_0 = f$ and $p \circ f_i = g_i$. Then the required map $\tilde{f}$ is obtained by putting $\tilde{f}(t_1, \dots, t_{n-1}, t) = f_i(t_1, \dots, t_{n-1})$. Q.E.D.

For the homology group of fibres with coefficient group G , we put

$$w_* = (w_\#^{-1})_*: H_n(F_{w(1)}; G) \rightarrow H_n(F_{w(0)}; G).$$

For the cohomology ring of fibres with coefficient ring R , we put

$$w_* = (w_\#)^*: H^n(F_{w(1)}; R) \rightarrow H^n(F_{w(0)}; R).$$

Then we have

THEOREM 2.8. With the above w_* , $\{H_n(F_b; G)\}_{b \in B}$ is a system of local coefficient groups over B and $\{H^n(F_b; R)\}_{b \in B}$ is a system of local coefficient rings over B . In a relative fibration $(E, E_0) \rightarrow B$, $\{H_n(F_b, F_{0,b}; G)\}$ and $\{H^n(F_b, F_{0,b}; R)\}_{b \in B}$ are systems of local coefficient groups and rings, respectively.

PROOF. First by Lemma 2.6(2) note that w_* is independent of the choice of S_w . For the constant path 0, choosing S_0 so that $S_0 u = u \circ \pi_1$, one has $0_* = 1$. By Lemma 2.6(2) one has that $w \simeq w'$ implies $w_* = w'_*$. When $w(1) = w'(0)$, we define $S_{w \ast w'} u$ for S_w and $S_{w'}$ by

$$S_{w \ast w'} u(t_1, t_2, \dots) = \begin{cases} S_w(2t_1, t_2, \dots) & (0 \leq t_1 \leq 1/2), \\ S_{w'}(2t_1 - 1, t_2, \dots) & (1/2 \leq t_1 \leq 1), \end{cases}$$

and then $w_\# \circ w'_\# = (w \ast w')_\#$, which implies $w_* \circ w'_* = (w \ast w')_*$. The case for relative fibrations is similar, observing that S_w is chosen so that $u \in Q_n(F_{0,b})$ implies $S_w u \in Q_{n+1}(E_0)$, $Du \in Q_{n+2}(E_0)$. Q.E.D.

LEMMA 2.9. The local coefficients are trivial in the following cases (and the same holds for relative fibrations):

- (1) B is 1-connected.
- (2) (E, p, B, F, G) is a fibre bundle with an arcwise connected structure group G .
- (3) The coefficient $\mathbf{k}$ is a field of characteristic 2 and the (co)homology $H(F_b; \mathbf{k})$ of each fibre is isomorphic to 0 or $\mathbf{k}$ in each degree.

PROOF. (1) is proved in Lemma 1.18(1). For the case (2) we construct S_w as follows. The bundle $w^\# E$ induced by $w: I \rightarrow B$ is equivalent to the trivial bundle $I \times F_{w(0)}$. Put $S_w u: I \times I^n \xrightarrow{1 \times u} I \times F_{w(0)} \cong w^\# E \xrightarrow{\tilde{w}} E$. Then $w_\# u$ is a composition of u and a homeomorphism $F_{w(0)} \cong F_{w(1)}$, where the

homeomorphism is induced by the action of an element g of G . Since G is arcwise connected, the action is homotopic to the identity map by a path connecting g and e . It follows that $w_{\#} \simeq 1$ (through the homeomorphism of F to $F_{w(0)}$ and $F_{w(1)}$ by the coordinate functions). For the case (3); if we consider the case k a prime field $Z_2 = Z/2Z$, $w_{\#}$ is an automorphism of $H(F_b; Z_2)$ and it is unique in this case. For general k , $w_{\#}$ is also unique as $H(F_b; k) \cong H(F_b; Z_2) \otimes k$. Q.E.D.

The Serre spectral sequence. We start with the case of a cohomology ring that will be used often in later sections. Let (E, p, B) be a fibration, R a commutative ring with 1, and $\underline{H}^*(F; R) = \{H^*(F_b; R)\}$ the system of local coefficient rings over the base space B defined above.

THEOREM 2.10 (Serre's fundamental theorem). *There exists a sequence $\{E_r^{p,q}\}$, $p, q \in \mathbb{Z}$, $r = 2, 3, \dots, \infty$ and a sequence $\{D^{p, n-p}\}$ of subgroups of $H^n(E; R)$ satisfying the following properties:*

2.10.1. $E_r^{p,q}$ is an R -module with the product

$$E_r^{p,q} \otimes E_r^{p',q'} \rightarrow E_r^{p+p', q+q'} \quad (\text{homomorphism}).$$

2.10.2. There is a homomorphism $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ satisfying

$$d_r \circ d_r = 0 \quad \text{and} \quad d_r(x \cdot y) = d_r x \cdot y + (-1)^p x \cdot d_r y \quad (x \in E_r^{p,q})$$

so that the cohomology ring $H(E_r^{p,q})$ with respect to d_r is isomorphic to $E_{r+1}^{p,q}$ (preserving the product).

2.10.3. $E_2^{p,q} \cong H^p(B; \underline{H}^q(F; R))$. The product on the left-hand side of the isomorphism given in 2.10.1 differs only by the sign $(-1)^{p'q}$ from that of the cohomology ring on the right-hand side.

2.10.4. $E_r^{p,q} = E_{\infty}^{p,q}$ ($r > \text{Max}(p, q+1)$).

2.10.5. The product in $H^*(E; R)$ induces a product

$$D^{p,q} \otimes D^{p',q'} \rightarrow D^{p+p', q+q'}.$$

2.10.6. $D^{p,q} \supset D^{p+1, q-1}$, $D^{p,q}/D^{p+1, q-1} \cong E_{\infty}^{p,q}$ (product preserving isomorphism).

2.10.7. $H^n(E; R) = D^{0,n} \supset D^{n+1, -1} = 0$.

2.10.8. A fibre map $f: (E, p, B) \rightarrow (E', p', B')$ between fibrations induces maps $f_r: E_r^{p,q} \rightarrow E_r^{p',q'}$ and $f_D: D_r^{p,q} \rightarrow D_r^{p',q'}$ which preserve products, and commute with d_r , $(f_r)^* = f_{r+1}^*$, and $(f_D)^* = f_{\infty}^*$. Also f_2 and $f_D|D^{0,n}$ are induced by the map f .

2.10.9. The homomorphism $i^*: H^n(E; R) \rightarrow H^n(F_b; R)$ induced by the inclusion $i: F_b \rightarrow E$ decomposes as follows:

$$\begin{aligned} H^n(E; R) &= D^{0,n} \xrightarrow{\text{proj}} E_{\infty}^{0,n} = E_{n+2}^{0,n} \subset E_{n+1}^{0,n} \subset \dots \subset E_2^{0,n} \\ &= H^0(B; \underline{H}^n(F; R)) \xrightarrow{i_b^0} H^0(b; H^n(F_b; R)) = H^n(F_b; R), \end{aligned}$$

where $i_0: b \rightarrow B$ is the inclusion. Since $d_r = 0: E_r^{-r, n+r-1} = 0 \rightarrow E_r^{0, n}$, we may regard $E_{r+1}^{0, n} = H(E_r^{0, n}) = \text{Ker}(d_r: E_r^{0, n} \rightarrow E_r^{r, n-r+1}) \subset E_r^{0, n}$.

2.10.10. The homomorphism $p^*: H^*(B, R) \rightarrow H^*(E; R)$ induced by the projection $p: E \rightarrow B$ decomposes as follows:

$$H^n(B; R) \xrightarrow{i} H^n(B; \underline{H^0(F; R)}) = E_2^{n, 0} \xrightarrow{\text{proj}} E_3^{n, 0} \xrightarrow{\text{proj}} \dots \xrightarrow{\text{proj}} E_{n+1}^{n, 0} \\ = E_\infty^{n, 0} = D^{n, 0} \subset H^n(E; R),$$

where $i: R \rightarrow H^0(F; R)$ is a map of the trivial local coefficient ring R given by $r \mapsto r \cdot 1_b \in H^0(F_b; R)$. Since $d_r = 0: E_r^{n, 0} \rightarrow E_r^{n+r, -r+1} = 0$ ($r \geq 2$), there is a projection $E_r^{n, 0} \rightarrow E_r^{n, 0}/d_r E_r^{n-r, r-1} = H(E_r^{n, 0}) = E_{r+1}^{n, 0}$.

2.10.11. Suppose that the local coefficient ring is trivial and that F and B are 0-connected, so we may assume $H^n(F_b; R) = E_2^{n, 0}$ and $H^{n+1}(B; R) = H^{n+1}(B, b; R) = E_2^{n+1, 0}$ ($n > 0$). Then in the diagram

$$\begin{array}{ccccc} H^n(F_b; R) & \xrightarrow{\delta^*} & H^{n+1}(E, F_b; R) & \xleftarrow{p^*} & H^{n+1}(B; R) \\ \parallel & & & & \parallel \\ E_2^{0, n} \supset E_{n+1}^{0, n} & \xrightarrow{d_{n+1}} & E_{n+1}^{n+1, 0} & \xleftarrow{\kappa} & E_2^{n+2, 0} \quad (\kappa: \text{projection}) \end{array}$$

we have $E_{n+1}^{0, n} = \{x \in H^n(F_b; R) \mid \delta^*(x) \in \text{Im } p^*\}$ and

$$\delta^*(x) = p^*(y) \text{ iff } d_{n+1}(x) = \kappa(y).$$

With respect to d_r , $E_r = \sum_{p, q} E_r^{p, q}$ of total degree $p + q$ is a chain complex and one obtains a sequence of cohomologies

$$H(E_2) = E_3, \quad H(E_3) = E_4, \quad \dots, \quad H(E_r) = E_{r+1}, \quad \dots$$

Such a sequence $\{E_r, d_r\}$ is generally called a (cohomology) spectral sequence. The one in Theorem 2.10 is called the Serre cohomology spectral sequence.

Ordinarily, one considers the case where the local coefficient ring is trivial and B and F are 0-connected. In this case, one has

2.11. (1) $\text{Im } i^* = E_\infty^{0, n}$ in (2.10.9),

i^* is epimorphic iff $E_{n+2}^{0, n} = E_2^{0, n}$ iff $d_r = 0: E_r^{0, n} \rightarrow E_r^{r, n-r+1}$ ($r = 2, 3, \dots, n+1$),

i^* is monomorphic iff $D^{1, n-1} = 0$ iff $E_\infty^{p, n-p} = 0$ ($p = 1, 2, \dots, n$).

(2) $\text{Im } p^* = D^{n, 0} = E_{n+1}^{n, 0}$ in (2.10.10),

p^* is epimorphism iff $D^{n, 0} = H^n(E; R)$ iff $E_\infty^{p, n-p} = 0$ ($p = 0, 1, \dots, n-1$),

p^* is monomorphic iff $E_2^{n, 0} = E_{n+1}^{n, 0}$ iff $d_r = 0: E_r^{n-r, r-1} \rightarrow E_r^{n, 0}$ ($r = 2, 3, \dots, n$).

DEFINITION. In 2.10.11 an element x of $H^n(F_b; R)$ satisfying $\delta^*(x) \in \text{Im } p^*$ is said to be transgressive and $y \in H^{n+1}(B; R)$ satisfying $\delta^*(x) = p^*(y)$ is called the transgression image of x and denoted by $y = \tau(x)$.

An element y for which there is an x satisfying $\delta^*(x) = p^*(y)$ is called *suspensive* and x is called the *suspension* of y , denoted by $x = \sigma(y)$.

2.12. If E is ∞ -connected, $\delta^* : H^n(F_b; R) \cong H^{n+1}(E, F_b; R)$ for $n > 0$ and so an element of $H^{n+1}(B; R)$ is always suspensive and $\sigma = \delta^{*-1} \circ p^*$.

Let $\alpha : H^*(; R) \rightarrow H^{*+k}(; R)$ be a stable cohomology operation of degree k . Since $\delta^*(x) = p^*(y)$ implies $\delta^*(\alpha(x)) = p^*(\alpha(y))$, one has

2.13. If x is transgressive then $\alpha(x)$ is transgressive and $\alpha(\tau(x)) = \tau(\alpha(x))$.

$\sum E_\infty^{p, n-p}$, which is obtained as a direct sum of coset groups of a sequence of subgroups $D^{p, n-p}$ of $H^n(E; R)$, is called the graded module associated with $H^n(E; R)$. We sometimes denote it by

$$E_\infty = \mathcal{G}r(H^n(E; R)), \quad (E_\infty = \sum E_\infty^{p, q}).$$

Also the process of reaching E_∞ by taking a sequence of cohomologies starting with E_2 is diagrammatically indicated by

$$E_2 = H^*(B; \overline{H^*(F; R)}) \Rightarrow H^*(E; R),$$

and the spectral sequence $\{E_r\}$ is said to *converge* to $H^*(E; R)$.

THEOREM 2.14 (The Serre exact sequence). *Let m and n be positive integers. If B and F are 0-connected in a fibration (E, p, B) , the system of local coefficient rings $\overline{H^*(F; R)}$ is trivial, $H^i(B; R) = 0$ for $0 < i < m$ and $H^j(F; R) = 0$ for $0 < j < n$, then we have the following exact sequence:*

$$\begin{array}{ccccccc} 0 \rightarrow H^1(B; R) & \xrightarrow{p^*} & H^1(E; R) & \xrightarrow{i^*} & & & \\ & \searrow \tau & \dots \rightarrow H^k(B; R) & \xrightarrow{p^*} & H^k(E; R) & \xrightarrow{i^*} & H^k(F; R) \xrightarrow{\tau} H^{k+1}(B; R) \xrightarrow{p^*} \\ & & \dots & \xrightarrow{p^*} & H^{m+n-1}(E; R) & \xrightarrow{i^*} & H^{m+n-1}(F; R). \end{array}$$

PROOF. By assumption one has $E_2^{p, 0} \cong H^p(B; R)$, $E_2^{0, q} \cong H^q(F; R)$, and $E_2^{p, q} = 0$, and hence $E_r^{p, q} = 0$ for $r \geq 2$ for $p \neq 0$, $q \neq 0$, $p + q < m + n$. Since $E_\infty^{p, q} = 0$ iff $D^{p, q} = D^{p+1, q-1}$, $k < m + n$ implies $D^{1, k-1} = D^{k, 0} = E_\infty^{k, 0}$ and $0 \rightarrow E_\infty^{k, 0} \subset H^k(E; R) \rightarrow E_\infty^{0, k} \rightarrow 0$ is exact. As the only possibility for $d_r : E_r^{p, q} \rightarrow E_r^{p+r, q-r+1}$ to be nontrivial is $d_r : E_r^{0, r-1} \rightarrow E_r^{r, 0}$ in the range $p + q < m + n - 1$, one has $E_2^{k, 0} = E_k^{k, 0}$, $E_{k+1}^{k, 0} = E_\infty^{k, 0}$, $E_2^{0, k-1} = E_k^{0, k-1}$, $E_{k+1}^{0, k-1} = E_\infty^{0, k-1}$ and $0 \rightarrow E_{k+1}^{0, k-1} \rightarrow E_k^{0, k-1} \xrightarrow{d_k} E_2^{k, 0} \rightarrow E_{k+1}^{k, 0} \rightarrow 0$ is exact for $k < m + n$. Combining these, we obtain the exact sequence of the theorem. Q.E.D.

LEMMA 2.15. *Suppose that the system of local coefficient rings $\overline{H^*(F; R)}$ is trivial, B and F are 0-connected, and either $H^*(B; R)$ or $\overline{H^*(F; R)}$*

is a finitely-generated free R -module in each dimension. Then the product $E_2^{p,0} \otimes E_2^{0,q} \rightarrow E_2^{p,q}$ induces an isomorphism

$$H^p(B; R) \otimes_R H^q(F; R) \cong E_2^{p,q}.$$

The proof can be obtained by decomposing into direct summands. In what follows the spectral sequence is used mostly in the case where the lemma holds. For $b \in H^p(B; R)$ and $f \in H^q(F; R)$ we denote the element corresponding to $b \otimes f$ by the same symbol $b \otimes f \in E_2^{p,q}$. The original elements b and f are identified with $b \otimes 1 \in E_2^{p,0} = H^p(B; R)$ and $1 \otimes f \in E_2^{0,q} = H^q(F; R)$ respectively. Also when $d_i(b \otimes f) = 0$ for $2 \leq i < r$, we sometimes denote the corresponding class by $\kappa_2'(b \otimes f) \in E_r^{p,q}$ but we will also often denote it by $b \otimes f \in E_r^{p,q}$, omitting κ_2' . For example, $\tau(f) = b$ iff $d_r(1 \otimes f) = b \otimes 1$ ($q = p - 1$), $d_r(b \otimes f) = d_r b \otimes f + (-1)^p b \otimes d_r f$, and so on.

THEOREM 2.16. *In the case of a relative fibration $(F, F_0) \xrightarrow{i} (E, E_0) \xrightarrow{p} B$, the statements 2.10.1 ~ 2.10.9 of the Serre fundamental theorem 2.10 hold with E and F replaced by (E, E_0) and (F, F_0) respectively. Moreover, if we denote by $\{E_r^{p,q}\}$ the resulting spectral sequence and by $\{E_r^{p,q}\}$ the one associated with (E, p, B) , then there are products $E_r^{p,q} \otimes E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$ and $D_r^{p,q} \otimes D_r^{p',q'} \rightarrow D_r^{p+p',q+q'}$ satisfying relations similar to those in Theorem 2.10 with products $H^q(F; R) \otimes H^{q'}(F, F_0) \rightarrow H^{q+q'}(F, F_0; R)$ and $H^n(E; R) \otimes H^{n'}(E, E_0; R) \rightarrow H^{n+n'}(E, E_0; R)$.*

The homology spectral sequence. There is also a spectral sequence for homology groups, dual to the cohomology one but less useful in applications, as a product structure can not, in general, be introduced.

THEOREM 2.17. *For a relative fibration $F \xrightarrow{i} (E, E_0) \xrightarrow{p} (B, B_0)$ and a coefficient ring R , there is a sequence $\{E_{p,q}^r\}$, where $p, q \in \mathbb{Z}$ and $r = 2, 3, \dots, \infty$ and a sequence of R -submodules $D_{p,n-p}$ of $H_n(B, B_0; R)$ satisfying the following conditions:*

2.17.1. *There is a homomorphism $d^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ such that $d^r \circ d^r = 0$ and $H(E_{p,q}^r) = E_{p,q}^{r+1}$ as homology groups with respect to d^r .*

2.17.2. $E_{p,q}^2 \cong H_p(B, B_0; H_q(F; R))$.

2.17.3. $E_{p,q}^r = E_{p,q}^\infty$ ($r > \max(p, q + 1)$).

2.17.4. $D_{p-1,q+1} \subset D_{p,q}$, $D_{p,q}/D_{p-1,q+1} \cong E_{p,q}^\infty$.

2.17.5. $H_n(E, E_0; R) \supset D_{n,0} \supset D_{-1,n+1} = 0$.

2.17.6. *A fibre map induces a natural morphism between the corresponding spectral sequences $\{E_{p,q}^r\}$.*

2.17.7. *If $B_0 = E_0 = \emptyset$, the homomorphism i_* induced by the inclusion*

$i: F \rightarrow E$ decomposes as follows:

$$\begin{aligned} H_n(F; R) &= H_0(b; H_n(F; R)) \xrightarrow{i_*} H_0(B; H_n(F; R)) \\ &\cong E_{0,n}^2 \xrightarrow{\text{proj}} E_{0,n}^3 \rightarrow \dots \xrightarrow{\text{proj}} E_{0,n}^{n+2} = E_{0,n}^\infty = D_{0,n} \subset H_n(E; R). \end{aligned}$$

2.17.8. The homomorphism p_* induced by the projection p decomposes as follows:

$$\begin{aligned} H_n(E, E_0; R) &= D_{n,0} \xrightarrow{\text{proj}} E_{n,0}^\infty = E_{n,0}^{n+1} \subset \dots \subset E_{n,0}^3 \subset E_{n,0}^2 \\ &= H_n(B, B_0; \underline{H_0(F; R)}) \xrightarrow{\text{proj}} H_n(B, B_0; R). \end{aligned}$$

2.17.9. There is a relation between

$$H_n(F_b; R) \xleftarrow{\partial} H_{n+1}(E, F_b; R) \xrightarrow{p_*} H_{n+1}(B; R)$$

and $d_{n+1}: E_{n+1,0}^{n+1} \rightarrow E_{0,n}^{n+1}$, dual to (2.10.11).

DEFINITION. A fibration (E, p, B) is called *principal* if the fibre F is an H -space acting on E and the action $\mu: E \times F \rightarrow E$ satisfies $p\mu(e, f) = p(e)$.

A principal bundle $(F = G)$ and a path fibration $(\Omega(X, A, x_0), p_0, A)$ ($F = \Omega(X)$) are examples of principal fibrations.

THEOREM 2.18. In a principal fibration the homology ring $H_*(F; R)$ acts on the spectral sequence

$$\{E'_{p,q}\}: E'_{p,q} \otimes H_s(F; R) \rightarrow E'_{p,q+s}$$

and the action has properties similar to those in the case of cohomology. (For example, $d'(x \cdot y) = d'(x) \cdot y$.)

Construction of the spectral sequence. We will give an outline of the proof of the Serre fundamental theorem. For details refer to the original paper [17].

Let (E, p, B) be a fibration. We introduce a sequence of chain subcomplexes $A_p = \sum_q A_{p,q}$ in the chain complex $C_*(E) = Q_*(E)/D_*(E)$ of the total space as follows:

DEFINITION. $A_{p,q}$ is the subgroup of $C_{p+q}(E)$ generated by $u: I^{p+q} \rightarrow E$ such that the value $(p \circ u)(t_1, \dots, t_{p+q})$ is independent of the last q coordinates $t_{p+1}, \dots, t_{p+q}$.

Then we have $A_p \supset A_{p-1}$ and $d(A_p) \subset A_p$.

For the case of the homology spectral sequence, we put $A'_{p,q} = A_{p,q} \otimes G$ and $\partial = d \otimes 1$ and define

$$\begin{aligned} Z'_{p,q} &= A'_{p,q} \cap \partial^{-1}(A'_{p-r, q+r-1}), & B'_{p,q} &= A'_{p,q} \cap \partial(A'_{p+r, q-r+1}), \\ E'_{p,q} &= Z'_{p,q} / (Z'^{-1}_{p-1, q+1} + B'^{-1}_{p,q}), \\ D_{p,q} &= \text{Im}(i_*: H_{p+q}(A_p; G) \rightarrow H_{p+q}(E; R)). \end{aligned}$$

We define d' as the map induced by $\partial: Z'_{p,q} \rightarrow B'_{p-r, q+r-1}$.

For the case of cohomology spectral sequence, we consider the subgroup

$$A^{p,q} = \{f \in \text{Hom}(C_{p+q}(E), R) | f(A_{p-1, q+1}) = 0\}$$

of $C^{p+q}(E; R) = \text{Hom}(C_{p+q}(E), R)$ (observe $A^{p,q} \subset A^{p-1, q+1}$) and define $\delta: A^{p,q} \rightarrow A^{p, q+1}$ by $\delta(f) = f \circ \partial$ and

$$\begin{aligned} Z_r^{p,q} &= A^{p,q} \cap \delta^{-1}(A^{p+r, q-r+1}), \quad B_r^{p,q} = A^{p,q} \cap \delta(A^{p-r, q+r-1}), \\ E_r^{p,q} &= Z_r^{p,q} / (Z_{r-1}^{p+1, q-1} + B_{r-1}^{p,q}), \\ D^{p,q} &= \text{Ker}(i^*: H^{p+q}(E; R) \rightarrow H^{p+q}(A_{p-1}; R)). \end{aligned}$$

We define d_r as the map induced by $\delta: Z_r^{p,q} \rightarrow B_r^{p+r, q-r+1}$.

Below we mainly consider the cohomology case. If one checks the definition of the cup product, one sees that the product $E_r^{p,q} \otimes E_{r'}^{p', q'} \rightarrow E_{r+r'}^{p+p', q+q'}$ is induced, using the property that $f \in A^{p,q}$, $g \in A^{p', q'}$ implies $f \cdot g \in A^{p+p', q+q'}$, and $\delta(f \cdot g) = \delta f \cdot g + (-1)^{p+q} f \cdot \delta g$. Then 2.10.i for $i = 1, 2, 4, 5, 6, 7$ are totally algebraic results. The most complicated part is the proof of 2.10.3. To begin with, the following is easily seen.

2.19. $E_0^{p,q} \cong C^{p+q}(A_p/A_{p-1}; R)$, $E_1^{p,q} \cong H^{p+q}(A_p/A_{p-1}; R)$ and d_1 is equivalent to the boundary homomorphism

$$\delta^*: H^{p+q}(A_p/A_{p-1}; R) \rightarrow H^{p+q+1}(A_{p+1}/A_p; R)$$

of a triple (A_{p+1}, A_p, A_{p-1}) .

DEFINITION. For $u: I^{p+q} \rightarrow E$ belonging to A_{p+q} we define

$$Bu: I^p \rightarrow B \text{ by } Bu(t_1, \dots, t_p) = (p \circ u)(t_1, \dots, t_p, *, \dots, *),$$

$$Fu: I^p \rightarrow F \text{ by } Fu(t_1, \dots, t_q) = u(0, \dots, 0, t_1, \dots, t_q),$$

where $b = Bu(0, \dots, 0)$.

Let $C^p(B; \underline{C}^q(F; G))$ be the R -module consisting of all functions f assigning to each $w: I^p \rightarrow B$ of $C_p(B)$ an element $f(w)$ of $C^q(F_b; R)$, where $b = w(0, \dots, 0)$, and make it a cochain complex by setting $(\delta_0 f)(w) = (-1)^p \delta(f(w))$. Then, by using the facts that $u \in A_{p-1, q+1}$ implies Bu degenerate and that $u \in A_{p, q+1}$ for $i \leq p$ implies $\partial_i^e u \in A_{p-1, q+1}$, one can see that a homomorphism

$$\varphi: C^p(B; \underline{C}^q(F; G)) \rightarrow E_0^{p,q} = \text{Hom}(A_{p,q}/A_{p-1, q+1}, R)$$

is defined by $(\varphi f)(u) = (f(Bu))(Fu)$ and satisfies $d_0 \circ \varphi = \varphi \circ \delta_0$. Hence φ induces a homomorphism

$$\Phi: C^p(B; \underline{H}^q(F; R)) \rightarrow E_1^{p,q} = H(E_0^{p,q}).$$

Here we regard the left-hand side as a cochain complex with local coefficients with respect to δ and the right-hand side with respect to d_1 .

LEMMA 2.20. $\Phi: C^*(B; \underline{H}^*(F; R)) \cong E_1 = \sum E_1^{p,q}$ is an isomorphism of cochain complexes, and $\Phi(f \cdot g) = (-1)^{p'q} \Phi(f) \cdot \Phi(g)$ holds for the product.

The proof of the lemma is modelled by that of Lemma 2.6. In order to construct an inverse morphism ψ such that $\psi \circ \varphi = 1$, it suffices to construct $K(w, v): I^{p+q} \rightarrow E$ from a pair (w, v) , where $w: I^p \rightarrow B$, $v: I^q \rightarrow F$, $b = w(0, \dots, 0)$, satisfying the similar conditions to those of Lemma 2.6(1):

- (i) $BK(w, v) = w$, $FK(w, v) = v$;
- (ii) $\partial_{p+i}^e K(w, v) = K(w, \partial_i^e v)$;
- (iii) $K(w' \circ \pi_k, v) = K(w', v) \circ \pi_k$, $K(w, v' \circ \pi_k) = K(w, v') \circ \pi_{p+k}$

since φ is constructed from the correspondence $u \mapsto (Bu, Fu)$. In order to show $\varphi \circ \psi \simeq 1$, it suffices to construct a homotopy D between u and $K(Bu, Fu)$. Here the equivalence corresponding to Lemma 2.7 is

2.21. Lemma 2.7 also holds for $(I^n, L) = (I^p \times I^q, I^p \times \partial I^q \cup 0 \times I^q)$, $= (I \times I^p \times I^q, \partial I \times I^p \times I^q \cup I \times I^p \times \partial I^q \cup I \times 0 \times I^q)$.

This can be proved by repeating Lemma 2.7 several times.

The other parts of the Serre fundamental theorem are directly verified, taking representative elements.

The case of a relative fibration (Theorem 2.16) is similarly proved, starting from

$$C^n(E, E_0; R) = \{f \in C^n(E; R) = \text{Hom}(C_n(E), R) \mid f(C_n(E_0)) = 0\}.$$

For the case of homology the arguments can be dually carried out, starting from $\varphi: E_0^{p,q} = (A_{p,q}/A_{p-1,q+1}) \otimes R \rightarrow C_p(B; \underline{C}_q(F; R))$ given by $u \times 1 \mapsto Bu \otimes Fu \otimes 1$.

§3. Applications of the Gysin exact sequence

The Thom isomorphism. Let

$$(F, F_0) \xrightarrow{i} (E, E_0) \xrightarrow{p} B$$

be a relative fibration, R a commutative ring with 1, and $(F_b, F_{0,b}) =$ (the fibre over $b \in B$).

THEOREM 3.1. Suppose $H^q(F_b, F_{0,b}; R) = 0$ for $q \neq n$, where $b \in B$. Then

(1) $H^p(B; \underline{H}^n(F, F_0; R)) \cong H^{p+n}(E, E_0; R)$ (the isomorphism of R -modules).

(2) If one further supposes $H^n(F_b, F_{0,b}; R) \cong R$, where $b \in B$, then the local coefficient system $\underline{H}^n(F, F_0; R)$ is trivial iff there exists $U \in H^n(E, E_0; R)$ such that $i_b^*(U)$ is a generator of the R -module $H^n(F_b, F_{0,b}; R)$ for every inclusion $i_b: (F_b, F_{0,b}) \rightarrow (E, E_0)$. For such U the correspondence $x \mapsto p^*(x) \cdot U$ for $p^*: H^n(B; R) \rightarrow H^n(E; R)$ gives rise to an isomorphism

$$H^p(B; R) \cong H^{p+n}(E, E_0; R).$$

PROOF. (1) We consider the spectral sequence

$$E_2 = H^*(B; \underline{H^*(F, F_0; R)}) \Rightarrow H^*(E, E_0; R),$$

associated with the relative fibration. From the assumption that $E_2^{p,q} = 0$ for $q \neq n$, we have $E_r^{p,q} = 0$ for $q \neq n$ and $r \geq 2$. In the context $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ for $r \geq 2$ one has $q \neq n$ or $q-r+1 \neq n$ and so $d_r = 0$ for $r \geq 2$ so $E_2^{p,n} = E_3^{p,n} = \dots = E_\infty^{p,n} \cong D^{p,n}/D^{p+1, n-1}$. On the other hand, $E_\infty^{p,q} = 0$ for $q \neq n$ implies $H^{p+n}(E, E_0; R) = D^{0, p+n} = D^{p,n}$ and $D^{p+1, n-1} = 0$. Thus

$$H^p(B; \underline{H^n(F, F_0; R)}) = E_2^{p,n} \cong H^{p+n}(E, E_0; R).$$

(2) One can prove the first part by applying Lemma 1.18(3) to each arcwise connected component of B . For the second part, by considering the product: $'E_r \otimes E_r \rightarrow E_r$ with the spectral sequence $\{'E_r\}$ of the fibration (E, p, B) , we can obtain a commutative diagram

$$\begin{array}{ccc} H^p(B; R) \otimes H^0(B; \underline{H^n(F, F_0; R)}) & \xrightarrow{p^* \otimes \varphi} & H^p(E; R) \otimes H^n(E, E_0; R) \\ \downarrow \mu & & \downarrow \mu \\ H^p(B; \underline{H^n(F, F_0; R)}) & \xrightarrow{\varphi} & H^{p+n}(E, E_0; R) \end{array}$$

involving the isomorphism φ in (1) and the product, where $j^*: H^n(B; R) \rightarrow H^n(B; \underline{H^0(F; R)})$ of 2.10.10 is used in the left column. If we choose $u \in H^0(B; \underline{H^n(F, F_0; R)}) = Z^0(B; \underline{H^n(F, F_0; R)})$ such that $\varphi(u) = U$, the assignment $r \mapsto r \cdot u(b) \in H^n(F_b, F_{0,b}; R)$ gives the triviality of the system of local coefficient rings, and the assignment $x \mapsto \mu(x \otimes u)$ gives an isomorphism $H^p(B; R) \cong H^p(B; \underline{H^n(F, F_0; R)})$ by Lemma 1.18(2) and its proof. By the commutativity of the diagram the assignment $x \mapsto p^*(x) \cdot \varphi(u) = p^*(x) \cdot U$ gives the isomorphism in (2). Q.E.D.

DEFINITION. The element $U \in H^n(E, E_0; R)$ prescribed by Theorem 3.1(2) is called the *Thom class*. The isomorphisms in (1) and (2) of Theorem 3.1 are called the *Thom isomorphisms*.

3.2. If there are bundle maps

$$f: (E, p, B) \rightarrow (E', p', B') \quad \text{and} \quad f|_{E_0}: (E_0, p_0, B) \rightarrow (E'_0, p'_0, B')$$

between relative fibre bundles $((E, E_0), p, B)$ and $((E', E'_0), p', B')$ with the same fibre (F, F_0) and the same structure group G having Thom classes U and U' respectively, then one can choose U' so that

$$f^* U' = U \text{ for } H^n(E', E'_0; R) \rightarrow H^n(E, E_0; R).$$

3.3. Given two relative bundles $((E, E_0), p, B, (F, F_0), G)$ and $((E', E'_0), p', B', (F', F'_0), G')$ with Thom classes U and U' respectively,

then a relative fibre bundle $((E \times E', E \times E'_0 \cup E_0 \times E'), p \times p', B \times B', (F \times F'_0 \cup F_0 \times F'), G \times G')$ is obtained as the direct product of them and the cross product $U \times U' \in H^{n+n'}(E \times E', E \times E'_0 \cup E_0 \times E'; R)$ of U and U' is its Thom class.

These two propositions can, in particular, be applied to the relative vector bundle

$$((E, E_0), p, B, (F^n, F^n - 0), G) \quad \text{where } G \subset GL(n, F),$$

obtained from the complement $E_0 = E - s_0(B)$ of the image of the zero-section $s_0: B \rightarrow E$ (i.e., $s_0(b) = 0 \in F_b^n$) in a vector bundle (E, p, B, F^n, G) where $G \subset GL(n, F)$. Here the system of local coefficient rings is trivial and the Thom class exists for the case $F = C, H$ and even for the case $F = R$ when Lemma 2.9 is applicable. For the case $F = R$ and $R = Z$, a vector bundle (E, p, B, R^n, G) is said to be *orientable* if the local coefficient ring $H^n(R^n, R^n - 0; Z)$ is trivial.

The Gysin exact sequence. By considering the mapping cylinder $\hat{E} = Z_p$ of the projection p in a fibre bundle (E, p, B, F, G) we have a relative fibre bundle

$$((\hat{E}, E), \pi, B, (\hat{F}, F), G)$$

by Lemma 2.5 and obtain

$$\begin{aligned} i^* &= \pi^{*-1}: H^*(\hat{E}; R) \cong H^*(B; R), \\ \delta^* &: H^{n-1}(F, \text{a point}; R) \cong H^n(\hat{F}, F; R), \end{aligned} \quad (3.4)$$

since B is a deformation retract of $\hat{E}$ and $\hat{F}$ is contractible.

DEFINITION. When the relative fibre bundle above has the Thom class $U \in H^n(\hat{E}, E; R)$, the image χ of U by $H^n(\hat{E}, E; R) \xrightarrow{j^*} H^n(\hat{E}; R) \xrightarrow{i^*} H^n(B; R)$ is called the *Euler class* of the fibre bundle (E, p, B, F, G) . The exact sequence in the following theorem is called the *Gysin exact sequence*.

THEOREM 3.5. If the system of local coefficient rings $H^n(F; R)$ is trivial and $H^*(F; R) \cong H^*(S^{n-1}; R)$ for $n \geq 1$ in a fibre bundle (E, p, B, F, G) , then there is an exact sequence

$$\cdots \rightarrow H^{i+n-1}(E; R) \xrightarrow{\lambda} H^i(B; R) \xrightarrow{\chi} H^{i+n}(B; R) \xrightarrow{p^*} H^{i+n}(E; R) \rightarrow \cdots$$

with the properties that $\chi \in H^n(B; R)$ is the Euler class and that $\lambda(p^*x \cdot y) = x \cdot \lambda(y)$.

PROOF. The Gysin exact sequence is obtained by using the Thom isomorphism and the first isomorphism of (3.4) in the cohomology exact sequence of the pair $(\hat{E}, E)$. Q.E.D.

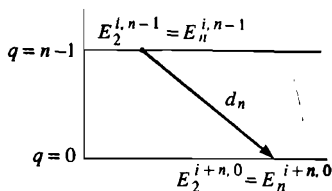


FIGURE 3.1

REMARK 3.6. Theorem 3.5 also holds for a fibration (E, p, B) if $n \geq 1$. The proof follows from the fact that $E_n^{i, n-1} \cong H^i(B; R)$, $E_n^{i+n, 0} \cong H^{i+n}(B; R)$, $0 \rightarrow E_\infty^{i+n, 0} \rightarrow H^{i+n}(E; R) \rightarrow E_\infty^{i+1, n-1} \rightarrow 0$ is exact, $E_\infty^{i+n, 0} = E_{n+1}^{i+n, 0}$, $E_\infty^{i, n-1} = E_{n+1}^{i, n-1}$ etc. in the spectral sequence of (E, p, B) . See Figure 3.1. λ corresponds to d_n .

A frequently used case is that of a sphere bundle (E, p, B, S^{n-1}, G) . By 3.2 one has

3.7. A bundle map $f: (E, p, B, S^{n-1}, G) \rightarrow (E', p', B', S^{n-1}, G)$ induces morphisms f^* and f^* such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{p^*} & H^{i+n-1}(E'; R) & \xrightarrow{\lambda'} & H^i(B'; R) & \xrightarrow{\lambda'^*} & H^{i+n}(B'; R) \xrightarrow{p'^*} \cdots \\ & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ \cdots & \xrightarrow{p^*} & H^{i+n-1}(E; R) & \xrightarrow{\lambda} & H^i(B; R) & \xrightarrow{\lambda^*} & H^{i+n}(B; R) \xrightarrow{p^*} \cdots \end{array}$$

Cohomology rings of classical groups and Stiefel manifolds. To study the cohomology of a Stiefel manifold $V_{m+k, k}(\mathbb{F}) = U(m+k, \mathbb{F})/U(m, \mathbb{F})$ the fibre bundle

$$U(m+l, \mathbb{F})/U(m, \mathbb{F}) \xrightarrow{i} U(m+k, \mathbb{F})/U(m, \mathbb{F}) \xrightarrow{p} U(m+k, \mathbb{F})/U(m+l, \mathbb{F}) \quad (3.8)$$

for $k > l > 0$ obtained from Corollary 2.12 of Chapter II is useful. In particular, we have

$$S^{d(m+l)-1} \xrightarrow{i} V_{m+k, k}(\mathbb{F}) \xrightarrow{p} V_{m+k, k-l}(\mathbb{F}) \quad (3.8')$$

for $l = 1$, since $U(m+1, \mathbb{F})/U(m, \mathbb{F}) \cong S^{d(m+1)-1}$ by (3.16) of Chapter II and the Gysin exact sequence

$$\begin{array}{c} \cdots \rightarrow H^i(V_{m+k, k-l}(\mathbb{F}); R) \xrightarrow{p^*} H^i(V_{m+k, k}(\mathbb{F}); R) \\ \xrightarrow{\lambda} H^{i-d(m+1)+1}(V_{m+k, k-l}(\mathbb{F}); R) \xrightarrow{\lambda^*} H^{i+1}(V_{m+k, k-l}(\mathbb{F}); R) \xrightarrow{p^*} \cdots \end{array} \quad (3.9)$$

associated with the fibering (3.8) is used, where

$$\chi \in H^{d(m+1)}(V_{m+k, k-l}(\mathbb{F}); R)$$

is the Euler class and R is the field $\mathbb{k}$ of characteristic 2 for $\mathbb{F} = \mathbb{R}$.

From now on we will omit $\mathbb{Z}$ when the coefficient ring is $R = \mathbb{Z}$ for simplicity.

THEOREM 3.10.

- (1) $H^*(U(m+k)/U(m)) = \Lambda(e_{2m+1}, e_{2m+3}, \dots, e_{2m+2k-1}),$
 $e_{2i-1} \in H^{2i-1};$
 (2) $H^*(Sp(m+k)/Sp(m)) = \Lambda(e_{4m+3}, e_{4m+7}, \dots, e_{4m+4k-1}),$
 $e_{4i-1} \in H_{4i-1}.$

PROOF. If $k = 1$, we have

$$H^*(V_{m+1,1}(\mathbb{F})) \cong H^*(S^{d(m+1)-1}) = \Lambda(e_{d(m+1)-1})$$

and the theorem is true. By induction on k , supposing $H^*(V_{m+k,k-1}(\mathbb{F})) = \Lambda(e_{d(m+2)-1}, \dots)$, we consider (3.9) for $R = \mathbb{Z}$. Since

$$\chi \in H^{d(m+1)}(V_{m+k,k-1}(\mathbb{F})) = 0,$$

the sequence

$$0 \rightarrow H^i(V_{m+k,k-1}(\mathbb{F})) \xrightarrow{p^*} H^i(V_{m+k,k}(\mathbb{F})) \xrightarrow{\lambda} H^{i-d(m+1)-1}(V_{m+k,k-1}(\mathbb{F})) \rightarrow 0$$

is exact, hence there exists $e_{d(m+1)-1}$ such that

$$\lambda(e_{d(m+1)-1}) = 1 \in H^0(V_{m+k,k-1}(\mathbb{F})).$$

$x \in H^*(V_{m+k,k}(\mathbb{F}))$ implies $\lambda(x - p^*(\lambda(x))e_{d(m+1)-1}) = \lambda(x) - \lambda(x) \cdot 1 = 0$ and so there exist y and $z \in H^*(V_{m+k,k-1}(\mathbb{F}))$ such that

$$x = p^*(y) + p^*(z)e_{d(m+1)-1},$$

and the above decomposition is unique. In addition, since $e_{d(m+1)-1}^2 = -e_{d(m+1)-1}^2$ for $d = 2, 4$ implies $e_{d(m+1)-1}^2 = 0$, we can see that the theorem is true by putting $p^*(e_{d(m+i)-1}) = e_{d(m+i)-1}$ for $i = 2, \dots, k$. Q.E.D.

Observe that

$$U(m+k)/U(m) \cong SU(m+k)/SU(m) \text{ for } m \geq 1. \quad (3.10')$$

COROLLARY 3.11.

$$\begin{aligned} H^*(U(m)) &= \Lambda(e_1, e_3, \dots, e_{2m-1}), \\ H^*(SU(m)) &= \Lambda(e_3, e_5, \dots, e_{2m-1}), \\ H^*(Sp(m)) &= \Lambda(e_3, e_7, \dots, e_{4m-1}). \end{aligned}$$

The above proof does not work for $\mathbb{F} = \mathbb{R}$. We now discuss the case $k = 2$: $V_{m+1,2}(\mathbb{R}) = O(m+2)/O(m) = SO(m+2)/SO(m)$ for $m \geq 1$.

LEMMA 3.12. In the fibration $S^m \xrightarrow{i} O(m+2)/O(m) \xrightarrow{p} S^{m+1}$ we have

$$\tau(s_m) = \chi = (1 + (-1)^{m+1})s_{m+1} \text{ for } m \geq 1,$$

where s_k is a generator of $H^k(S^k) \cong \mathbb{Z}$.

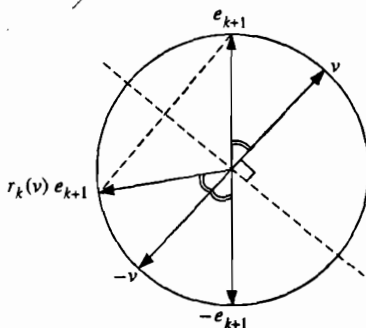


FIGURE 3.2

PROOF. For $v \in S^k \subset \mathbb{R}^{k+1}$, let $r_k(v) \in O(k+1)$ be the reflection with respect to the hyperplane perpendicular to v , that is, $r_k(v)(v) = -v$ and $v' \perp v$ implies $r_k(v)(v') = v'$. Expressing this in a formula we have

$$r_k(v)(v') = v' - 2(v, v')v.$$

Define $\rho_k: S^k \rightarrow S^k$ by $\rho_k(v) = -v$, the point symmetry with respect to the center of S^k . We denote the upper (lower) hemisphere of S^k by $E_+^k(E_-^k) = \{v = (x_1, \dots, x_{k+1}) \in S^k | x_{k+1} \geq 0 (\leq 0)\}$, where $E_+^k \cap E_-^k = S^{k-1}$. The projection $p: O(k+1) \rightarrow S^k = O(k+1)/O(k)$ is given by $p(r) = r(e_{k+1})$, where $e_{k+1} = (0, \dots, 0, 1)$. With this notation we will show the following:

3.12. $r_{\pm} = p \circ (r_k|E_{\pm}^k): (E_{\pm}^k, S^{k-1}) \rightarrow (S^k, e_{k+1})$ is a relative homeomorphism. $r_k \circ \rho_k = r_k$, $r_k|S^{k-1} = r_{k-1}$.

The last two formulae are obvious. The point $r_k(v)e_{k+1}$ lies in the two-dimensional plane containing v and e_{k+1} and is the point extending the arc doubly on a great circle from $-e_{k+1}$ to $-v$ (see Figure 3.2). The first assertion of 3.12' follows.

Now we consider the composition:

$$s: (E_+^{m+1}, S^m) \xrightarrow{r_{m+1}} (O(m+2), O(m+1)) \xrightarrow{\text{proj}} (O(m+2)/O(m), S^m).$$

In the commutative diagram

$$\begin{array}{ccc} H^m(S^m) & \xrightarrow{\delta} & H^{m+1}(O(m+2)/O(m), S^m) \xrightarrow{p^*} H^{m+1}(S^{m+1}, e_{m+1}) \\ \downarrow r_m^* & & \downarrow s^* \quad \swarrow r_+^* \\ H^m(S^m) & \xrightarrow{\delta_0} & H^{m+1}(E_+^{m+1}, S^m) \end{array}$$

$\delta_0, r_+^*, p^*, s^*$ are isomorphisms. This implies that the assertion of the lemma on $\tau = p^{*-1} \circ \delta$ is equivalent to $r_m^*(s_m) = \pm(1 + (-1)^{m+1})s_m$.

We denote by $e_{\pm} \in H^m(E_{\pm}^m, S^{m-1})$ the generator corresponding to $s_m \in H^m(S^m)$ under the isomorphism $H^m(S^m, E_{\mp}^m) \cong H^m(S^m)$ and the excision isomorphism $H^m(S^m, E_{\mp}^m) \cong H^m(E_{\pm}^m, S^{m-1})$. By (3.12') $r_{\pm}^*: H^m(S^m) \rightarrow H^m(E_{\pm}^m, S^{m-1})$ is an isomorphism, which implies $r_{\pm}^*(s_m) = a_{\pm} e_{\pm}$ where $a_{\pm} \in \mathbb{Z}$, $|a_{\pm}| = 1$. We have $\rho_m^*(s_m) = (-1)^{m+1} s_m$, since ρ_m changes the signs of $(m+1)$ coordinates. We see $a_- = (-1)^{m+1} a_+$, as the three isomorphisms in the above commute with ρ_m^* . When we decompose $H^m(S^m, S^{m-1})$ into $H^m(E_+^m, S^{m-1}) \oplus H^m(E_-^m, S^{m-1})$, $j^*: H^m(S^m, S^{m-1}) \rightarrow H^m(S^m)$ satisfies $j^*(e_+ \oplus e_-) = s_m$. $r_m(S^{m-1}) = e_{m+1}$ implies r_m^* decomposes through $H^m(S^m) \rightarrow H^m(S^m, S^{m-1}) \rightarrow H^m(S^m)$ and hence $r_m^*(s_m) = j^*(a_+ e_+ \oplus a_- e_-) = a_+(1 + (-1)^{m+1}) s_m$. Q.E.D.

COROLLARY 3.13. $H_m(SO(m+2)/SO(m)) \cong \mathbb{Z}_2$ for m odd and is isomorphic to $\mathbb{Z}$ for m even.

REMARK 3.13'. (1) $a_- = 1$, $a_+ = (-1)^{m+1}$ in the above.

(2) If one operates the ρ_m on S^m , then $S^m/\rho_m = O(m+1)/O(m) \times O(1) \cong P_m(\mathbb{R})$. It is seen by (3.10') that r_m defines a continuous injection $r_m: P_m(\mathbb{R}) \rightarrow O(m+1)$ satisfying $r_m|_{P_{m-1}(\mathbb{R})} = r_{m-1}: P_{m-1}(\mathbb{R}) \rightarrow O(m)$.

(3) If we denote the image of an m -cell $e_m = P_m(\mathbb{R}) - P_{m-1}(\mathbb{R})$ by r_m by the same symbol $e_m \subset O(m+1)$, we can define a $\sum i_j$ -cell $e_{i_1} \cdots e_{i_j}$ for $m \leq i_1 < \cdots < i_j < m+k$ in $O(m+k)$ and $O(m+k)/O(m)$ using the product in $O(m+k)$. Then the Stiefel manifold $O(m+k)/O(m) = SO(m+k)/SO(m)$ for $m \geq 1$ is decomposed by these cells and the 0-cell, $O(m)/O(m)$, into a finite CW-complex.

NOTATION. $A = \Delta_R(x_1, \dots, x_m) = \Delta(x_1, \dots, x_m)$ denotes a graded algebra over R with a basis $\{x_{i_1} \cdots x_{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq m\}$ over R .

If we add relations $x_i^2 = 0$, A becomes an exterior algebra $\Lambda(x_1, \dots, x_m)$. In particular, the characteristic of $R \neq 2$ and the degree of each x_i odd implies $A = \Lambda(x_1, \dots, x_m)$.

THEOREM 3.14. (1) If $\mathbf{k}$ is a field of characteristic 2, then

$$H^*(SO(m+k)/SO(m); \mathbf{k}) = \Delta(e_m, e_{m+1}, \dots, e_{m+k-1}), \quad e_{m+i} \in H^{m+i}.$$

(2) If $\mathbf{k}$ is a field of characteristic $\neq 2$, then

$$H^*(SO(m+k)/SO(m); \mathbf{k})$$

$$= \begin{cases} \Lambda(e_{2m+1}, e_{2m+5}, \dots, e_{2m+2k-3}) & \text{for } m \text{ odd, } k \text{ even,} \\ \Lambda(e_{2m+1}, e_{2m+5}, \dots, e_{2m+2k-5}, e_{m+k-1}) & \text{for } m, k \text{ odd,} \\ \Lambda(e_m, e_{2m+3}, e_{2m+7}, \dots, e_{2m+2k-3}) & \text{for } m \text{ even, } k \text{ odd,} \\ \Lambda(e_m, e_{2m+3}, e_{2m+7}, \dots, e_{2m+2k-5}, e_{m+k-1}) & \text{for } m, k \text{ odd.} \end{cases}$$

PROOF. (1) In this case the proof of Theorem 3.10 is applicable, since $\chi = 0$ by Lemma 3.12, where e_{m+1}^2 is not in general 0 (cf. Theorem 6.6).

(2) By induction on k . The case $k = 1$ is clear since $SO(m+1)/SO(m) = S^m$. We can use the inductive hypothesis on $SO(m+k)/SO(m+1)$ in the sphere bundle

$$S^m \xrightarrow{i} SO(m+k)/SO(m) \xrightarrow{p} SO(m+k)/SO(m+1).$$

For m even, the proof is similar to that of Theorem 3.10, since $\chi \in H^{m+1}(SO(m+k)/SO(m+1); \mathbf{k}) = 0$.

For m odd, $\chi \in H^{m+1}(SO(m+k)/SO(m+1); \mathbf{k}) = \{e_{m+1}\}$, $k > 1$. If we consider the above sphere bundle for

$$k = 2: SO(m+2)/SO(m) \xrightarrow{p} SO(m+2)/SO(m+1) = S^{m+1},$$

we see that $\chi = \pm 2e_{m+1}$ by Lemma 3.12. If we put

$$H^*(SO(m+k)/SO(m+1); \mathbf{k}) = \Lambda(e_{m+1}) \otimes A$$

with $A \cong H^*(SO(m+k)/SO(m+2); \mathbf{k})$ in the homomorphism

$$\chi: H^*(SO(m+k)/SO(m+1); \mathbf{k}) \rightarrow H^*(SO(m+k)/SO(m+1); \mathbf{k}),$$

then we have $\text{Ker}(\chi) = \{e_{m+1}\} \otimes A$ and $\text{Coker}(\chi) \cong A$, since the characteristic of $\mathbf{k} \neq 2$. By the Gysin exact sequence, the sequence

$$0 \rightarrow A \xrightarrow{p^*} H^*(SO(m+k)/SO(m); \mathbf{k}) \xrightarrow{i^*} \{e_{m+1}\} \otimes A \rightarrow 0$$

is exact. We take e_{2m+1} so that $\lambda(e_{2m+1}) = e_{m+1}$. $2m+1$ is odd so $e_{2m+1}^2 = 0$. Similarly for the remainder of the proof of Theorem 3.10

$$H^*(SO(m+k)/SO(m); \mathbf{k}) \cong \Lambda(e_{2m+1}) \otimes A.$$

(e_{2m+1} is not necessarily unique.) Q.E.D.

COROLLARY 3.15. (1) If the characteristic of $\mathbf{k}$ is 2, then

$$H^*(SO(m); \mathbf{k}) = \Delta(e_1, e_2, \dots, e_{m-1}).$$

(2) If the characteristic $\mathbf{k} \neq 2$, then

$$H^*(SO(m); \mathbf{k}) = \begin{cases} \Lambda(e_3, e_7, \dots, e_{2m-3}) & \text{for } m \text{ odd,} \\ \Lambda(e_3, e_7, \dots, e_{2m-5}, e_{m-1}) & \text{for } m \text{ even.} \end{cases}$$

(3) $H^*(O(m); \mathbf{k}) \cong H^*(SO(m); \mathbf{k}) \otimes H^*(SO^-(m); \mathbf{k})$, where $SO^-(m) = \{A \in O(m) | \det A = -1\} \cong SO(m)$ (homeomorphism).

Cohomology of the classifying spaces of classical groups. Let G be a Lie group and H a closed subgroup. By §6 of Chapter II we can take classifying spaces BG and BH , which are CW-complexes, unique up to homotopy equivalence. Now we consider the fibre bundle in 6.18 of the same chapter:

$$G/H \xrightarrow{i} BH \xrightarrow{p} BG. \quad (3.16)$$

THEOREM 3.17. (1) $H^*(BU(n)) = \mathbb{Z}[c_1, c_2, \dots, c_n]$, $c_i \in H^{2i}$.

(2) $H^*(BSU(n)) = \mathbb{Z}[c_2, c_3, \dots, c_n]$, $c_i \in H^{2i}$.

(3) $H^*(BSp(n)) = \mathbb{Z}[q_1, q_2, \dots, q_n]$, $q_i \in H^{4i}$.

PROOF. The proof is by induction on n . We have $H^*(BG) = \mathbb{Z}$ taking $BG = EG$ and $G = \{e\}$ for the case $n = 0$ in (1), (3) and the case $n = 1$ in (2). We will only show (1), as the proofs of (2) and (3) are similar. If we take $G = U(n)$ and $H = U(n-1)$ so that $G/H = S^{2n-1}$, we have the Gysin exact sequence

$$\begin{aligned} \dots \xrightarrow{\lambda} H^j(BU(n)) \xrightarrow{\cdot \chi} H^{j+2n}(BU(n)) \xrightarrow{p^*} H^{j+2n}(BU(n-1)) \\ \xrightarrow{\lambda} H^{j+1}(BU(n)) \rightarrow \dots \end{aligned}$$

For the generator c_i ($1 \leq i \leq n-1$) of $H^*(BU(n))$, $\lambda(c_i) \in H^{2i-2n+1}(BU(n)) = 0$, which implies that c_i is the image of p^* . Thus, p^* is an epimorphism and the following sequence is exact:

$$0 \rightarrow H^j(BU(n)) \xrightarrow{\cdot \chi} H^{j+2n}(BU(n)) \xrightarrow{p^*} H^{j+2n}(BU(n-1)) \rightarrow 0.$$

We put $c_i = p^{*-1}(c_i)$ for $1 \leq i < n$ and $c_n = \chi$. For an arbitrary $x \in H^*(BU(n))$ if $p^*(x) = f(c_1, \dots, c_{n-1})$, then $p^*(x - f(c_1, \dots, c_{n-1})) = 0$ and hence there exists $g \in H^j(BU(n))$ such that $x = f(c_1, \dots, c_n) + g \cdot c_n$. By induction on j the element x is expressed uniquely as a polynomial in $c_1, \dots, c_n$. Thus $H^*(BU(n)) = \mathbb{Z}[c_1, \dots, c_n]$. Q.E.D.

For the case $G = SO(n)$ and $H = SO(n-1)$; since $SO(n)$ is connected, the system of local coefficient rings is trivial and we can use the Gysin exact sequence. For the case $G = O(n)$ and $H = O(n-1)$; if $n > 1$ and the characteristic of $\mathbf{k}$ is 2, the local coefficient ring $H^*(S^{n-1}; \mathbf{k})$ is trivial by Lemma 2.9(3). If $n = 1$, $H^*(S^0; \mathbf{k})$ is not trivial but $H^*(D^1, S^0; \mathbf{k})$ is trivial when $\mathbf{k}$ is the field of characteristic 2 and hence the Thom isomorphism holds and the Gysin exact sequence can be used.

THEOREM 3.18. (1) $H_0(P_n(\mathbb{R})) \cong \mathbb{Z}$, $H_i(P_n(\mathbb{R})) \cong \mathbb{Z}_2$ for i odd and $0 < i < n$, $H_n(P_n(\mathbb{R})) \cong \mathbb{Z}$ for n odd and $H_i(P_n(\mathbb{R})) = 0$ for the other cases.

(2) For $\mathbf{k}$ a field of characteristic 2,

$$H^*(P_n(\mathbb{R})) = \mathbf{k}[t]/(t^{n+1}), \quad t \in H^1;$$

for $\mathbf{k}$ a field of characteristic $\neq 2$,

$$H^*(P_n(\mathbb{R})) = \begin{cases} \mathbf{k} & \text{for } n \text{ even,} \\ \Lambda(s), s \in H^n & \text{for } n \text{ odd.} \end{cases}$$

(3) For $\mathbf{k}$ a field of characteristic 2,

$$H^*(BO(1); \mathbf{k}) = \mathbf{k}[t], \quad t \in H^1,$$

for $\mathbf{k}$ a field of characteristic $\neq 2$,

$$H^*(BO(1); \mathbf{k}) = \mathbf{k}.$$

PROOF. (1) If we put $e^n = P_n(\mathbf{R}) - P_{n-1}(\mathbf{R})$, we have a cellular decomposition $P_n(\mathbf{R}) = e^0 \cup e^1 \cup \dots \cup e^n$. By virtue of Remark 3.13(2) the proof of Lemma 3.12 shows $\partial e^m = (1 + (-1)^m)e^{m-1}$. (1) follows from this.

The universal coefficient theorem shows that (2) and (3) are true additively, and hence they are true if the characteristic of $\mathbf{k} \neq 2$. If $\mathbf{k}$ is of characteristic 2, the Gysin exact sequence can be applied to the fibre bundles $O(1) \rightarrow S^n \rightarrow P_n(\mathbf{R})$ and $O(1) \rightarrow EO(1) \rightarrow BO(1)$. In the former case $\chi: H^i(P_n(\mathbf{R}); \mathbf{k}) \rightarrow H^{i+1}(P_n(\mathbf{R}); \mathbf{k})$ is a monomorphism for $0 \leq i < n$. Putting $\chi = t$, we obtain the first half of (2). The first half of (3) is similar. Q.E.D.

THEOREM 3.19. (1) If $\mathbf{k}$ is a field of characteristic 2, then there are $w_i \in H^i$ such that

$$H^*(BO(n); \mathbf{k}) = \mathbf{k}[w_1, w_2, \dots, w_n],$$

$$H^*(BSO(n); \mathbf{k}) = \mathbf{k}[w_2, w_3, \dots, w_n].$$

(2) If $\mathbf{k}$ is a field of characteristic $\neq 2$, then there are $p_i \in H^{4i}$ such that

$$H^*(BSO(n); \mathbf{k}) = \begin{cases} \mathbf{k}[p_1, p_2, \dots, p_n], & \text{for } n = 2m + 1 \text{ odd,} \\ \mathbf{k}[p_1, p_2, \dots, p_{n-1}, \chi], & \text{for } n = 2m \text{ even, } \chi \in H^{2m}. \end{cases}$$

PROOF. (1) The method is same as in the proof of Theorem 3.17 but we need to show $\chi = w_n \neq 0$ (which implies $\lambda(w_{n-1}) = 0$), since $\lambda(w_{n-1}) \in H^0 \cong \mathbf{k}$.

We consider the associated vector bundle $\xi_1: \mathbf{R} \rightarrow E_1 \xrightarrow{\pi} S^1$ and the associated segment bundle $D^1 \rightarrow M \rightarrow S^1$, where M is the Möbius band, of the $O(1)$ -bundle $O(1) \rightarrow S^1 \rightarrow P_1(\mathbf{R}) = S^1$. Since i^* is equivalent to $p^*: H^1(S^1; \mathbf{k}) \rightarrow H^1(S^1; \mathbf{k})$ in the exact sequence

$$H^1(M, \partial M; \mathbf{k}) \xrightarrow{j^*} H^1(M; \mathbf{k}) \xrightarrow{i^*} H^1(\partial M; \mathbf{k})$$

and since $p^*(x) = 2x = 0$, the image j^*U_1 of the Thom class $U_1 \in H^1(M, \partial M; \mathbf{k})$ is a generator of $H^1(M; \mathbf{k}) \cong \mathbf{k}$ and so the Euler class χ_1 of the $O(1)$ -bundle is a generator of $H^1(S^1; \mathbf{k})$. We see from (3.3) that the Euler class of the associated sphere bundle $\xi: S^{n-1} \rightarrow E_n \rightarrow S^1 \times \dots \times S^1$ of the direct product $\xi_1 \times \dots \times \xi_1$ of n copies of ξ_1 is $\chi_1 \times \dots \times \chi_1 \neq 0$. Since $S^{n-1} \rightarrow BO(n-1) \rightarrow BO(n)$ is the universal fibre bundle, there is a continuous map $f: S^1 \times \dots \times S^1 \rightarrow BO(n)$ inducing ξ . By 3.7 $f^*(\chi) = \chi_1 \times \dots \times \chi_1 \neq 0$, which implies $\chi = w_n \neq 0$.

Therefore it is proved inductively, starting from Lemma 3.18(3), that $H^*(BO(n); \mathbf{k}) = \mathbf{k}[w_1, w_2, \dots, w_n]$.

As for $BSO(n)$, first $SO(2) \cong U(1)$ implies

$$H^1(BSO(2); \mathbf{k}) \cong H^1(U(1); \mathbf{k}) = 0.$$

Also for $n > 2$, $H^1(BSO(n); \mathbf{k}) = 0$ follows easily from the Gysin exact sequence. Since $O(n)/SO(n) \cong O(1)$, there is a Gysin exact sequence

$$\dots \xrightarrow{\lambda} H^i(BO(1); \mathbf{k}) \xrightarrow{\chi} H^{i+1}(BO(n); \mathbf{k}) \xrightarrow{p^*} H^{i+1}(BSO(n); \mathbf{k}) \xrightarrow{\lambda} \dots$$

associated with $O(1) \rightarrow BSO(n) \rightarrow BO(n)$. $H^1(BSO(n); \mathbf{k}) = 0$ implies $\chi \neq 0$, so $\chi = w_1$ and $\chi \cdot$ is a monomorphism and hence p^* is an epimorphism. Thus

$$\begin{aligned} H^*(BSO(n); \mathbf{k}) &\cong H^*(BO(n); \mathbf{k}) / \text{Im}(\chi \cdot) = \mathbf{k}[w_1, w_2, \dots, w_n] / (w_1) \\ &= \mathbf{k}[w_2, \dots, w_n]. \end{aligned}$$

Putting $p^*w_i = w_i$ for $2 \leq i \leq n$ we obtain the latter half of (1).

For the case n odd in (2), putting $n = 2m + 1$, we argue by induction on $m \geq 1$. We have

$$H^*(SO(2m+1)/SO(2m-1); \mathbf{k}) \cong H^*(S^{4m-1}; \mathbf{k})$$

by Theorem 3.14(2) applied to the fibre bundle

$$SO(2m+1)/SO(2m-1) \rightarrow BSO(2m-1) \rightarrow BSO(2m+1).$$

Similarly to the case of $BSp(m)$, $H^*(BSO(2m+1); \mathbf{k})$ is obtained from the exact sequence

$$\begin{aligned} 0 \rightarrow H^2(BSO(2m+1); \mathbf{k}) &\xrightarrow{\chi} H^{i+4m}(BSO(2m+1); \mathbf{k}) \\ &\xrightarrow{p^*} H^{i+4m}(BSO(2m-1); \mathbf{k}) \rightarrow 0. \end{aligned}$$

Now p factors through $BSO(2m-1) \xrightarrow{p'} BSO(2m) \rightarrow BSO(2m+1)$ and p^* is an epimorphism and hence p'^* is an epimorphism in the Gysin exact sequence associated with the sphere bundle $S^{2m-1} \rightarrow BSO(2m-1) \rightarrow BSO(2m)$. Thus we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow H^i(BSO(2m); \mathbf{k}) &\xrightarrow{\chi} H^{i+2m}(BSO(2m); \mathbf{k}) \\ &\xrightarrow{p'^*} H^{i+2m}(BSO(2m-1); \mathbf{k}) \rightarrow 0 \end{aligned}$$

from which $H^*(BSO(2m); \mathbf{k})$ is obtained as in Theorem 3.17. Q.E.D.

REMARK 3.20. For $\mathbf{k}$ a field of characteristic $\neq 2$,

$$H^*(BO(n); \mathbf{k}) \cong H^*(BSO(n); \mathbf{k}).$$

The cohomology ring of $BO(1, \mathbb{F}) = P_\infty(\mathbb{F})$ is already obtained. The finite dimensional case is given in Theorem 3.18 and as follows:

$$\begin{aligned} H^*(P_n(\mathbb{C})) &= \mathbb{Z}[t]/(t^{n+1}), & t \in H^2. \\ H^*(P_n(\mathbb{H})) &= \mathbb{Z}[t]/(t^{n+1}), & t \in H^4. \end{aligned} \tag{3.21}$$

These are clear from the fact that $P_n(\mathbb{F})$ is the dn -skeleton of $P_\infty(\mathbb{F})$ but can also be shown by using the Gysin exact sequence of the sphere bundle $S^{n-1} \rightarrow S^{dn-1} \rightarrow P_n(\mathbb{F})$.

§4. The Leray-Hirsch theorem and the action of the Weyl group

Conditions which ensure that the spectral sequence collapses

DEFINITION. A spectral sequence $\{E_r, d_r\}$ is said to *collapse* if $d_r = 0$ for $r \geq 2$.

NOTATION. $H^+(X; R) = \sum_{i \geq 0} H^i(X; R)$;

$$f^+ = f^*|H^+(Y; R): H^+(Y; R) \rightarrow H^+(X; R) \quad (f: X \rightarrow Y).$$

4.1. A spectral sequence $\{E_r\}$ collapses iff $E_2 = E_3 = \dots = E_\infty$.

THEOREM 4.2. Let B and F be 0-connected. Suppose that the Serre spectral sequence of a fibration (E, p, B) with trivial local coefficient ring $H^*(F; R)$ collapses. Then

(1) $p^*: H^*(B; R) \rightarrow H^*(E; R)$ is a monomorphism.

(2) $i^*: H^*(E; R) \rightarrow H^*(F; R)$ is an epimorphism and the ideal $(\text{Imp}^+) \subset \text{Ker } i^*$.

Moreover, if we suppose that $H^i(F; R)$ is a finitely-generated R -free module for every i , then

(3) $H^*(E; R)$ is free as an Imp^* -module with a basis $\{e_\alpha\}$ such that $\{i^*e_\alpha\}$ is a homogeneous basis of $H^*(F; R)$ as an R -module.

(4) Imp^* is a direct summand of $H^*(E; R)$ and

$$H^*(B; R) \cong H^*(E; R)/(\text{Imp}^+).$$

PROOF. (1) and (2) are obtained directly from 2.11. As for (3), we have $H^p(B; R) \otimes_R H^q(F; R) \cong E_\infty^{p,q}$ by Lemma 2.15 and 4.1, and the isomorphism is given by the product of $E_\infty^{p,0} \cong H^p(B; R)$ and $E_\infty^{0,q} \cong H^q(F; R)$ and induced from the product $D^{p,0} \otimes D^{0,q} \rightarrow D^{p,q}$. Here 2.10.10 implies $\text{Imp}^* = E_\infty^{*,0} = D^{*,0}$. By (2.10.9) $i^*e_\alpha \in E_\infty^{0,p}$ can be considered the class of $e_\alpha \in D^{0,p} = H^p(E; R)$. Then $D^{p,*}/D^{p+1,*} \cong E_\infty^{p,*}$ is an Imp^* -free module with basis $\{i^*e_\alpha \in E_\infty^{0,p}\}$ and hence, by induction on p , $D^{p,*}$ is an Imp^* -free module with basis $\{e_\alpha \in D^{i,0} \text{ for } i \leq p\}$. Thus (3) is obtained. (4) follows from the fact that one can take the first element of e_α to be a generator 1 of $H^0(E; R)$. Q.E.D.

COROLLARY 4.3. If $H^i(B; R)$ and $H^j(F; R)$ are finitely-generated R -free modules for every i and j in Theorem 4.2, then so is $H^k(E; R)$ for every k , and the Poincaré polynomials satisfy

$$P_R(E, t) = P_R(B, t) \cdot P_R(F, t).$$

THEOREM 4.4. If $i^*: H^*(E; R) \rightarrow H^*(F; R)$ is an epimorphism in a fibration (E, p, B) with B 0-connected, then the system of local coefficient rings

$H^*(F; R)$ is trivial. In addition, if the assumption of Lemma 2.15 is satisfied, then the Serre spectral sequence collapses and Theorem 4.2 is applicable.

PROOF. By (2.10.9), $E_{\infty}^{0,n} = E_{n+1}^{0,n} = \dots = E_2^{0,n}$ and

$$i_0^*: H^0(B; H^q(F; R)) \rightarrow H^0(b; H^q(F_b; R)) = H^q(F_b; R)$$

is an epimorphism if i^* is an epimorphism. By the latter half of Lemma 1.18(3) the system of local coefficient rings is trivial. The equality in the first half is equivalent to $d_r = 0: E_r^{0,n} \rightarrow E_r^{r,n-r+1}$ for $r \geq 2$. For the case where Lemma 2.15 is applicable, we have $E_r^{p,q} \otimes E_r^{p',q'} \cong E_r^{p+p',q+q'}$ by inductively supposing that $E_2^{p,q} = E_r^{p,q}$. Clearly $d_r = 0: E_r^{0,q} \rightarrow E_r^{p+r,1-r}$ for $r \geq 2$ and hence $d_r(b \otimes f) = d_r b \otimes f + (-1)^p b \otimes d_r f = 0$. Therefore $d_r = 0$, which implies $E_{r+1}^{p,q} = E_r^{p,q} = E_2^{p,q}$. Thus the latter half is obtained by induction. Q.E.D.

DEFINITION. If $i^*: H^*(E; R) \rightarrow H^*(F_b; R)$ is an epimorphism, then the fibre F_b is said to be *totally nonhomologous to zero* with respect to the coefficient ring R .

Theorems 4.2 and 4.4 are called the *Leray-Hirsch theorem*.

If R is a field and E and F_b are of finite type, then i^* is an epimorphism iff $i_*: H_*(F_b; R) \rightarrow H_*(E; R)$ is a monomorphism.

This is the original meaning of the statement that the fibre F_b is totally nonhomologous to zero.

LEMMA 4.5. *The Serre spectral sequence (for which Lemma 2.15 is applicable) collapses in the following cases:*

(1) $H^i(B; R) = H^i(F; R) = 0$ for i odd;

(2) R is a field or $\mathbb{Z}$, and $H^*(B; R)$, $H^*(F; R)$, and $H^*(E; R)$ are finitely-generated R -free modules for each dimension such that

$$P_R(E, t) = P_R(B, t) \cdot P_R(F, t).$$

PROOF. For case (1); that p or q is odd implies $E_2^{p,q} = 0$ and hence $E_r^{p,q} = 0$ for $r \geq 2$. One of $p, q, p+r, q-r+1$ is odd in $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ so $d_r = 0$.

For case (2);

$$P_R(B, t) \cdot P_R(F, t) = \sum_{p,q} \text{rank } E_2^{p,q} \cdot t^{p+q}$$

and

$$P_R(E, t) = \sum_n \text{rank } H^n(E; R) \cdot t^n = \sum_{p,q} (\text{rank } E_{\infty}^{p,q}) \cdot t^{p+q}.$$

In general, $\text{rank } E_r^{p,q} \geq \text{rank } E_{\infty}^{p,q}$. $d_r \neq 0: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ implies $\text{rank } E_r^{p,q} > \text{rank } E_{r+1}^{p,q}$ and hence it is necessary that $d_r = 0$. Q.E.D.

The cohomology of G/T^n (G : a compact connected classical group). Consider the standard maximal torus T^n of $G = SO(2n)$, $SO(2n+1)$, $U(n)$, $Sp(n)$, $SU(n+1)$ defined in §3 of Chapter I. We will study below the relations between the homogeneous space G/T^n and the classifying spaces BG , BT^n .

THEOREM 4.6. *If $G = U(n)$, $Sp(n)$, $SU(n+1)$, then $H^*(G/T^n)$ is a finitely-generated free module, multiplicatively generated by $H^2(G/T^n)$ and its Poincaré polynomial is given by*

$$P(U(n)/T^n, t) = \prod_{i=1}^n \frac{1-t^{2i}}{1-t^2},$$

$$P(SU(n+1)/T^n, t) = P(U(n+1)/T^{n+1}, t),$$

$$P(Sp(n)/T^n, t) = \prod_{i=1}^n \frac{1-t^{4i}}{1-t^2}.$$

PROOF. The base space $U(n+1)/U(n) \times T^1$ in the fibre bundle

$$U(n) \times T^1/T^{n+1} = U(n)/T^n \xrightarrow{1} U(n+1)/T^{n+1} \rightarrow p \gg U(n+1)/U(n) \times T^1$$

is homeomorphic with $P_n(\mathbb{C}) = U(n+1)/U(1) \times U(n)$ and hence is identified with $P_n(\mathbb{C})$. By (3.21) $H^*(P_n(\mathbb{C})) = \mathbb{Z}[t]/(t^{n+1})$, where $t \in H^2(P_n(\mathbb{C}))$. Since the theorem is obvious for $U(1)/T^1 =$ a point, Lemma 4.5(1) can be applied assuming the theorem valid for $U(n)/T^n$. By Corollary 4.3

$$\begin{aligned} P(U(n+1)/T^{n+1}, t) &= P(P_n(\mathbb{C}), t) \cdot P(U(n)/T^n, t) \\ &= \frac{1-t^{2n+2}}{1-t^2} \prod_{i=1}^n \frac{1-t^{2i}}{1-t^2} = \prod_{i=1}^{n+1} \frac{1-t^{2i}}{1-t^2}. \end{aligned}$$

Since $i^*(e_\alpha)$ in Theorem 4.2(3) is expressible by a polynomial in the elements of $H^2(U(n)/T^n)$, e_α can be chosen so that it can be expressed as a polynomial in the elements $\{x_\beta\}$ in $H^2(U(n+1)/T^{n+1})$. Thus $H^*(U(n+1)/T^{n+1})$ is multiplicatively generated by $\{p^*(t), x_\beta\}$. By induction on n the theorem is true for $G = U(n)$.

Consider $T^n = SU(n+1) \cap T^{n+1}$ and $T^1 \times 1 \subset T^{n+1}$. Since $T^1 \times 1$ and $SU(n+1)$ generate $U(n+1)$, $SU(n+1)$ acts on $U(n+1)/T^{n+1}$ and the isotropy group is T^n . Thus $SU(n+1)/T^n \cong U(n+1)/T^{n+1}$ (a homeomorphism).

The case $G = Sp(n)$ is similar to the case $G = U(n)$, showing that

$$Sp(n+1)/Sp(n) \times T^1 \cong P_{2n+1}(\mathbb{C}) \text{ (a homeomorphism).}$$

The inclusion

$$c': Sp(n+1) \rightarrow U(2n+2)$$

induces a map

$$\begin{aligned}\varphi: Sp(n+1)/1 \times Sp(n) &= V_{n+1,1}(\mathbf{H}) \rightarrow U(2n+2)/1 \times U(2n+1) \\ &= V_{2n+2,1}(\mathbf{C}),\end{aligned}$$

which are the sets of vectors with length 1 respectively in $\mathbf{H}^{n+1}$ and $\mathbf{C}^{2n+2}$. If we identify them by $c': \mathbf{H}^{n+1} \cong \mathbf{C}^{2n+2}$ in §2 of Chapter I, then $\varphi = 1$. Thus φ is a homeomorphism and hence

$$\begin{aligned}Sp(n+1)/Sp(n) \times T^1 &\cong Sp(n+1)/T^1 \times Sp(n) \\ &\cong U(2n+1)/T^1 \times U(2n+1) = P_{2n+1}(\mathbf{C}),\end{aligned}$$

since the actions of $T^1 = U(1) \subset Sp(1)$ correspond under φ . Q.E.D.

LEMMA 4.7. $H^*(SO(2n+2)/SO(2n) \times T^1)$ is a free module and

$$\begin{aligned}P(SO(2n+2)/SO(2n) \times T^1, t) &= (1+t^{2n})(1-t^{2n+2})(1-t^2)^{-1} \\ &= (1+t^{2n})(1+t^2+t^4+\cdots+t^{2n}).\end{aligned}$$

PROOF. We have $V_{2n+2,2}(\mathbf{R}) = SO(2n+2)/SO(2n) \times 1 = O(2n+2)/O(2n) \times 1$. Since $\chi = 0$ in Lemma 3.12, it follows easily from the Gysin exact sequence that

$$H^*(V_{2n+2,2}(\mathbf{R})) \cong H^*(S^{2n} \times S^{2n+1}) = \Lambda(x_{2n}, x_{2n+1}). \quad (4.7')$$

Putting $V = V_{2n+2,2}(\mathbf{R})$ and $B = SO(2n+2)/SO(2n) \times T^1$, we consider the exact sequence

$$\cdots \rightarrow H^{i+1}(V) \xrightarrow{\lambda} H^i(B) \xrightarrow{t^*} H^{i+2}(B) \xrightarrow{p^*} H^{i+2}(V) \xrightarrow{\lambda} \cdots$$

associated with the sphere bundle $T^1 \rightarrow V \rightarrow B$, where $\chi = t \in H^2(B)$. Firstly, $H^0(V) \cong \mathbf{Z}$ and $H^i(V) = 0$ for $i \neq 0$, $i < 2n$, imply $H^{2j}(B) \cong \mathbf{Z}$ and $H^{2j+1}(B) = 0$ for $0 \leq j \leq n-1$. B is a $4n$ -dimensional manifold so $H^i(B) = 0$ for $i > 4n$ and $0 \rightarrow H^{4n+1}(V) \rightarrow H^{4n}(B) \rightarrow 0$ is exact so $H^{4n}(B) \cong \mathbf{Z}$. Secondly, $H^i(V) = 0$ for $2n+1 < i < 4n+1$ which implies $H^{2j}(B) \cong \mathbf{Z}$ and $H^{2j-1}(B) = 0$ for $n+1 \leq j \leq 2n$. Lastly, the exact sequence $0 \rightarrow H^{2n+1}(V) \rightarrow H^{2n}(B) \rightarrow H^{2n+2}(B) \rightarrow 0$ implies $H^{2n}(B) \cong \mathbf{Z} \oplus \mathbf{Z}$. Thus the lemma is proved. Q.E.D.

REMARK 4.8. $H^*(SO(2n+2)/SO(2n) \times T^1) = \mathbf{Z}[t, s]/(t^{n+1} - 2ts, s^2)$, where $t \in H^2$, $s \in H^{2n}$.

THEOREM 4.9. $H^*(SO(2n)/T^n)$ and $H^*(SO(2n+1)/T^n)$ are finitely-generated modules and their Poincaré polynomials are given by

$$\begin{aligned}P(SO(2n)/T^n, t) &= \frac{1}{1+t^{2n}} \prod_{i=1}^n \frac{1-t^{4i}}{1-t^2}, \\ P(SO(2n+1)/T^n, t) &= \prod_{i=1}^n \frac{1-t^{4i}}{1-t^2}.\end{aligned}$$

PROOF. The theorem is trivial for $SO(2)/T^1 = \text{a point}$. If we suppose that the theorem is true for $SO(2n)/T^n$, then by Lemma 4.7 we can apply Lemma 4.5(1) to the two fibre bundles

$$\begin{aligned} SO(2n)/T^n &\rightarrow SO(2n+1)/T^n \rightarrow SO(2n+1)/SO(2n) = S^n; \\ SO(2n)/T^n &\rightarrow SO(2n+2)/T^{n+1} \rightarrow SO(2n+2)/SO(2n) \times T^1. \end{aligned}$$

The theorem is obtained inductively by Corollary 4.3. Q.E.D.

In general, it is known (Theorem 4.21 of Chapter VI) that $H^*(G/T)$ is a free module and $H^{\text{odd}}(G/T) = 0$ for a compact Lie group G and its maximal torus T . Theorems 4.6 and 4.9 express this concretely.

NOTATION. $\mathbf{Z}_2^n = O(n) \cap D(n, \mathbf{R}) = \{\varepsilon_1 \otimes \cdots \otimes \varepsilon_n | \varepsilon_i = \pm 1\} \subset O(n)$

$S\mathbf{Z}_2^n = SO(n) \cap \mathbf{Z}_2^n \subset SO(n)$.

Obviously,

$$\begin{aligned} \mathbf{Z}_2^n &= \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 = O(1) \times \cdots \times O(1) \quad (n \text{ copies}), \\ S\mathbf{Z}_2^n &\cong \mathbf{Z}_2^{n-1} \quad (\text{a homeomorphism}). \end{aligned}$$

Let $i_j: \mathbf{Z}_2 \rightarrow \mathbf{Z}_2^n$ be the inclusion into the j th factor and let

$$\rho_j: B\mathbf{Z}_2 \rightarrow B\mathbf{Z}_2^n \quad \text{for } j = 1, \dots, n$$

be the projection of the bundle in (3.16) ($G = \mathbf{Z}_2^n$, $H = \mathbf{Z}_2$), where the inclusion $\mathbf{Z}_2 \subset \mathbf{Z}_2^n$ is given by i_j . By Lemma 3.18 and the Künneth formula one has

4.10. $H^*(B\mathbf{Z}_2^n; \mathbf{k}) = \mathbf{k}[x_1, x_2, \dots, x_n]$, $x_i \in H^1$, for a field $\mathbf{k}$ of characteristic 2. Here x_i is determined by $\rho_j^*(x_i) = \delta_{ij}x$ for $\rho_j^*: H^1(B\mathbf{Z}_2^n; \mathbf{k}) \rightarrow H^1(B\mathbf{Z}_2; \mathbf{k}) = \mathbf{k}[x]$.

THEOREM 4.11. $SO(n)/S\mathbf{Z}_2^n \cong O(n)/\mathbf{Z}_2^n$ (a homeomorphism). If $\mathbf{k}$ is a field of characteristic 2, $H^*(O(n)/\mathbf{Z}_2^n; \mathbf{k})$ is multiplicatively generated by $H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k})$ and its Poincaré polynomial is given by

$$P_k(O(n)/\mathbf{Z}_2^n, t) = P_k(SO(n)/S\mathbf{Z}_2^n, t) = \prod_{i=1}^n \frac{1-t^i}{1-t}.$$

PROOF. Suppose inductively that the theorem is true for $O(n)/\mathbf{Z}_2^n$. To apply Theorem 4.4 to the fibre bundle

$$O(n)/\mathbf{Z}_2^n \xrightarrow{i} O(n+1)/\mathbf{Z}_2^{n+1} \xrightarrow{p} O(n+1)/O(n) \times \mathbf{Z}_2 \cong P_n(\mathbf{R})$$

it suffices to show that

$$i^*: H^j(O(n+1)/\mathbf{Z}_2^{n+1}; \mathbf{k}) \rightarrow H^j(O(n)/\mathbf{Z}_2^n; \mathbf{k})$$

is an epimorphism. To that end it suffices to see that i^* is an epimorphism for $j = 1$, since $H^*(O(n)/\mathbf{Z}_2^n; \mathbf{k})$ is generated by $H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k})$.

Considering the exact sequence

$$H^1(P_n(\mathbf{R}); \mathbf{k}) \xrightarrow{p^*} H^1(O(n+1)/\mathbf{Z}_2^{n+1}; \mathbf{k}) \xrightarrow{i^*} H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k})$$

in Theorem 2.14, it suffices to show that the dimension of the second term is not less than n over $\mathbf{k}$, since the dimensions of the first term and the third term are 1 and $n-1$ respectively over $\mathbf{k}$. This is seen by comparing the dimensions over $\mathbf{k}$, after applying 3.19(1) and 4.10 to the exact sequence

$$0 \rightarrow H^1(BO(n+1); \mathbf{k}) \rightarrow H^1(B\mathbf{Z}_2^{n+1}; \mathbf{k}) \rightarrow H^1(O(n+1)/\mathbf{Z}_2^{n+1}; \mathbf{k}).$$

Thus by Theorem 4.4 we can apply Theorem 4.2 to the first fibre bundle and so the results in the theorem are obtained similarly to the proof of Theorem 4.6. Q.E.D.

The action of the Weyl group. Let G be a Lie group and T be an abelian closed subgroup of G .

NOTATION. $W(G:T)$ denotes $N_G(T)/T$, where

$$N_G(T) = \{g \in G | gT = Tg\}.$$

The action of $W(G:T)$ on T is defined by

$$\varphi \cdot t = n^{-1}tn \quad (t \in T),$$

where $n \in N_G(T)$ represents the class $\varphi = \{n\} = nT = Tn \in W(G:T)$. The definition is independent of the choice of the representative $n \cdot t'$, $t' \in T$, of φ , since T is abelian.

On the other hand, as we have $BG = EG/G$ in the universal G -bundle (EG, p, BG, G) , the bundle (EG, p, BT, T) with $BT = EG/T$ is the universal T -bundle. The action of $W(G:T)$ on BT : $W(G:T) \times BT \rightarrow BT$ is defined by

$$\varphi \cdot eT = enT \quad (= eTn),$$

for $\varphi = \{n\}$ in the above and $eT \in BT$ ($e \in EG$). It is obvious that the definition is independent of the choice of n .

The action of $W(G:T)$ on T induces one on $H^*(T)$ and on $H^*(BT)$.

NOTATION. $H^*(BT; R)^{W(G:T)}$ = (the subring formed by the invariant elements under the action of $W(G:T)$).

Consider the homomorphism

$$\rho^*: H^*(BG; R) \rightarrow H^*(BT; R)$$

induced by the projection $\rho = \rho(T, G)$ of the fibre bundle in (3.16) where $H = T$.

LEMMA 4.12. For $\varphi \in W(G:T)$, $\rho \circ (\varphi \cdot) = \rho$. In particular, $\text{Im } \rho^* \subset H^*(BT; R)^{W(G:T)}$.

PROOF. $\rho(eT) = eG$ implies $\rho(\varphi \cdot eT) = \rho(enT) = enG = eG = \rho(eT)$. Thus, $\rho \circ (\varphi \cdot) = \rho$ and so $(\varphi \cdot)^* \rho^*(x) = \rho^*(x)$ for $x \in H^*(BG; R)$. Thus, $\rho^*(x) \in H^*(BT; R)^{W(G:T)}$. Q.E.D.

Suppose G is 0-connected in the universal G -bundle $G \rightarrow EG \rightarrow BG$. EG is ∞ -connected so $H^*(EG; R) \cong R$ and hence by Theorem 2.14 one has

4.13. $H^1(BG; R) = 0$ and the transgression is an isomorphism:

$$\tau_b: H^1(G_b; R) \cong H^2(BG; R), \quad b \in BG.$$

4.13 holds even if one replaces G by the 0-connected T , and then the transgression

$$\tau_b: H^1(T_b; R) \xrightarrow{\delta^*} H^2(EG, T_b; R) \xleftarrow{p^*} H^2(BT, b; R) \xrightarrow{j^*} H^2(BT; R)$$

depends on $b \in BT$. We take $b \in \rho^{-1}(\bar{b})$ ($= (G/T)_b$), define a homeomorphism $\lambda_e: T \rightarrow T_b$ for $e \in T_b$ by

$$\lambda_e(t) = e \cdot t, \quad t \in T,$$

and put

$$\tau = \tau_b \circ \lambda_e^{*-1}: H^1(T; R) \cong H^2(BT; R), \quad b = p(e). \quad (4.13')$$

LEMMA 4.14. Let G and T be both 0-connected, where T is an abelian closed subgroup of G . Then τ in (4.13') is determined independently of the choice of $e \in G_b$ and the action of $W(G: T)$ commutes with τ .

PROOF. Let $e' \in G_b = e \cdot G$. Consider $\tau' = \tau_{b'} \circ \lambda_{e'}^{*-1}$, where $b' = p(e')$. Since $(G/T)_e \cong G/T$ is a connected manifold, one can choose a path $w: I \rightarrow (G/T)_e \subset BT$ connecting b and b' such that the image $W = w(I)$ is contractible. By the covering homotopy property there is a path $w': I \rightarrow G_b$ connecting e and a point e'_1 of $T_{b'}$ such that $p \circ w' = \bar{w}$, where one can write $e'_1 = e' \cdot t$ with $t \in T$. Consider a path $l: I \rightarrow T$ connecting e and t and put $w(\theta) = w'(\theta) \cdot l(\theta)$. Then $w: I \rightarrow G_b$ is a path connecting e and e' such that $p \circ w = \bar{w}$, hence the image $W = w(I)$ is homeomorphic with $\bar{W}$ and W is contractible. Putting $\bar{T} = p^{-1}\bar{W}$, by the action of T , $W \times I$ and $\bar{T}$ are homeomorphic and T_b and $T_{b'}$ are deformation retracts of $\bar{T}$, and $\lambda_e \simeq \lambda_{e'}: T \rightarrow \bar{T}$. It follows that the following diagram is commutative and each morphism is an isomorphism, and hence it is shown that $\tau = \tau'$.

$$\begin{array}{ccccccc}
 & H^1(T_b; R) & \xrightarrow{\delta^*} & H^2(EG, T_b; R) & \xleftarrow{p^*} & H^2(BT, b; R) & \\
 & \lambda_e^* \nearrow & \uparrow & \uparrow & \uparrow j^* & \searrow j^* & \\
 H^1(T; R) & H^1(\bar{T}_b; R) & \xrightarrow{\delta^*} & H^2(EG, \bar{T}_b; R) & \xleftarrow{p^*} & H^2(BT, \bar{W}; R) & H^2(BT; R) \\
 & \lambda_{e'}^* \searrow & \downarrow & \downarrow & \downarrow j^* & \nearrow j^* & \\
 & H^1(T_{b'}; R) & \xrightarrow{\delta^*} & H^2(EG, T_{b'}; R) & \xleftarrow{p^*} & H^2(BT, b'; R) &
 \end{array}$$

Next, let $\varphi = \{n\} \in W(G: T)$, where $n \in N_G(T)$, $\varphi(b) = b'$, $e \in T_b$, and $e' = en \in T_{b'}$. We define $f: EG \rightarrow EG$ by $f(x) = x \cdot n$; as for the

action of φ on T , $\lambda_e(\varphi \cdot t) = e' \cdot n^{-1}tn = e \cdot tn = \lambda_e(t) \cdot n = f(\lambda_e(t))$; as for the action of φ on BT , $\varphi \cdot p(x) = \varphi \cdot xT = xTn = x \cdot nT = p(x \cdot n) = p(f(x))$. Thus the diagram

$$\begin{array}{ccccccc} H^1(T; R) & \xrightarrow{\lambda_e^*} & H^1(T_{b'}; R) & \xrightarrow{\delta^*} & H^2(EG, T_{b'}; R) & \xleftarrow{p^*} & H^2(BT, b'; R) & \xrightarrow{j^*} & H^2(BT; R) \\ \downarrow \varphi^* & & \downarrow f^* & & \downarrow f^* & & \downarrow \varphi^* & \nearrow j^* & \\ H^1(T; R) & \xrightarrow{\lambda_e^*} & H^1(T_{b'}; R) & \xrightarrow{\delta^*} & H^2(EG, T_{b'}; R) & \xleftarrow{p^*} & H^2(BT, b'; R) & \xrightarrow{j^*} & H^2(BT; R) \end{array}$$

is commutative. This shows that the action of φ commutes with τ , that is, $\tau \circ \varphi^* = \varphi^* \circ \tau$. Q.E.D.

The Weyl groups considered in §3 of Chapter I are the following:

$$W(G) = W(G; T^n) = N_G(T^n)/T^n,$$

$$\text{for } G = U(n), SU(n+1), Sp(n), SO(2n), SO(2n+1).$$

Applying the Künneth formula 1.10 to $H^*(T) = H^*(U(1)) = H^*(S^1) = \Lambda(x)$, where $x \in H^1$, one has

4.15. $H^*(T^n) = \Lambda(x_1, x_2, \dots, x_n)$, where $x_i \in H^1(T^n)$ is determined by the following condition: $i_j^*(x_i) = \delta_{ij}x$ for the inclusion $i_j: T^1 \rightarrow T^n = T \times \dots \times T$ into the j th factor. $H^1(T^n)$ has a basis $\{x_1, x_2, \dots, x_n\}$.

LEMMA 4.16. (1) The set of all the actions of $W(U(n))$ on $H^1(T^n)$ coincides with the set of permutations of the basis elements $\{x_1, x_2, \dots, x_n\}$.

(2) The set of all the actions of $W(Sp(n))$ on $H^1(T^n)$ coincides with the set of permutations of the basis elements $\{x_1, x_2, \dots, x_n\}$ after changing the signs of any number of them.

(3) The action of $W(SO(2n+1))$ on $H^1(T^n)$ is same as that of $W(Sp(n))$.

(4) The set of all the actions of $W(SO(2n))$ on $H^1(T^n)$ coincides with the set of permutations of the basis elements $\{x_1, x_2, \dots, x_n\}$ after changing the signs of an even number of them.

PROOF. It can be verified by using 4.15 that the permutation of λ_i in the action of $W(G)$ given in Theorem 3.13 of Chapter I corresponds to that of x_i . Also, by the correspondence $\lambda \mapsto \lambda^{-1}$, $T = U(1) = S^1$ is mapped to itself with reversed orientation and hence x is mapped to $-x$ in $H^1(T)$, that is, x_i is mapped to $-x_i$ by the correspondence $\lambda \mapsto \lambda^{-1}$. The lemma follows from these observations. Q.E.D.

COROLLARY 4.17. The maximal torus T^n in $SU(n+1)$ is given by $T^n = \{t_1 \otimes \dots \otimes t_{n+1} \in T^{n+1} \subset U(n+1) | t_1 \dots t_{n+1} = 1\}$. Therefore the inclusion $i: T^n \rightarrow T^{n+1}$ induces an isomorphism

$$H^*(T^n) \cong \Lambda(x_1, \dots, x_{n+1}) / \left(\sum x_i \right),$$

and $H^1(T^n)$ is given by the generators $\{x_1, \dots, x_{n+1}\}$ with the relation $x_1 + \dots + x_{n+1} = 0$. Using these generators, the action of $W(SU(n+1))$ induces the permutations of $\{x_1, \dots, x_{n+1}\}$.

NOTATION. $W_2(O(n)) = W(O(n) : \mathbb{Z}_2^n)$, $W_2(SO(n)) = W(SO(n) : S\mathbb{Z}_2^n)$

LEMMA 4.18. (1) $W_2(O(n))$ is isomorphic with the n th symmetric group and each element is represented by a permutation matrix $P_\sigma = \{\delta_{i\sigma(j)}\}$.

(2) Let $\mathbf{k}$ be a field of characteristic 2. Then $W_2(O(n))$ acts on $H^*(B\mathbb{Z}_2^n; \mathbf{k}) = \mathbf{k}[x_1, \dots, x_n]$ via the permutations of $\{x_1, \dots, x_n\}$.

PROOF. (1) Consider $A = (a_{ij}) \in N_{O(n)}(\mathbb{Z}_2^n) \subset O(n)$. For any i , let $\sigma(i)$ be the minimal j such that $a_{ij} \neq 0$. For $D = I_{\sigma(i)} \oplus (-I_{n-\sigma(i)})$ there exists $D' = \varepsilon_1 \oplus \dots \oplus \varepsilon_n$ such that $AD = D'A$ and so $a_{i\sigma(i)} = \varepsilon_i a_{i\sigma(i)} \neq 0$, and $a_{ij} = -\varepsilon_i a_{ij}$ for $j > \sigma(i)$, which implies $a_{ij} = 0$ for $j \neq \sigma(i)$. Thus, there is $D_0 \in O(n) \cap D(n, \mathbb{R}) = \mathbb{Z}_2^n$ such that $A = P_\sigma D_0$. (1) is obtained since obviously $P_\sigma \in N_{O(n)}(\mathbb{Z}_2^n)$.

(2) The following diagram is commutative and δ^* and j^* are isomorphisms (the coefficient field $\mathbf{k}$ is omitted):

$$\begin{array}{ccccccc} H^0((\mathbb{Z}_2^n)_b, e_b) & \xrightarrow{\delta^*} & H^1(EO(n), (\mathbb{Z}_2^n)_b) & \xleftarrow{p^*} & H^1(B\mathbb{Z}_2^n, b) & \xrightarrow{j^*} & H^1(B\mathbb{Z}_2^n) \\ \downarrow i_j^* & & \downarrow i^* & & \downarrow \rho_j^* & & \downarrow \rho_j^* \\ \mathbf{k} \cong H^0((\mathbb{Z}_2)_a, e_a) & \xrightarrow{\delta^*} & H^1(EO(n), (\mathbb{Z}_2)_a) & \xleftarrow{p^*} & H^1(B\mathbb{Z}_2, a) & \xrightarrow{j^*} & H^1(B\mathbb{Z}_2) \cong \mathbf{k} \end{array}$$

where we use the notation i_j and ρ_j as in 4.10. One can directly verify that p^* in the lower sequence is an isomorphism (p^* in the upper sequence is not an isomorphism). Putting $\Sigma = \delta^{*-1} \circ p^* \circ j^{*-1}$ (the suspension) we obtain the commutative diagram

$$\begin{array}{ccc} H^0((\mathbb{Z}_2^n)_b, e_b; \mathbf{k}) & \xleftarrow{\Sigma} & H^1(B\mathbb{Z}_2^n; \mathbf{k}) \\ \downarrow i_j^* & & \downarrow \rho_j^* \\ \mathbf{k} \cong H^0((\mathbb{Z}_2)_a, e_a; \mathbf{k}) & \xleftarrow[\cong]{\Sigma} & H^1(B\mathbb{Z}_2; \mathbf{k}). \end{array}$$

By considering the action of an element $\varphi_\sigma = \{P_\sigma\}$ of $W_2(O(n))$ following the latter half of the proof of Lemma 4.14, there is $f = f_\sigma : (\mathbb{Z}_2^n)_b \rightarrow (\mathbb{Z}_2^n)_{\varphi(b)}$ such that $f_\sigma^* \circ \Sigma = \Sigma \circ \varphi_\sigma^*$ on $H^1(B\mathbb{Z}_2^n; \mathbf{k})$ and $f_\sigma \circ i_j = i_{\sigma(j)} \circ g_\sigma$, where $g_\sigma = f_\sigma|_{(\mathbb{Z}_2)_a} : (\mathbb{Z}_2)_a \rightarrow (\mathbb{Z}_2)_{a'}$ and $\rho_{\sigma(j)}(a') = \varphi(b)$. Now observe that g_σ is a homeomorphism and that g_σ^* gives a morphism between the generators 1 of $H^0(\quad; \mathbf{k}) \cong \mathbf{k}$, since $\mathbf{k}$ is of characteristic 2. By 4.10 the generators of $H^1(B\mathbb{Z}_2^n; \mathbf{k})$ satisfy $\Sigma \rho_j^* x_k = \Sigma \delta_{jk} x = \delta_{jk} \in \mathbf{k}$. It follows that

$$\Sigma \rho_j^* \varphi_\sigma^*(x_k) = i_j^* \Sigma \varphi_\sigma^*(x_k) = i_j^* f_\sigma^* \Sigma x_k = g_\sigma^* i_{\sigma(j)}^* \Sigma x_k = g_\sigma^* \Sigma \rho_{\sigma(j)}^* x_k,$$

which shows that $\varphi_{\sigma}^*(x_k) = x_{\sigma^{-1}(k)}$. Thus we see that $W_2(O(n))$ acts via the permutations of $\{x_1, \dots, x_n\}$. Q.E.D.

COROLLARY 4.19. $W_2(SO(n))$ is isomorphic with $W_2(O(n))$, each element is represented by $P_{\sigma}(I_{n-1} \otimes \text{Sign } \sigma)$ and the action on $H^*(BSZ_2^n; \mathbf{k}) = \mathbf{k}[x_1, \dots, x_n]/(x_1 + \dots + x_n)$ is via the permutations of $\{x_1, \dots, x_n\}$.

§5. Characteristic classes and the relations among them

Characteristic classes

DEFINITION. If each fibre bundle $\xi = (E, p, B, F, G)$ with the structure group G and fibre F such that B is a CW-complex we assign an element

$$\alpha(\xi) \in H^*(B; R), \quad \text{where } R \text{ is a fixed coefficient ring,}$$

of the cohomology ring of the base space B , then $\alpha = \{\alpha(\xi)\}$ is called the *characteristic class* of the fibre bundle with respect to (F, G) , if $\alpha(\xi)$ is natural with respect to bundle maps $f: \xi' = (E', p', B', F, G) \rightarrow \xi$, i.e.,

$$\tilde{f}^*(\alpha(\xi)) = \alpha(\xi') \quad \text{for } \tilde{f}^*: H^*(B; R) \rightarrow H^*(B'; R).$$

Obviously a characteristic class may be defined by the following properties:

- 5.1. (1) If $\xi \equiv \xi'$, then $\alpha(\xi) = \alpha(\xi')$,
 (2) $g^*(\alpha(\xi)) = \alpha(g^*\xi)$ for $\xi = (E, p, B, F, G)$ and $g: B' \rightarrow B$.

If there is a characteristic class α such that $\alpha(\xi) \neq \alpha(\xi')$, then ξ and ξ' are not equivalent. This property is useful for studying the structure of fibre bundles.

THEOREM 5.2. If G acts effectively on F and if there is a classifying space BG that is a CW-complex, then the characteristic classes of fibre bundles with respect to (F, G) are in one to one correspondence with the elements of $H^*(BG; R)$.

PROOF. Consider the universal fibre bundle $\gamma = (EG \times_G F, p, BG, F, G)$ and assign to the characteristic class $\alpha = \{\alpha(\xi)\}$ an element $\alpha(\gamma) \in H^*(BG; R)$. Conversely, if $x \in H^*(BG; R)$ is given, we choose $g: B \rightarrow BG$ by the classifying theorem (Theorem 6.1 of Chapter II) so that $\xi \equiv g^*\gamma$ for every ξ , and put $\alpha_x(\xi) = g^*(x)$. Then the correspondence $x \mapsto \alpha_x(\xi)$ is determined, since $g^*\gamma \equiv g'^*\gamma$ implies $g \simeq g'$ and hence $g^*(x) = g'^*(x)$. Considering the map $\tilde{f}: B' \rightarrow B$ for a bundle map $f: \xi \rightarrow \xi'$ we have $\alpha_x(\xi') = (g \circ \tilde{f})^*(x) = \tilde{f}^*(g^*(x)) = \tilde{f}^*(\alpha_x(\xi))$, since $\xi' \equiv \tilde{f}^*\xi \equiv \tilde{f}^*g^*\gamma = (g \circ \tilde{f})^*\gamma$. Thus $\alpha_x = \{\alpha_x(\xi)\}$ is a characteristic class.

If we take $g = \mathbf{1}_{BG}$, then $\alpha_x(\gamma) = x$. Conversely, if we put $x = \alpha(\gamma)$ for a characteristic class α , by 5.1(2) we have $\alpha(\xi) = g^*\alpha(\gamma) = g^*(x) = \alpha_x(\xi)$ for $g: B \rightarrow BG$ in the above and $\xi = g^*\gamma$. Hence $\alpha = \alpha_x$. This shows that the assignment $x \mapsto \alpha_x$ is a one-to-one correspondence. Q.E.D.

COROLLARY 5.3. *Under the conditions of Theorem 5.2, a fibre bundle and its associated G -bundle have the same characteristic classes.*

Now we consider the case when G is a classical group. In $H^*(BG; R)$ we define Chern classes, Stiefel-Whitney classes, Pontrjagin classes, the Euler class, symplectic Pontrjagin classes, etc. The characteristic classes of fibre bundles corresponding to the elements above are called by the same names.

Chern classes and symplectic Pontrjagin classes. Put $G = U(n)$ and $H = T^n$ in (3.16) and consider the fibre bundle

$$U(n)/T^n \xrightarrow{\iota} BT^n \xrightarrow{\rho} BU(n).$$

Here the cohomology ring of BT^n is obtained as follows. (By (6.20) of Chapter II we may consider $BG \times BG' = B(G \times G')$, which we will use without notice in what follows.)

Since by Theorem 3.17 we have $H^*(BT^1) = H^*(BU(1)) = \mathbb{Z}[t]$ with $t \in H^2$, by the Künneth formula $H^*(BT^n) \cong H^*(BT^1 \times \cdots \times BT^1) \cong H^*(BT^1) \otimes \cdots \otimes H^*(BT^1) \cong \mathbb{Z}[t_1, \dots, t_n]$ with $t_i \in H^2$. On the other hand, by (4.13') we have $\tau: H^1(T^n) \cong H^2(BT^n)$. Therefore

5.4. $H^*(BT^n) = \mathbb{Z}[t_1, \dots, t_n]$, where $\tau(x_i) = t_i$ ($i = 1, \dots, n$) for the generator $x_i \in H^1(T^n)$ in 4.15.

THEOREM 5.5. (1) $\rho^*: H^*(BU(n)) \rightarrow H^*(BT^n) = \mathbb{Z}[t_1, \dots, t_n]$ is a monomorphism and

$$\text{Im } \rho^* = H^*(BT^n)^{W(U(n))} = \mathbb{Z}[\sigma_1, \dots, \sigma_n],$$

where $\sigma_i = \sigma_i(t_1, \dots, t_n)$ is the i th elementary symmetric polynomial.

(2) There are unique $c_i \in H^{2i}(BU(n))$ satisfying $\rho^*(\sum_i c_i) = \prod_{j=1}^n (1 + t_j)$ such that $H^*(BU(n)) = \mathbb{Z}[c_1, c_2, \dots, c_n]$.

(3) $i^*: H^*(BT^n) \rightarrow H^*(U(n)/T^n)$ is an epimorphism and induces the following isomorphism:

$$H^*(U(n)/T^n) \cong H^*(BT^n)/(\text{Im } \rho^+) = \mathbb{Z}[t_1, \dots, t_n]/(\sigma_1, \dots, \sigma_n).$$

PROOF. By Theorem 4.6, 5.4, and Theorem 3.17 the cohomologies of $U(n)/T^n$, BT^n , and $BU(n)$ are free modules and have no elements of odd degree. In addition, the local coefficient ring is trivial, since $U(n)$ is connected. Therefore by Lemma 4.5(1) we can apply Theorem 4.2 and it suffices to show the part $\text{Im } \rho^* = H^*(BT^n)^{W(U(n))}$ in order to prove the theorem. By Lemma 4.12 we have $\text{Im } \rho^* \subset H^*(BT^n)^{W(U(n))}$. By Lemmas 4.14 and 4.16 the action of $W(U(n))$ on $H^*(BT^n)$ is given by the permutations of $t_1, \dots, t_n$. Thus $H^*(BT^n)^{W(U(n))}$ is the set of all symmetric polynomials which, as is well known, can be uniquely expressed as polynomials of the elementary symmetric polynomials σ_i , that is, $\text{Im } \rho^* \subset \mathbb{Z}[\sigma_1, \dots, \sigma_n]$. Also

by Theorem 4.2, ρ^* is a monomorphism onto a direct summand. By these facts, $\text{Im } \rho^* = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$. Q.E.D.

Consider the fibre bundle

$$Sp(n)/T^n \xrightarrow{i} BT^n \xrightarrow{\rho} BSp(n),$$

in (3.16) for $(G, H) = (Sp(n), T^n)$.

THEOREM 5.6. (1) $\rho^*: H^*(BSp(n)) \rightarrow H^*(BT^n)$ is a monomorphism,

$$\text{Im } \rho^* = \mathbb{Z}[\sigma_1, \dots, \sigma_n] = H^*(BT^n)^{W(Sp(n))}, \quad \sigma_i = \sigma_i(t_1^2, \dots, t_n^2).$$

(2) There are unique $q_i \in H^{4i}(BSp(n))$ satisfying $\rho^*(\sum q_i) = \prod(1 + t_j^2)$ such that

$$H^*(BSp(n)) = \mathbb{Z}[q_1, \dots, q_n].$$

(3) $i^*: H^*(BT^n) \rightarrow H^*(Sp(n)/T^n)$ is an epimorphism and induces the following isomorphism:

$$H^*(Sp(n)/T^n) \cong \mathbb{Z}[q_1, \dots, q_n]/(\sigma_1, \dots, \sigma_n).$$

PROOF. Since the action of $W(Sp(n))$ on $H^*(BT^n)$ consists of changes of signs and permutations, the invariants under it are symmetric polynomials of $t_1^2, \dots, t_n^2$. The rest is similar to Theorem 5.5. Q.E.D.

DEFINITION. The element c_i in Theorem 5.5 is called the i th Chern class. The element q_i in Theorem 5.6 is called the i th symplectic Pontrjagin class.

The definition depends on the choice of the generator x of $H^1(T^1)$. If x is replaced by $-x$, then c_i are replaced by $(-1)^i c_i$, but there is no essential difference.

Now consider T^n in $SU(n+1)$, by the inclusion

$$i: T^n \subset T^{n+1} \subset U(n+1)$$

in Corollary 4.17. By the same symbol denote the image of t_i of the epimorphism $H^*(BT^{n+1}) \rightarrow H^*(BT^n)$ induced from $BT^n \rightarrow BT^{n+1}$ and then we have $H^*(BT^n) = \mathbb{Z}[t_1, \dots, t_{n+1}]/(\sigma_1)$, where $\sigma_1 = t_1 + \dots + t_{n+1}$.

COROLLARY 5.7. We obtain a similar result to Theorem 5.5 for

$$U(n+1)/T^{n+1} = SU(n+1)/T^n \xrightarrow{i} BT^n \xrightarrow{\rho} BU(n+1).$$

The morphism p^* induced by the projection $p: BSU(n+1) \rightarrow BU(n+1)$ gives rise to an isomorphism $\mathbb{Z}[c_1, \dots, c_{n+1}]/(c_1) = \mathbb{Z}[c_2, \dots, c_{n+1}] \cong H^*(BSU(n+1))$. If one puts $c_i = p^*(c_i) \in H^*(BSU(n+1))$, then the above isomorphism changes to equality, and $\rho^* \sum c_i \equiv \prod(1 + t_j) \pmod{\sigma_1}$.

From the inclusions $q: U(n) \subset Sp(n)$ and $c': Sp(n) \subset SU(2n) \subset U(2n)$ we obtain the projections

$$\rho_q: BU(n) \rightarrow BSp(n), \quad \rho_{c'}: BSp(n) \rightarrow BU(2n).$$

Also, $\oplus: U(m) \times U(n) \subset U(m+n)$, $Sp(m) \times Sp(n) \subset Sp(m+n)$ give rise to

$$\begin{aligned}\rho_{m,n}: BU(m) \times BU(n) &\rightarrow BU(m+n), \\ BSp(m) \times BSp(n) &\rightarrow BSp(m+n).\end{aligned}$$

The following theorem holds for these maps.

THEOREM 5.8. (1) $\rho_q^*(q_i) = \sum_{j+k=2i} (-1)^{i+j} c_j c_k$, $\rho_q^*: H^*(BSp(n)) \rightarrow H^*(BU(n))$.

(2) $\rho_c^*(c_{2j-1}) = 0$, $\rho_c^*(c_{2j}) = (-1)^j q_j$, $\rho_c^*: H^*(BU(2n)) \rightarrow H^*(BSp(n))$.

(3) $\rho_{m,n}^*(c_i) = \sum_{j+k=i} c_j \times c_k$, $\rho_{m,n}^*(q_i) = \sum_{j+k=i} q_j \times q_k$.

PROOF. (1) Since $U(n)$ and $Sp(n)$ have T^n in common, we distinguish between the monomorphism ρ^* in Theorem 5.5 and the one in Theorem 5.6 and denote them by ρ_1^* and ρ_2^* respectively, and then $\rho_1^* \circ \rho_q^* = \rho_2^*$. We have

$$\begin{aligned}\rho_1^* \left(\rho_q^* \sum (-1)^i q_i \right) &= \rho_2^* \sum (-1)^i q_i = \prod (1 - t_j^2) = \prod (1 - t_j)(1 + t_j) \\ &= \rho_1^* \left(\sum (-1)^i c_i \right) \rho_1^* \left(\sum c_i \right) = \rho_1^* \left(\sum (-1)^i c_i \right) \left(\sum c_i \right).\end{aligned}$$

Since ρ_1^* is a monomorphism, we have

$$\rho_q^* \sum (-1)^i q_i = \left(\sum (-1)^i c_i \right) \left(\sum c_i \right) = \sum_i \sum_{j+k=2i} (-1)^j c_j c_k,$$

so

$$\rho_q^*(q_i) = (-1)^i \sum_{j+k=2i} c_j c_k.$$

(2) Since $A \in T^n$ implies $c'(A) = A \oplus \bar{A} \in T^{2n}$ we see that $\rho^*(t_i) = \rho^*(-t_{n+i}) = t_i$ for $i = 1, 2, \dots, n$ in the commutative diagram of the projections:

$$\begin{array}{ccc} BT^n & \xrightarrow{\rho} & BT^{2n} \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ BSp(n) & \xrightarrow{\rho_c'} & BU(2n) \end{array}$$

Therefore

$$\begin{aligned}\rho_1^* \rho_c^* \sum c_i &= \rho^* \rho_2^* \sum c_i = \rho^* \prod (1 + t_j) \\ &= \prod (1 + t_j)(1 - t_j) = \prod (1 - t_j^2) = \rho_1^* \sum (-1)^i q_i,\end{aligned}$$

which implies $\rho_c^* \sum c_i = \sum (-1)^i q_i$. We obtain (2), by comparing the parts of the same degree.

(3) If we consider in $H^*(BT^m \times BT^n) \cong H^*(BT^{m+n}) = \mathbb{Z}[t_1, \dots, t_{m+n}]$ using ρ^* , then c_i and $c_j \times c_k$ are replaced by $\sigma_i(t_1, \dots, t_{m+n})$ and

$\sigma_j(t_1, \dots, t_m) \cdot \sigma_k(t_{m+1}, \dots, t_{m+n})$ respectively. We have

$$\begin{aligned} \sum \sigma_i(t_1, \dots, t_{m+n}) &= \prod_{i=1}^{m+n} (1 + t_i) = \prod_{i=1}^m (1 + t_i) \prod_{i=m+1}^{m+n} (1 + t_i) \\ &= \sum \sigma_j(t_1, \dots, t_m) \sum \sigma_k(t_{m+1}, \dots, t_{m+n}), \end{aligned}$$

from which the first half of (3) is obtained. The latter half is similar. Q.E.D.

Stiefel-Whitney classes. Let $\mathbf{k}$ be a field of characteristic 2. Consider the fibre bundle

$$O(n)/\mathbf{Z}_2^n \xrightarrow{i} B\mathbf{Z}_2^n \xrightarrow{\rho} BO(n)$$

in (3.16) $(G, H) = (O(n), \mathbf{Z}_2^n)$. We have $H^*(B\mathbf{Z}_2^n; \mathbf{k}) = \mathbf{k}[x_1, \dots, x_n]$ for the generator $x_i \in H^1(B\mathbf{Z}_2^n; \mathbf{k})$ in 4.10. Consider the action of $W_2(O(n)) = W(O(n); \mathbf{Z}_2^n)$.

THEOREM 5.9. *Let $\mathbf{k}$ be a field of characteristic 2.*

(1) $\rho^*: H^*(BO(n); \mathbf{k}) \rightarrow H^*(B\mathbf{Z}_2^n; \mathbf{k})$ is a monomorphism, and $\text{Im } \rho^* = H^*(BO(n); \mathbf{k})^{W_2(O(n))} = \mathbf{k}[\sigma_1, \dots, \sigma_n]$, $\sigma_i = \sigma_i(x_1, \dots, x_n)$.

(2) There exists unique $w_i \in H^1(BO(n); \mathbf{k})$ satisfying $\rho^* \sum w_i = \prod (1 + t_j)$ such that

$$H^*(BO(n); \mathbf{k}) = \mathbf{k}[w_1, w_2, \dots, w_n].$$

(3) $i^*: H^*(B\mathbf{Z}_2^n; \mathbf{k}) \rightarrow H^*(O(n)/\mathbf{Z}_2^n; \mathbf{k})$ is an epimorphism and induces the following isomorphism:

$$H^*(O(n)/\mathbf{Z}_2^n; \mathbf{k}) = \mathbf{k}[x_1, \dots, x_n]/(\sigma_1, \dots, \sigma_n).$$

PROOF. The theorem is proved similarly to Theorem 5.5, using Theorems 3.19 and 4.11, 4.10 and Lemma 9.18, but as the structure group $O(n)$ is not connected and has elements of odd degree, it suffices to prove the assumption:

$$i^*: H^*(B\mathbf{Z}_2^n; \mathbf{k}) \rightarrow H^*(O(n)/\mathbf{Z}_2^n; \mathbf{k})$$

is an epimorphism of Theorem 4.4 instead of Lemma 4.5. To that end it suffices to show that

$$i^*: H^1(B\mathbf{Z}_2^n; \mathbf{k}) \rightarrow H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k})$$

is an epimorphism, since $H^*(O(n)/\mathbf{Z}_2^n; \mathbf{k})$ is generated by the elements of $H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k})$ by Theorem 4.11. On the other hand, since the structure group of the fibre bundle $O(n)/\mathbf{Z}_2^n = SO(n)/SZ_2^n \xrightarrow{i} BSZ_2^n \xrightarrow{\rho} BSO(n)$ is connected, the first sequence is exact (Theorem 2.14) and p^* is an epimorphism in the following commutative diagram:

$$\begin{array}{ccccc} 0 = H^1(BSO(n); \mathbf{k}) & \longrightarrow & H^1(BSZ_2^n; \mathbf{k}) & \xrightarrow{i_0^*} & H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k}) \\ & & \uparrow p^* & \nearrow i^* & \\ & & H^1(B\mathbf{Z}_2^n; \mathbf{k}) & & \end{array}$$

Theorem 4.11 implies $\dim_{\mathbf{k}} H^1(O(n)/\mathbf{Z}_2^n; \mathbf{k}) = n - 1$. Since

$$H^*(BS\mathbf{Z}_2^n; \mathbf{k}) = \mathbf{k}[x_1, \dots, x_n]/(x_1 + \dots + x_n),$$

we have $\dim_{\mathbf{k}} H^1(BS\mathbf{Z}_2^n; \mathbf{k}) = n - 1$. Thus the monomorphism i_0^* is an isomorphism and so i^* is an epimorphism. Q.E.D.

COROLLARY 5.10. *A result similar to Theorem 5.9 holds with $(O(n), \mathbf{Z}_2^n)$ replaced by $(SO(n), S\mathbf{Z}_2^n)$. The homomorphism p^* induced by the projection $p: BSO(n) \rightarrow BO(n)$ gives an isomorphism*

$$\mathbf{k}[w_1, w_2, \dots, w_n]/(w_1) = \mathbf{k}[w_2, \dots, w_n] \cong H^*(BSO(n); \mathbf{k}).$$

DEFINITION. We call $w_i \in H^i(BO(n); \mathbf{k})$ in Theorem 5.9 the *i*th Stiefel-Whitney class. We also call $w_i = p^*w_i \in H^*(BSO(n); \mathbf{k})$ in Corollary 5.10 the *i*th Stiefel-Whitney class.

From the inclusions $c: O(n) \subset U(n)$ and $r: U(n) \subset SO(2n) \subset O(2n)$ we obtain the projections

$$\rho_c: BO(n) \rightarrow BU(n) \quad \text{and} \quad \rho_r: BU(n) \rightarrow BO(2n).$$

Also $\oplus: O(m) \times O(n) \rightarrow O(m+n)$ gives rise to a map

$$\rho_{m,n}: BO(m) \times BO(n) \rightarrow BO(m+n).$$

We denote the image of c_i by the natural map $\mathbf{Z} \rightarrow \mathbf{k}$ of coefficient rings by the same symbol $c_i \in H^{2i}(BU(n); \mathbf{k})$. Let the characteristic of $\mathbf{k}$ be 2 in the following:

THEOREM 5.11. (1) $\rho_c^*(c_i) = w_i^2$, $\rho_c^*: H^*(BU(n); \mathbf{k}) \rightarrow H^*(BO(n); \mathbf{k})$.
 (2) $\rho_r^*(w_{2i-1}) = 0$, $\rho_r^*(w_{2i}) = c_i$, $\rho_r^*: H^*(BO(2n); \mathbf{k}) \rightarrow H^*(BU(n); \mathbf{k})$.
 (3) $\rho_{m,n}^*(w_i) = \sum_{j+k=i} w_j \times w_k$.

PROOF. (1) Since we have $\mathbf{Z}_2^n = O(n) \cap T^n$, we obtain the commutative diagram

$$\begin{array}{ccc} B\mathbf{Z}_2^n & \xrightarrow{\rho} & BO(n) \\ \downarrow p & & \downarrow \rho_c \\ BT^n & \xrightarrow{\rho} & BU(n). \end{array}$$

To give a similar proof as that of Theorem 5.8, it suffices to prove $p^*(t_i) = x_i^2$ for $H^*(BT^n; \mathbf{k}) = \mathbf{k}[t_1, \dots, t_n]$ and $H^*(B\mathbf{Z}_2^n; \mathbf{k}) = \mathbf{k}[x_1, \dots, x_n]$. Then we have $\rho_c^*(c_i) = \sum_{j+k=2i} \pm w_j w_k = w_i^2$. To this end, by considering the inclusion $T^1 \rightarrow T^n$ to each factor, one can see that it suffices to show

$$p^*(t) = x^2 \quad \text{for } p^*: H^2(BT^1; \mathbf{k}) \rightarrow H^2(B\mathbf{Z}_2; \mathbf{k}), \quad (5.11')$$

for the case $n = 1$. Observe that $T/\mathbf{Z}_2 = O(1)/\mathbf{Z}_2$ is homeomorphic with S^1 . We have the Gysin exact sequence

$$H^2(BT; \mathbf{k}) \xrightarrow{p^*} H^2(B\mathbf{Z}_2; \mathbf{k}) \xrightarrow{\lambda} H^1(BT; \mathbf{k})$$

associated with the S^1 -bundle $S^1 \rightarrow B\mathbb{Z}_2 \xrightarrow{p} BT$. We obtain (5.11'), since $H^1(BT; \mathbf{k}) = 0$ and since $H^2(BT; \mathbf{k})$ and $H^2(B\mathbb{Z}_2; \mathbf{k})$ are generated by t and x^2 respectively. (Consider first the case when $\mathbf{k}$ is a prime field.)

(2) $\varphi_0 = r \circ c: O(n) \rightarrow O(2n)$ satisfies $\varphi_0(A) = A \oplus A$ and $\varphi_0(\mathbb{Z}_2^n) \subset \mathbb{Z}_2^{2n}$, and we obtain the commutative diagram

$$\begin{array}{ccc} B\mathbb{Z}_2^n & \xrightarrow{\rho_1} & B\mathbb{Z}_2^{2n} \\ \downarrow \rho & & \downarrow \rho \\ BO(n) & \xrightarrow{\rho_0} & BO(2n). \end{array}$$

The property of $\rho_0^*: H^*(BO(2n); \mathbf{k}) \rightarrow H^*(BO(n); \mathbf{k})$ is easily seen by $\rho_1^*: H^*(B\mathbb{Z}_2^{2n}; \mathbf{k}) \rightarrow H^*(B\mathbb{Z}_2^n; \mathbf{k})$ after composing it with ρ^* , where $\rho_1^*(x_i) = \rho_1^*(x_{i+n}) = x_i$ for $i = 1, \dots, n$. We have

$$\begin{aligned} \rho^* \left(\rho_0^* \sum_{i=0}^{2n} w_i \right) &= \rho_1^* \rho^* \left(\sum w_i \right) = \rho_1^* \prod_{i=1}^{2n} (1 + x_i) \\ &= \prod_{i=1}^n (1 + x_i)^2 = \left(\rho^* \sum_{i=0}^n w_i \right)^2 = \rho^* \sum_{i=0}^n w_i^2. \end{aligned}$$

Therefore

$$\rho^*(w_{2i-1}) = 0 \quad \text{and} \quad \rho^*(w_{2i}) = w_i^2.$$

Obviously $\rho = \rho_r \circ \rho_c$ implies $\rho_0^* = \rho_c^* \circ \rho_r^*$. $H^{\text{odd}}(BU(n); k) = 0$ implies $\rho_r^*(w_{2i-1}) = 0$. (1) implies $\rho_c^*(\rho_r^*(w_{2i})) = \rho_0^*(w_{2i}) = w_i^2 = \rho_r^*(c_i)$. Also (1) shows that ρ_c^* is a monomorphism, and hence $\rho_r^*(w_{2i}) = c_i$.

The proof of (3) is similar to that of Theorem 5.8(3). Q.E.D.

The Wu formula. The cohomology ring with coefficients $\mathbf{k} = \mathbb{Z}_2$ a prime field of characteristic 2 is equipped with the Steenrod squaring operation $Sq^i: H^n(X; \mathbb{Z}_2) \rightarrow H^{n+i}(X; \mathbb{Z}_2)$ having the properties 1.16. We have the following Wu formula which determines the action of Sq^i on the characteristic classes w_i , c_i , and q_i defined so far. To generalize the binomial coefficient $\binom{a}{b} = a!/b!(a-b)!$ we adopt the following convention:

$$\binom{-1}{0} = 1$$

THEOREM 5.12. (1) $Sq^j w_i = \sum_{k=0}^j \binom{i-k-1}{j-k} w_{i+j-k} w_k \quad (0 \leq j \leq i)$.

(2) $Sq^{2j} c_i = \sum_{k=0}^j \binom{i-k-1}{j-k} c_{i+j-k} c_k \quad (0 \leq j \leq i)$.

(3) $Sq^{4j} q_i = \sum_{k=0}^j \binom{i-k-1}{j-k} q_{i+j-k} q_k \quad (0 \leq j \leq i)$.

PROOF. (1) Since $\rho^*: H^*(BO(n); \mathbb{Z}_2) \rightarrow H^*(BT^n; \mathbb{Z}_2)$ is a monomorphism, it suffices to show the equality

$$Sq^j \sigma_i = \sum_{k=0}^j \binom{i-k-1}{j-k} \sigma_{i+j-k} \sigma_k, \quad \sigma_i = \sigma_i(x_1, \dots, x_n),$$

which is obtained by applying ρ^* to the equality (1). We define a polynomial $\sigma_{a,b} = \sigma_{a,b}(x_1, \dots, x_n)$ in $x_1, \dots, x_n$ by the equality

$$\prod_{i=1}^n (1 + x_i t + x_i^2 u) = \sum_{a,b} \sigma_{a,b} t^a u^b.$$

$\sigma_{a,b}$ is a minimal polynomial of the form $x_1 \cdots x_a x_{a+1}^2 \cdots x_{a+b}^2 + \cdots$ such that $\sigma_{a,0} = \sigma_a$ and $\sigma_{0,b} = \sigma_b^2$. By the formulas $Sq^0(x_i) = x_i$, $Sq^1(x_i) = x_i^2$, and $Sq^j(x_i) = 0$ for $j \neq 0, 1$, and by the Cartan formula we have

$$Sq^j \sigma_i = Sq^j \sum x_{k_1} \cdots x_{k_i} = \sum_{\varepsilon_1 + \cdots + \varepsilon_i = j} Sq^{\varepsilon_1} x_{k_1} \cdots Sq^{\varepsilon_i} x_{k_i} = \sigma_{i-j,j}.$$

Therefore it suffices to prove the following equality (*) by induction on n :

$$\sigma_{a,b} = \sum_{k=0}^b \binom{a+b-k-1}{b-k} \sigma_{a+2b-k} \sigma_k \quad (0 \leq b \leq i). \quad (*)$$

This is trivial if $n = 1$. Suppose (*) for $n \geq 1$, denote by $\sigma'_{a,b}$ and σ'_i the $\sigma_{a,b}$ and the σ_i with respect to the variables $x_1, \dots, x_n, x_{n+1}$ and put $x_{n+1} = t$. Then we have

$$\sigma'_i = \sigma_i + \sigma_{i-1} t, \quad \sigma'_{a,b} = \sigma_{a,b} + \sigma_{a-1,b} t + \sigma_{a,b-1} t^2.$$

Therefore (*) is proved inductively as follows:

$$\begin{aligned} & \sum_{k=0}^b \binom{a+b-k-1}{b-k} \sigma'_{a+2b-k} \sigma'_k \\ &= \sum_{k=0}^b \binom{a+b-k-1}{b-k} (\sigma_{a+2b-k} + \sigma_{a+2b-k-1} t) (\sigma_k + \sigma_{k-1} t) \\ &= \sum_{k=0}^b \binom{a+b-k-1}{b-k} \sigma_{a+2b-k} \sigma_k \\ & \quad + \sum_{k=0}^{b-1} \left\{ \binom{a+b-k-1}{b-k} + \binom{a+b-k-2}{b-k-1} \right\} \sigma_{a+2b-k-1} \sigma_k t \\ & \quad + \sigma_{a+b-1} \sigma_b t + \sum_{k=0}^{b-1} \binom{a+b-k-2}{b-k-1} \sigma_{a+2(b-1)-k} \sigma_k t^2 \\ &= \sigma_{a,b} + \sum_{k=0}^b \binom{a+b-k-2}{b-k} \sigma_{a+2b-k-1} \sigma_k t + \sigma_{a,b-1} t^2 \\ &= \sigma_{a,b} + \sigma_{a-1,b} t + \sigma_{a,b-1} t^2 = \sigma'_{a,b}. \end{aligned}$$

(2) and (3) are similarly obtained from $Sq^0 = \text{id}$, $Sq^2(t_i) = t_i^2$, $Sq^4(t_i^2) = t_i^4$, $Sq^j(t_i) = 0$ for $j \neq 0, 2$ and $Sq^j(t_i^2) = 0$ for $j \neq 0, 4$. Q.E.D.

The Euler class and Pontrjagin classes. The Euler class $\chi \in H^m(B; R)$ of a sphere bundle $S^{m-1} \rightarrow E \rightarrow B$ in §3 depends on the choice of a generator of $H^m(D^m, S^{m-1}; R) \cong H^{m-1}(S^{m-1}; R) \cong R$. If $R = \mathbb{Z}$ in the following lemma we choose the generator by making a suitable sign change if necessary. If $R = \mathbf{k}$ is a field of characteristic 2, then there is a unique generator corresponding to 1 in the prime field $\mathbb{Z}_2$.

LEMMA 5.13. (1) $H^{2n}(BU(n)) \ni c_n = \chi$ the Euler class of the sphere bundle $S^{2n-1} = U(n)/U(n-1) \rightarrow BU(n-1) \xrightarrow{p} BU(n)$. Similarly for SU instead of U .

(2) $H^{4n}(BSp(n)) \ni q_n = \chi$ the Euler class of the sphere bundle $S^{4n-1} = Sp(n)/Sp(n-1) \rightarrow BSp(n-1) \xrightarrow{p} BSp(n)$.

(3) $H^n(BO(n); \mathbf{k}) \ni w_n = \chi$ the Euler class of the sphere bundle $S^{n-1} = O(n)/O(n-1) \rightarrow BO(n-1) \xrightarrow{p} BO(n)$, where $\mathbf{k}$ is a field of characteristic 2. Similarly for SO instead of O .

PROOF. First, for each of the projections p in (1), (2), and (3) we will show

$$\begin{aligned} p^*(c_i) &= c_i, & p^*(q_i) &= q_i, & p^*(w_i) &= w_i & \text{for } 0 \leq i \leq n-1, \\ p^*(c_n) &= 0, & p^*(q_n) &= 0, & p^*(w_n) &= 0. \end{aligned} \quad (5.13')$$

For the case (1); by the monomorphism ρ^* of Theorem 5.5 it suffices to show $\rho_0^*(\sigma_i(t_1, \dots, t_n)) = \sigma_i(t_1, \dots, t_{n-1})$, where $\rho_0: BT^{n-1} \rightarrow BT^n$. (Recall $\sigma_n(t_1, \dots, t_{n-1}) = 0$.) This is clear from $\rho_0^*(t_i) = t_i$ for $1 \leq i \leq n-1$ and $\rho_0^*(t_n) = 0$. The cases (2) and (3) are similar.

The lemma is now obtained from the Gysin exact sequence of each sphere bundle. Q.E.D.

For the homomorphism

$$\rho_c^*: H^*(BU(n)) \rightarrow H^*(BSO(n)),$$

induced by the projection $\rho_c: BSO(n) \rightarrow BU(n)$ which is obtained from the inclusion $c: SO(n) \subset U(n)$, we have:

DEFINITION. $p_i = (-1)^i \rho_c^*(c_{2i}) \in H^{4i}(BSO(n))$ is called the i th Pontrjagin class.

It follows from Theorem 5.11(1) that

$$p_i \equiv w_{2i}^2 \pmod{2}. \quad (5.14)$$

Concerning the projections

$$\rho: BT^m \rightarrow BSO(n), \quad \rho_r: BU(m) \rightarrow BSO(n),$$

induced from the inclusion $T^m \subset SO(n)$ and $r: U(m) \subset SO(n)$ respectively, the following lemma holds, where $m = [n/2]$.

LEMMA 5.15. (1) $\rho^*(p_i) = \sigma_i = \sigma_i(t_1^2, \dots, t_m^2) \in H^{4i}(BT^m)$.

(2) $\rho_r^*(p_i) = \sum_{j+k=i} (-1)^j c_j c_k \in H^{4i}(BU(m))$.

PROOF. First note the following:

5.15'. Let $T_0^m = g^{-1}T^mg$ with $g \in G$ be a subgroup conjugate to T^m . Define $h_g: T^m \rightarrow T_0^m$ by $h_g(t) = g^{-1}tg$. Then there is a map $\bar{h}_g: BT^m \rightarrow BT_0^m$ such that the following diagram is commutative:

$$\begin{array}{ccc} BT^m & \xrightarrow{\rho} & BG \\ \downarrow h_g & \parallel & \\ BT_0^m & \xrightarrow{\rho} & BG \end{array} \quad \begin{array}{ccc} H^1(T_0^m) & \xrightarrow{\tau} & H^2(BT_0^m) \\ \downarrow h_g^* & & \downarrow h_g^* \\ H^1(T^m) & \xrightarrow{\tau} & H^2(BT^m) \end{array}$$

In fact, $\cdot g: EG \rightarrow EG$ induces $\bar{h}_g$ satisfying $\rho \circ \bar{h}_g = \rho$ such that $(\cdot g) \circ \lambda_e = \lambda_e \circ h_g$ for λ_e in (4.13') and hence we get the second commutative diagram.

It follows from (5.13') that we can put $n = 2m$ in the lemma. To distinguish T^m in $U(m)$ from the one in $SO(2m)$, we use the notation $T_U^m \subset U(m)$.

$$\begin{array}{ccccccc} BT_U^m & \xrightarrow{h_{g'}} & BT^m & \xrightarrow{h_g} & BT_0^m & \xrightarrow{\rho} & BT_U^{2m} \\ \downarrow \rho & & \downarrow \rho & & & & \downarrow \rho \\ BU(m) & \xrightarrow{\rho_r} & BSO(2m) & \xrightarrow{\rho_c} & BU(2m). & & \end{array}$$

Consider the above commutative diagram, where $g' \in SO(2m)$ is some permutation matrix P_σ and $h_{g'}: T_0^m \rightarrow T^m$ coincides with the isomorphism $r: U(1) \cong SO(2)$ on each component. The element $g \in U(m)$ such that $T_0^m = g^{-1}T^mg$ is the one given by $g^{-1} = P_\sigma \eta$ with $\eta_m = \eta$ in 2.7(3) of Chapter I and satisfies $h_g(h_{g'}(A)) = A \oplus \bar{A}$ where $A \in T^m$. By virtue of the diagram and 5.15' we have $\rho^*((-1)^i p_i) = \rho^* \rho_c^*(c_{2i}) = h_g^* \rho^* \rho^*(c_{2i}) = h_g^* \rho^* \sigma_{2i}(t_1, \dots, t_{2n}) = \sigma_{2i}(t_1, \dots, t_n, -t_1, \dots, -t_n)$. Similarly to the proof of Theorem 5.8(2), one has $\sigma_{2i}(t_1, \dots, t_n, -t_1, \dots, -t_n) = (-1)^i \sigma_i(t_1^2, \dots, t_n^2) = (-1)^i \sigma$. The proof of (2) is similar to that of Theorem 5.8(1). Q.E.D.

THEOREM 5.16. Let $\mathbf{k}$ be a field of characteristic $\neq 2$ and $m = [n/2]$.

(1) $\rho^*: H(BSO(n); \mathbf{k}) \rightarrow H^*(BT^m; \mathbf{k})$ is a monomorphism and

$$\text{Im } \rho^* = H^*(BT^m; \mathbf{k})^{w(SO(n))} = \begin{cases} \mathbf{k}[\sigma_1, \dots, \sigma_m] & \text{if } n = 2m + 1, \\ \mathbf{k}[\sigma_1, \dots, \sigma_{m-1}, \sigma_m] & \text{if } n = 2m. \end{cases}$$

(2) There exist unique $p_i \in H^{4i}(BSO(n); \mathbf{k})$ satisfying $\rho^*(\sum_{i=0}^m p_i) = \prod_{j=1}^m (1 + t_j^2)$ and $\chi \in H^{2m}(BSO(2m); \mathbf{k})$ satisfying $\rho^*(\chi) = \sigma_m = t_1 \cdots t_m$

if $n = 2m$ such that

$$\begin{aligned} H^*(BSO(2m+1); \mathbf{k}) &= \mathbf{k}[p_1, \dots, p_m], \\ H^*(BSO(2m); \mathbf{k}) &= \mathbf{k}[p_1, \dots, p_{m-1}, \chi], \quad \chi^2 = p_m. \end{aligned}$$

(3) $i^*: H^*(BSO(n); \mathbf{k}) \rightarrow H^*(SO(n)/T^m; \mathbf{k})$ is an epimorphism and induces an isomorphism

$$\mathbf{k}[t_1, \dots, t_m]/(p_1, \dots, p_{m-1}, (p_m \text{ or } \chi)) \cong H^*(SO(n)/T^m; \mathbf{k}).$$

PROOF. It is proved similarly to Theorem 5.5. Q.E.D.

We remark that Lemma 5.15 characterizes p_i .

THEOREM 5.17. (1) There exists a unique $\chi \in H^{2m}(BSO(2m))$ satisfying $\rho^*(\chi) = t_1 t_2 \cdots t_m$ and $\chi \equiv w_m^2 \pmod{2}$. Then $\rho_r^*(\chi) = c_m$.

(2) There is a direct decomposition $H^*(BSO(n)) = F \oplus T_2$, where

(i) F is a free subgroup such that

$$F = \begin{cases} \mathbf{Z}[p_1, \dots, p_m] & \text{if } n = 2m+1, \\ \mathbf{Z}[p_1, \dots, p_{m-1}, \chi] & \text{if } n = 2m. \end{cases}$$

(ii) T_2 is an elementary 2-group (i.e., each element is of order 2) and is mapped isomorphically onto the image of $Sq^1: H^{*-1}(BSO(n); \mathbf{Z}_2) \rightarrow H^*(BSO(n); \mathbf{Z}_2)$ by the mod 2 reduction $\rho_2: H^*(BSO(n)) \rightarrow H^*(BSO(n); \mathbf{Z}_2)$.

PROOF. First consider (2). Applying the results in Lemma 1.15 to Theorem 5.16 we see that $H^*(BSO(n))$ is p -torsion free for $p \neq 2$, that $F \subset H^*(BSO(n))$ for F in (i), and that $H^*(BSO(n))/F = (2 \text{ torsion})$. If one can show

$$H_\beta(H^*(BSO(n); \mathbf{Z}_2)) \cong F \otimes \mathbf{Z}_2 \quad (*)$$

with respect to $\beta = \beta_2 = Sq^1$ by Lemma 1.15(3), one sees that $H^*(BSO(n)) = F \oplus T_2$ where T_2 is an elementary 2-group. Then (ii) is obtained directly from the Bockstein exact sequence.

Putting $j = 1$ in the Wu formula (Theorem 5.2(1)), one gets

$$\beta(w_i) = Sq^1(w_i) = w_i w_1 + (i+1)w_{i+1}. \quad (5.18)$$

In $H^*(BSO(n); \mathbf{Z}_2) = \mathbf{Z}_2[w_2, \dots, w_n]$ where $w_1 = 0$ one gets

$$\beta(w_{2j}) = w_{2j+1}, \quad \beta(w_{2j+1}) = 0 \text{ for } j \geq 1. \quad (5.18')$$

Thus if $n = 2m+1$, then by using the Künneth formula one has

$$H_\beta(H^*(BSO(n); \mathbf{Z}_2)) = H_\beta(\otimes \mathbf{Z}_2[w_{2j}, w_{2j+1}]) \cong H_\beta(\mathbf{Z}_2[w_{2j}, \beta(w_{2j})]).$$

Here, since one has $\beta(x^{2i}(\beta x)^j) = 0$ and $\beta(x^{2i+1}(\beta x)^j) = x^{2i}(\beta x)^{j+1}$ in $Z_2[x, \beta x]$, one obtains $H_\beta(Z_2[x, \beta x]) \cong Z_2[x^2]$. Thus

$$H_\beta(H^*(BSO(2m+1); Z_2)) \cong Z_2[w_{2j}^2] \cong Z_2[p_1, \dots, p_m] \cong F \otimes Z_2,$$

$$H_\beta(H^*(BSO(2m); Z_2)) \cong H_\beta(H^*(BSO(2m-1); Z_2)) \otimes Z_2[w_{2m}]$$

$$\cong Z_2[p_1, \dots, p_{m-1}, \chi] \cong F \otimes Z_2.$$

As for (1), denoting by χ' the Euler class, one has $\rho_r^*(\chi') = \pm c_m$ and $\chi' \equiv w_{2m} \pmod{2}$ and hence, if one puts $\chi = \pm \chi'$, it is unique by (2). Q.E.D.

We will call χ determined by this theorem the *Euler class* in $BSO(2m)$.

§6. Cohomology of symmetric spaces of classical type

On symmetric pairs and symmetric spaces. Let G be a (connected) Lie group and K a subgroup.

DEFINITION. (G, K) is called a *symmetric pair* if there is an automorphism $\sigma: G \rightarrow G$ such that $\sigma^2 = 1_G$ and $G^\sigma \supset K \supset (G^\sigma)_e = (\text{the component of the identity } e)$, where $G^\sigma = \{g \in G | \sigma(g) = g\}$.

M is called a (compact) *symmetric space* if M is a connected (compact) Riemannian manifold and there is an isometric transformation $\sigma_x: M \rightarrow M$ for each $x \in M$ such that $\sigma_x^2 = 1_M$ and $(d\sigma_x)_x: T_x M \rightarrow T_x M$ is a point symmetry ($v \mapsto -v$).

In a symmetric space M , σ_x reflects a geodesic through x with x at the center. If we denote by x the midpoint of two points y and z on a geodesic, then $\sigma_x(y) = z$. Thus the Lie group $I(M)$ of all isometric transformations of M (which are known to form a Lie group) acts on M transitively, and so $M \cong I(M)/K_p$, where $K_p = \{g \in I(M) | g(p) = p\}$. In addition, if one defines

$$\sigma: I(M) \rightarrow I(M) \quad \text{by} \quad \sigma(g) = \sigma_p g \sigma_p^{-1}$$

for $\sigma_p: M \rightarrow M$, then obviously $\sigma^2 = 1$. We have $K_p \subset I(M)^\sigma$. In fact, $g \in K_p$ iff $g(p) = p$, which implies $\sigma(g)(p) = \sigma_p g \sigma_p^{-1}(p) = p$, so $\sigma(g) \in K_p$. Conversely, $g \in I(M)^\sigma$ implies $\sigma_p(g(p)) = \sigma_p g \sigma_p^{-1}(p) = \sigma(g)(p) = g(p)$. Here, if G is close enough to the identity e , then $g(p)$ is fixed by σ_p and close to p , and hence $g(p) = p$, which implies $g \in K_p$. It follows that:

6.1. *For a given symmetric space M , there is a symmetric pair (G, K) such that $M \cong G/K$.*

Conversely, if one gives to G in a symmetric pair (G, K) a G -invariant Riemannian metric and introduces in G/K the Riemannian metric induced from it, then the action of G on G/K is an isometric transformation. One can define an isometric transformation $\sigma_x: G/K \rightarrow G/K$ by

$$\sigma_x(y) = (g\sigma g^{-1})y \quad (y \in G/K)$$

for each point $x = gK$ of G/K .

Obviously $\sigma_x^2 = 1_{G/K}$. In addition, using dL_g the behaviour of $d\sigma_x$ is reduced to that of $d\sigma_e$, and $(d\sigma_e)_e: T_e(G/K) \rightarrow T_e(G/K)$ has a fixed point only at the origin and must be a point symmetry, since $(d\sigma_e)^2 = 1$. Thus we have

6.2. For a given symmetric pair (G, K) , G/K is a symmetric space.

If one defines $\sigma(g, h) = (h, g)$ in the direct product $G \times G$ of a Lie group G , one has $G^\sigma = \{(g, g) | g \in G\}$ and $(G \times G)/G^\sigma \cong G$ (a diffeomorphism). Thus

6.3. A Lie group is a symmetric space.

The direct product of two symmetric spaces is obviously a symmetric space. A symmetric space that is not decomposable into the direct product of symmetric spaces is said to be *irreducible*. If in a covering space $(\tilde{M}, p, M)$ M is a symmetric space, then so is $\tilde{M}$.

E. Cartan classified 1-connected, irreducible symmetric spaces $M = G/K$. Here G is a simple Lie group. A symmetric space G/K is said to be of *classical type* if G is a classical group. The *irreducible symmetric spaces of classical type* $M = G/K$ are classified into the following 8 types (where the notation on the left side is due to E. Cartan):

- (AI) $SU(n)/c(SO(n)); \sigma(A) = \bar{A}$;
- (AII) $SU(2n)/c'(Sp(n)); (\sigma(A) = J_n A J_n^{-1}, J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix})$;
- (AIII) $U(p+q)/U(p) \times U(q); \sigma(A) = J A J^{-1}, J = (-I_p) \oplus I_q$;
- (BDI) $SO(p+q)/SO(p) \times SO(q); \sigma(A) = J A J^{-1}, J = (-I_p) \oplus I_q$;
- (BDII) $SO(p+1)/SO(p)$ ($q = 1$ in the above);
- (CI) $Sp(n)/q(U(n)); \sigma(A) = (iI_n)A(-iI_n)$;
- (CII) $Sp(p+q)/Sp(p) \times Sp(q); \sigma(A) = J A J^{-1}, J = (-I_p) \oplus I_q$;
- (DIII) $SO(2n)/r(Sp(n)); \sigma(A) = J_n A J_n^{-1}, J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

We will discuss the cohomology rings of these spaces below. Notice that properties of symmetric spaces are not necessary for the calculations.

Relations among the cohomology rings of classical groups, classifying spaces, and homogeneous spaces. The domains of definition of the transgression τ and the suspension σ

$$H^i(F; R) \xrightleftharpoons[\sigma]{\tau} H^{i+1}(B; R) \quad (i > 0)$$

in a fibration (E, p, B) with fibre F are restricted and their values are determined only modulo a suitable subgroup. However so long as they are

defined they are inverse to each other and satisfy the naturality condition

$$\begin{array}{ccc} H^i(F; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(B; R) \\ \uparrow \mathcal{F} & & \uparrow \mathcal{F} \\ H^i(F'; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(B; R) \end{array} \quad \begin{pmatrix} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow f & & \downarrow f & & \downarrow f \\ F & \xrightarrow{i'} & E' & \xrightarrow{p'} & B' \end{pmatrix} \quad (6.4)$$

with respect to a fibre map $(f, \tilde{f})$ (the commutativity of the above diagram).

Commutative diagrams of this kind associated with classifying spaces are shown below (cf. §6 of Chapter II):

$$\begin{array}{ccc} H^i(H; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(BH; R) \\ \uparrow i^* & & \uparrow \rho^* \\ H^i(G; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(BG; R) \end{array} \quad \begin{pmatrix} H & \rightarrow & EH & \rightarrow & BH \\ \downarrow i & & \parallel & & \downarrow \rho \\ G & \rightarrow & EG & \rightarrow & BG \end{pmatrix} \quad (6.4.1)$$

where $\rho = Bi$.

$$\begin{array}{ccc} H^i(G; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(BG; R) \\ \uparrow p^* & & \parallel \\ H^i(G/H; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(BG; R) \end{array} \quad \begin{pmatrix} G & \rightarrow & EG & \rightarrow & BG \\ \downarrow p & & \downarrow & & \parallel \\ G/H & \rightarrow & BH & \rightarrow & BG \end{pmatrix} \quad (6.4.2)$$

where $BH = EG/H$.

$$\begin{array}{ccc} H^i(H; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(BH; R) \\ \uparrow i^* & & \parallel \\ H^i(G; R) & \xrightleftharpoons[\sigma]{\tau} & H^{i+1}(BH; R) \end{array} \quad \begin{pmatrix} H & \rightarrow & EG & \rightarrow & BH \\ \downarrow i & & \downarrow i_1 & & \parallel \\ G & \xrightarrow{\pi} & G/H & \xrightarrow{p'} & BH \end{pmatrix} \quad (6.4.3)$$

where $G/H \cong EG \times G/H$.

THEOREM 6.5. (1) $H^*(U(n)) = \Lambda(e_1, e_3, \dots, e_{2n-1})$,

$H^*(SU(n)) = \Lambda(e_3, e_5, \dots, e_{2n-1})$, where $e_{2i-1} = \sigma(c_i) \in H^{2i-1}$.

(2) $H^*(Sp(n)) = \Lambda(e_3, e_7, \dots, e_{4n-1})$, where $e_{4i-1} = \sigma(q_i) \in H^{4i-1}$.

(3) If $\mathbf{k}$ is a field of characteristic $\neq 2$,

$$H^*(SO(2m+1); \mathbf{k}) = \Lambda(e_3, e_7, \dots, e_{4m-1}),$$

$$H^*(SO(2m); \mathbf{k}) = \Lambda(e_3, e_7, \dots, e_{4m-5}, s_{2m-1}),$$

where $e_{4i-1} = \sigma(p_i) \in H^{4i-1}$, $s_{2m-1} = \sigma(\chi) \in H^{2m-1}$.

(4) The homomorphism induced from the inclusion $i: G(n-1) \rightarrow G(n)$, where $G = U, SU, Sp, SO$, maps a generator to the one with the same symbol and to 0 if there is no corresponding symbol. The elements e_{2n-1}, e_{4n-1} ,

s_{2m-1} ($n = 2m$) are the images of a generator of $H^{dn-1}(S^{dn-1})$ under p^* for the projection $p: G(n) \rightarrow S^{dn-1}$.

PROOF. The first half of (4) is obtained from the naturality of σ (cf. (5.13')). The latter half of (4) is obtained from the Serre exact sequence. (1) and (2) are inductively obtained from the triviality of the spectral sequence associated with $G(n-1) \rightarrow G(n) \rightarrow S^{dn-1}$ for $G = U, SU, Sp$. Similarly for (3) using $SO(2m-1) \rightarrow SO(2m) \rightarrow S^{2m-1}$ and $SO(2m-1) \rightarrow SO(2m+1) \rightarrow V_{2m+1,2}(\mathbb{R})$. Q.E.D.

THEOREM 6.6. Let $\mathbf{k}$ be a field of characteristic 2.

(1) For $e_i = \sigma(w_{i+1}) \in H^i(SO(n); \mathbf{k})$, one has

$$H^*(SO(n); \mathbf{k}) = \Delta(e_1, e_2, \dots, e_{n-1}).$$

(2) $e_i^2 = e_{2i}$, ($e_i^2 = 0$ if $2i \geq n$), hence

$$H^*(SO(n); \mathbf{k}) = \mathbf{k}[e_1, e_3, \dots, e_{2m-1}]/(e_1^{a_1}, \dots, e_{2m-1}^{a_m}) \quad \text{for } m = [n/2],$$

where a_i is the least integer such that $2^{a_i}(2i-1) \geq n$.

(3) If $\mathbf{k} = \mathbb{Z}_2$, then $Sq^j e_i = \binom{i}{j} e_{i+j}$ for $0 \leq j \leq i$.

PROOF. (1) is similar to Theorem 6.5. The suspension σ commutes with Sq^j and satisfies $\sigma(xy) = 0$ for elements x and y of positive degree. By the Wu formula one has

$$\begin{aligned} Sq^j e_i &= Sq^j(\sigma(w_{i+1})) = \sigma(Sq^j w_{i+1}) \\ &= \sigma\left(\binom{i}{j} w_{i+j+1} + \binom{i-1}{j-1} w_{i+j} w_1 + \dots\right) = \binom{i}{j} e_{i+j} \quad \text{for } 0 \leq j \leq i, \end{aligned}$$

which implies (3). In particular one has (2), since $e_i^2 = Sq^i e_i = e_{2i}$. Q.E.D.

THEOREM 6.7. (1) The homomorphism c^* induced from the inclusion $c': Sp(n) \subset SU(2n) \subset U(2n)$ is an epimorphism and satisfies $c^*(e_{4i-3}) = 0$ and $c^*(e_{4i-1}) = (-1)^i e_{4i-1}$ for the generators given in (1), (2) of Theorem 6.5. The suspension images $e_{4i-3} = \sigma(c_{2i-1}) \in H^{4i-3}(U(2n)/Sp(n))$ of $c_{2i-1} \in H^{4i-2}(BU(2n))$ are uniquely determined by $p^*(e_{4i-3}) = e_{4i-3} \in H^{4i-3}(U(2n))$ for the projection $p: U(2n) \rightarrow U(2n)/Sp(n)$ and

$$H^*(U(2n)/Sp(n)) = \Lambda(e_1, e_5, e_9, \dots, e_{4n-3}).$$

The result for $SU(n)/Sp(n)$ is obtained if one deletes e_i .

(2) Let $\mathbf{k}$ be a field of characteristic $\neq 2$. The homomorphism i^* induced from the inclusion $i: SO(n) \subset SU(n) \subset U(n)$ satisfies $i^*(e_{4i-3}) = 0$ and $i^*(e_{4i-1}) = (-1)^i e_{4i-1}$ for the generators given in (1), (3) of Theorem 6.5, and there is $e_{2m} \in H^{2m}(U(2m)/SO(2m); \mathbf{k})$ satisfying $i^*(e_{2m}) = \chi$, if $n = 2m$. The suspension images $e_{4i-3} = \sigma(c_{2i-1}) \in H^{4i-3}(U(n)/SO(n); \mathbf{k})$

of $c_{2i-1} \in H^{4i-2}(BU(n); \mathbf{k})$ are uniquely determined by $p^*(e_{4i-3}) = e_{4i-3} \in H^{4i-3}(U(n); \mathbf{k})$ and

$$H^*(U(n)/SO(n); \mathbf{k}) = \begin{cases} \Lambda(e_1, e_5, e_9, \dots, e_{4m+1}), & \text{if } n = 2m+1, \\ \Lambda(e_1, e_5, \dots, e_{4m-2}) \otimes \Delta(e_{2m}), & \text{if } n = 2m. \end{cases}$$

The result for $SU(n)/SO(n)$ is obtained if one deletes e_1 .

(3) Let $\mathbf{k}$ be a field of characteristic 2. The inclusion i and the projection p in (2) induce

$$i^* = 0: H^*(U(n); \mathbf{k}) \rightarrow H^*(SO(n); \mathbf{k})$$

and

$$p^*: H^*(U(n)/SO(n); \mathbf{k}) \rightarrow H^*(U(n); \mathbf{k}).$$

If one puts $e_i = i^*(w_i) \in H^i(U(n)/SO(n); \mathbf{k})$ for the inclusion $i: U(n)/SO(n) \rightarrow BSO(n)$, one has

$$H^*(U(n)/SO(n); \mathbf{k}) = \Lambda(e_1, e_2, \dots, e_n).$$

The result for $SU(n)/SO(n)$ is obtained if one deletes e_1 .

PROOF. (1) The result for c^* is obtained from Theorem 5.8(2), taking $(G, H) = (U(2n), Sp(n))$ in (6.4.1), and (6.4.2) implies $p^*(e_{4i-3}) = e_{4i-3} = \sigma(c_{2i-1})$. We have $E_2 = \mathbf{Z}_2[q_1, \dots, q_n] \otimes \Lambda(\sigma(c_1), \dots, \sigma(c_n))$ in the spectral sequence associated with the fibre bundle

$$U(n) \xrightarrow{i} EU(2n) \times_{Sp(n)} U(2n) \rightarrow BSp(n).$$

By (6.4.3), $d_{4i}(1 \otimes \sigma(c_{2i})) = q_i \otimes 1$ and $d_r(1 \otimes \sigma(c_{2i-1})) = 0$ imply $E_\infty = \Lambda(\sigma(c_1), \sigma(c_3), \dots, \sigma(c_{2i-1}))$.

The inclusion i is equivalent to

$$p: U(2n) \rightarrow U(2n)/Sp(n)$$

so there is e_{4i-3} such that $p^*(e_{4i-3}) = \sigma(c_{2i-1})$ and $H^*(U(2n)/Sp(n)) = \Lambda(e_1, e_5, \dots, e_{4n-3})$. By (6.4.2) the element e_{4i-3} is the suspension image of c_{2i-1} . Obviously p^* is a monomorphism and so e_{4i-3} is unique.

(2) The case $n = 2m + 1$ is dealt with similarly to (1). For the case $n = 2m$; E_2 in (1) is replaced by

$$E_2 = \mathbf{k}[p_1, \dots, p_{m-1}, \chi] \otimes \Lambda(\sigma(c_1), \dots, \sigma(c_{2n-1})).$$

Noticing that $d_{4n}(1 \otimes \sigma(c_{2n})) = p_m \otimes 1 = \chi^2 \otimes 1$, we have

$$H^*(U(2m)/SO(2m); \mathbf{k}) = \Lambda(e_1, e_5, \dots, e_{4m-3}) \otimes \Delta(e_{2m})$$

with $i^*(e_{2m}) = \chi$.

(3) Consider the spectral sequence

$$E_2 = H^*(BSO(n); \mathbf{k}) \otimes H^*(U(n); \mathbf{k}) \Rightarrow H^*(U(n)/SO(n); \mathbf{k}),$$

similarly to (1). By (6.4.3) we have $\tau(\sigma(c_i)) = \rho_c^*(c_i)$ in

$$E_2 = \mathbf{k}[w_1, \dots, w_n] \otimes \mathbf{k}[\sigma(c_1), \dots, \sigma(c_n)],$$

where we have $\tau(\sigma(c_i)) = w_i^2$ by using Theorem 5.11(1), that is,

$$d_{2i}(1 \otimes \sigma(c_i)) = w_i^2 \otimes 1.$$

Using this, $E_\infty = \Lambda(w_1, \dots, w_n)$, which implies $H^*(U(n)/SO(n); \mathbf{k}) = \Delta(e_1, e_2, \dots, e_n)$. $\rho_c \circ \iota = 0$ (constant map), so $0 = (\iota^* \circ \rho_c^*)(c_i) = \iota^*(w_i^2) = e_i^2$. Thus $H^*(U(n)/O(n); \mathbf{k}) = \Lambda(e_1, \dots, e_n)$. Q.E.D.

The cohomology ring of G/K (where K is of maximal rank)

DEFINITION. Let T^n be the standard maximal torus of G . A closed subgroup K of G is said to be of *maximal rank* if there exists $g \in G$ such that $T = g^{-1}T^n g \subset K$.

In this case the following two fibre bundles are often used:

$$K/T \xrightarrow{i} G/T \xrightarrow{p} G/K, \quad G/K \xrightarrow{\iota} BK \xrightarrow{p} BG. \quad (6.8)$$

THEOREM 6.9. (1) $\iota^*: H^*(BU(n)) \rightarrow H^*(Sp(n)/U(n))$ is an epimorphism inducing an isomorphism:

$$H^*(Sp(n)/U(n)) \cong \mathbf{Z}[c_1, \dots, c_n] / \left(\sum_{i+j=2k} (-1)^i c_i c_j; k \geq 1 \right).$$

(2) $\iota^*: H^*(B(U(m) \times U(n))) \rightarrow H^*(U(m+n)/U(m) \times U(n))$ is an epimorphism inducing an isomorphism:

$$H^*(U(m+n)/U(m) \times U(n)) \cong (\mathbf{Z}[c_1, \dots, c_m] \otimes \mathbf{Z}[c_1, \dots, c_n]) / \left(\sum_{i+j=k} c_i \otimes c_j; k \geq 1 \right).$$

(3) $\iota^*: H^*(B(Sp(m) \times Sp(n))) \rightarrow H^*(Sp(m+n)/Sp(m) \times Sp(n))$ is an epimorphism inducing an isomorphism:

$$H^*(Sp(m+n)/Sp(m) \times Sp(n)) \cong (\mathbf{Z}[q_1, \dots, q_m] \otimes \mathbf{Z}[q_1, \dots, q_n]) / \left(\sum_{i+j=k} q_i \otimes q_j; k \geq 1 \right).$$

PROOF. (1) By the commutative diagram

$$\begin{array}{ccc} U(n)/T^n & \xrightarrow{i} & BT^n \\ \downarrow i & & \parallel \\ Sp(n)/T^n & \xrightarrow{i} & BT^n \end{array}$$

and by the epimorphism $\iota^*: H^*(BT^n) \rightarrow H^*(U(n)/T^n)$ (Theorem 5.5(3)) we see that

$$\iota^*: H^*(Sp(n)/T^n) \rightarrow H^*(U(n)/T^n)$$

is an epimorphism. By Theorem 4.4,

$$p^*: H^*(Sp(n)/U(n)) \rightarrow H^*(Sp(n)/T^n)$$

is a monomorphism. By Theorem 4.6, $H^*(Sp(n)/U(n))$ is free and 0 in odd degrees. Since the same holds for $H^*(BSp(n))$ (Theorem 5.6), the spectral sequence associated with $Sp(n)/U(n) \xrightarrow{i} BU(n) \xrightarrow{p} BSp(n)$ collapses by Lemma 4.5(1) and hence, applying Theorem 4.2, we have $H^*(Sp(n)/U(n)) \cong H^*(BU(n)/(\text{Im } \rho^+)) = \mathbb{Z}[c_1, \dots, c_m]/(\rho^*(q_k); k \geq 1)$, where $\rho^*(q_k) = (-1)^k \sum_{i+j=2k} (-1)^i c_i c_j$ (Theorem 5.8(1)).

The proofs of (2) and (3) are similar. Q.E.D.

LEMMA 6.10. (1) Consider the exact sequence of a pair $(X, A) \dots \xrightarrow{\delta^*} \xrightarrow{j^*} \xrightarrow{i^*} \dots$, the mod p reduction $\rho_p: H^n(X) \rightarrow H^n(X; \mathbb{Z}_p)$, and the Bockstein homomorphism $\beta_p: H^n(X, A; \mathbb{Z}_p) \rightarrow H^{n+1}(X, A; \mathbb{Z}_p)$. For $x \in H^n(X, A; \mathbb{Z}_p)$ and $y \in H^n(X)$ such that $\rho_p(y) = j^*(x) \in H^n(X, A; \mathbb{Z}_p)$ there exists $z \in H^n(A)$ such that

$$i^*(y) = p \cdot z, \quad \delta^* \rho_p(z) = \rho_p \delta^*(z) = -\beta_p(x).$$

(2) Let (E, p, B) be a fibration. If $x \in H^n(B; \mathbb{Z}_p)$ and $y \in H^n(E)$ satisfy $p^*(x) = \rho_p(y) \in H^n(E; \mathbb{Z}_p)$, then there is $z \in H^n(F)$ with $\rho_p(z)$ transgressive such that

$$i^*(y) = p \cdot z \quad \text{and} \quad \tau(\rho_p(z)) = -\beta_p(x).$$

PROOF. (1) Consider the exact sequence $0 \rightarrow C^*(X, A) \xrightarrow{j^\#} C^*(X) \xrightarrow{i^\#} C^*(A) \rightarrow 0$ of cochain complexes. If $\bar{x} \in C^n(X, A)$ and $\bar{y} \in C^n(X)$ are representatives of x and y respectively, then by the assumption $j^\#(\bar{x}) = \bar{y} - p \cdot y_1 + \delta y_2$, where $y_1 \in C^n(X)$, $y_2 \in C^{n-1}(X)$, so $0 = i^\# j^\#(\bar{x}) = i^\# \bar{y} - p \cdot i^\# y_1 + \delta i^\# y_2$. $\delta y = 0$ implies $0 = p \cdot \delta(i^\# y_1)$ so $\delta(i^\# y_1) = 0$. Therefore if we denote the class of $i^\# y_1$ by $z \in H^n(A)$, then $i^*(y) = p \cdot z$. On the other hand, $\delta^*(z)$ is represented by $\bar{z} \in C^{n+1}(X, A)$ satisfying $j^\# \bar{z} = \delta y_1$, and $\beta_p(x)$ is by $\bar{w} \in C^{n+1}(X, A)$ satisfying $\delta \bar{x} = p \bar{w}$, where $j^\#(p \cdot \bar{w}) = j^\# \delta \bar{x} = \delta j^\# \bar{x} = \delta \bar{y} - p \cdot \delta y_1 = -j^\#(p \cdot \bar{z})$. Since $j^\#$ is a monomorphism and $C^*(X, A)$ is torsion free, $j^\#(p \cdot \bar{w}) = -j^\#(p \cdot \bar{z})$, which implies $\bar{w} = -\bar{z}$, that is, $\delta^* \rho_p(z) = \rho_p \delta^*(z) = -\beta_p(x)$.

(2) Considering the following commutative diagram, we obtain the equality $\rho_p(p_0^* y) = j^*(p^* x)$.

$$\begin{array}{ccccccc} H^{*-1}(F; \mathbb{Z}_p) & \xrightarrow{\delta^*} & H^*(E, F; \mathbb{Z}_p) & \xrightarrow{j^*} & H^*(E; \mathbb{Z}_p) & \xrightarrow{i^*} & \\ & \searrow \tau & \uparrow p^* & & \nearrow p_0^* & & \\ & & H^*(B; \mathbb{Z}_p) & & & & \end{array}$$

Applying (1) to it, we obtain $z \in H^n(X)$ such that $i^*(y) = p \cdot z$ and $\delta^*(\rho_p z) = -\beta_p(p^*x) = \rho^*(-\beta_p(x))$, that is, $\tau(\rho_p z) = -\beta_p(x)$. Q.E.D.

THEOREM 6.11. For $i^*: H^*(BU(n)) \rightarrow H^*(SO(2n)/U(n))$ there exists $e_{2i} \in H^{2i}(SO(2n)/U(n))$ satisfying $i^*(c_i) = 2e_{2i}$ so that $e_{2i} = 0$ for $i \geq n$ and

$$\begin{aligned} H^*(SO(2n)/U(n)) &= \mathbb{Z}[e_2, e_4, \dots, e_{2n-2}] / \left(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i} \right) \\ &= \Delta(e_2, e_4, \dots, e_{2n-2}) \quad (\text{a free module}). \end{aligned}$$

PROOF. For the same reason as in the proof of Theorem 6.9 $i^*: H^*(SO(2n)/T^n) \rightarrow H^*(U(n)/T^n)$ is an epimorphism. By Theorem 4.4, $p^*: H^*(SO(2n)/U(n)) \rightarrow H^*(SO(2n)/T^n)$ is a monomorphism. Using Theorem 4.9, $H^*(SO(2n)/U(n))$ is free without any elements of odd degree.

Let $\mathbf{k}$ be a field of characteristic $\neq 2$. If we use Lemma 5.15(2) and Theorem 5.17(1) instead of Theorem 5.8(1) in the proof of Theorem 6.9, we have

$$\begin{aligned} H^*(SO(2n)/U(n); \mathbf{k}) &\cong \mathbf{k}[c_1, \dots, c_n] / \left(\sum_{i+j=2k} (-1)^i c_i c_j \ (k \geq 1), c_n \right) \\ &\cong \mathbf{k}[c_1, \dots, c_{n-1}] / \left(\sum_{i+j=2k} (-1)^i c_i c_j \ (k \geq 1) \right). \end{aligned}$$

This shows that the quotient ring of $H^*(SO(2n)/U(n))$ by the subring generated by $i^*c_1, \dots, i^*c_{n-1}$ is of 2-torsion.

Next consider the case $\mathbf{k} = \mathbb{Z}_2$. We deduce $i^*(c_i) = 0$ from the fact that $i^* \circ \rho_r^* = 0: H^*(BSO(2n); \mathbb{Z}_2) \rightarrow H^*(BU(n); \mathbb{Z}_2) \rightarrow H^*(SO(2n)/U(n); \mathbb{Z}_2)$ and that $\rho_r^*(w_{2i}) = c_i$ (Theorem 5.11(2)). With integral coefficients this shows the existence of $e_{2i} \in H^{2i}(SO(2n)/U(n))$ such that $i^*(c_i) = 2e_{2i}$. If we use Lemma 6.10(2) here, we have $\tau(e_{2i}) = \beta_2(w_{2i}) = Sq^1 w_{2i} = w_{2i+1}$ (the Wu formula) for $e_{2i} \in H^{2i}(SO(2n)/U(n); \mathbb{Z}_2)$. If we use (6.4.1) and (6.4.2) we have $i^*(e_{2i-1}) = i^*(\sigma(w_{2i})) = \sigma(c_i) = e_{2i-1}$ and $p^*(e_{2i}) = \sigma(w_{2i+1}) = e_{2i}$ in the sequence

$$H^*(SO(2n)/U(n); \mathbb{Z}_2) \xrightarrow{p^*} \Delta(e_1, e_2, \dots, e_{2n-1}) \xrightarrow{i} \Lambda(e_1, e_3, \dots, e_{2n-1})$$

induced from $U(n) \xrightarrow{i} SO(2n) \xrightarrow{p} SO(2n)/U(n)$ where $i = r$.

Therefore we obtain

$$H^*(SO(2n)/U(n); \mathbb{Z}_2) = \Delta(e_2, e_4, \dots, e_{2n-2})$$

(Theorem 4.2).

Thus we have seen that the elements $e_{2i} = \frac{1}{2} \cdot i^*(c_i)$ multiplicatively generate $H^*(SO(2n)/U(n))$.

Since $H^*(SO(2n)/U(n))$ is torsion free, we have

$$\begin{aligned} 0 &= \sum_{i+j=2k} (-1)^i c_i c_j = 2e_{2k} + \sum_{i=1}^{2k-1} (-1)^i c_i c_{2k-i} \\ &= 4 \left(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i} \right), \end{aligned}$$

and hence $e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i} = 0$. This equality gives a formula to rewrite e_{2i}^2 as a linear combination of elementary monomials and shows that

$$H^*(SO(2n)/U(n)) = \Delta(e_2, e_4, \dots, e_{2n-2}).$$

We also see that we have enough relations. Q.E.D.

CHAPTER IV

The Periodicity of KF -Groups and Homotopy Groups

In this chapter we discuss the homotopy theory of classical groups and homogeneous spaces. In particular, the primary object is to prove the so-called Bott periodicity theorem without using Morse theory (Chapter VI) but with (co)homology theory as the main tool. We devote §1 to a review of homotopy theory. The reader should pay attention to the Serre $\mathcal{E}$ -theory and its application to homotopy theory. Although we discuss homology rings and Hopf algebras of H -spaces in §2, we only state the results needed in §3, since we return to them in Chapter VII. We define symmetric spaces of infinite dimension in §3 as certain limits of symmetric spaces, and we determine their (co)homology rings. In §4 we obtain KF -groups by the method of Grothendieck from the abelian semigroups formed by vector bundles, which are generalizations of vector spaces over a field F , and state their fundamental properties. After the above preliminaries we establish the periodicity of KF -groups in §5. In §6 we calculate some homotopy groups of classical groups and homogeneous spaces but those whose computation requires more advanced results are omitted.

§1. Fundamental notions of the homotopy theory

Homotopy groups and homology groups. The definition of the homotopy group $\pi_n(X, A)$ was given in Chapter II, §3 and that of the (integral) homology group $H_n(X, A)$ in Chapter III. Relating them we have the *Hurewicz homomorphism*

$$\Xi: \pi_n(X, A) \rightarrow H_n(X, A),$$

which is defined as follows: We fix a generator ι_n of $H_n(I^n, \partial I^n) \cong \mathbb{Z}$ and put

$$\Xi[f] = f_*(\iota_n) \in H_n(X, A) \quad (f_*: H_n(I^n, \partial I^n) \rightarrow H_n(X, A)),$$

for an element $[f] \in [I^n, \partial I^n, J^{n-1}; X, A, x_0] = \pi_n(X, A)$. Ξ is well defined since $[f] = [g]$ implies $f \simeq g$ and hence $f_* = g_*$. One can prove that Ξ is a homomorphism if $n > 2$ or if $n = 1$ and $A = x_0$, using the properties of homology groups.

Now suppose that (X, A) is k -connected (i.e., $\pi_0(A) = \pi_i(X, A) = 0$ for $0 \leq i \leq k$).

We define $C_n(X, A, k) = \{\text{the subgroup of } C_n(X) \text{ generated by } (u: (I^n, \text{the } k\text{-skeleton of } I^n) \rightarrow (X, A))\}$. Using the k -connectedness of (X, A) one can construct a homotopy u_t for arbitrary $u: I^n \rightarrow X$ such that $u_0 = u$ and $u_1 \in C_n(X, A, k)$. The homotopy u_t can be constructed by induction on n so that $u \in C_n(X, A, k)$ implies $u_t = u$, $(\partial_i^e u)_t = \partial_i^e(u_t)$, and it induces a chain homotopy. By putting $\eta(u) = u_1$, one obtains the following:

1.1. If (X, A) is k -connected, then the inclusion $i: C_*(X, A, k) \rightarrow C_*(X, A)$ is a chain equivalence, i.e., $\eta \circ i = 1$ and $i \circ \eta \simeq 1$ for the chain map $\eta: C_*(X, A) \rightarrow C_*(X, A, k)$. So $C_*(X, A, k)/C_*(A) \rightarrow C_*(X)/C_*(A)$ is also a chain equivalence.

Obviously $C_n(X, A, k) = C_n(A)$ for $n < k$, which implies the following:

1.2. If (X, A) is k -connected, then

$$H_n(X, A; G) = H^n(X, A; G) = 0 \quad (n < k)$$

for any coefficient group G .

The situation when $n = k + 1$ and $G = \mathbb{Z}$ is the following Hurewicz theorem:

THEOREM 1.3. Let (X, A) be k -connected.

(1) $\Xi: \pi_{k+1}(X, A) \rightarrow H_{k+1}(X, A)$ is an epimorphism.

(2) If (X, A) is $(k + 1)$ -simple and $k > 1$, then $\Xi: \pi_{k+1}(X, A) \rightarrow H_{k+1}(X, A)$ is an isomorphism.

(3) If X is arcwise connected, $\Xi: \pi_1(X) \rightarrow H_1(X)$ is an epimorphism with kernel $\text{Ker } \Xi = [\pi_1(X), \pi_1(X)]$ (the commutator group), that is $H_1(X)$ is the abelianization of $\pi_1(X)$.

We omit the proof of the theorem—the proofs for (1) and (3) are relatively easy. One can directly show (2) by using the homotopy addition theorem except for the case $\pi_1(A) = 0$, which will be proved in a general mod $\mathcal{C}$ form later as an application of the homology spectral sequence based on (3).

Let X and Y be arcwise connected and consider the following two conditions on $f: X \rightarrow Y$:

$$\begin{aligned} f_*: \pi_i(X) \rightarrow \pi_i(Y) \text{ is an isomorphism for } i \leq k \\ \text{and an epimorphism for } i = k + 1 \end{aligned} \quad (1.4.1)_k$$

$$\begin{aligned} f_*: H_i(X) \rightarrow H_i(Y) \text{ is an isomorphism for } i \leq k \\ \text{and an epimorphism for } i = k + 1. \end{aligned} \quad (1.4.2)_k$$

THEOREM 1.5 (J. H. C. Whitehead). If $f: X \rightarrow Y$ in the above satisfies either

$$(\alpha) \quad \pi_1(X) = 0 \text{ or}$$

(β) X and Y are H -spaces and f is an H -map (i.e., f preserves multiplication in the homotopy sense), then the two conditions $(1.4.1)_k$ and $(1.4.2)_k$ are equivalent.

PROOF. We may suppose that f is the inclusion, by replacing Y with the mapping cylinder of f . From the exact sequences of homotopy and homology groups, one has

$(1.4.1)_k$ is equivalent to $(1.5.1)_k$: $\pi_i(Y, X) = 0$ for $i \leq k+1$,

$(1.4.2)_k$ is equivalent to $(1.5.2)_k$: $H_i(Y, X) = 0$ for $i \leq k+1$.

The following is shown inductively, using Theorem 1.3.

$(1.5)'$. If (Y, X) is simple, then $(1.5.1)_k$ and $(1.5.2)_k$ are equivalent.

If the assumption (α) is satisfied, (Y, X) is obviously simple and so the theorem is proved. If (β) is satisfied, the simplicity of (Y, X) is also provable (the details are omitted). Q.E.D.

Combining this with 6.3(1) of Chapter II we obtain the following corollary.

COROLLARY 1.6. Suppose that $f: X \rightarrow Y$ satisfies the assumption (α) or (β) in Theorem 1.5 and that X and Y are homotopy equivalent to CW -complexes. Then the following conditions are equivalent:

(1) f is a homotopy equivalence.

(2) f is a weak homotopy equivalence, i.e., $f_*: \pi_i(X) \cong \pi_i(Y)$ for all i .

(3) $f_*: H_*(X) \cong H_*(Y)$.

Serre's $\mathcal{E}$ -theory

DEFINITION. A set $\mathcal{E}$ of isomorphism classes of abelian groups is called a (Serre) class if the following axiom (I) is satisfied:

(I) (1) $\mathcal{E} \neq \emptyset$;

(2) If $A, B \in \mathcal{E}$, and $A \rightarrow C \rightarrow B$ is exact then $C \in \mathcal{E}$.

Axiom (I) is equivalent to the following:

(I') (1) $0 = \{e\} \in \mathcal{E}$;

(2) If $A \subset B$ and $B \in \mathcal{E}$ then $A \in \mathcal{E}$ and $B/A \in \mathcal{E}$;

(3) If $A, B \in \mathcal{E}$ and $0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0$ is exact then $C \in \mathcal{E}$.

DEFINITION. Let $f: A \rightarrow B$ be a homomorphism of abelian groups.

f is called a $\mathcal{E}$ -epimorphism if $\text{Coker } f = B/\text{Im } f \in \mathcal{E}$.

f is called a $\mathcal{E}$ -monomorphism if $\text{Ker } f \in \mathcal{E}$.

f is called a $\mathcal{E}$ -isomorphism if f is a $\mathcal{E}$ -epimorphism as well as a $\mathcal{E}$ -monomorphism.

Abelian groups B_1 and B_2 are called $\mathcal{E}$ -isomorphic if for some A there are $\mathcal{E}$ -isomorphisms $f_i: A \rightarrow B_i$ for $i = 1, 2$.

Further we consider the following axioms:

(II_A) If $A, B \in \mathcal{E}$ then $A \otimes B \in \mathcal{E}$, $\text{Tor}(A, B) \in \mathcal{E}$.

(II_B) If $A \in \mathcal{E}$ then $A \otimes B \in \mathcal{E}$ for any abelian group B .

(III) If $A \in \mathcal{E}$ then $H_i(A)$ (the homology of a group) $\in \mathcal{E}$ for $i > 0$.

Usually the homology $H_i(A)$ of a group A is algebraically defined but it is known that it is isomorphic with the homology $H_i(K(A, 1))$ of an Eilenberg-Mac Lane complex $K(A, 1)$ defined below. In this book we define $H_i(A) = H_i(K(A, 1))$.

DEFINITION. Let G be a group and n a positive integer. X is called an Eilenberg-Mac Lane space of type (G, n) if $\pi_i(X) = 0$ for $i \neq n$ and $\pi_n(X) \cong G$. In particular, if X is a CW-complex, we call X an Eilenberg-Mac Lane complex of type (G, n) , and write $X = K(G, n)$.

Sometimes we take $K(G, 0) = G$. If $n > 1$, G must be an abelian group. The proof of the following theorem is omitted.

THEOREM 1.7. Let G be an abelian group and n a positive integer.

(1) There exists a $K(G, n)$ such that there is a weak homotopy equivalence $K(G, n) \rightarrow X$ for any Eilenberg-Mac Lane space X of type (G, n) . Thus $K(G, n)$ is unique up to homotopy equivalence.

(2) $K(G, n)$ is homotopy equivalent to $\Omega(K(G, n+1))$ and hence $K(G, n)$ is a homotopy commutative H -space.

(3) Let $u_n \in H^n(K(G, n); G) \cong \text{Hom}(H_n(K(G, n)), G)$ be the class determined by the inverse homomorphism

$$H_n(K(G, n)) \rightarrow \pi_n(K(G, n)) \cong G$$

of the Hurewicz isomorphism (which we call the fundamental class). For any CW-complex X the assignment $[f] \mapsto f^*(u_n)$ gives rise to an isomorphism $[X, K(G, n)] \cong H^n(X; G)$.

(4) The set of cohomology operations Φ of type $(n, G, n+q, H)$ is in one to one correspondence with $H^{n+q}(K(G, n); H)$ under the assignment $\Phi \mapsto \Phi(u_n)$.

NOTATION.

$\mathcal{F}$ denotes the class of finitely-generated abelian groups;

$\mathcal{O}$ denotes the class consisting only of $\{e\}$;

$\mathcal{E}_0$ denotes the class of finite abelian groups;

$\mathcal{E}_p$ denotes the class of finite abelian groups without p -primary components (p a prime);

$\mathcal{T}_0$ denotes the class of abelian groups whose elements are of finite order (torsion groups);

$\mathcal{T}_p$ denotes the class of abelian groups whose elements are of finite order prime to p .

1.8. (1) $\mathcal{F}, \mathcal{O}, \mathcal{E}_0, \mathcal{E}_p, \mathcal{T}_0$, and $\mathcal{T}_p$ satisfy (II_A) and (III) .

(2) $\mathcal{F}, \mathcal{O}, \mathcal{T}_0$, and $\mathcal{T}_p$ satisfy (II_B) .

(3) $\mathcal{F} \cap \mathcal{T}_0 = \mathcal{E}_0, \mathcal{F} \cap \mathcal{T}_p = \mathcal{E}_p, \bigcap \mathcal{T}_p = \bigcap \mathcal{E}_p = \mathcal{O}$.

Observe that an $\mathcal{O}$ -epimorphism, $\mathcal{O}$ -monomorphism, etc. are an ordinary epimorphism, monomorphism, etc.

THEOREM 1.9. Suppose F is 0-connected, and the local coefficient group $H_*(F)$ is trivial in a fibration (E, p, B) . Let $B_0 \subset B$ and $E_0 = p^{-1}(B_0)$.

(1) Let $\mathcal{C}$ be a class satisfying (II_A) . If $H_i(B, B_0) = 0$ for $i = 0, 1$, $H_i(B, B_0) \in \mathcal{C}$ for $i < n$, and $H_j(F) \in \mathcal{C}$ for $0 < j < m$ then $p_*: H_k(E, E_0) \rightarrow H_k(B, B_0)$ is a $\mathcal{C}$ -isomorphism for $k \leq \text{Min}(n, m+1)$ and a $\mathcal{C}$ -epimorphism for $k = \text{Min}(n, m+1) + 1$.

(2) Let $\mathcal{C}$ be a class satisfying (II_B) . If $H_i(B, B_0) \in \mathcal{C}$ for $0 < i < n$ and $H_j(F) \in \mathcal{C}$ for $0 < j < m$ then $p_*: H_k(E, E_0) \rightarrow H_k(B, B_0)$ is a $\mathcal{C}$ -isomorphism for $k \leq m+n-1$ and a $\mathcal{C}$ -epimorphism for $k = m+n$.

PROOF. In the homology spectral sequence

$$E^2 = H_*(B, B_0; \underline{H_*(F)}) \Rightarrow H_*(E, E_0),$$

we have $E_{i,j}^2 = H_i(B, B_0) \otimes H_j(F) \oplus \text{Tor}(H_{i-1}(B, B_0), H_j(F))$. If we put $s = \text{Min}(n, m+1)$ in the case (1) and $s = m+n-1$ in the case (2), then for $i \neq 0$, $i+j \leq s$ by the assumption implies $E_{i,j}^2 \in \mathcal{C}$ and hence $E_{i,j}^r \in \mathcal{C}$. So, for $k \leq s+1$, $d^r(E_{k,0}^r) \subset E_{k-r,r-1}^r \in \mathcal{C}$, which implies $E_{k,0}^{r+1} \hookrightarrow E_{k,0}^r$ is a $\mathcal{C}$ -isomorphism and hence $E_{k,0}^\infty = H_k(E, E_0)/D_{k-1,1} \hookrightarrow E_{j,0}^2 = H_k(B, B_0)$ is a $\mathcal{C}$ -isomorphism. Thus, for $k \leq s$, $E_{k-j,j}^\infty \in \mathcal{C}$. Hence $D_{k-1,1} \in \mathcal{C}$, and the theorem is proved. Q.E.D.

COROLLARY 1.10. Let $\mathcal{C}$ be a class satisfying (II_A) .

(1) If X is 1-connected and $H_i(X) \in \mathcal{C}$ for $0 < i < m$, then $H_i(\Omega X) \in \mathcal{C}$ for $0 < i < m-1$ and the homology suspension $\Sigma: H_{m-1}(\Omega X) \rightarrow H_m(X)$ is a $\mathcal{C}$ -isomorphism, where $m > 1$.

(2) If $\mathcal{C}$ satisfies (III), then $A \in \mathcal{C}$ implies $H_i(K(A, n)) \in \mathcal{C}$ for $i > 0$.

PROOF. (1) is obtained by induction on m , applying Theorem 1.9(1) to the fibration $\Omega X \rightarrow PX \rightarrow X$. (2) is obtained from (1) and Theorem 1.7 by induction on n . Q.E.D.

Application of $\mathcal{C}$ -theory to homotopy groups

THEOREM 1.11 (The mod $\mathcal{C}$ Hurewicz theorem). (1) Let $\mathcal{C}$ be a class satisfying (II_A) and (III) and let X be 1-connected. If $H_i(X) \in \mathcal{C}$ for $0 < i < n$ then $\Xi: \pi_i(X) \rightarrow H_i(X)$ is a $\mathcal{C}$ -isomorphism for $i \leq n$.

(2) Let $\mathcal{C}$ be a class satisfying (II_B) and (III). Assume that $A \subset X$, A is 1-connected, and (X, A) is 2-connected. If $H_i(X, A) \in \mathcal{C}$ for $i \leq n$ then $\Xi: \pi_i(X, A) \rightarrow H_i(X, A)$ is a $\mathcal{C}$ -isomorphism for $i \leq n$.

PROOF. First we consider (2). In the fibration $\Omega X \rightarrow (PX, \Omega(X, A)) \rightarrow (X, A)$, $\Omega(X, A)$ is 1-connected and X is 1-connected hence ΩX is 0-connected. By Theorem 1.9(2) $p_*: H_k(PX, \Omega(X, A)) \rightarrow H_k(X, A)$ is a

$\mathcal{E}$ -isomorphism for $k \leq n$. By the commutativity of the diagram

$$\begin{array}{ccccc} \pi_i(X, A) & \xleftarrow[p_*]{\cong} & \pi_i(PX, \Omega(X, A)) & \xrightarrow[\cong]{\partial_*} & \pi_{i-1}(\Omega(X, A)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_i(X, A) & \xleftarrow[p_*]{} & H_i(PX, \Omega(X, A)) & \xrightarrow[\cong]{\partial_*} & H_{i-1}(\Omega(X, A)) \end{array}$$

one sees that (2) is reduced to (1).

Since $H_i(\Omega X) \in \mathcal{E}$ for $0 < i < n-1$ by Corollary 1.10(1),

$$p_*: H_i(PX, \Omega X) \rightarrow H_i(X, x_0)$$

is a $\mathcal{E}$ -isomorphism for $i \leq n$ by Theorem 1.9(1). (1) for the case $n = 2$ is the ordinary Hurewicz isomorphism $\pi_1(\Omega X) \cong H_1(\Omega X)$. If $n > 2$ and X is 2-connected, then (1) is reduced to the case one degree lower. Thus it suffices for the case $n > 2$ to show that X can be replaced by a 2-connected space. By attaching cells e_α^m for $m = 4, 5, \dots$ successively to X we kill the homotopy groups of dimensions ≥ 3 and construct an Eilenberg-Mac Lane space X^* of $(\pi_2(X), 2)$ type which contains X . $\pi_2(X) \cong H_2(X) \in \mathcal{E}$ implies $H_i(X^*) \in \mathcal{E}$ for $i > 0$ hence $H_i(\Omega X^*) \in \mathcal{E}$ for $i > 0$. Applying Theorem 1.9(1) to the fibration $\Omega X^* \rightarrow \Omega(X^*, X) \xrightarrow{p_1} X$ we see that $p_1^*: H_i(\Omega(X^*, X)) \rightarrow H_i(X)$ is a $\mathcal{E}$ -isomorphism for $i \leq n$. Thus (1) is reduced to the case where X is 2-connected. Q.E.D.

THEOREM 1.12 (The mod $\mathcal{E}$ J. H. C. Whitehead theorem). *Let $f: X \rightarrow Y$ be a map between 1-connected spaces (such that $f_*: \pi_2(X) \rightarrow \pi_2(Y)$ is an epimorphism).*

(1) *If $\mathcal{E}$ is a class satisfying (Π_B) and (III) , then the following conditions are equivalent:*

- (a) $f_*: \pi_i(X) \rightarrow \pi_i(Y)$ is a $\mathcal{E}$ -isomorphism for $i \leq n$ and a $\mathcal{E}$ -epimorphism for $i = n+1$.
- (b) $f_*: H_i(X) \rightarrow H_i(Y)$ is a $\mathcal{E}$ -isomorphism for $i \leq n$ and a $\mathcal{E}$ -epimorphism for $i = n+1$.

(2) *Let X and Y be of finite type and $\mathcal{E} = \mathcal{E}_p$. Then (a), (b), and the following (c), (c)* are equivalent:*

- (c) $f_*: H_i(X; \mathbb{Z}_p) \rightarrow H_i(Y; \mathbb{Z}_p)$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$.
- (c*) $f^*: H^i(X; \mathbb{Z}_p) \rightarrow H^i(Y; \mathbb{Z}_p)$ is an isomorphism for $i \leq n$ and a monomorphism for $i = n+1$.

Here the same also holds for the case $\mathcal{E} = \mathcal{E}_0$ replacing $\mathbb{Z}_p$ by $\mathbb{Q}$.

PROOF. One can replace f by an inclusion similarly to the proof of Theorem 1.5. Then (a), (b), (c), (c*) are equivalent to $\pi_i(Y, X) \in \mathcal{E}$, $H_i(Y, X) \in \mathcal{E}$, $H_i(Y, X; \mathbb{Z}_p) = 0$, $H^i(Y, X; \mathbb{Z}_p) = 0$ respectively for

$i < n + 1$. Thus (1) follows from Theorem 1.11(2) and (2) from the universal coefficient theorem. (The theorem holds even if $f_*|\pi_2$ is not an epimorphism). Q.E.D.

THEOREM 1.13. (The mod $\mathcal{E}$ Blakers–Massey theorem). *Let $\mathcal{E}$ be a class satisfying (II_B) and (III) , X a CW-complex, and A and B subcomplexes of X such that $x_0 \in A \cap B$, $A \cap B$ is 1-connected, and A and $A \cup B$ are 2-connected. Let $m, n > 2$. Then, if $H_i(A, A \cap B) \in \mathcal{E}$ for $i \leq m$ and $H_j(B, A \cap B) \in \mathcal{E}$ for $j \leq n$, the excision homomorphism $i_*: \pi_k(A, A \cap B) \rightarrow \pi_k(A \cup B, B)$ is a $\mathcal{E}$ -isomorphism for $k < m + n$ and a $\mathcal{E}$ -epimorphism for $k = m + n$.*

PROOF. The inclusion $i: (A, A \cap B) \rightarrow (A \cup B, B)$ induces a fibre map $f: (PA, p, A) \rightarrow (P(A \cup B), p, A \cup B)$, which induces a homomorphism $f^*: E'_{i,j} \rightarrow 'E'_{i,j}$ between the spectral sequences as follows:

$$f_{i,j}^2: E_{i,j}^2 = H_j(A, A \cap B; H_j(\Omega A)) \rightarrow 'E_{i,j}^2 = H_j(A \cup B, B; H_j(\Omega(A \cup B))),$$

where $f_{i,j}^2$ is induced by the excision isomorphism $i_*: H_i(A, A \cap B) \cong H_i(A \cup B, B)$ and a homomorphism of the coefficient group $\Omega i_*: H_j(\Omega A) \rightarrow H_j(\Omega(A \cup B))$.

$H_j(A \cup B, B) \cong H_j(B, A \cap B) \in \mathcal{E}$ for $j \leq n$ implies $\pi_j(A \cup B, B) \in \mathcal{E}$ for $j \leq n$. Thus $\pi_j(\Omega(A \cup B), \Omega B) \in \mathcal{E}$ for $j < n$, which implies $H_j(\Omega(A \cup B), \Omega B) \in \mathcal{E}$ for $j < n$, so Ωi_* is a $\mathcal{E}$ -isomorphism for $j < n - 1$ and a $\mathcal{E}$ -epimorphism for $j = n - 1$. Since $H_i(A, A \cap B), H_i(A \cup B, B) \in \mathcal{E}$ for $i < m + 1$, one can see by using the universal coefficient theorem that $f_{i,j}^2$ is a $\mathcal{E}$ -isomorphism for $i + j < m + n$ and a $\mathcal{E}$ -epimorphism for $i + j = m + n$. By a simple consideration of the spectral sequence one sees that this holds for each f^* and so in the limit

$$f_*: H_k(PA, \Omega(A, A \cap B)) \rightarrow H_k(P(A \cup B), \Omega(A \cup B, B))$$

is a $\mathcal{E}$ -isomorphism for $k < m + n$ and a $\mathcal{E}$ -epimorphism for $k = m + n$. This also holds for $\Omega i_*: H_{k-1}(\Omega(A, A \cap B)) \rightarrow H_{k-1}(\Omega(A \cup B, B))$, since PA and $P(A \cup B)$ are contractible. By Theorem 1.12 we have the same result replacing H_{k-1} by π_{k-1} . Therefore the theorem is proved by using the isomorphism $\pi_k(A, A \cap B) \cong \pi_{k-1}(\Omega(A, A \cap B))$, etc. Q.E.D.

COROLLARY 1.14 (The mod $\mathcal{E}$ Freudenthal theorem). *Let $\mathcal{E}$ be a class satisfying (II_B) and (III) . If X is a 1-connected CW-complex such that $H_i(X) \in \mathcal{E}$ for $0 < i \leq n$, then the suspension homomorphism*

$$S: \pi_k(X) \rightarrow \pi_{k+1}(SX)$$

is a $\mathcal{E}$ -isomorphism for $k < 2n$ and a $\mathcal{E}$ -epimorphism for $k = 2n + 1$.

PROOF. Take A (and B) to be the upper (and lower) half of SX in Theorem 1.13. Q.E.D.

REMARK 1.15. If a complex is of finite type in Theorem 1.13 and Corollary 1.14, then (II_B) can be replaced by (II_A) .

General homotopy groups. We assign every topological space X a distinguished point x_0 and call it the *base point* of X . We consider only (continuous) maps and homotopies preserving the base points and denote the set of homotopy classes by

$$[X, Y]_0 = [X, x_0; Y, y_0] = \{[f]: (X, x_0) \rightarrow (Y, y_0) \text{ (the homotopy class)}\}$$

(cf. §3 of Chapter II).

By using the identification $\psi: I^n \rightarrow I^n/\partial I^n = S^n$, we obtain a bijection

$$\psi^*: [S^n, Y]_0 \cong \pi_n(Y, y_0) = [I^n, \partial I^n; Y, y_0].$$

Therefore $[X, Y]_0$ is regarded as a generalization of $\pi_n(Y, y_0)$. There are two ways to introduce a natural group structure on $[X, Y]_0$.

DEFINITION. Let X and Y be topological spaces with base points x_0 and y_0 respectively. The subspace

$$X \vee Y = X \times y_0 \cup x_0 \times Y,$$

of $X \times Y$ is called the *wedge* of X and Y and the quotient space

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

is called the *smash product* of X and Y .

In particular, when $\{1\}$ is the base point of $I = [0, 1]$,

$$SX = X \wedge I \cong (X \times I)/(X \times 1 \cup x_0 \times I)$$

is called the *reduced cone* on X ; when $e_1 = (1, 0)$ is the base point of S^1 ,

$$CX = X \wedge S^1 \cong (X \times I)/(X \times \partial I \cup x_0 \times I)$$

is called the *reduced suspension* of X . (Sometimes they are simply referred to as the *cone* and the *suspension*.)

Since the product map $f \times g: X \times Y \rightarrow X' \times Y'$ of $f: (X, x_0) \rightarrow (X', x'_0)$ and $g: (Y, y_0) \rightarrow (Y', y'_0)$ maps $X \vee Y$ to $X' \vee Y'$, $f \times g$ induces the smash product of the maps

$$f \wedge g: X \wedge Y \rightarrow X' \wedge Y'.$$

Also for reduced cones and reduced suspensions, taking g to be the identity map of I and S^1 respectively, the reduced cone and the reduced suspension of a map f are defined by:

$$Cf = f \wedge 1_I: CX \rightarrow CX', \quad Sf = f \wedge 1_{S^1}: SX \rightarrow SX'.$$

When we consider the smash product $X \wedge Y$, it is more convenient and practical to suppose that X and Y are Hausdorff spaces and that one of them is compact. In this case we have a natural homeomorphism $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ and so, using this identification, we have associativity $(f \wedge g) \wedge h = f \wedge (g \wedge h)$ of the smash product of maps. This is based on the following lemma.

LEMMA 1.16. Let $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ be identification maps. If f satisfies one of the conditions below, then $f \times f': X \times X' \rightarrow Y \times Y'$ is an identification map.

- (i) $X = Y$ is a locally compact space and $f = 1_X$.
- (ii) X is compact and Y is a Hausdorff space.

A notion dual to that of an H -space is a co- H -space.

DEFINITION. A space X with the base point x_0 is called a co- H -space if there is a comultiplication $\phi: (X, x_0) \rightarrow (X \vee X, (x_0, x_0))$ with the following property: for the projections $p_1, p_2: X \vee X \rightarrow X$ (i.e., $p_1(x, y) = x$, $p_2(x, y) = y$, where $(x, y) \in X \vee X$), $p_1 \circ \phi \simeq 1_X \simeq p_2 \circ \phi: (X, x_0) \rightarrow (X, x_0)$.

The comultiplication ϕ is called homotopy associative if

$$(\phi \vee 1_X) \circ \phi \simeq (1_X \vee \phi) \circ \phi.$$

The comultiplication ϕ is called homotopy commutative if

$$T \circ \phi \simeq \phi (T(x, y) = (y, x) \in X \vee X).$$

If X is a co- H -space, then the product $f * g$ of $f, g: (X, x_0) \rightarrow (Y, y_0)$ is defined by

$$(f * g)(x) = \begin{cases} f(x') & \text{if } \phi(x) = (x', x_0), \\ g(x'') & \text{if } \phi(x) = (x_0, x''). \end{cases}$$

The product induces one between homotopy classes.

If Y is an H -space with multiplication $\mu: (Y \times Y, (y_0, y_0)) \rightarrow (Y, y_0)$, the product $f \cdot g: (X, x_0) \rightarrow (Y, y_0)$ of maps $f, g: (X, x_0) \rightarrow (Y, y_0)$ is defined by

$$(f \cdot g)(x) = \mu(f(x), g(x)).$$

The product induces one between homotopy classes.

The composition of maps $f: (X, x_0) \rightarrow (Y, y_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$

$$g \circ f: (X, x_0) \rightarrow (Z, z_0)$$

induces composition $[g] \circ [f] = [g \circ f] \in [X, Z]_0$ between homotopy classes. If we define

$$g_*: [X, Y]_0 \rightarrow [X, Z]_0 \quad \text{by } g_*[f] = [g \circ f],$$

and

$$f^*: [Y, Z]_0 \rightarrow [X, Z]_0 \quad \text{by } f^*[g] = [g \circ f],$$

then the following equalities are obtained from the associativity of composition of maps:

$$(g \circ g')_* = g_* \circ g'_*, \quad (f \circ f')^* = f'^* \circ f^*.$$

If we denote by $0 \in [X, Y]_0$ the class of the constant map $X \rightarrow y_0$, obviously we have

$$g_*(0) = 0, \quad f^*(0) = 0.$$

THEOREM 1.17. (1) If X is a co- H -space, the product $*$ in $[X, Y]_0$ satisfies:

$$\alpha * 0 = 0 * \alpha = \alpha, \quad g_*(\alpha * \beta) = g_*(\alpha) * g_*(\beta) \quad \text{for } g: (Y, y_0) \rightarrow (Y', y'_0).$$

Conversely, if $[X, Y]_0$ has such a product $*$ for arbitrary Y , there is a comultiplication on X , unique up to homotopy, inducing $*$.

(2) If Y is an H -space, the product $\cdot$ in $[X, Y]_0$ satisfies

$$\alpha \cdot 0 = 0 \cdot \alpha = \alpha, \quad f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta) \quad \text{for } f: (X', x'_0) \rightarrow (X, x_0).$$

Conversely, if $[X, Y]_0$ has such a product $\cdot$ for arbitrary X , there is a multiplication on Y , unique up to homotopy, inducing $\cdot$.

(3) If X is a co- H -space and Y is an H -space, then the products of (1) and (2) in $[X, Y]_0$ coincide and are abelian ($\alpha\beta = \beta\alpha$).

(4) A comultiplication on a co- H -space is homotopy associative (or homotopy commutative) iff the product in $[X, Y]_0$ is associative (or abelian). Similarly for an H -space.

(Proof omitted.)

DEFINITION. A co- H -space X is called *homotopy invertible* if there is $\iota: (X, x_0) \rightarrow (X, x_0)$ such that $\iota * 1_X \simeq 0 \simeq 1_X * \iota$. A homotopy associative, homotopy invertible co- H -space is called a *co- H -group*.

An H -space X is called *homotopy invertible* if there is $\iota: (X, x_0) \rightarrow (X, x_0)$ such that $\iota \cdot 1_X \simeq 0 \simeq 1_X \cdot \iota$.

A homotopy associative, homotopy invertible H -space is called an *H -group*.

THEOREM 1.18. Let X be a co- H -space (or Y an H -space). X (or Y) is homotopy invertible iff every element of $[X, Y]_0$ has an inverse.

COROLLARY 1.19. If X is a co- H -group or Y is an H -group then $[X, Y]_0$ is a group. If X is a co- H -group and Y is an H -group then $[X, Y]_0$ is an abelian group.

A reduced suspension is a typical example of a co- H -space. Taking $S^1 = I/\partial I$ and using the identification map $\psi: I \rightarrow S^1$, we consider the identification map

$$d_X: X \times I \rightarrow SX \quad (\text{i.e., } d_X(x, t) = \text{the class of } (x, \psi(t)) \in X \wedge S^1).$$

Observe that d_X is a map contracting $X \times \partial I \cup x_0 \times I$ to a point. If we define

$$\phi: SX \rightarrow SX \vee SX \quad \text{by } \phi(d_X(x, t)) = \begin{cases} d_X(x, 2t) & \text{if } 0 \leq t \leq 1/2, \\ d_X(x, 2t-1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

then SX is a co- H -space.

1.20. SX is a co- H -group and hence $[SX, Y]_0$ is a group. If X is a co- H -space, then SX is homotopy commutative. In particular, $[S(SX), Y]_0$ is an abelian group.

We have $S^1 \wedge S^1 = S^2$ in $S(SX) = (X \wedge S^1) \wedge S^1 = X \wedge (S^1 \wedge S^1)$ and so we may regard it as $S^2X = X \wedge S^2 = S(SX)$. In general we put

$$S^n X = X \wedge S^n, \quad S^n f = f \wedge 1_{S^n}: S^n X \rightarrow S^n X$$

and we have $S^{n+1}X = S(S^n X)$ and $S^{n+1}f = S(S^n f)$.

As examples of H -spaces other than topological groups, we have loop spaces $\Omega Y = \{w: (I, \partial I) \rightarrow (Y, y_0)\}$. The multiplication is of course given by the product of loops.

1.21. ΩY is an H -group, hence $[X, \Omega Y]_0$ is a group. If Y is an H -space, ΩY is homotopy commutative. In particular, $[X, \Omega(\Omega Y)]_0$ is an abelian group.

$\Omega(\Omega Y)$ can be identified with $\Omega^2 Y = \{f: (I^2, \partial I^2) \rightarrow (Y, y_0)\}$.

DEFINITION. $F: (SX, *) \rightarrow (Y, y_0)$ and $f: (X, x_0) \rightarrow (\Omega Y, y_0)$ are called *adjoint maps* of each other if $F(d_X(x, t)) = f(x)(t)$ for $x \in X$, $t \in I$.

THEOREM 1.22. The adjoint maps are in one to one correspondence, which induces an isomorphism $[SX, Y]_0 \cong [X, \Omega Y]_0$. Under this isomorphism the following are commutative diagrams of homomorphisms:

$$\begin{array}{ccc} [SX, Y]_0 \cong [X, \Omega Y]_0 & & [SX', Y]_0 \cong [X', \Omega Y]_0 \\ \downarrow g_* & \downarrow \Omega g_* & \downarrow sh_* \quad \downarrow h_* \\ [SX, Y']_0 \cong [X, \Omega Y']_0 & & [SX, Y]_0 \cong [X, \Omega Y]_0. \end{array}$$

LEMMA 1.23. If a homotopy associative H -space has the homotopy type of a CW -complex and if $\pi_0(X)$ forms a group (with respect to the multiplication of X), then X is an H -group.

PROOF. One may suppose that a given H -space X is a connected CW -complex. Using the multiplication $\mu: X \times X \rightarrow X$ we define $f: X \times X \rightarrow X \times X$ by $f(x, y) = (x, \mu(x, y))$. The homomorphism f_* of homotopy groups induced by f maps $(\alpha, \beta) \in \pi_i(X) \oplus \pi_i(X) \cong \pi_i(X \times X)$ to $(\alpha, \alpha + \beta)$ and hence is an isomorphism. By Corollary 1.6 there exists $g': X \times X \rightarrow X \times X$ such that $f \circ g' \simeq 1$ and $g' \circ f \simeq 1$. Let $p_1, p_2: X \times X \rightarrow X$ be the projections given by $p_1(x, y) = x$, $p_2(x, y) = y$. $p_1 \circ f = p_1$ implies $p_1 \circ g' = p_1 \circ f \circ g' \simeq p_1$. By the covering homotopy property with respect to p_1 there is g such that $p_1 \circ g = p_1$ and $g \simeq g'$, and it can be expressed as $g(x, y) = (x, h(x, y))$, where $h: X \times X \rightarrow X$. If we put $\iota(x) = h(x, e)$, then $\mu(x, \iota(x)) = p_2(x, \mu(x, \iota(x))) = p_2 f g(x, e)$, which implies $\mu \circ (1 \times \iota) \circ d = p_2 \circ f \circ g \circ i_1 \simeq p_2 \circ i_1 = 0$. Moreover

$\mu \circ (1 \times 1) \circ d \simeq \mu \circ ((\mu \circ (0 \times 1) \circ d) \times 1) \circ d = \mu \circ (\mu \times 1) \circ (0 \times 1 \times 1) \circ (d \times 1) \circ d \simeq$
 $\mu \circ (1 \times \mu) \circ (0 \times 1 \times 1) \circ (1 \times d) \circ d = \mu \circ (0 \times (\mu \circ (1 \times 1) \circ d)) \circ d = \mu \circ (0 \times 0) \circ d = 0.$
 Thus X is homotopy invertible. Q.E.D.

The homotopy exact sequence

REVISION. A sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ of based sets is called *exact* if $\text{Im } \varphi = \text{Ker } \psi$, where $\text{Ker } \psi = \psi^{-1}$ (a base point).

We take the base point of $[X, Y]_0$ to be 0 (the identity element) the class of the constant map.

We consider the identification map $\pi: X \rightarrow X/A$ contracting a subspace A of X to a point. The base point of X/A is taken to be $*$ = A/A . From the definition of the identification we obtain the following bijection:

$$\pi^*: [X/A, Y]_0 \cong [X, A; Y, y_0]. \quad (1.24)$$

THEOREM 1.25. *The following sequence is exact:*

$$[SY, A]_0 \xrightarrow{i_*} [SY, X]_0 \xrightarrow{j_*} [CY, Y; X, A]_0 \xrightarrow{\partial_*} [Y, A]_0 \xrightarrow{i_*} [Y, X]_0,$$

where $i: A \subset X$, $j: (X, x_0) \rightarrow (X, A)$ are the inclusions, $\partial_*[f] = [f|Y]$, and we identify $(\pi^*:)[CY, Y; X, x_0] = [SY, X]_0$ using $\pi: CY \rightarrow CY/Y = SY$.

PROOF. One can prove this directly. Observe that the exact sequence above is equivalent to the exact sequence $\pi_1(A') \rightarrow \pi_1(X') \rightarrow \pi_1(X', A') \rightarrow \pi_0(A') \rightarrow \pi_0(X')$, where we take $X' = \text{Map}(Y, X) \supset A' = \text{Map}(Y, A)$. Q.E.D.

THEOREM 1.26. *Let (E, p, B) be a fibration, $F = p^{-1}(b_0)$, K a CW-complex. Then $p_*: [CK, K; E, F] \rightarrow [CK, K; B, b_0] \cong [SK, B]_0$ is a bijection and the following sequence is exact:*

$$[SK, E]_0 \xrightarrow{p_*} [SK, B]_0 \xrightarrow{\Delta} [K, F]_0 \xrightarrow{i_*} [K, E]_0 \xrightarrow{p_*} [K, B]_0,$$

where $\Delta = \partial_* \circ p_*^{-1}$.

PROOF. The theorem is proved similarly as in Theorem 3.10 and Corollary 3.11 of Chapter II, using the fact that a fibration has the covering homotopy property for a CW-complex. Q.E.D.

We define the *mapping cone* C_f of a map $f: (X, x_0) \rightarrow (Y, y_0)$ by

$$C_f = (Y \amalg CX) / \sim; \quad (x, 0) \sim f(x) \in Y \quad (x \in X).$$

DEFINITION. A pair (X, A) is said to have the *homotopy extension property* if for an arbitrary map $f: X \rightarrow Y$ and homotopy $g_t: A \rightarrow Y$ satisfying $g_0 = f|A$, there is a homotopy $f_t: X \rightarrow Y$ such that $f_0 = f$ and $f_t|A = g_t$.

The following property is easily verified.

1.27. (1) (C_f, Y) has the homotopy extension property.

(2) A CW -pair (K, L) (i.e., a pair of a CW -complex K and its subcomplex L) has the homotopy extension property.

THEOREM 1.28 (The Puppe exact sequence). (1) The following sequence is exact:

$$[C_f, W]_0 \xrightarrow{j^*} [Y, W]_0 \xrightarrow{f^*} [X, W]_0,$$

where C_f is the mapping cone of $f: X \rightarrow Y$ and $j: Y \hookrightarrow C_f$ is the inclusion.

(2) If (X, A) has the homotopy extension property, then the identification map $\pi: X \cup CA (= C_i) \rightarrow X/A$ contracting CA to a point is a homotopy equivalence where $i: A \hookrightarrow X$.

(3) Under the assumption of (2), if we define

$$\delta^* = \pi^{*-1} \circ p^*: [SA, W]_0 \rightarrow [X \cup CA, W]_0 \cong [X/A, W]_0$$

using the bijection $\pi^*: [X/A, W]_0 \cong [X \cup CA, W]_0$ and the map $p: X \cup CA \rightarrow SA$ contracting X to a point, then the following sequence is exact:

$$\begin{aligned} \dots \xrightarrow{S\pi^*} [SX, W]_0 \\ \xrightarrow{Si^*} [SA, W]_0 \xrightarrow{\delta^*} [X/A, W]_0 \xrightarrow{\pi^*} [X, W]_0 \xrightarrow{i^*} [A, W]_0 \end{aligned}$$

where $\pi = \pi|_X: X \rightarrow X/A$.

PROOF. (1) is easily provable.

(2) By the assumption and the contractibility of CA , there exists a homotopy $h_t: X \cup CA \rightarrow X \cup CA$ such that $h_0 = 1$ and $h_t(a, s) = (a, s + t - st)$. Then there is $h: XA \rightarrow X \cup CA$ such that $h_1 = h \circ \pi$. It is easily shown that $h \circ \pi \cong 1_{X \cup CA}$ and $\pi \circ h \cong 1_{X/A}$.

(3) By (1) and (2) we have $\text{Ker } i^* = \text{Im } j^* = \text{Im}(j^* \circ \pi^*) = \text{Im } \pi^*$ for $j: X \hookrightarrow X \cup CA$. If we replace i by j , then π is equivalent to p and hence $[SA, W]_0 \xrightarrow{p^*} [X \cup CA, W]_0 \xrightarrow{j^*} [X, W]_0$ is exact. If we replace it by the bijection π^* , we obtain the exact sequence $\xrightarrow{\delta^*} \xrightarrow{\pi^*}$.

We consider the mapping cone $C_p = SA \amalg C(X \cup CA)/\sim$ of p . Since $p(X) = x_0$, $C_p \supset (x_0 \amalg CX)/\sim = SX$. Further, the part $C(CA)$ is homeomorphic with $(CA \times I)/(* \times I)$ with a suitably changed parameter, and C_p is seen to be homeomorphic to the space obtained by contracting $* \times I$ to a point in the mapping cylinder Z_{Si} of $Si: SA \rightarrow SX$. Therefore SX is a deformation retract of C_p , and it is shown that the inclusion $k: SA \hookrightarrow C_p$ is equivalent to $-Si: SA \rightarrow SX$ (i.e., $(-Si)(d_A(a, t)) = d_X(i(a), 1 - t)$). Thus $\text{Ker } \delta^* = \text{Ker } p^* = \text{Im } k^* = \text{Im}(-Si)^* = \text{Im}(Si)^*$. Q.E.D.

If one puts $W = K(G, n)$ in the theorem, one obtains the exact sequence of cohomology groups

$$\begin{aligned} \dots \rightarrow H^{n-1}(X; G) \xrightarrow{i^*} H^{n-1}(A; G) \xrightarrow{\delta^*} H^n(X/A; G) \\ \xrightarrow{\pi^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \rightarrow \dots \end{aligned}$$

where one has the following by Theorem 1.22, since $\Omega W = K(G, n-1)$.

$$1.29. [SA, K(G, n)]_0 \cong [A, K(G, n-1)]_0 \cong H^{n-1}(A; G) \text{ for } n > 1.$$

§2. Homology rings and Hopf algebras of H -spaces

Homology rings of H -spaces. Let X be an H -space with multiplication μ . Then $H_*(X; R) = \sum H_n(X; R)$ is a (graded) algebra (not necessarily associative) over R with respect to the Pontrjagin product

$$\mu_* \circ \times : H_p(X; R) \otimes_R H_q(X; R) \xrightarrow{\times} H_{p+q}(X \times X; R) \xrightarrow{\mu_*} H_{p+q}(X; R).$$

- 2.1. (1) If (X, μ) is homotopy associative then $H_*(X; R)$ is associative.
 (2) If (X, μ) is homotopy commutative then $H_*(X; R)$ is abelian

$$\alpha\beta = (-1)^{pq}\beta\alpha, \quad \alpha \in H_p, \quad \beta \in H_q.$$

If an H -space X acts on a space Y by $\mu_Y: Y \times X \rightarrow Y$, we obtain a right action (a homomorphism of R -modules) of $H_*(X; R)$ on $H_*(Y; R)$ similarly by

$$\mu_Y^* \circ \times : H_p(Y; R) \otimes_R H_q(X; R) \rightarrow H_{p+q}(Y \times X; R) \rightarrow H_{p+q}(Y; R).$$

If the action of X on Y is homotopy associative (i.e., $\mu_Y \circ (\mu_Y \times 1_X) \simeq \mu_Y \circ (1_Y \times \mu)$), then $H_*(Y; R)$ is a right $H_*(X; R)$ -module.

To calculate homology rings we use the spectral sequence

$$E_{*,*}^2 = H_*(B; \underline{H_*(F; R)}) \Rightarrow H_*(E; R)$$

of a principal fibration (E, p, B, F) (Theorem 2.18 of Chapter III).

Analogously to Lemma 2.9(2) of Chapter III, the following holds:

2.2. If in the principal fibration (E, p, B, F) the H -space F is arcwise connected, then the system of local coefficient rings $\underline{H_*(F; R)} = \{H_*(F_b; R)\}$ is trivial.

Also, corresponding to Lemma 2.15 of Chapter III, the following holds:

2.3. (1) If in the fibration (E, p, B) the system of local coefficient rings $\underline{H_*(F; R)}$ is trivial and if either $H_*(B; R)$ or $H_*(F; R)$ is an R -free module for each degree, then there is an isomorphism of R -modules (R is a principal ring):

$$H_p(B; R) \otimes_R H_q(F; R) \cong E_{p,q}^2.$$

(2) If (E, p, B) is a principal fibration in the above, then the isomorphism in (1) is an isomorphism of $H_*(F; R)$ -modules given by the action of $H_q(F; R)$ such that

$$d^r(x \cdot y) = d^r(x) \cdot y, \quad x \in E'_{p,q}, \quad y \in H_q(F; R).$$

Let (E, p, B) be a fibration in which the base space is arcwise connected. Then the map

$$\Omega p: \Omega(E, F) \rightarrow \Omega B$$

induced by p gives a homomorphism $\Omega p_*: \pi_i(\Omega(E, F)) \rightarrow \pi_i(\Omega B)$, which is equivalent to $p_*: \pi_{i+1}(E, F) \rightarrow \pi_{i+1}(B)$. Since p_* is an isomorphism by Theorem 3.10 of Chapter II, Ωp is a weak homotopy equivalence. Since homotopy groups and (co)homology groups are invariant under weak homotopy equivalences, one may identify weak homotopy equivalent spaces in the calculation of (co)homology groups. While doing so, the fibre $\Omega(E, F)$ is replaced by ΩB , $P(E, F)$ by its deformation retract F , and p_1 by the inclusion $i: F \hookrightarrow E$ in a fibration $(P(E, F), p_1, E)$. Further, by considering two fibrations

$$(\Omega(E, F), p_0, F) \quad \text{and} \quad (P(\Omega(E, F), \Omega E), p_0, \Omega E),$$

one can obtain the following:

2.4. *If (E, p, B) is a fibration, then upon substituting suitable weak equivalences,*

$$(F, i, E, \Omega B), \quad (\Omega B, d, F, \Omega E), \quad \text{and} \quad (\Omega E, \Omega p, \Omega B, \Omega F)$$

become principal fibrations, where $d: \Omega B \xleftarrow{\Omega p} \Omega(E, F) \xrightarrow{p_0} F$.

THEOREM 2.5 (The Wang exact sequence). *Suppose that in a fibration (E, p, B) , $H_*(B; R) \cong H_*(S^{n+1}; R)$, where R is a principal ring, and that the system of local coefficient rings $\underline{H}_*(F; R)$ is trivial. Then we have the following exact sequence.*

$$\cdots \rightarrow H_{q+1}(E; R) \xrightarrow{\Delta} H_{q-n}(F; R) \xrightarrow{\theta} H_q(F; R) \xrightarrow{i_*} H_q(E; R) \xrightarrow{\Delta} \cdots$$

In particular, in the case of a principal fibration this is an exact sequence of $H_*(F; R)$ -modules and

$$\theta(x) = \theta(1) \cdot x.$$

PROOF. By the assumption, $E_{n+1,q}^2 \cong E_{0,q}^2 \cong H_q(F; R)$, $E_{p,q}^2 = 0$ for $p \neq 0, n+1$, and $E_{p,q}^r = 0$ for $r \geq 2, p \neq 0, n+1$. Therefore $d^r = 0$ for $r \neq n+1$ so $E_{p,q}^2 = E_{p,q}^{n+1}$, $E_{p,q}^{n+2} = E_{p,q}^\infty$, and the sequence

$$0 \rightarrow E_{n+1,q-n}^\infty \rightarrow E_{n+1,q-n}^{n+1} \xrightarrow{d^{n+1}} E_{0,q}^{n+1} \rightarrow E_{0,q}^\infty \rightarrow 0$$

is exact. Using $E_{0,q}^\infty = D_{0,q}$, $E_{n+1,q-n}^\infty = D_{n+1,q-n}/D_{0,q+1}$, and $H_s(E; R) = D_{n+1,s-n-1}$, we obtain the exact sequence of the theorem. The last part is clear, since the spectral sequence is one of $H_*(F; R)$ -modules. Q.E.D.

COROLLARY 2.6. *If B is 1-connected and $H_*(B; R) \cong H_*(S^{n+1}; R)$, then*

$$H_*(\Omega B; R) = R[x], \quad \text{where } x = \theta(1) \in H_n(\Omega B; R).$$

PROOF. Since in the principal fibration $(PB, p, B, \Omega B)$ PB is contractible, $\theta: H_{q-n}(\Omega B; R) \rightarrow H_q(\Omega B; R)$ for $q > 0$ is an isomorphism given by multiplying by $x = \theta(1)$. Q.E.D.

Cohomology rings of classical groups and their loop spaces

THEOREM 2.7. (1) $H_*(U(n)) = \Delta(e_1, e_3, \dots, e_{2n-1})$,

$$H_*(SU(n)) = \Delta(e_3, e_5, \dots, e_{2n-1});$$

(2) $H_*(Sp(n)) = \Delta(e_3, e_7, \dots, e_{4n-1})$;

(3) for k a field of characteristic $\neq 2$,

$$H_*(SO(2n+1); k) = \Delta(e_3, e_7, \dots, e_{4n-1}),$$

$$H_*(SO(2n); k) = \Delta(e_3, e_7, \dots, e_{4n-5}, e_{2n-1});$$

(4) for k a field of characteristic $= 2$,

$$H_*(SO(n); k) = \Delta(e_1, e_2, \dots, e_{n-1}).$$

PROOF. It follows from results on the cohomology groups in Chapter III that the results above are true for the underlying module structures. One obtains the theorem by induction on n , using the Wang sequence. Q.E.D.

REMARK 2.7'. It will be shown in Theorem 2.17 that Δ can be replaced by Λ in the above. So these homology groups are commutative.

THEOREM 2.8. $H_*(\Omega SU(n)) = \mathbb{Z}[e_2, e_4, \dots, e_{2n-2}]$, $e_{2i} \in H_{2i}$;

$$H_*(\Omega Sp(n)) = \mathbb{Z}[e_2, e_6, \dots, e_{4n-2}], \quad e_{4i-2} \in H_{4i-2}.$$

PROOF. By Theorem 4.12 of Chapter II, $SU(n)$ is 1-connected thus $\Omega SU(n)$ is arcwise connected. So we can apply (2.3)(2) to the homology spectral sequence

$$E_{*,*}^2 = H_*(\Omega S^{2n+1}) \otimes H_*(\Omega SU(n)) \Rightarrow H_*(\Omega SU(n+1))$$

of the principal fibration

$$(\Omega SU(n+1), \Omega p, \Omega S^{2n+1}, \Omega SU(n)),$$

where $p: SU(n+1) \rightarrow SU(n+1)/SU(n) = S^{2n+1}$. Since the theorem is clear for $SU(1) = \text{a point}$, one has

$$E_{*,*}^2 = \mathbb{Z}[e_{2n}] \otimes \mathbb{Z}[e_2, \dots, e_{2n-2}],$$

where e_{2n} is a generator of $H_{2n}(\Omega S^{2n+1})$ by the inductive hypothesis on n and by Corollary 2.6. $E_{p,q}^2 = 0$ for $p+q$ odd implies $d^r = 0$ for $r \geq 2$ and hence $E_{*,*}^2 = E_{*,*}^\infty$, that is, the spectral sequence collapses, $i_*: H_*(\Omega SU(n)) \rightarrow H_*(\Omega SU(n+1))$ is a monomorphism,

$$\Omega p_*: H_*(\Omega SU(n+1)) \rightarrow H_*(\Omega S^{2n+1})$$

is an epimorphism and these are homomorphisms of rings. If we choose $e_{2n} \in H_{2n}(\Omega SU(n+1)) = D_{2n,0}$ such that $\Omega p_*(e_{2n}) = e_{2n}$, its class is $[e_{2n}] = e_{2n} \otimes 1 \in E_{2n,0}^\infty = D_{2n,0}/D_{2n-1,1}$. Considering the operation of $H_*(\Omega SU(n))$

on $e_{2n}^k \in D_{2kn,0} = H_{2kn}(\Omega SU(n+1))$, one can see that the class of $e_{2n}^k \cdot \alpha \in D_{2nk,q}$ is

$$e_{2n}^k \otimes \alpha \quad \text{for } \alpha \in H_q(\Omega SU(n)) = \mathbb{Z}[e_2, \dots, e_{2n-2}].$$

It follows that

$$\mathbb{Z}[e_{2n}] \otimes H_*(\Omega SU(n)) \cong H_*(\Omega SU(n+1))$$

is an isomorphism of $H_*(\Omega SU(n))$ -modules.

Observe that $SU(n+1)$ is an H -space. Thus, $\Omega SU(n+1)$ is a homotopy commutative H -space, so the homology ring $H_*(\Omega SU(n+1))$ is commutative, and hence the above isomorphism is an isomorphism of rings. Thus the first part of the theorem is proved. The proof of the latter half is similar. Q.E.D.

Since $U(n)$ and $SO(n)$ are not 1-connected, their loop spaces break up into several arcwise components.

DEFINITION. For each element α of the fundamental group $\pi_1(X)$ of a topological space X , we denote by $\Omega_\alpha X$ the arcwise component of a loop w which represents α ($\alpha = [w]$) in the loop space ΩX and call it the α -component of ΩX . In particular, $\Omega_0 X$ is the 0-component containing the constant loop (the identity).

LEMMA 2.9. *All the arcwise components $\Omega_\alpha X$ of a loop space ΩX are homotopy equivalent to each other. Moreover, if X is a topological group, then the spaces $\Omega_\alpha X$ are homeomorphic to each other.*

PROOF. Denote the product of loops by $*$: $\Omega X \times \Omega X \rightarrow \Omega X$. If one assigns $l * w$ to $l \in \Omega X$, then one obtains a morphism $*w: \Omega_0 X \rightarrow \Omega_\alpha X$; its inverse morphism is given by the same construction using the inverse loop $\bar{w}$ of w . The relation $w * \bar{w} \simeq 0$ implies $(*\bar{w}) \circ (*w) \simeq 1_{\Omega X}$. Similarly $(*w) \circ (*\bar{w}) \simeq 1_{\Omega X}$. Thus $\Omega_0 X$ and $\Omega_\alpha X$ are homotopy equivalent. For the case X is a topological group, we define a product $\cdot$: $\Omega X \times \Omega X \rightarrow \Omega X$ of loops by $(w \cdot w')(t) = w(t)w'(t)$ for $t \in I$ and consider $\cdot w: \Omega_0 X \rightarrow \Omega_\alpha X$. Define w^{-1} by $w^{-1}(t) = (w(t))^{-1}$ and then we have $(\cdot w^{-1}) \circ (\cdot w) = 1_{\Omega X}$ and $(\cdot w) \circ (\cdot w^{-1}) = 1_{\Omega X}$ and thus $\cdot w$ is a homeomorphism. Q.E.D.

LEMMA 2.10. *Suppose that a map $f: X \rightarrow Y$ induces an isomorphism $f_*: \pi_i(X) \rightarrow \pi_i(Y)$ for $i > 1$. If $\alpha \in \pi_1(X)$ and $\beta = f_*(\alpha) \in \pi_1(Y)$, then $\Omega f: \Omega_\alpha X \rightarrow \Omega_\beta Y$ is a weak homotopy equivalence and so induces an isomorphism of (co)homology groups. In particular, if in a covering space $(\tilde{Y}, p, Y)$ Y is locally 1-connected (i.e., each point of Y has a sufficiently small, 1-connected neighborhood), then $\Omega_\alpha \tilde{Y} \rightarrow \Omega_\beta Y$ is a homeomorphism.*

PROOF. f_* is equivalent to $\Omega f_*: \pi_{i-1}(\Omega_0 X) \rightarrow \pi_{i-1}(\Omega_0 Y)$. Using the morphisms $*w$ and $*(f \circ w)$ in the proof of the previous lemmas, we show that Ωf_* is equivalent to $\Omega f_*: \pi_{i-1}(\Omega_\alpha X) \rightarrow \pi_{i-1}(\Omega_\beta Y)$. Since it is an

isomorphism for $i > 1$ and is trivial for $i = 1$, $\Omega f: \Omega_\alpha X \rightarrow \Omega_\beta Y$ is a weak homotopy equivalence.

For the case of a covering space, from the local 1-connectedness of Y one can prove that ΩY is locally arcwise connected. One can lift the evaluation map $e: \Omega_\beta Y \times I \rightarrow Y$ to $\tilde{e}: \Omega_\beta Y \times I \rightarrow \tilde{Y}$ by using Lemma 1.7 of Chapter II. The map $\tilde{e}$ induces an inverse map $\Omega_\beta Y \rightarrow \Omega_\alpha \tilde{Y}$ of $\Omega p: \Omega_\alpha \tilde{Y} \rightarrow \Omega_\beta Y$ and so Ωp is seen to be a homeomorphism. Q.E.D.

THEOREM 2.11. (1) $H_*(\Omega_0 U(n)) = \mathbb{Z}[e_2, e_4, \dots, e_{2n-2}]$, $e_{2i} \in H_{2i}$.

(2) If $\mathbf{k}$ is a field of characteristic $\neq 2$, then

$$H_*(\Omega_0 SO(2n+1); \mathbf{k}) = \mathbf{k}[e_2, e_6, \dots, e_{4n-2}];$$

$$H_*(\Omega_0 SO(2n); \mathbf{k}) = \mathbf{k}[e_2, e_6, \dots, e_{4n-2}, e_{2n}]$$

PROOF. (1) $SU(n)$ is the fibre of a fibration $(U(n), \det, U(1))$, where $\pi_j(U(1)) \cong \pi_j(S^1) = 0$ for $j > 1$. It follows that $i_*: \pi_j(SU(n)) \cong \pi_j(U(n))$ for $j > 1$ and hence $\Omega i: \Omega SU(n) \rightarrow \Omega_0 U(n)$ is a weak homotopy equivalence by Lemma 2.10. Thus (1) is reduced to the first half of Theorem 2.8.

(2) By the latter half of Lemma 2.10, one has

2.12. $\Omega p: \Omega \text{Spin}(n) \rightarrow \Omega_0 SO(n)$ is a homeomorphism for $n \geq 3$ in the universal covering group $(\text{Spin}(n), p, SO(n))$.

On the other hand, if one considers the universal covering map $p': \text{Spin}(n+1) \rightarrow SO(n+1)$ and the inclusion $i: SO(n) = SO(n) \times 1 \hookrightarrow SO(n+1)$, the map $i \circ p: \text{Spin}(n) \rightarrow SO(n+1)$ can be uniquely lifted to $\tilde{i}: \text{Spin}(n) \rightarrow \text{Spin}(n+1)$ so that it preserves the identity element. In addition, $\tilde{i}$ is a monomorphism and so $\text{Spin}(n)$ may be considered as a subgroup of $\text{Spin}(n+1)$. In the resulting sequence

$$\text{Spin}(n) \subset \text{Spin}(n+1) \subset \dots \subset \text{Spin}(m) \quad \text{for } m > n \geq 3,$$

the following is easily shown:

$$\text{Spin}(m)/\text{Spin}(n) \cong SO(m)/SO(n) \quad (\text{a homeomorphism}). \quad (2.13)$$

Now by (2.12) it suffices to show (2) for $\Omega \text{Spin}(n)$. (2) is true for $n = 1$, since one has $\text{Spin}(3) \cong Sp(1)$ and $\text{Spin}(4) \cong Sp(1) \times Sp(1)$ by 5.18 and 5.19 of Chapter II. By induction on n , applying a proof similar to that of Theorem 2.8 to the principal fibrations

$$\Omega \text{Spin}(2n+1) \rightarrow \Omega \text{Spin}(2n+2) \rightarrow \Omega S^{2n+1},$$

$$\Omega \text{Spin}(2n+1) \rightarrow \Omega \text{Spin}(2n+3) \rightarrow \Omega(SO(2n+3)/SO(2n+1)),$$

one proves (2) with the help of

$$H_*(SO(2n+3)/SO(2n+1); \mathbf{k})$$

$$\cong H^*(SO(2n+3)/SO(2n+1); \mathbf{k}) \cong \Lambda(e_{4n+3}),$$

which follows from Theorem 3.14(1) of Chapter III. Q.E.D.

Hopf algebras

DEFINITION. A graded module $A = \sum_{i \geq 0} A_i$ over a graded commutative ring R ($R = \sum R_i$, $R_i = 0$ for $i \neq 0$) with 1 is called a *quasi Hopf algebra* if A has a *product* $\mu: A \otimes_R A \rightarrow A$, a *coproduct* (a *diagonal map*) $\phi: A \rightarrow A \otimes_R A$, a *unit* (element) $\eta: R \rightarrow A$ and a *counit* (element) $\varepsilon: A \rightarrow R$ satisfying the following condition (*).

(*) μ, ϕ, η , and ε are homomorphisms of graded modules such that the following diagrams are commutative:

$$\begin{array}{ccccc}
 & & A \otimes_R R & & \\
 & \nearrow 1_A \otimes \eta & \downarrow \cong & \nwarrow 1_A \otimes \varepsilon & \\
 A \otimes_R A & \xrightarrow{\mu} & A & \xrightarrow{\phi} & A \otimes_R A \\
 & \nwarrow \eta \otimes 1_A & \uparrow \cong & \nearrow \varepsilon \otimes 1_A & \\
 & & R \otimes_R A & &
 \end{array}$$

$$\begin{array}{ccccc}
 R \otimes_R R & \xrightarrow{1_R} & R = R \otimes_R R & & \\
 \eta \otimes \eta \downarrow & \searrow \eta & \nearrow \varepsilon & \uparrow \varepsilon \otimes \varepsilon & \\
 A \otimes_R A & \xrightarrow{\mu} & A & \xrightarrow{\phi} & A \otimes_R A \\
 \phi \otimes \phi \downarrow & & & \uparrow \mu \otimes \mu & \\
 A \otimes_R A \otimes_R A \otimes_R A & \xrightarrow{1_A \otimes T \otimes 1_A} & A \otimes_R A \otimes_R A \otimes_R A & &
 \end{array}$$

where $T: A_i \otimes A_j \rightarrow A_j \otimes A_i$ is the map $T(a \otimes a') = (-1)^{ij} a' \otimes a$.

DEFINITION. $A = \sum_{i \geq 0} A_i$ is called *connected* if $A_0 = R$.

If we denote the product by $a \cdot a' = \mu(a \otimes a')$ in the definition of a connected quasi Hopf algebra (A, μ, ϕ) , the condition (*) takes the following simplified form:

(**) μ and ϕ are homomorphisms of graded modules, $1 \in R = A_0$ is the unit ($a \cdot 1 = 1 \cdot a = a$), ϕ is a homomorphism ($\phi(aa') = \phi(a)\phi(a')$), $\phi(1) = 1 \otimes 1$, and $\phi(a) = a \otimes 1 + 1 \otimes a + \sum a'_i \otimes a''_i$ for $a \in A_+$ = $\sum_{i > 0} A_i$, where $a'_i, a''_i \in A_+$.

Note that the product in $A \otimes_R A$ is given by $(1_A \otimes T \otimes 1_A)(\mu \otimes \mu)$.

DEFINITION. For a quasi Hopf algebra (A, μ, ϕ) ,

- A is called *associative* if $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$;
- A is called *coassociative* if $(\phi \otimes 1)\phi = (1 \otimes \phi)\phi$;
- A is called *commutative* if $\mu T = \mu$;
- A is called *cocommutative* if $T\phi = \phi$.

DEFINITION. (A, μ, ϕ) is called a *Hopf algebra* if (A, μ, ϕ) is an associative quasi Hopf algebra.

DEFINITION. The *dual module* $M^* = \sum M_i^*$ of a graded R -module $M = \sum M_i$ is defined by $M_i^* = \text{Hom}_R(M_i, R)$.

The *dual (homomorphism)* $f^*: N^* \rightarrow M^*$ of a homomorphism $f: M \rightarrow N$ of graded R -modules is the map defined by $f^*(\varphi) = \varphi \circ f$.

A graded R -module M is called *free of finite type* if each M_i is a free R -module having a finite basis.

For $x \in M_n$ and $\varphi \in M_n^* = \text{Hom}_R(M_n, R)$ we use the notation

$$\langle \varphi, x \rangle = \varphi(x) \in R$$

for the Kronecker product. If M is free of finite type, then $(M^*)^* = M$. The

dual basis $\{x_i^*\}$ of a (homogeneous) basis $\{x_i\}$ of M is given by

$$\langle x_i^*, x_j \rangle = \delta_{ij}.$$

If M and N are free of finite type, then one can regard $(M \otimes N)^* = M^* \otimes N^*$ by giving the Kronecker product by

$$\langle \varphi \otimes \psi, x \otimes y \rangle = (-1)^{mn} \langle \varphi, x \rangle \langle \psi, y \rangle, \quad \text{where } x \in M_m, \psi \in N_n^*.$$

The following two theorems are easily obtained from the definitions above.

THEOREM 2.14. *Let A be free of finite type. (A, μ, ϕ) is a quasi Hopf algebra with unit η and counit ε if and only if (A^*, ϕ^*, μ^*) is a quasi Hopf algebra with the unit ε^* and the counit η^* . Moreover*

- A is associative iff A^* is coassociative;
- A is coassociative iff A^* is associative;
- A is commutative iff A^* is cocommutative;
- A is cocommutative iff A^* is commutative.

THEOREM 2.15. *Let X be an H -space of finite type with a multiplication μ such that $H_*(X; R)$ is R -free. Then $H^*(X; R)$ is a commutative Hopf algebra whose product is the cup product and whose coproduct is*

$$\mu^*: H^*(X; R) \rightarrow H^*(X \times X; R) \xleftarrow[\cong]{\times} H^*(X; R) \otimes H^*(X; R),$$

and $H_*(X; R) \cong (H^*(X; R))^*$ is a coassociative and cocommutative quasi Hopf algebra whose product is μ_* and whose coproduct is the dual of the cup product. If an H -space (X, μ) is homotopy associative (or homotopy commutative) then $H^*(X; R)$ is a coassociative (or commutative) Hopf algebra, and $H_*(X; R)$ is a Hopf algebra (or commutative quasi Hopf algebra).

Hopf algebras will be discussed also in §1 of Chapter VII. Here we note only the necessary facts concerning them which are needed for this chapter.

DEFINITION. An element $a \in A_+$ of a connected quasi Hopf algebra A is called *primitive* if $\phi(a) = a \otimes 1 + 1 \otimes a$ and $a \neq 0$.

LEMMA 2.16. *In a principal fibration (E, p, B, F) such that the total space E is ∞ -connected, the base space B is 1-connected, and the cohomology $H^*(F; R)$ of the fibre F is free of finite type, a (nonzero) element in the image of the suspension $\sigma: H^{n+1}(B; R) \rightarrow H^n(F; R)$ is primitive for $n > 0$.*

PROOF. Let $\mu: (E \times F, F \times F) \rightarrow (E, F)$ be the action of F . If we rewrite μ^* by using the Künneth isomorphism, we obtain the following commutative diagram (where the coefficient ring R is omitted):

$$\begin{array}{ccccccc} H^*(B) & \xrightarrow{\mu^*} & H^*(E, F) & \xleftarrow[\cong]{\delta^*} & H^*(F) \\ \downarrow \cong \otimes 1 & & \downarrow \mu^* & & \downarrow \mu^* \\ H^*(B) \otimes R & \xrightarrow{\mu^* \otimes 1} & H^*(E, F) \otimes H^*(F) & \xleftarrow[\cong]{\delta^* \otimes 1} & H^*(F) \otimes H^*(F) & \xleftarrow[\cong]{\iota^* \otimes 1} & H^*(E) \otimes H^*(F). \end{array}$$

Then, since $\sigma = \delta^{*-1} \circ p^*$, $(\delta^* \otimes 1)(\mu^*(\sigma(x))) = \mu^*p^*(x) = p^*(x) \otimes 1 = (\delta^* \otimes 1)(\sigma(x) \otimes 1)$. Therefore there is $y \in H^*(F)$ such that $\mu^*(\sigma(x)) = \sigma(x) \otimes 1 + 1 \otimes y$. It is clear that $y = \sigma(x)$. Q.E.D.

THEOREM 2.17. (1) *If a quasi Hopf algebra A , which is free of finite type, has a simple system of primitive generators: $A = \Delta(x_1, x_2, \dots)$, then A^* is a commutative Hopf algebra and $A^* = \Lambda(x_1^*, x_2^*, \dots)$ for x_i^* dual to x_i with respect to the basis $\{x_{i_1} \cdots x_{i_r}; i_1 < \cdots < i_r\}$ of A .*

(2) *Theorem 2.7 holds if one replaces Δ with Λ , while the generators are replaced by $e_{2i-1} = \sigma(c_i)^*$, $e_{4i-1} = \sigma(q_i)^*$, $e_{4i-1} = \sigma(p_i)^*$, $e_{2n-1} = \sigma(\chi)^*$, $e_i = \sigma(w_{i+1})^*$ respectively.*

PROOF. (1) First, $\phi(x_i) = x_i \otimes 1 + 1 \otimes x_i$ implies $T\phi(x_i) = \phi(x_i)$ and $(\phi \otimes 1)\phi(x_i) = (1 \otimes \phi)\phi(x_i)$. Since ϕ is a homomorphism of rings and since A is generated by $\{x_i\}$, A is cocommutative and coassociative hence A^* is a commutative Hopf algebra. If we put $x_J = x_1^{j_1} x_2^{j_2} \cdots$ for a sequence $J = (j_1, j_2, \dots)$ with only a finite number of nonzero j_i , where j_i is 0 or 1, then $\{x_J\}$ is a basis of A . $\phi(x_i) = x_i \otimes 1 + 1 \otimes x_i$ implies $\phi(x_J) = \sum_{J+K=I} \pm x_J \otimes x_K$. Similarly when we put $x_J^* = (x_1^*)^{j_1} (x_2^*)^{j_2} \cdots$, we will show $\langle x_J^*, x_I \rangle = \pm \delta_{JI}$ by induction on the size $|J| = \sum j_i$ of J . If $x_L = x_M x_S$, then $\langle x_L^*, x_I \rangle = \langle \phi^*(x_M^* \otimes x_S^*), x_I \rangle = \langle x_M^* \otimes x_S^*, \phi(x_I) \rangle = \sum \pm \langle x_M^*, x_J \rangle \langle x_S^*, x_K \rangle$, where $\langle x_S^*, x_K \rangle$ is not 0 only if $x_S = x_K$. In addition when $x_S = x_K$, by the inductive hypothesis, $\langle x_M^*, x_J \rangle \neq 0$ only if $M = J$ whence $L = I$. Therefore $\langle x_J^*, x_I \rangle = \pm \delta_{JI}$ and so $\{\pm x_J^*\}$ is a dual basis thus $A^* = \Delta(x_1^*, x_2^*, \dots)$.

$\langle (x_i^*)^2, x_I \rangle = \sum_{J+K=I} \pm \langle x_i^*, x_J \rangle \langle x_i^*, x_K \rangle$ is not 0 only if $x_J = x_K = x_i$, but there is no such I that $I = J + K$, $J = K = i$. Therefore $(x_i^*)^2 = 0$ from which it follows that $A^* = \Lambda(x_1^*, x_2^*, \dots)$.

(2) is deduced from (1) by using Lemma 2.16 and also the results on $H^*(G; R)$ in Chapter III. Q.E.D.

DEFINITION. We denote by

$$s_n(\sigma_1, \dots, \sigma_n) \in \mathbb{Z}[\sigma_1, \dots, \sigma_n]$$

the polynomial through which the symmetric polynomial $t_1^n + t_2^n + \cdots + t_m^n$ ($1 \leq n \leq m$) is expressed in terms of elementary symmetric polynomials σ_i ($1 + \sum_{i=1}^m \sigma_i = \prod_{i=1}^m (1 + t_i)$). (The definition is independent of m .)

THEOREM 2.18. *Let the polynomial ring $A = R[x_1, x_2, \dots]$ be a commutative Hopf algebra with $\phi(x_i) = \sum_{j+k=i} x_j \otimes x_k$ ($x_0 = 1$), where $x_i \in A_{\text{id}}$ for $i = 1, 2, \dots$; $d > 0$. Then*

(1) *if one takes $y_i = (x_i^1)^*$ for the dual basis with respect to the monomials in x_i , one has*

$$A^* = R[y_1, y_2, \dots], \quad \mu^*(y_i) = \sum_{j+k=i} y_j \otimes y_k \quad \text{where } y_0 = 1.$$

(2) $s_n(x_1, \dots, x_n)$ is primitive. Any primitive element of A is a constant multiple of $s_n(x_1, \dots, x_n)$ if $R = \mathbb{Z}$ or $\mathbb{Q}$.

PROOF. By the form of ϕ , A is coassociative and cocommutative hence A^* is a commutative Hopf algebra. Let $I = (\phi)$ or $I = (i_1, \dots, i_r)$ be a sequence of integers $1 \leq i_1 \leq \dots \leq i_r$. If we put $x_I = x_{i_1} \cdots x_{i_r}$, $\{x_I\}$ is a basis of A . Similarly we put $y_J = y_{j_1} \cdots y_{j_s}$ for $J = (j_1, \dots, j_s)$.

We define the correspondence $I \mapsto \lambda(I)$ inductively as follows. Define $\lambda(\phi) = \phi$. Let $I = (i_1, \dots, i_r)$; if $\lambda(I_r) = (j_1, \dots, j_s)$ for a sequence I_r obtained by deleting the zero terms from $(i_1 - 1, \dots, i_r - 1)$, we define $\lambda(I) = (j_1, \dots, j_s, r)$. It is easy to see that λ is a one to one correspondence. Moreover we introduce an order by

$$I > I' = (i'_1, \dots, i'_t) \text{ if } t < r \text{ or, } t = r \text{ and there is } k \text{ such that} \\ i_r = i'_r, \dots, i_{k+1} = i'_{k+1}, i_k > i'_k.$$

Then $\phi(x_I) = x_I \otimes x'_I + (\text{other terms})$, $I > I'$ implies that $\phi(x_{I'}) = x_{I'} \otimes x'_I + (\text{other terms})$ or does not contain a term of the form $* \otimes x'_I$. If there is I'_r ($t = r$), $I > I'$ implies $I_r > I'_r$. These facts show inductively that

$$\text{if } \langle y_{\lambda(I)}, x_I \rangle = 1, \text{ and } I > I', \text{ then } \langle y_{\lambda(I)}, x_{I'} \rangle = 0.$$

This shows that $\{y_J\}$ forms a basis of A^* , since λ is bijective.

We have $\langle \mu^*(y_i), x_J \otimes x_K \rangle = \langle y_i, x_J x_K \rangle$. This is not 0, only if $x_J = x_i^j$ and $x_K = x_i^k$ with $j + k = i$, and we have $\mu^*(y_i) = \sum_{j+k=i} y_j \otimes y_k$.

(2) First we recall the following Newton formula for $s_n = s_n(x_1, \dots, x_n)$:

$$s_n = \sum_{i=1}^{n-1} (-1)^{i-1} x_i s_{n-i} + (-1)^{n-1} n x_n. \quad (2.18')$$

Obviously $s_1 = x_1$ is primitive. By induction on n we have

$$\begin{aligned} \phi(s_n) &= \sum_{i=1}^{n-1} (-1)^{i-1} \phi(x_i) \phi(s_{n-i}) + (-1)^{n-1} \phi(x_n) \\ &= \sum_{i=1}^{n-1} \sum_{j+k=i} [(-1)^{j-1} x_j s_{n-i} \otimes (-1)^k x_k + (-1)^k x_k \otimes (-1)^{j-1} x_j s_{n-i}] \\ &\quad + \sum_{k=1}^{n-1} (-1)^k [(-1)^{n-k-1} (n-k) x_{n-k} \otimes x_k k \\ &\quad \quad \quad + x_{\otimes} (-1)^{n-k-1} (n-k) x_{n-k}] \\ &= s_n \otimes 1 + 1 \otimes s_n + \sum_{k=0}^{n-1} (-1)^k (A_{n-k} \otimes x_k + x_k \otimes A_{n-k}), \end{aligned}$$

where

$$A_{n-k} = \sum_{j=0}^{n-k-1} (-1)^{j-1} x_j s_{n-k-j} + (-1)^{n-k-1} (n-k) x_{n-k}$$

and $A_{n-k} = 0$ by (2.18'). So s_n is primitive. When $R = \mathbf{Q}$, we have $A = \mathbf{Q}[s_1, s_2, \dots]$ by (2.18'). Then any primitive element is a scalar multiple of s_i , since $\phi(s_i s_j)$ contains the term $s_i \otimes s_j$ but the term does not come out of any monomials different from $s_i s_j$. Restricting to $\mathbf{Z}[x_1, x_2, \dots]$ we obtain the latter half of (2). Q.E.D.

Cohomology of Spinor groups. Since the inclusion $SO(n) = (\text{a point}) \times_G \text{Spin}(n) \rightarrow EZ_2 \times \text{Spin}(n)$ in the universal Z_2 -bundle ($G = Z_2$) is a homotopy equivalence, the associated bundle $\text{Spin}(n) \rightarrow EZ_2 \times_G \text{Spin}(n) \rightarrow BZ_2$ may be considered as a principal fibration of the following form:

$$\text{Spin}(n) \xrightarrow{p} SO(n) \xrightarrow{f} BZ_2,$$

where p is the covering map and f is a map inducing the covering map.

As is already known,

$$H^*(BZ_2; \mathbf{Z}_2) = \mathbf{Z}_2[t] \quad \text{and} \quad H^*(SO(n); \mathbf{Z}_2) = \Delta(e_1, e_2, \dots, e_{n-1}),$$

where $e_i = \sigma w_{i+1}$. We denote $e_i = p^* e_i \in H^*(\text{Spin}(n); \mathbf{Z}_2)$ using the same notation.

THEOREM 2.19. (1) Let $s = s(n)$ be an integer such that $2^{s-1} < n \leq 2^s$. Then there is $z \in H^{2^{s-1}}(\text{Spin}(n); \mathbf{Z}_2)$ satisfying $\tau(z) = t^{2^s}$ such that

$$H^*(\text{Spin}(n); \mathbf{Z}_2) = \Delta(z, e_i \text{ for } i < n, i \neq 2^r).$$

In addition $Sq^j e_i = \binom{j}{i} e_{i+j}$, in particular $e_i^2 = e_{2i}$, where we let $e_i = 0$ for $i \geq n$ or $i = 2^r$.

(2) Let $\mathbf{k}$ be a field of characteristic $\neq 2$, then

$$p^*: H^*(SO(n); \mathbf{k}) \cong H^*(\text{Spin}(n); \mathbf{k}).$$

PROOF. (1) By induction on $n \geq 3$. The case $n = 3$ is clear, since $\text{Spin}(3) \cong Sp(1)$. Consider the spectral sequence (the coefficient field $\mathbf{Z}_2$ is omitted) associated with the fibre bundle $\text{Spin}(n) \xrightarrow{i} \text{Spin}(n+1) \xrightarrow{\pi} S^n$:

$$'E_2 = H^*(S^n) \otimes H^*(\text{Spin}(n)) \Rightarrow H^*(\text{Spin}(n+1)).$$

Since e_i for $i < n$ is in the image of $i^*: H^*(SO(n+1)) \rightarrow H^*(SO(n))$, so is $e_i \in H^*(\text{Spin}(n))$ and hence $'d_r(1 \otimes e_i) = 0$. Therefore the spectral sequence can be computed, once $'d_r(1 \otimes z)$ is determined.

First we consider the case $n < 2^s$ and suppose $'d_r(1 \otimes z) \neq 0$. Then we have $(\Lambda(s_n) \otimes \Delta(e_i \text{ for } i < n, i \neq 2^r))/(x) \cong 'E_\infty = \mathcal{E}rH^*(\text{Spin}(n+1))$ for degrees $< 2^s + 2$, where $\deg x = 2^s$. Now we consider the spectral sequence

$$E_2 = \mathbf{Z}_2[t] \otimes H^*(\text{Spin}(n+1)) \Rightarrow H^*(SO(n+1)) = \Delta(e_1, e_2, \dots, e_n)$$

associated with $\text{Spin}(n+1) \xrightarrow{p} \text{SO}(n+1) \xrightarrow{f} \text{BZ}_2$. If we compare the Poincaré polynomials we can see $E_2 = E_\infty$ for degrees $< 2^s + 2$. Therefore $f^*: H^{2^s}(\text{BZ}_2) \rightarrow H^{2^s}(\text{SO}(n+1))$ is a monomorphism. Obviously $f^*(t) = e_1$ so $f^*(t^{2^s}) = e_1^{2^s} = e_{2^s} = 0$ for $n < 2^s$. This is a contradiction, hence $'d_r(1 \otimes z) = 0$, so $\{E_r\}$ collapses, and hence $H^*(\text{Spin}(n+1)) = \Delta(z, \pi^* s_n, e_i \text{ for } i < n, i \neq 2^r)$. Again by considering the spectral sequence $\{E_r\}$, it is seen that we can replace $\pi^* s_n$ by e_n and choose z so as to satisfy $\tau(z) = t^{2^s}$. $Sq^j e_i = \binom{j}{i} e_{i+j}$ holds in $\text{SO}(n+1)$, and hence holds in $\text{Spin}(n+1)$.

Next consider the case $n = 2^s$. If $'d_r(1 \otimes z) = 0$, the spectral sequence $\{E_r\}$ collapses, and $H^*(\text{Spin}(n+1)) = \Delta(\pi^* s_n, z, e_i \text{ for } i < n, i \neq 2^r)$. Observe that π decomposes as $\text{Spin}(n) \xrightarrow{p} \text{SO}(n) \xrightarrow{\pi'} S^n$. We have $\pi^* s_n - e_n \in \Delta(e_i \text{ for } i < n) \text{ in } H^*(\text{SO}(n+1))$. $n = 2^s$ implies $p^*(e_n) = 0$, whence $\pi^* s_n \in \Delta(e_i \text{ for } i < n, i \neq 2^r)$. This is a contradiction and so $'d_r(1 \otimes z) \neq 0$. Similarly to the case $n < 2^s$, we have $H^*(\text{Spin}(n+1)) \cong \Delta(\pi^* s_n, e_i \text{ for } i < n, i \neq 2^r)/(y)$ for degrees $< 2^s + 2$. By the above consideration, $y = \pi^* s_n + \text{decomposable}$ and $'d_n(1 \otimes z) = s_n \otimes 1 + \dots$ which implies $'E_{n+1} = \Lambda(s_n \otimes z) \otimes \Delta(e_i \text{ for } i < n, i \neq 2^r)$. If we further had $'d_n(s_n \otimes z) \neq 0$, we would get a contradiction, by a similar discussion in degrees $< 2^{s+1} + 2$ to the case $n < 2^s$. Hence $'d_r(s_n \otimes z) = 0$, so $'E_{n+1} = E_\infty$, and hence $H^*(\text{Spin}(n+1)) = \Delta(\bar{z}, e_i \text{ for } i < n, i \neq 2^r)$. Observe that $\bar{z} \in H^{2n-1}$ corresponds to $s_n \otimes z$ and $2n-1 = 2^{s+1} = s^{s(n+1)} - 1$. Considering $\{E_r\}$, we obtain the result for the case $\text{Spin}(2^s + 1)$.

(2) is obtained from $H^*(\text{BZ}_2; \mathbf{k}) \cong \mathbf{k}$. Q.E.D.

LEMMA 2.20. *In the range of degrees $< n-1$ the following holds:*

(1) $H^*(\Omega \text{Spin}(n); \mathbf{Z}_2) \cong H^*(\Omega_0 \text{SO}(n); \mathbf{Z}_2) = \mathbf{Z}_2[x_2, x_6, \dots, x_{4k-2}, \dots]$ where $x_{4k-2} = \sigma(e_{4k-1})$ ($e_{4k-1} = \sigma(w_{4k})$).

(2) $H^*(\Omega_0 \text{SO}(n))$ is a free module and its Poincaré polynomial is

$$\prod_{k \geq 1} \frac{1}{1 - t^{4k-2}}.$$

PROOF. (1) We prove the lemma for $\Omega \text{Spin}(n)$, since $\Omega \text{Spin}(n) \cong \Omega_0 \text{SO}(n)$. By Theorem 2.19 we have

$$H^*(\text{Spin}(n); \mathbf{Z}_2) = \mathbf{Z}_2[e_3, e_5, e_7, \dots]$$

($e_{2i} = e_i^2$) for degrees $< n-1$. We will show that the natural map

$$\Delta(x_2, x_4, x_6, \dots) \rightarrow H^*(\Omega \text{Spin}(n); \mathbf{Z}_2)$$

is an isomorphism for degrees $< n-3$, where $x_{2i} = \sigma(e_{2i+1})$. To this end we need the comparison theorem (Theorem 3.15 of Chapter I, Mimura-Toda

[14] or Theorem 2.4 of Chapter VII). We introduce d_r in the formal spectral sequence

$$'E_2 = \mathbb{Z}_2[e_3, e_5, e_7, \dots] \otimes \Delta(x_2, x_4, x_6, \dots)$$

by the distributive law and by $'d_r(1 \otimes x_{2i}) = e_{2i+1} \otimes 1$ and then we have $'E_\infty \cong \mathbb{Z}_2$. Comparing it with the spectral sequence of $\Omega \text{Spin}(n) \rightarrow P \text{Spin}(n) \rightarrow \text{Spin}(n)$ we see that $H^*(\Omega \text{Spin}(n); \mathbb{Z}_2) = \Delta(x_2, x_4, x_6, \dots)$ for degrees $< n-3$. In addition we have $x_{2i}^2 = Sq^{2i} x_{2i} = Sq^{2i} \sigma(e_{2i+1}) = \sigma(Sq^{2i} e_{2i+1}) = \sigma(\binom{2i+1}{2i} e_{4i+1}) = \sigma(e_{4i+1}) = x_{4i}$ and hence we obtain (1).

(2) is obtained by applying the universal coefficient theorem to (1) and Theorem 2.11(2). Q.E.D.

COROLLARY 2.21. $H_*(\Omega_0 SO(n); \mathbb{Z}_2) = \Lambda(x_2^*, x_4^*, x_6^*, \dots)$ for degrees $< n-3$.

PROOF. The element $x_{2i} = \sigma(e_{2i+1})$ is primitive by Lemma 2.16 in

$$H^*(\Omega_0 SO(n); \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_6, x_{10}, \dots] = \Delta(x_2, x_4, x_6, \dots)$$

of Lemma 2.20. Since the proof of Lemma 2.17(1) holds even restricting the degree, the corollary is obtained. Q.E.D.

§3. Symmetric spaces of infinite dimension

Classical groups of infinite dimension. The definitions of classical groups and symmetric spaces of classical type studied in §6 of Chapter III can be extended to the infinite dimensions. Among their important properties is that they are H -spaces, and loop spaces of some other spaces under a suitable cyclic process so that the Bott periodicity theorem holds with period 2 or 8 and that they have a close relation with the K -theory.

In this section G denotes one of U, SU, O, SO, Sp . For example, $G(n)$ represents $U(n), SU(n), O(n), So(n), sp(n)$, as fitting.

By means of the correspondence $G(n) \ni A \mapsto A \oplus I_1 \in G(n+1)$, we obtain a sequence of compact Hausdorff spaces

$$G(1) \subset G(2) \subset \dots \subset G(n) \subset G(n+1) \subset \dots$$

DEFINITION. $G = \bigcup_n G(n)$ with the weak topology is called a *classical group of infinite dimension* (e.g., the *unitary group of infinite dimension*, the *orthogonal group of infinite dimension*, and so on).

From the definition of the weak topology (i.e., X is an open set of G iff $X \cap G(n)$ is an open set of $G(n)$) one obtains:

3.1. (1) F is a closed set in G iff $F \cap G(n)$ is a closed set in $G(n)$.

(2) $f: G \rightarrow Y$ is continuous iff $f|G(n): G(n) \rightarrow Y$ is continuous for each n .

(3) $f_i: G \rightarrow Y$ is a homotopy iff $f_i|G(n)$ is a homotopy for each n .

(4) C is a compact subset of G iff $C \subset G(n)$ for some n .

Since the restriction of the multiplication μ_{n+1} of $G(n+1)$ on $G(n)$ is the multiplication μ_n in $G(n)$ we can define a multiplication in G

$$\mu: G \times G \rightarrow G$$

by using 3.1(2) with $\mu|_{G(n) \times G(n)} = \mu_n$ so that μ is continuous. Similarly for the inverse element, one has

3.2. G is a topological group with multiplication μ above.

DEFINITION. Given an injection $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ between natural numbers we define $\psi_\alpha: G \rightarrow G$ by

$$\psi_\alpha(A) = B = (b_{ij}) \in G(n) \quad \text{for } A = (a_{kl}) \in G(m),$$

where

$$n \geq \text{Max}\{\alpha(i); 1 \leq i \leq m\} \text{ and } b_{ij} = \begin{cases} a_{kl} & (i = \alpha(k), j = \alpha(l)), \\ \delta_{ij} & (i \notin \text{Im } \alpha \text{ or } j \notin \text{Im } \alpha). \end{cases}$$

LEMMA 3.3. The maps ψ_α are monomorphisms and they are homotopic to one another preserving the base point (the identity). In particular, $\psi_\alpha \simeq 1_G: G \rightarrow G$.

PROOF. It is obvious that each ψ_α is injective. One easily checks that $\psi_\alpha(AB) = \psi_\alpha(A)\psi_\alpha(B)$ for $A, B \in G(n)$ and so ψ_α is a homomorphism. The continuity of ψ_α is by 3.1(2).

For the latter half it suffices to show $\psi_\alpha \simeq 1_G$. We call a bijection $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ a finite permutation if $j > m$ implies $\sigma(j) = j$ for sufficiently large m . The σ may be identified with a permutation of $\{1, \dots, m\}$. For a given α we choose a finite sequence of permutations $\sigma_1, \sigma_2, \dots$ satisfying the following conditions:

- (i) $\sigma_i(j) = j$ for $j < i$,
- (ii) σ_i is an even permutation,
- (iii) if we take sufficiently large $n = n(j)$ for given j , then $\sigma_n \cdots \sigma_2 \sigma_1(\alpha(j)) = j$.

The construction of σ_i is done inductively. Suppose we construct them so that $\sigma_n \cdots \sigma_2 \sigma_1(\alpha(j)) = j$ for $j \leq n = n(j)$. We can take an even permutation σ_{n+1} so that $\sigma_{n+1} \sigma_n \cdots \sigma_2 \sigma_1(\alpha(n+1)) = n+1$ and $\sigma_{n+1}(j) = j$ for $j \leq n$, since $\sigma_n \cdots \sigma_2 \sigma_1(\alpha(n+1)) \leq n+1$. We can then repeat this process.

If σ_i can be identified with a permutation of $\{1, \dots, m_i\}$, by (i) and (ii) we have $P_i \in 1 \times SO(m_i - i) \subset SO(m_i) \subset G(m_i)$, where $P_i = (\delta_{j\sigma_i(j)})$. Since $SO(m_i - i)$ is 0-connected, there is a path $w_i: I \rightarrow 1 \times SO(m_i - i) \subset G(m_i)$ connecting P_i and the identity. Consider P_i and w_i in G . We define a homotopy h_i by

$$h_i(A) = w_1(t) \cdots w_k(t) \psi_\alpha(A) w_k(t)^{-1} \cdots w_1(t)^{-1}, \quad (k \leq n)$$

for $A \in G(m)$ and $\psi_\alpha(A) \in G(n)$. Since $w_k(t)\psi_\alpha(A)w_k(t)^{-1} = \psi_\alpha(A)$ for $k > n$, h_t is determined independently of the choice of k and so induces a homotopy $h_t: G \rightarrow G$. Then $\psi_\alpha \simeq 1_G$, as $h_1 = \psi_\alpha$ and $h_0 = 1_G$ are easily checked. Q.E.D.

DEFINITION. For two injections $\alpha, \beta: N \rightarrow N$ we define

$$\psi_{\alpha, \beta}: G \times G \rightarrow G \quad \text{by } \psi_{\alpha, \beta} = \mu \circ (\psi_\alpha \times \psi_\beta).$$

THEOREM 3.4. (1) The maps $\psi_{\alpha, \beta}$ are homotopic to each other. In particular $\psi_{\alpha, \beta} \simeq \mu$.

(2) If $\text{Im } \alpha \cap \text{Im } \beta = \emptyset$, $\psi_{\alpha, \beta}$ is an injection commuting with the multiplication (a homomorphism).

(3) $(G, \psi_{\alpha, \beta})$ is a homotopy commutative H -group.

PROOF. (1) is easy from Lemma 3.3. In particular, $\alpha = \beta = 1_N$ implies $\psi_{\alpha, \beta} = \mu$.

(2) Clearly $\text{Im } \alpha \cap \text{Im } \beta = \emptyset$ implies $\psi_{\alpha, \beta}$ is an injection. In addition, since $\text{Im } \psi_\alpha$ and $\text{Im } \psi_\beta$ commute elementwise, we have

$$\begin{aligned} \psi_{\alpha, \beta}(xy) &= \psi_{\alpha, \beta}((x_1, x_2)(y_1, y_2)) = \psi_\alpha(x_1 y_1) \psi_\beta(x_2 y_2) \\ &= \psi_\alpha(x_1) \psi_\alpha(y_1) \psi_\beta(x_2) \psi_\beta(y_2) = \psi_\alpha(x_1) \psi_\beta(x_2) \psi_\alpha(y_1) \psi_\beta(y_2) \\ &= \psi_{\alpha, \beta}(x) \psi_{\alpha, \beta}(y) \end{aligned}$$

for $x = (x_1, x_2) \in G \times G$ and $y = (y_1, y_2) \in G \times G$.

(3) (1) implies $\psi_{\alpha, \beta} \simeq \psi_{\beta, \alpha}$, which proves homotopy commutativity. If $\text{Im } \alpha$, $\text{Im } \beta$, and $\text{Im } \gamma$ do not intersect one another, then

$$\psi_{\alpha, \beta} \circ (1 \times \psi_{\beta, \gamma}) = \psi_{\beta, \gamma} \circ (\psi_{\alpha, \beta} \times 1)$$

holds. So we can obtain homotopy associativity by (1). By Lemma 1.23 $(G, \psi_{\alpha, \beta})$ is an H -group. (One can use $\mu \circ (1 \times \mu) = \mu \circ (\mu \times 1)$ but we prove it in this form for later use.) Q.E.D.

Symmetric spaces of infinite dimension. A symmetric space of infinite dimension is obtained as the limit of symmetric spaces of classical type in §6 of Chapter III. We define symmetric spaces of infinite dimension considering two distinct cases.

Let $H(n) \subset G(n)$ be one of the following inclusions:

$$SO(n) \subset SU(n) \subset U(n) \subset Sp(n), \quad SO(n) \subset O(n) \subset U(n).$$

Then we have

$$H(n) = G(n) \cap H(n+1)$$

for $H(n) = H(n) \times 1 \subset H(n+1)$ and $G(n) = G(n) \times 1 \subset G(n+1)$ and hence the natural map $G(n)/H(n) \rightarrow G(n+1)/H(n+1)$ is injective. By this map $G(n)/H(n)$ may be considered as a closed subspace of $G(n+1)/H(n+1)$, since the former is a compact Hausdorff space.

DEFINITION. The space $\bigcup_n G(n)/H(n)$ with weak topology is called the quotient space of G by H , and denoted by $G/H = \bigcup_n G(n)/H(n)$.

The inclusion $H(n) \rightarrow G(n)$ and the projection $G(n) \rightarrow G(n)/H(n)$ induce

$$i: H \rightarrow G \quad \text{and} \quad \pi: G \rightarrow G/H$$

respectively. They are also called the *inclusion* and the *projection*. We use the following notations for the particular G and H to distinguish:

$$i_C: O \rightarrow U, \quad SO \rightarrow SU, \quad SO \rightarrow U, \\ i_H: U \rightarrow Sp, \quad SU \rightarrow Sp.$$

THEOREM 3.5. Let α and $\beta: N \rightarrow N$ be monotone increasing.

(1) $\psi_\alpha: G \rightarrow G$ induces $\psi_\alpha: G/H \rightarrow G/H$ so that the diagram

$$\begin{array}{ccccc} H & \xrightarrow{i} & G & \xrightarrow{\pi} & G/H \\ \downarrow \psi_\alpha & & \downarrow \psi_\alpha & & \downarrow \psi_\alpha \\ H & \xrightarrow{i} & G & \xrightarrow{\pi} & G/H \end{array}$$

is commutative. The maps $\psi_\alpha: G/H \rightarrow G/H$ are homotopic to each other and in particular $\psi_\alpha \simeq 1_{G/H}$.

(2) If $\text{Im } \alpha \cap \text{Im } \beta = \emptyset$, then $\psi_{\alpha, \beta}: G \times G \rightarrow G$ induces $\psi_{\alpha, \beta}: (G/H) \times (G/H) \rightarrow G/H$, and the maps $\psi_{\alpha, \beta}$ are homotopic to each other in G/H .

(3) $(G/H, \psi_{\alpha, \beta})$ is a homotopy commutative H -group, and $i: H \rightarrow G$ and $\pi: G \rightarrow G/H$ are H -maps.

PROOF. (1) Since

$$\psi_\alpha(n) = \psi_\alpha|G(n): G(n) \rightarrow G(\alpha(n))$$

satisfies $\psi_\alpha(n)(H(n)) \subset H(\alpha(n))$ it induces

$$\psi_\alpha(n): G(n)/H(n) \rightarrow G(\alpha(n))/H(\alpha(n))$$

so that $\psi_\alpha(n+1)|G(n)/H(n) = \psi_\alpha(n)$. Therefore there exists $\psi_\alpha: G/H \rightarrow G/H$ satisfying $\psi_\alpha|G(n)/H(n) = \psi_\alpha(n)$ so that the diagram (1) is commutative. We prove $\psi_\alpha \simeq 1_{G/H}$ following that of Lemma 3.3. One notices here, however, that a homotopy $h_t: G/H \rightarrow G/H$ is given by $h_t(\{A\}) = \{w_1(t) \cdots w_k(t)\psi_\alpha(A)\}$, since $w_i(t) \in SO(m_i) \subset H(m_i)$.

The proof of (2) and (3) is similar to that of Theorem 3.4. One only observes that the equality

$$\psi_{\alpha, \beta}(\{A\}, \{A'\}) = \{\psi_\alpha(A)\psi_\beta(A')\}$$

defining $\psi_{\alpha, \beta}: G/H \times G/H \rightarrow G/H$ is independent of the choice of representatives $A, A' \in G(n)$. We have

$$\begin{aligned} \psi_\alpha(AB)\psi_\beta(A'B') &= \psi_\alpha(A)\psi_\beta(A')\psi_\alpha(B)\psi_\beta(B') \\ &\equiv \psi_\alpha(A)\psi_\beta(A') \bmod H(\text{Max}(\alpha(n), \beta(n))), \end{aligned}$$

since $\psi_\alpha(B)$ commutes with $\psi_\beta(A')$ for $B, B' \in H(n)$. Q.E.D.

The second cases are the limits of $G(2m)/G(m) \times G(m)$, $SO(2m)/U(m)$, $SU(2m)/Sp(m)$, etc., obtained from the inclusions

$$G(m) \times G(m) \rightarrow G(2m), \quad r: U(m) \rightarrow SO(2m), \\ c': Sp(m) \rightarrow SU(2m)$$

considered in §2 of Chapter I. For these cases they do not commute with the ordinary inclusion $G(m) = G(m) \times 1 \subset G(m+k)$ and so we perform the operation to interchange the coordinates.

NOTATION. $\tau_m = \psi_\alpha: G(2m) \rightarrow G(2m)$, where $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ is an injection satisfying $\alpha(i) = 2i - 1$ and $\alpha(m+i) = 2i$ for $1 \leq i \leq m$.

Observe that $\tau = \alpha\{1, 2, \dots, 2m\}$ is a permutation of $\{1, 2, \dots, 2m\}$. We have

$$\tau_m(A) = P_\tau^{-1} A P_\tau, \quad A \in G(2m)$$

for a permutation matrix $P_\tau = (\delta_{i\tau(j)})$ and hence τ_m is an automorphism. Also we have

$$\begin{aligned} \tau_{m+1}((A \oplus a) \oplus (B \oplus b)) &= \tau_m(A \oplus B) \oplus a \oplus b, \\ A, B &\in G(m), a, b \in G(1); \\ \tau_{m+1}(r(A \oplus a)) &= \tau_m(r(A)) \oplus a \oplus a, A \in U(m), a \in O(1); \\ \tau_{m+1}(c'(A \oplus a)) &= \tau_m(c'(A)) \oplus a \oplus \bar{a}, \in Sp(m), a \in U(1). \end{aligned} \quad (3.6)$$

Let $K(m)$ be a subgroup of $G(2m)$, which is of the following:

$$\begin{aligned} K(m) &= \tau_m(G(m) \times G(m)), & G &= SO, O, SU, U, Sp; \\ K(m) &= \tau_m(r(H(m))), & G &= O, SO; \quad H = U, SU; \\ K(m) &= \tau_m(c'(H(m))), & G &= U, SU; \quad H = Sp. \end{aligned}$$

We can then regard $G(2m)/K(m) \subset G(2m+2)/K(m+1)$, since we have

$$K(m) = G(2m) \cap K(m+1)$$

by (3.6).

DEFINITION. The space $\bigcup_m G(2m)/K(m)$ with the weak topology is called the quotient space of G by $G \times G$ or H , denoted as follows:

$$\begin{aligned} BG = G/H &= \bigcup_m G(2m)/\tau_m(G(m) \times G(m)) \\ &\quad (\text{the classifying space of } G); \quad H \cong G \times G \\ G/H &= \bigcup_m G(2m)/\tau_m(r(H(m))), \quad G = O, SO; \quad H \cong U, SU \\ G/H &= \bigcup_m G(2m)/\tau_m(c'(H(m))), \quad G = U, SU; \quad H \cong Sp. \end{aligned}$$

If we consider $\tau_m: G(m) \times G(m) \rightarrow G(2m)$, $\tau_m \circ r: H(m) \rightarrow G(2m)$, and $\tau_m \circ c': H(m) \rightarrow G(2m)$ in accordance with the above cases, by (3.6) they

induce inclusions $i_G, i_R, i_{C'}$, and the projections π in infinite dimension:

$$G \times G \xrightarrow{i_G} G \xrightarrow{\pi} BG; H \xrightarrow{i_R} G \xrightarrow{\pi} G/H, \text{ for } G = O, SO;$$

$$H = USU; \quad H \xrightarrow{i_{C'}} G \xrightarrow{\pi} G/H, \text{ for } G = U, SU; \quad H = Sp.$$

THEOREM 3.7. Let $\alpha, \beta: \mathbb{N} \rightarrow \mathbb{N}$ be monotone increasing. We define $\bar{\gamma}: \mathbb{N} \rightarrow \mathbb{N}$ by $\bar{\gamma}(2i-1) = 2\gamma(i)-1$, $\bar{\gamma}(2i) = 2\gamma(i)$ for $\gamma = \alpha, \beta$. For the case BG , we take $H = G \times G$, $BG = G/H$ and regard $\psi_\alpha \times \psi_\alpha: G \times G \rightarrow G \times G$ as ψ_α .

(1) $\psi_\alpha: G \rightarrow G$ induces $\psi_\alpha: G/H \rightarrow G/H$ so that the diagram

$$\begin{array}{ccccc} H & \xrightarrow{i} & G & \xrightarrow{\pi} & G/H \\ \downarrow \psi_\alpha & & \downarrow \psi_\alpha & & \downarrow \psi_\alpha & i = i_G, i_R, \text{ or } i_{C'} \\ H & \xrightarrow{i} & G & \xrightarrow{\pi} & G/H \end{array}$$

is commutative. The maps $\psi_\alpha: G/H \rightarrow G/H$ are homotopic to each other and in particular $\psi_\alpha \simeq 1_{G/H}$.

(2) If $\text{Im } \alpha \cap \text{Im } \beta = \emptyset$, then $\psi_{\alpha, \bar{\beta}}$ induces $\psi_{\alpha, \bar{\beta}}: (G/H) \times (G/H) \rightarrow G/H$ and the maps $\psi_{\alpha, \bar{\beta}}$ are homotopic to each other in G/H .

(3) $(G/H, \psi_{\alpha, \bar{\beta}})$ is a homotopy commutative H -group, $i: H \rightarrow G$, $\pi: G \rightarrow G/H$ are H -maps.

PROOF. Similar to the proof of Theorem 3.5. Q.E.D.

Among the spaces of infinite dimension defined so far, the following

$$\begin{array}{l} SU/SO, \quad U/SO, \quad U/O, \quad SU/Sp, \quad U/Sp, \\ BO, \quad BU, \quad BSp, \quad SO/U, \quad SO/SU, \quad U/Sp, \quad SU/Sp \end{array}$$

are generally called *symmetric spaces of infinite dimension*, since they are obtained as limits of the symmetric spaces (of classical type) in §6 of Chapter III. These spaces have the homotopy type of CW -complexes, as they are defined as unions of compact manifolds with weak topology (Lemma 6.4 of Chapter II). Thus a weak homotopy equivalence between them is a homotopy equivalence.

H-maps of the classifying spaces of the classical groups of infinite dimension. The map

$$\rho: H(2n)/H(n) \times H(n) \rightarrow G(2n)/G(n) \times G(n)$$

induced from the inclusion $H(n) \subset G(n)$ considered in Theorem 3.5 commutes with τ_m . Using (3.6) the map ρ induces a map

$$\rho: BH = \bigcup_n H(2n)/\tau_n(H(n) \times H(n)) \rightarrow BG = \bigcup_n G(2n)/\tau_n(G(n) \times G(n)).$$

In particular we use the following notation:

$$\rho_C: \mathbf{BO} \rightarrow \mathbf{BU}, \quad \mathbf{BSO} \rightarrow \mathbf{BU}, \quad \mathbf{BSO} \rightarrow \mathbf{BSU};$$

$$\rho_H: \mathbf{BU} \rightarrow \mathbf{BSp}, \quad \mathbf{BSU} \rightarrow \mathbf{BSp}.$$

Similarly, we see that the composition $\tau_{2n} \circ (\tau_n \times \tau_n) \circ \sigma_n \circ \varphi \circ \tau_n^{-1}: H(2n) \rightarrow G(4n)$ induces

$$\rho: \mathbf{BH} = \bigcup_n H(2n)/\tau_n(H(n) \times H(n)) \rightarrow \mathbf{BG} = \bigcup_n G(4n)/\tau_{2n}(G(2n) \times G(2n)),$$

where $\varphi = \mathbf{r}: H(n) \subset G(2n)$ (for $H = U, SU$; $G = O, SO$), $\varphi = \mathbf{bg}: H(n) \subset G(2n)$ (for $H = Sp$; $G = U, SU$) and $\sigma_n: G(4n) \rightarrow G(4n)$ is given by $\sigma_n(A) = S_n A S_n^{-1} = S_n A S_n$, ($A \in G(4n)$) with

$$S_n = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} \in O(4n).$$

In particular we use the following notation:

$$\rho_R: \mathbf{BU} \rightarrow \mathbf{BSO} (\rightarrow \mathbf{BO}), \quad \mathbf{BSU} \rightarrow \mathbf{BSO} (\rightarrow \mathbf{BO});$$

$$\rho_{C'}: \mathbf{BSp} \rightarrow \mathbf{BSU} (\rightarrow \mathbf{BU}).$$

Since $\varphi: H(n) \subset G(2n)$ in the above satisfies $\varphi(G(n)) \subset G(n) \times G(n)$, we see by (3.6) that $\tau_n \circ \varphi$ induces

$$\mathbf{j}: \mathbf{H}/\mathbf{G} = \bigcup_n H(n)/G(n) \rightarrow \mathbf{BG} = \bigcup_n G(2n)/\tau_n(G(n) \times G(n)).$$

In particular,

$$\mathbf{j}_C: \mathbf{U}/\mathbf{O} \rightarrow \mathbf{BO}, \quad \mathbf{U}/\mathbf{SO} (\mathbf{SU}/\mathbf{SO}) \rightarrow \mathbf{BSO};$$

$$\mathbf{j}_H: \mathbf{Sp}/\mathbf{U} \rightarrow \mathbf{BU}, \quad \mathbf{Sp}/\mathbf{SU} \rightarrow \mathbf{BSU}.$$

Next we consider the inclusion $O(2n) \subset U(n)$ in $\mathbf{r}(U(n))$. If we put $\eta_n = (1/\sqrt{2}) \begin{pmatrix} I_n & I_n \\ iI_n & I_n \end{pmatrix}$, then $A \in U(n)$ implies $\eta_n(\mathbf{r}A)\eta_n^{-1} = A \oplus \bar{A}$ (2.7(3) of Chapter I). Thus, if we define $j_n: O(2n) \rightarrow U(2n)$ by $j_n(A) = \eta_n A \eta_n^{-1}$, then $\tau_n \circ j_n \circ \tau_n^{-1}$ is a map $O(2n)/\tau_n(\mathbf{r}U(n)) \rightarrow U(2n)/\tau_n(U(n) \times U(n))$ and hence we can define

$$\mathbf{j}_R: \mathbf{O}/\mathbf{U} (\mathbf{SO}/\mathbf{U}) \rightarrow \mathbf{BU}, \quad \mathbf{SO}/\mathbf{SU} \rightarrow \mathbf{BSU}$$

as its limits, where the equality

$$\tau_{n+1}[\eta_{n+1}(\tau_{n+1}^{-1}(A \oplus I_2))\eta_{n+1}^{-1}] = \tau_n[\eta_n(\tau_n^{-1}A)\eta_n^{-1}] \oplus I_2$$

is used in place of (3.6). By similar considerations for the inclusion $U(2n) \subset Sp(n)$, $\mathbf{c}(Sp(n))$, $\eta_n = (1/\sqrt{2}) \begin{pmatrix} I_n & I_n \\ iI_n & I_n \end{pmatrix}$ and $j_n: U(2n) \rightarrow Sp(2n)$ given by $j_n(A) = \eta_n A \eta_n^{-1}$, we can define

$$\mathbf{j}_{C'}: \mathbf{U}/\mathbf{Sp} (\mathbf{SU}/\mathbf{Sp}) \rightarrow \mathbf{BSp}.$$

Similarly to Theorems 3.5 and 3.7 the following holds:

3.8. $\rho_C, \rho_H, \rho_R, \rho_{C'}, \mathbf{j}_C, \mathbf{j}_H, \mathbf{j}_R, \mathbf{j}_{C'}$ are H -maps.

Now it is necessary to relate $\mathbf{BH}$ and $BH(n)$. For the present $BH(n)$ can be defined as $\bigcup_n H(n+N)/H(n) \times H(N)$ but if $H(n) \subset G(an)$ it is also defined as $\bigcup_n G(an+N)/H(n) \times G(N)$, and they are homotopy equivalent by a natural map (Lemma 6.4 and (6.3) of Chapter II). We denote by

$$i: BG(n) = \bigcup_N G(n+N)/G(n) \times G(N) \rightarrow \mathbf{BG} = \bigcup_n G(2n)/\tau_n(G(n) \times G(n))$$

the natural map of $BG(n)$ to $\mathbf{BG}$ induced from $\psi_\alpha: G(n+N) \rightarrow G(2n+N)$ where $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ is given by $\alpha(i) = 2i - 1$ for $1 \leq i \leq n$ and $\alpha(2i) = 2(i - n)$ for $i > n$.

By putting $a = 1$ for $\rho = \rho_C, \rho_H$ and $a = 2$ for $\rho = \rho_R, \rho_{C'}$, we obtain the following commutative diagram:

$$\begin{array}{ccc} BH(n) = \bigcup_N \frac{H(n+N)}{H(n) \times H(N)} & \xrightarrow{\quad} & \bigcup_N \frac{G(an+N)}{H(an) \times G(N)} \xrightarrow{\rho} BG(n) = \bigcup_N \frac{G(an+N)}{G(an) \times G(N)} \\ \downarrow i & & \downarrow i \\ \mathbf{BH} = \bigcup_n H(2n)/\tau_n(H(n) \times H(n)) & \xrightarrow{\rho} & \mathbf{BG} = \bigcup_n G(2an)/\tau_{an}(G(an) \times G(an)). \end{array} \quad (3.9)$$

As for $\mathbf{j}: \mathbf{G}/\mathbf{H} \rightarrow \mathbf{BH}$, by putting $a = 1$ for $\mathbf{j} = \mathbf{j}_C, \mathbf{j}_H$ and $a = 2$ for $\mathbf{j} = \mathbf{j}_R, \mathbf{j}_{C'}$, we obtain the following homotopy commutative diagram:

$$\begin{array}{ccc} G(an)/H(n) \xrightarrow{i} \bigcup_N \frac{G(an+N)}{H(n) \times G(N)} & \xrightarrow{\simeq} & BH(n) = \bigcup_N \frac{H(n+N)}{H(n) \times H(N)} \\ \downarrow i & & \downarrow i \\ \mathbf{G}/\mathbf{H} = \bigcup G(an)/H(n) & \xrightarrow{j} & \mathbf{BH} = \bigcup H(2n)/\tau_n(H(n) \times H(n)). \end{array} \quad (3.9')$$

Here, $G(2n)/H(n)$ is taken to be $G(2n)/\tau_n(\varphi H(n))$ for $\mathbf{j} = \mathbf{j}_R, \mathbf{j}_{C'}$ but there is no essential difference.

DEFINITION. $\sigma_{G(n)}: G(n) \rightarrow \Omega BG(n)$ is defined by

$$\sigma_{G(n)}(A)(t) = \{S_{t/2}(n)(A \oplus I_n)S_{t/2}(n)^{-1}\} \in G(2n)/G(n) \times G(n) \subset BG(n)$$

for $A \in G(n)$ and $t \in I$. $\sigma_G: \mathbf{G} \rightarrow \Omega(\mathbf{BG})$ is defined by

$$\sigma_G(A)(t) = \tau_n(\sigma_{G(n)}(A)(t)) \in G(2n)/\tau_n(G(n) \times G(n)) \subset \mathbf{BG},$$

where

$$S_t(n) = \begin{pmatrix} \cos \pi t I_n & -\sin \pi t I_n \\ \sin \pi t I_n & \cos \pi t I_n \end{pmatrix} \in SO(2n) \subset G(2n).$$

We define the suspension homomorphism $\sigma: H^{i+1}(\mathbf{BG}; R) \rightarrow H^i(\mathbf{G}; R)$ in $\mathbf{BG}$ by the composition $\sigma_G^* \circ \sigma: H^{i+1}(\mathbf{BG}; R) \rightarrow H^i(\Omega \mathbf{BG}; R) \rightarrow H^i(\mathbf{G}; R)$.

THEOREM 3.10. (1) $\sigma_{G(n)}$ and σ_G are (weak) homotopy equivalences. σ_G is an H -map.

(2) The following diagram is commutative:

$$\begin{array}{ccccc} G(n) & \xrightarrow{\sigma_{G(n)}} & \Omega BG(n) & & H^{i+1}(\mathbf{BG}; R) \xrightarrow{\sigma} H^i(\mathbf{G}; R) \\ \downarrow i & & \downarrow \Omega i & & \downarrow i^* \\ \mathbf{G} & \xrightarrow{\sigma_G} & \Omega \mathbf{BG} & & H^{i+1}(BG(n); R) \xrightarrow{\sigma} H^i(G(n); R). \end{array}$$

PROOF. (1) Since we have the commutative diagram

$$\begin{array}{ccc} \pi_{i+1}(BG(n)) & \xrightarrow[\cong]{\Delta} & \pi_i(G(n)) \\ & \searrow \Delta & \downarrow \sigma_{G(n)*} \\ & & \pi_i(\Omega BG(n)) \end{array}$$

$\sigma_{G(n)}$ is a weak homotopy equivalence. Similarly for σ_G .

The latter half is similar to that of Theorem 3.5(3).

(2) is easy. Q.E.D.

Cohomology of symmetric spaces of infinite dimension

LEMMA 3.11. *The homomorphisms induced from the inclusions i below are isomorphisms of homotopy groups $\pi_j(\quad)$, homology groups $H_j(\quad; R)$ and cohomology groups $H^j(\quad; R)$ for $j < N$ and the epimorphisms of homotopy groups and homology groups and monomorphisms of cohomology groups for $j = N$, where N is the integer given below for each i .*

- (1) $i: G(n) \rightarrow G$, $N = d(n+1) - 1$ ($d = 1$ for $G = O, SO$, $d = 2$ for $G = SU, U$, and $d = 4$ for $G = Sp$).
- (2) $i: G(n)/H(n) \rightarrow G/H$, $N = d(n+1) - 1$ ($d = 1$ for $G = U, SU$, $H = O, SO$, and $d = 2$ for $G = Sp$, $H = U, SU$).
- (3) $i = \tau_n: G(2n)/H(n) \rightarrow G/H$, $N = d(2n+1) - 1$ ($d = 1$ for $G = O, SO$, $H = U, SU$, and $d = 2$ for $G = U, SU$, $H = Sp$).
- (4) $i: BG(n) \rightarrow BG$, $N = d(n+1) - 1$ (d is same as in (1)).

PROOF. If we prove the results for homotopy groups, then the ones for (co)homology groups are easily deduced from Theorem 1.12(1). By 3.1, $\pi_i(G)$ is isomorphic to $\pi_i(G(m))$ for sufficiently large m . Similarly for G/H . Thus it suffices to show the lemma replacing G and G/H by $G(n+1)$, $G(n+1)/H(n+1)$, $G(2n+2)/H(n+1)$. (1) is clear from Theorem 3.15 of Chapter II. By Theorem 1.22, (1) implies (4).

To prove (2) we apply the 5-lemma to the following commutative and exact diagram:

$$\begin{array}{ccccc} \pi_j(H(m)) & \longrightarrow & \pi_j(G(m)) & \longrightarrow & \pi_j(G(m)/H(m)) \\ \downarrow i_{H*} & & \downarrow i_{G*} & & \downarrow i_* \\ \pi_j(H(m+1)) & \longrightarrow & \pi_j(G(m+1)) & \longrightarrow & \pi_j(G(m+1)/H(m+1)) \\ & \xrightarrow{\partial} & \pi_{j-1}(H(m)) & \longrightarrow & \pi_{j-1}(G(m)) \\ & & \downarrow i_{H*} & & \downarrow i_{G*} \\ & \xrightarrow{\partial} & \pi_{j-1}(H(m+1)) & \longrightarrow & \pi_{j-1}(G(m+1)) \end{array}$$

Similarly for (3). Q.E.D.

By means of this lemma one can generalize the results of Chapter III to the infinite-dimensional case. We use the same notation to denote elements corresponding to each other under isomorphisms in the lemma. One can also discuss results in the infinite-dimensional case within the framework of finite dimension. Observe that the isomorphism (1) in the lemma is one of Hopf algebras.

It follows from Theorems 6.5 and 6.6 of Chapter III and Lemma 2.16 that

3.12. *Every generator in the following is primitive:*

$$\begin{aligned} H^*(U) &= \Lambda(e_1, e_3, \dots, e_{2n-1}, \dots); \\ H^*(SU) &= \Lambda(e_3, e_5, \dots, e_{2n-1}, \dots); \\ H^*(Sp) &= \Lambda(e_3, e_7, \dots, e_{4n-1}, \dots); \\ H^*(SO; k) &= \Lambda(e_3, e_7, \dots, e_{4n-1}, \dots) \\ &\quad \text{for } k \text{ a field of characteristic } \neq 2; \\ H^*(SO; Z_2) &= Z_2[e_1, e_3, e_5, \dots, e_{2n-1}, \dots] \text{ with } e_{2i} = e_i^2. \end{aligned}$$

By applying Theorem 2.17 to these we obtain:

$$\begin{aligned} 3.12^* \quad H_*(U) &= \Lambda(e_1^*, e_3^*, \dots, e_{2n-1}^*, \dots); \\ H_*(SU) &= \Lambda(e_3^*, e_5^*, \dots, e_{2n-1}^*, \dots); \\ H_*(Sp) &= \Lambda(e_3^*, e_7^*, \dots, e_{4n-1}^*, \dots); \\ H_*(SO; k) &= \Lambda(e_3^*, e_7^*, \dots, e_{4n-1}^*, \dots) \\ &\quad \text{for } k \text{ a field of characteristic } \neq 2; \\ H_*(SO; Z_2) &= \Lambda(e_1^*, e_2^*, e_3^*, \dots, e_{n-1}^*, \dots). \end{aligned}$$

It follows from Theorems 2.8 and 2.11, Lemma 2.10, and Corollary 2.21 that

$$\begin{aligned} 3.13. \quad H_*(\Omega SU) &= Z[e_2, e_4, e_6, \dots, e_{2n}, \dots]; \\ H_*(\Omega Sp) &= Z[e_2, e_6, \dots, e_{4n-2}, \dots]; \\ H_*(\Omega_0 SO, k) &= Z[e_2, e_6, \dots, e_{4n-2}, \dots] \\ &\quad \text{for } k \text{ a field of characteristic } \neq 2. \end{aligned}$$

$$3.13^* \quad H_*(\Omega_0 SO; Z_2) = Z_2[e_2, e_6, \dots, e_{4n-2}, \dots].$$

3.14. $H_*(\Omega SO)$ is free of finite type and its Poincaré series is

$$\prod_{n \geq 1} 1/(1 - t^{4n-2}).$$

LEMMA 3.15. $\Omega_{\mathbb{C}}: \Omega_0 SO \rightarrow \Omega SU$ induces

$$\Omega_{\mathbb{C}}^*(H^+(\Omega SU)) \subset 2H^+(\Omega_0 SO).$$

PROOF. By Corollary 2.21, $H^*(\Omega_0 SO; Z_2) = \Lambda(e_2^*, e_4^*, \dots, e_{2n}^*, \dots)$. Considering

$$\Omega_{\mathbb{C}}: H_*(\Omega_0 SO; Z_2) \rightarrow H_*(\Omega SU; Z_2) = Z_2[e_2, e_4, \dots, e_{2n}, \dots]$$

we have $(\Omega i_C^*(e_{2n}^*))^2 = \Omega i_C^*(e_{2n}^{*2}) = 0$. Thus

$$\Omega i_C^*(e_{2n}^*) = 0 \quad \text{and} \quad \Omega i_C^*(H_+(\Omega_0 \text{SO}; \mathbb{Z}_2)) = 0.$$

By duality, $\Omega i_C^*(H^+(\Omega \text{SU}; \mathbb{Z}_2)) = 0$ hence $\Omega i_C^*(H^+(\Omega \text{SU})) \equiv 0 \pmod{2}$.
Q.E.D.

By Theorem 6.7 of Chapter III,

3.16.

$$\begin{aligned} H^*(\text{U}/\text{Sp}) &= \Lambda(e_1, e_3, \dots, e_{4n-3}, \dots); \\ H^*(\text{U}/\text{SO}; \mathbf{k}) &= \Lambda(e_1, e_3, \dots, e_{4n-3}, \dots) \\ &\quad \text{for } \mathbf{k} \text{ a field of characteristic } \neq 2; \\ H^*(\text{U}/\text{SO}; \mathbf{k}) &= \Lambda(e_1, e_2, \dots, e_n, \dots) \\ &\quad \text{for } \mathbf{k} \text{ a field of characteristic } 2. \end{aligned}$$

Here one deletes e_1 when U is replaced by SU . Since e_{4n-3} can be the image of a primitive generator e_{4n-3} in $H^*(\text{U})$, we have

3.16*.

$$\begin{aligned} H_*(\text{U}/\text{Sp}) &= \Lambda(e_1^*, e_3^*, \dots, e_{4n-3}^*, \dots); \\ H_*(\text{U}/\text{SO}; \mathbf{k}) &= \Lambda(e_1^*, e_3^*, \dots, e_{4n-3}^*, \dots) \\ &\quad \text{for } \mathbf{k} \text{ a field of characteristic } \neq 2. \end{aligned}$$

By the relations in the result

$$H^*(U(2n)/U(n) \times U(n)) = \mathbb{Z}[c_1, \dots, c_n, c'_1, \dots, c'_n] / \left(\sum_{i+j=k} c_i c'_j \right),$$

c'_i can be rewritten in terms of c_i and the ring is isomorphic to $\mathbb{Z}[c_1, \dots, c_n]$ for degrees $\leq 2n$. In addition, by the natural map

$$U(2n)/U(n) \times U(n) \rightarrow BU(n) = \bigcup_N U(n+N)/U(n) \times U(N),$$

c_i in the above is the image of $c_i \in H^{2i}(BU(n))$. It follows that c_i in $U(2n+2)/U(n+1) \times U(n+1)$ corresponds to c_i in $U(2n)/U(n) \times U(n)$ for $i \leq n$. Similarly for U replaced by SU, Sp, SO, O , and hence the results of Chapter III can be rewritten as follows:

3.17:

$$\begin{aligned} H^*(\text{BU}) &= \mathbb{Z}[c_1, c_2, \dots, c_n, \dots], \quad \mu^*(c_i) = \sum_{j+k=i} c_j \otimes c_k; \\ H^*(\text{BSp}) &= \mathbb{Z}[q_1, q_2, \dots, q_n, \dots], \quad \mu^*(q_i) = \sum_{j+k=i} q_j \otimes q_k; \\ H^*(\text{BSO}; \mathbf{k}) &= \mathbf{k}[p_1, p_2, \dots, p_n, \dots], \quad \mu^*(p_i) = \sum_{j+k=i} p_j \otimes p_k \\ &\quad \text{for } \mathbf{k} \text{ a field of characteristic } \neq 2; \end{aligned}$$

$$H^*(\text{BSO}; \mathbf{k}) = \mathbf{k}[w_1, w_2, \dots, w_n, \dots],$$

$$\mu^*(w_i) = \sum_{j+k=i} w_j \otimes w_k, \text{ for } \mathbf{k} \text{ a field of characteristic } 2.$$

It follows from Theorem 2.18 that

3.17*. *The results in 3.17 are formally the same in the homology ring, where the generators are $(c_1^n)^*$, $(q_1^n)^*$, $(p_1^n)^*$, $(w_1^n)^*$.*

REMARK. As the dual of $H^*(\mathbf{BSO}; \mathbf{Z}_2) = \mathbf{Z}_2[w_2, \dots, w_n, \dots]$ one has

$$H_*(\mathbf{BSO}; \mathbf{Z}_2) = \mathbf{Z}_2[w_2^*, w_3^*, (w_2^2)^*, (w_2 w_3)^*, \dots].$$

It follows from Theorems 6.9 and 6.11 that

3.18.

$$\begin{aligned} H^*(\mathbf{Sp}/\mathbf{U}) &= \mathbf{Z}[c_1, c_i, \dots, c_n, \dots] / \left(\sum_{i+j=k} (-1)^i c_i c_j \right) \\ &= \Delta(c_1, c_2, \dots, c_n, \dots), \\ H^*(\mathbf{Sp}/\mathbf{SU}) &= H^*(\mathbf{Sp}/\mathbf{U}) / (c_1) = \Delta(c_2, c_3, \dots, c_n, \dots), \\ H^*(\mathbf{SO}/\mathbf{U}) &= \mathbf{Z}[e_2, e_4, \dots, e_{2n}, \dots] / (e_{4k} + \sum_{i=1}^{2k-1} e_{2i} e_{4k-2i}) \\ &= \Delta(e_2, e_4, \dots, e_{2n}, \dots) \text{ (a free module),} \end{aligned}$$

where c_i is the image of c_i by j_H^* and $2e_{2i} = j_R^*(c_i)$.

§4. Vector bundles and KF-groups

Vector bundles over a field $\mathbf{F}$. One can say that an m -dimensional vector bundle over $\mathbf{F}$ ($= \mathbf{R}, \mathbf{C}, \mathbf{H}$)

$$\xi = (E_\xi, p, X, \mathbf{F}^m, GL(m, \mathbf{F}))$$

with base space X a topological space is a locally trivial fibration $(E_\xi, p, X, \mathbf{F}^m)$, each fibre $\mathbf{F}_x^m$ of which has a structure of a vector space over $\mathbf{F}$ such that the vector addition and the right action of $\mathbf{F}$ are continuous (cf. Theorem 2.2 of Chapter II). We call $n = \dim \xi$ the *dimension* of ξ .

Consider another n -dimensional vector bundle $\eta = (E_\eta, p', Y, \mathbf{F}^n, GL(n, \mathbf{F}))$ in addition to the above ξ . The *direct product*

$$\xi \times \eta = (E_\xi \times E_\eta, p \times p', X \times X', \mathbf{F}^{m+n}, GL(m+n, \mathbf{F}))$$

of ξ and η is clearly an $(m+n)$ -dimensional vector bundle.

If $X = Y$, the vector bundle induced from the diagonal map $d: X \rightarrow X \times X$ is called the *Whitney sum* of ξ and η , denoted by

$$\xi \oplus \eta = d^*(\xi \times \eta) = (E_{\xi \oplus \eta}, p'', X, \mathbf{F}^{m+n}, GL(m+n, \mathbf{F})).$$

The Whitney sum $\xi \oplus \eta$ can be viewed as constructed by collecting the direct sums $\mathbf{F}_x^m \oplus \mathbf{F}_x^n$ of fibres $\mathbf{F}_x^m$ and $\mathbf{F}_x^n$ over x of E_ξ and E_η .

NOTATION. ε_F^k = a trivial k -dimensional vector bundle

$$(X \times \mathbf{F}^k, p, X, \mathbf{F}^k, GL(k, \mathbf{F})).$$

In general, one can construct the direct sum (Whitney sum), tensor product, conjugate, Hom, exterior product, and so on of vector bundles corresponding to the direct sum (Whitney sum), tensor product, conjugate, Hom, exterior product, and so on of vector spaces. The foundation of these constructions is Theorem 2.1 of Chapter II; one only needs to construct suitable coordinate transformations.

Following this method we later define the Whitney sum, tensor product, and conjugate.

Take a common base space X and coordinate neighborhoods $\{U_\alpha\}$ and let

$g_{\alpha\beta}$ be the coordinate transformation of an m -dimensional vector bundle ξ over F ;

$'g_{\alpha\beta}$ be the coordinate transformation of an m -dimensional vector bundle η over $'F$.

The Whitney sum $\xi \oplus \eta$ is defined when $F = 'F$, the coordinate transformation $\bar{g}_{\beta\alpha}$ is given by $\bar{g}_{\beta\alpha}(x) = g_{\beta\alpha}(x) \oplus 'g_{\beta\alpha}(x) \in GL(m+n, F)$ and $\xi \oplus \eta$ is an $(m+n)$ -dimensional vector bundle over F .

If $F = R$ or $'F = R$, the tensor product $\xi \otimes_R \eta$ over R is defined according as $F^m \otimes_R R^n \cong F^{mn}$ or $R^m \otimes_R 'F^n \cong 'F^{mn}$ and is an mn -dimensional vector bundle over F or $'F$. If one takes a basis $\{e_1, \dots, e_{mn}\}$ of $F^m \otimes_R 'F^n$ given by $e_{(i-1)n+j} = e_i \otimes 'e_j$ for a basis $\{e_1, \dots, e_m\}$ of F^m and a basis $\{'e_1, \dots, 'e_n\}$ of $'F^n$, the coordinate transformations $\bar{g}_{\beta\alpha}$ of $\xi \otimes \eta$ are given by $\bar{g}_{\beta\alpha}(x) = g_{\beta\alpha}(x) \otimes 'g_{\beta\alpha}(x)$, where the tensor product $A \otimes B$ of matrices $A = (a_{ij}) \in GL(m, F)$ and $B = (b_{kl}) \in GL(n, 'F)$ is defined by

$$A \otimes B = (c_{st}) \in GL(mn, F \otimes_R 'F), \quad c_{(i-1)n+k, (j-1)n+l} = a_{ij} b_{kl}.$$

If $F = 'F = C$, then the tensor product $\xi \otimes_C \eta$ over C is defined similarly.

The conjugate vector bundle $\bar{\xi}$ of a vector bundle ξ over C is determined by coordinate functions $\bar{g}_{\beta\alpha}$ given by $\bar{g}_{\beta\alpha}(x) = \overline{g_{\beta\alpha}(x)}$.

The following equivalence relations hold:

$$4.1. \quad \xi \oplus \eta \equiv \eta \oplus \xi, \quad (\xi \oplus \eta) \oplus \zeta \equiv \xi \oplus (\eta \oplus \zeta), \\ (\bar{\xi} \oplus \bar{\eta}) \equiv \bar{\xi} \oplus \bar{\eta} \text{ and } \bar{\bar{\eta}} \equiv \eta \text{ over } C.$$

$$4.2. \quad \text{If the tensor products make sense, then } \xi \otimes \eta \equiv \eta \otimes \xi, \quad (\xi \otimes \eta) \otimes \zeta \equiv \xi \otimes (\eta \otimes \zeta), \\ (\xi \oplus \eta) \otimes \zeta \equiv (\xi \otimes \zeta) \oplus (\eta \otimes \zeta), \quad \bar{\xi} \otimes_C \bar{\eta} \equiv \bar{\xi} \otimes_C \bar{\eta} \text{ over } C.$$

If we consider the monomorphism

$$GL(n, R) \xrightarrow{c} GL(n, C) \xrightarrow{q} GL(n, H)$$

corresponding to the extensions $R \subset C \subset H$ of coefficient fields, we obtain the complexification ξ_C (a vector bundle of the same dimension over C) of a vector bundle ξ over R and the bundle ξ_H (a vector bundle of the same

dimension over $\mathbf{H}$) of a vector bundle ξ over $\mathbf{C}$. One may regard $\xi_{\mathbf{C}} = \xi \otimes_{\mathbf{R}} \varepsilon_{\mathbf{C}}^1$. Conversely, if we consider $r: GL(n, \mathbf{C}) \rightarrow GL(2n, \mathbf{R})$ obtained by regarding $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$, we obtain the bundle $\xi_{\mathbf{R}}$ (a $2m$ -dimensional vector bundle over $\mathbf{R}$) of an m -dimensional vector bundle over $\mathbf{C}$; if we consider $c': GL(n, \mathbf{H}) \rightarrow GL(2n, \mathbf{C})$ obtained by regarding $\mathbf{H}^n = \mathbf{C}^n + j\mathbf{C}^n$, we obtain a $2m$ -dimensional vector bundle $\xi_{\mathbf{C}'}$ over $\mathbf{C}$ for a given m -dimensional vector bundle ξ over $\mathbf{H}$. Concerning them the following hold:

$$4.3. (1) (\xi \oplus \eta)_{\mathbf{C}} \equiv \xi_{\mathbf{C}} \oplus \eta_{\mathbf{C}}, (\xi \oplus \eta)_{\mathbf{H}} \equiv \xi_{\mathbf{H}} \oplus \eta_{\mathbf{H}},$$

$$(\xi \oplus \eta)_{\mathbf{R}} \equiv \xi_{\mathbf{R}} \oplus \eta_{\mathbf{R}}, (\xi \oplus \eta)_{\mathbf{C}'} \equiv \xi_{\mathbf{C}'} \oplus \eta_{\mathbf{C}'}.$$

$$(2) \xi_{\mathbf{C}} \otimes_{\mathbf{C}} \eta_{\mathbf{C}} \equiv (\xi \otimes_{\mathbf{R}} \eta)_{\mathbf{C}},$$

$$\xi_{\mathbf{C}} \otimes_{\mathbf{R}} \eta \equiv (\xi \otimes_{\mathbf{R}} \eta)_{\mathbf{C}}, \xi_{\mathbf{H}} \otimes_{\mathbf{R}} \eta \equiv (\xi \otimes_{\mathbf{R}} \eta)_{\mathbf{H}},$$

$$\xi_{\mathbf{R}} \otimes_{\mathbf{R}} \eta \equiv (\xi \otimes_{\mathbf{R}} \eta)_{\mathbf{R}}, \xi_{\mathbf{C}'} \otimes_{\mathbf{R}} \eta \equiv (\xi \otimes_{\mathbf{R}} \eta)_{\mathbf{C}'}.$$

The following can also be shown by using 2.3, 2.4(3), (2.13), and 2.14(3) of Chapter I:

$$4.4. (1) (\xi_{\mathbf{C}})_{\mathbf{R}} \equiv \xi \oplus \bar{\xi}, (\xi_{\mathbf{R}})_{\mathbf{C}} \equiv \xi \oplus \bar{\xi},$$

$$(2) (\xi_{\mathbf{R}})_{\mathbf{C}'} \equiv \xi \oplus \bar{\xi}, (\xi_{\mathbf{C}'})_{\mathbf{R}} \equiv \xi \oplus \bar{\xi}.$$

KF-groups

NOTATION. $\text{Vect}_{\mathbf{F}}(X) = \{\text{the equivalence classes of vector bundles over } \mathbf{F} \text{ with base space } X \text{ a topological space}\}.$

The Whitney sum $\xi \oplus \eta$ of elements ξ, η of $\text{Vect}_{\mathbf{F}}(X)$ is commutative and associative (by 4.1). Let 0 be the class of the 0 -dimensional vector bundle $(X, 1_X, X, \mathbf{F}_0 = \{0\}, U(0, \mathbf{F}) = \{e\})$. Then $\xi \oplus 0 = 0 \oplus \xi = \xi$. Thus $\text{Vect}_{\mathbf{F}}(X)$ forms an abelian semigroup with respect to $\oplus$. We denote by $\text{KF}(X)$ the abelian group obtained from $\text{Vect}_{\mathbf{F}}(X)$ by the method of the Grothendieck construction, and call it the *KF-group* of X , that is:

DEFINITION. $\text{KF}(X) = F(\text{Vect}_{\mathbf{F}}(X)) / (\xi \oplus \eta - \xi - \eta)$, where $F(\text{Vect}_{\mathbf{F}}(X))$ is the free abelian group with basis $\text{Vect}_{\mathbf{F}}(X)$ and $(\xi \oplus \eta - \xi - \eta)$ is the subgroup generated by the elements of the form $\xi \oplus \eta - \xi - \eta$ for all $\xi, \eta \in \text{Vect}_{\mathbf{F}}(X)$.

The following notation will be used (conventionally) for various $\mathbf{F}$.

NOTATION.

$$K(X) = KU(X) = KC(X), \quad KO(X) = KR(X), \quad KSp(X) = KH(X).$$

We also use the expression $KG(X)$ for $G = O, U, Sp$.

The tensor product of vector bundles is associative and distributive with respect to the Whitney sum. Let $1_{\mathbf{F}}$ be the class of the 1 -dimensional vector bundle $\varepsilon_{\mathbf{F}}^1 = (X \times \mathbf{F}, p, X, \mathbf{F}, U(1, \mathbf{F}))$. Then we have $\xi \otimes_{\mathbf{R}} 1_{\mathbf{R}} = 1_{\mathbf{R}} \otimes_{\mathbf{R}} \xi = \xi$, and $\xi \otimes_{\mathbf{C}} 1_{\mathbf{C}} = 1_{\mathbf{C}} \otimes_{\mathbf{C}} \xi = \xi$ over $\mathbf{C}$. The tensor product is naturally extendable over the *KF-group*, and the following holds:

4.5. (1) $\text{KF}(X)$ forms a commutative ring with the unit $1_{\mathbf{F}}$ with respect to the tensor product $\otimes_{\mathbf{F}}: \text{KF}(X) \times \text{KF}(X) \rightarrow \text{KF}(X)$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}$.

(2) $KF(X)$ is a two-sided $KR(X)$ -module with respect to the tensor product $\otimes_R: KR(X) \times KF(X) \rightarrow KF(X)$, $KF(X) \times KR(X) \rightarrow KF(X)$ and the right action coincides with the left action.

If we put $r(\xi) = \xi_R$, $c(\xi) = \xi_C$, $c'(\xi) = \xi_{C'}$, $q(\xi) = \xi_H$, $t(\xi) = \bar{\xi}$, these induce the following homomorphisms of KF -groups by 4.3(1) and 4.1:

$$\begin{aligned} r: KO(X) &\rightarrow KO(X), & c: KO(X) &\rightarrow K(X), \\ c': KSp(X) &\rightarrow K(X), & q: K(X) &\rightarrow KSp(X), & t: K(X) &\rightarrow K(X). \end{aligned}$$

One obtains the following properties by 4.4, 4.3, and 4.2.

- 4.6. (1) $rc = 2$, $cr = 1 + t$, $c'q = 1 + t$, $qc' = 2$, $tt = 1$, $tr = r$.
 (2) c is a homomorphism of rings.
 (3) r , c' , q , and t are homomorphisms of $KO(X)$ -modules.

The vector bundle $f^*\xi$ with base space X is induced from a vector bundle ξ with base space Y by a (continuous) map $f: X \rightarrow Y$ and thus defines a morphism $f^*: \text{Vect}_F(Y) \rightarrow \text{Vect}_F(X)$. It is easy to see that f^* induces naturally a homomorphism

$$f^*: KF(Y) \rightarrow KF(X)$$

with the following properties:

4.7. (1) $KF(\)$ is a contravariant functor from the category of topological spaces to the category of abelian groups: $1_X^* = 1_{KF(X)}$, $(g \circ f)^* = f^* \circ g^*$.

(2) $f \simeq g$ (X a paracompact, Hausdorff) implies $f^* = g^*$.

(3) For $F = R, C$, f^* is a homomorphism of rings $(f^*(\xi \otimes_F \eta) = f^*\xi \otimes_F f^*\eta)$.

(4) $f^*(\xi \otimes_R \eta) = f^*\xi \otimes_R f^*\eta$ (if $\otimes_R$ is defined).

(5) r, c, c', q, t commute with f^* .

NOTATION. $\widetilde{KF}(X) = \text{Ker}(i_0^*: KF(X) \rightarrow KF(x_0))$, where X has a base point x_0 and $i_0: x_0 \rightarrow X$ is the inclusion.

A vector bundle over x_0 is trivial. And, $\pi^*: KF(x_0) \rightarrow KF(X)$ induced by the constant map $\pi: X \rightarrow x_0$ satisfies $i_0^* \circ \pi^* = 1$. It follows that

4.8. (1) The morphism $\dim: KF(X) \rightarrow \mathbb{Z}$, which assigns to a vector bundle (the algebraic sum of) its dimension is equivalent to i_0^* , in particular $KF(x_0) \cong \mathbb{Z}$.

(2) $KF(X) \cong \widetilde{KF}(X) \oplus \mathbb{Z}$ (a direct sum).

(3) $\otimes_R, \otimes_C, r, c, c', q, t$ are closed in $\widetilde{KF}$.

(4) A map $f: X \rightarrow Y$ preserving base points induces $f^*: \widetilde{KF}(Y) \rightarrow \widetilde{KF}(X)$, and 4.7 holds regarding this.

KF-groups and classifying spaces of infinite dimension. In what follows, when we consider vector bundles, the base spaces will be CW -complexes of paracompact Hausdorff spaces having the same homotopy type as a CW -complex. Also in the latter case we simply call such a space a CW -complex. By Corollary 2.17 of Chapter II one may assume that every vector bundle has the structure group $U(n, F)$. By the classifying theorem (Theorem 6.10 of Chapter II) one has

4.9. *There is a universal vector bundle γ_n over $BU(n, F)$ such that the equivalence classes of n -dimensional vector bundles ξ over F with base space X a CW -complex are in one to one correspondence with the elements $[f]$ of $[X, BU(n, F)]$ by $f^*(\gamma_n) \equiv \xi$.*

DEFINITION. Vector bundles ξ and η over F with the same base space are called *stably equivalent* if there are k and $l \geq 0$ such that $\xi \oplus \varepsilon_F^k \equiv \eta \oplus \varepsilon_F^l$. Then we denote $\xi \equiv_s \eta$.

LEMMA 4.10. *Let X be a CW -complex of finite dimension. The stable equivalence classes of vector bundles over F with base space X are in one to one correspondence with the elements of $[X, BG]$, where $G = O$ for $F = \mathbb{R}$, $G = U$ for $F = \mathbb{C}$, and $G = Sp$ for $F = \mathbb{H}$.*

PROOF. Since $G(2N)/G(N) \times G(N)$ is $(d(N+1)-2)$ -universal (Lemma 6.8 of Chapter II), we take N so large that $[X, G(2N)/G(N) \times G(N)]$, $[X, BG(N)]$ and $[X, BG]$ correspond bijectively to each other by the natural morphism (§3). Then one can take $[f] \in [X, BG(N)] \cong [X, BG]$ corresponding to $\xi \oplus \varepsilon_F^{N-n}$, which is stably equivalent to ξ . Q.E.D.

THEOREM 4.11. (1) *The correspondence in Lemma 4.10 induces a natural isomorphism $\widehat{KF}(X) \cong [X, BG]$, where X is a CW -complex of finite dimension and the group structure of $[X, BG] \cong [X, BG]_0$ is the one induced from that of the H -group BG .*

(2) *$KG(X) \cong [X, BG \times \mathbb{Z}]$ (an isomorphism) for a connected CW -complex X of finite dimension.*

PROOF. BG is 0-connected. Hence $[X, BG]_0 \cong [X, BG]$. An element of $KG(X)$ is expressible in the form $\xi - \eta$ by separating the terms with positive coefficients and those with negative ones in $\sum a_i \xi_i$ and replacing them by the Whitney sum. An element of $\widehat{KG}(X)$ is expressed as $\xi - \eta$ with $\dim \xi = \dim \eta$. Since BG is an H -group, $[X, BG]_0$ has inverse elements. By Lemma 4.10 for η there is η' such that $\eta \oplus \eta' \equiv_s \varepsilon_F^n$. It follows that an element of $\widehat{KG}(X)$ is expressible in the form such that $\xi - \varepsilon_F^n$ with $n = \dim \xi$. In addition, considering the definition we see that $\xi - \varepsilon_F^n$ and $\eta - \varepsilon_F^m$ represent the same element of $\widehat{KG}(X)$ iff $\xi \equiv_s \eta$. Thus $\widehat{KG}(X) \cong [X, BG]$. Therefore (1) is proved, since it is easy to see that the product of BG corresponds to the Whitney sum of vector bundles.

(2) is proved by assigning an element of $\mathbb{Z}$ to the formal dimension of a vector bundle. Q.E.D.

COROLLARY 4.12. *For a vector bundle ξ over a CW-complex of finite dimension, there is η such that $\xi \oplus \eta$ is trivial.*

By the isomorphism in Theorem 4.11, $f^*: \widetilde{KG}(Y) \rightarrow \widetilde{KG}(X)$ corresponds to $f^*: [Y, \mathbf{BG}]_0 \rightarrow [X, \mathbf{BG}]_0$ and so by Theorem 1.23

4.13. *For a CW-pair (X, A) of finite dimension, the following sequence is exact:*

$$\dots \xrightarrow{s\pi^*} \widetilde{KG}(SX) \xrightarrow{Si^*} \widetilde{KG}(SA) \xrightarrow{\delta^*} \widetilde{KG}(X/A) \xrightarrow{\pi_*} \widetilde{KG}(X) \xrightarrow{i^*} \widetilde{KG}(A).$$

REMARK 4.13'. By putting $KG^n(X, A) = \widetilde{KG}(S^{8N-n}(X/A))$ for sufficiently large N , using the periodicity $\widetilde{KG}(X) \cong \widetilde{KG}(S^8 X)$ of $\widetilde{KG}$ -groups in §5, we obtain a generalized cohomology theory $\widetilde{KG}(\quad)$.

For F, F' , and G such that the tensor product $\otimes$ (or $\otimes_R$): $KF(X) \otimes KF'(X) \rightarrow KG(X)$ can be defined, we will define the cross product $\hat{\otimes}$ and the smash product $\wedge$ as follows.

The cross product $\hat{\otimes}: KF(A) \otimes KF'(B) \rightarrow KG(A \times B)$ is defined by

$$\alpha \hat{\otimes} \beta = p_A^* \alpha \otimes p_B^* \beta,$$

where $p_A: A \times B \rightarrow A$ and $p_B: A \times B \rightarrow B$ are the projections. The cross product is obviously natural and associative:

$$(f \times g)^*(\alpha \hat{\otimes} \beta) = f^* \alpha \hat{\otimes} g^* \beta, \quad (\alpha \hat{\otimes} \beta) \hat{\otimes} \gamma = \alpha \hat{\otimes} (\beta \hat{\otimes} \gamma). \quad (4.14)$$

Let $i_A: A = A \times b_0 \subset A \vee B$ and $i_B: B = a_0 \times B \subset A \vee B$ be the inclusions. Then

$$(i_A^*, i_B^*): \widetilde{KG}(A \vee B) \rightarrow \widetilde{KG}(A) \oplus \widetilde{KG}(B)$$

is obviously an isomorphism. Let $i: A \vee B \rightarrow A \times B$ be the inclusion. Consider the composition

$$\widetilde{KG}(A) \oplus \widetilde{KG}(B) \xrightarrow{p_A^* + p_B^*} \widetilde{KG}(A \times B) \xrightarrow{i^*} \widetilde{KG}(A \vee B) \xrightarrow{(i_A^*, i_B^*)} \widetilde{KG}(A) \oplus \widetilde{KG}(B).$$

This is seen to be the identity map by $p_A \circ i \circ i_A = 1$, $p_A \circ i \circ i_B = 0$, and so on. Therefore i^* is an epimorphism and $\text{Ker } i^*$ is a direct summand.

Let A and B be CW-complexes of finite dimension. If we apply 4.13, we obtain an isomorphism

$$\pi^*: \widetilde{KG}(A \wedge B) \cong \text{Ker}(i^*: \widetilde{KG}(A \times B) \rightarrow \widetilde{KG}(A \vee B)). \quad (4.15)$$

Considering the cross product $\alpha \hat{\otimes} \beta \in \widetilde{KG}(A \times B)$ of $\alpha \in \widetilde{KF}(A) \subset KF(A)$ and $\beta \in \widetilde{KF}'(B) \subset KF'(B)$, we have $i^*(\alpha \hat{\otimes} \beta) = 0$ and so

$$\alpha \hat{\otimes} \beta \in \pi^*(\widetilde{KG}(A \wedge B)),$$

since $(i_A^* \circ i^*)(\alpha \hat{\otimes} \beta) = 0^* \alpha \hat{\otimes} \beta = 0$ and $(i_B^* \circ i^*)(\alpha \hat{\otimes} \beta) = \alpha \hat{\otimes} 0^* \beta = 0$. Thus we can define the smash product $\wedge: \widehat{KF}(A) \otimes \widehat{KF}'(B) \rightarrow \widehat{KG}(A \wedge B)$ by

$$\alpha \wedge \beta = \pi^{*-1}(\alpha \hat{\otimes} \beta).$$

The smash product is natural and associative:

$$(f \wedge g)^*(\alpha \wedge \beta) = f^*(\alpha) \wedge g^*(\beta), \quad (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \quad (4.14)'$$

We use the same symbols $\hat{\otimes}$ and $\wedge$ in the case of $\otimes_{\mathbb{R}}$. They have properties similar to $\hat{\otimes}$ and $\wedge$ above, and associativity holds if it can be defined at all.

Chern character. The following lemma is useful for studying Chern classes of vector bundles.

LEMMA 4.16. *For complex vector bundles ξ and η over a CW-complex X , there exist a space Y and a map $\rho: Y \rightarrow X$ such that*

(1) $\rho^*\xi = \xi_1 \oplus \cdots \oplus \xi_n$ and $\rho^*\eta = \eta_1 \oplus \cdots \oplus \eta_m$, where ξ_i and η_j are 1-dimensional complex vector bundles.

(2) $\rho^*: H^*(X) \rightarrow H^*(Y)$ is a monomorphism.

PROOF. Consider the map $f: X \rightarrow BU(n)$ corresponding to ξ by (4.9): $\xi = f^*\gamma_n$. Let $U(n)/T^n \xrightarrow{i} Y' \xrightarrow{g} X$ be the induced fibre bundle by f from the fibre bundle $U(n)/T^n \xrightarrow{i} BT^n \xrightarrow{p} BU(n)$. Since $p^*\gamma_n$ is the universal vector bundle over BT^n , it is equivalent to the direct product of the universal vector bundles over BT . Therefore it is decomposed as $p^*\gamma_n = \gamma_1^{(1)} \oplus \cdots \oplus \gamma_1^{(n)}$ the Whitney sum of one-dimensional complex vector bundles. Let $\tilde{f}: Y' \rightarrow BT^n$ be the bundle map over f . Then $g^*\xi = g^*f^*\gamma_n = \tilde{f}^*p^*\gamma_n = \xi_1 \oplus \cdots \oplus \xi_n$, where $\xi_i = \tilde{f}^*(\gamma_1^{(i)})$.

Next, $i^*: H^*(BT^n) \rightarrow H^*(U(n)/T^n)$ is an epimorphism by Theorem 5.5 of Chapter III. Since the map i is decomposed as $U(n)/T^n \xrightarrow{i} Y' \xrightarrow{j} BT^n$, $i^*: H^*(Y') \rightarrow H^*(U(n)/T^n)$ is also an epimorphism. By Theorems 4.4 and 4.2 of Chapter III, g^* is a monomorphism.

If η is given, we consider a map $Y' \rightarrow BU(m)$ corresponding to $g^*\eta$ and obtain a map $g': Y \rightarrow Y'$ in a similar way; we can put $\rho = g \circ g'$. Q.E.D.

LEMMA 4.17. *Putting $c_1(\xi_i) = x_i$, $c_1(\eta_j) = y_j \in H^2(Y)$ in Lemma 4.16, we have*

$$(1) \rho^* \sum_{i=0}^n c_i(\xi) = \prod_{i=1}^n (1 + x_i), \quad \rho^* \sum_{j=0}^m c_j(\eta) = \prod_{j=1}^m (1 + y_j),$$

$$(2) \rho^* \sum_{k=0}^{m+n} c_k(\xi \oplus \eta) = \prod_{i=1}^n (1 + x_i) \prod_{j=1}^m (1 + y_j)$$

and hence $c_k(\xi \oplus \eta) = \sum_{i+j=k} c_i(\xi)c_j(\eta)$,

$$(3) \rho^* \sum_{k=0}^{mn} c_k(\xi \otimes \eta) = \prod_{i,j} (1 + x_i + y_j).$$

PROOF. (1) By definition of the Chern class (§5 of Chapter III), we have $\rho^*(\sum^n c_i) = \prod^n (1 + t_i)$. Pulling both sides back by the bundle map

$$(\tilde{f}, f): (Y', g, X) \rightarrow (BT^n, p, BU(n)),$$

we obtain the first half of (1) since we have $f^*(c_i) = c_i(\xi)$ and $\tilde{f}^*(t_i) = x_i$. The latter half is similar.

(2) is clear from $\rho^*(\xi \otimes \eta) = \xi_1 \oplus \dots \oplus \xi_n \oplus \eta_1 \oplus \dots \oplus \eta_m$.

(3) To prove (3) it suffices to show $c_1(\xi_i \times \eta_j) = c_1(\xi_i) + c_1(\eta_j)$, since $\rho^*(\xi \otimes \eta) = \bigoplus_{i,j} \xi_i \otimes \eta_j$. For this we can consider $c_1 = c_1(\xi_1)$ on the classifying space $BU(1) = P_\infty(\mathbb{C})$, further on $S^2 = P_1(\mathbb{C})$. If ξ and η correspond to f and $g: S^2 \rightarrow BU(1)$, then $\xi \otimes \eta$ is represented by the multiplication in $U(1)$ under the isomorphism $\pi_2(BU(1)) \cong \pi_1(U(1))$ (cf. the proof of Lemma 4.2). Thus we obtain $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$. Q.E.D.

DEFINITION. $ch_n(\xi) \in H^{2n}(X; \mathbb{Q})$ is defined by setting $\rho^*ch_n(\xi) = \sum_{i=1}^n (x_i^n/n!)$. We call $ch_n(\xi)$ the n th Chern character of ξ and $ch(\xi) = \sum_{n \geq 0} ch_n(\xi) \in H^{**}(X; \mathbb{Q})$ the total Chern character, where $H^{**}(X; \mathbb{Q}) = \prod H^i(X; \mathbb{Q})$.

If we put $ch_n = ch_n(\gamma_n) \in H^{2n}(BU(n); \mathbb{Q})$, then

$$\xi = f^*\gamma_n \text{ implies } ch_n(\xi) = f^*(ch_n).$$

Making use of the Newton formula (2.18'), we have

$$4.18. (1) ch_n = \frac{1}{n!} s_n(c_1, \dots, c_n) = \frac{1}{n!} c_1^n + \dots + (-1)^{n-1} \frac{1}{(n-1)!} c_n.$$

$$(2) \rho^*ch(\xi) = e^{x_1} + \dots + e^{x_n}.$$

THEOREM 4.19. (1) $ch(\xi \oplus \eta) = ch(\xi) + ch(\eta)$, $ch(\xi \otimes \eta) = ch(\xi)ch(\eta)$.

(2) ch and ch_n can be extended to $ch: K(X) \rightarrow H^{**}(X; \mathbb{Q})$ and $ch_n: K(X) \rightarrow H^{2n}(X; \mathbb{Q})$ respectively, and ch is a homomorphism of rings satisfying $f^{**} \circ ch = ch \circ f^*$.

PROOF. We can study ch using the monomorphism ρ^* :

$$\begin{aligned} \rho^*ch(\xi \oplus \eta) &= e^{x_1} + \dots + e^{x_n} + e^{y_1} + \dots + e^{y_m} \\ &= \rho^*ch(\xi) + \rho^*ch(\eta) = \rho^*(ch(\xi) + ch(\eta)); \end{aligned}$$

$$\begin{aligned} \rho^*ch(\xi \otimes \eta) &= \sum_{i,j} e^{x_i + y_j} = \sum_i e^{x_i} \sum_j e^{y_j} \\ &= \rho^*ch(\xi) \rho^*ch(\eta) = \rho^*(ch(\xi)ch(\eta)). \end{aligned}$$

(2) One can extend ch over $K(X)$ using $ch(\xi \oplus \eta) = ch(\xi) + ch(\eta)$.

The rest is also clear. Q.E.D.

REMARK. The definition of the total Chern class $c = \sum c_i$ cannot be extended over $K(X)$, since $c(\xi \oplus \eta) = c(\xi)c(\eta)$.

Representations by a map of the smash product. The tensor product

$$\otimes: G(n) \times O(m) \rightarrow G(mn)$$

is a monomorphism of Lie groups. When we have

$$BG(mn) = EG(mn)/G(mn),$$

$EG(mn)/\otimes(G(n) \times O(m))$ is the classical space of $G(n) \times O(m)$, and hence there is a homotopy equivalence

$$BG(n) \times BO(m) \rightarrow EG(mn)/\otimes(G(n) \times O(m)).$$

We denote by

$$\otimes BG(n) \times BO(m) \rightarrow BG(mn)$$

the homotopy equivalence followed by the projection into $BG(mn)$. Then

4.20. If $f: A \rightarrow BG(n)$ and $g: B \rightarrow BO(m)$ correspond to ξ and η respectively ($\xi \equiv f^* \gamma_n$, $\eta \equiv g^* \gamma_m$), $\otimes \circ (f \times g)$ corresponds to $\xi \otimes \eta$.

If we consider a further embedding $BG(mn)$ into $\mathbf{BG}$, then

LEMMA 4.21. Let A and B be CW -complexes of finite dimension. Take m and n so large that $\widetilde{KG}(A) \cong [A, BG(n)]_0$ and $\widetilde{KO}(B) \cong [B, BO(m)]_0$. Then there exists a map

$$\wedge: BG(n) \wedge BO(m) \rightarrow \mathbf{BG}$$

such that if $\alpha \in \widetilde{KG}(A)$ and $\beta \in \widetilde{KO}(B)$ correspond to $f: A \rightarrow BG(n)$ and $g: B \rightarrow BO(m)$ respectively, then $\wedge \circ (f \wedge g)$ corresponds to $\alpha \wedge \beta \in \widetilde{KG}(A \wedge B)$.

PROOF. Let p_1 and p_2 be projections of $BG(n) \times BO(m)$ onto $BG(n)$ and $BO(m)$ respectively followed by the inclusions to $BG(n) \wedge BO(m)$. Then by 4.20, $\otimes \circ p_1 \circ (f \times g)$ and $\otimes \circ p_2 \circ (f \times g)$ correspond to $\xi \otimes 1$ and $1 \otimes \eta$ respectively. If we put $h(x) = \mu(x, \mu((1 \circ \otimes \circ p_1)x, (1 \circ \otimes \circ p_2)x))$ using the multiplication μ and the inverse ι in $\mathbf{BG}$, then $h \circ (f \times g)$ represents $\xi \otimes \eta - \xi \otimes 1 - 1 \otimes \eta = \pi^*(\xi \wedge \eta) - 1 \otimes 1$. We have

$$\begin{aligned} h|BG(n) &= \mu \circ (\otimes \times (\mu \circ (\iota \times 0) \circ d) \circ \otimes) \circ d \\ &= \mu \circ (\otimes \times (\iota \circ \otimes)) \circ d = \mu \circ (1 \times \iota) \circ d \circ \otimes \simeq 0 \circ \otimes = 0. \end{aligned}$$

Similarly $h|BO(m) \simeq 0$. By using the homotopy extension property, we have $h \simeq h'$ and $h'(BG(n) \vee BO(m)) = e$. Then $\wedge = h' \circ \pi^{-1}$ is the required map. Q.E.D.

Similarly,

4.21'. Lemma 4.2 holds also for $\wedge: \tilde{K}(A) \otimes \tilde{K}(B) \rightarrow \tilde{K}(A \wedge B)$ and a map $\wedge: BU(n) \wedge BU(m) \rightarrow BU$.

More concretely, consider the case $A = S^{k+1}$ and $B = S^1$. By means of the isomorphisms $\tilde{K}G(S^{k+1}) \cong [S^{k+1}, \mathbf{BG}] = \pi_{k+1}(\mathbf{BG}) \cong \pi_k(\mathbf{G})$, we construct a vector bundle ξ_f corresponding to $f: S^k \rightarrow G(n) \subset \mathbf{G}$ as follows. Let $G(n) = U(n, \mathbf{F})$. We regard S^{k+1} as $D^{k+1} \times \{0, 1\} / \sim$, where $S^k \times 0$ is identified with $S^k \times 1$ by the relation $(x, 0) \sim (x, 1)$. We denote by E_f the space obtained by introducing the relation

$$(x, 0, v) \sim (x, 1, v'), \quad v' = f(x) \cdot v,$$

between $S^k \times 0 \times \mathbf{F}^n$ and $S^k \times 1 \times \mathbf{F}^n$ in $D^{k+1} \times \{0, 1\} \times \mathbf{F}^n$. If we put $p(x, \varepsilon, v) = (x, v)$, then $\xi_f = (E_f, p, S^{k+1})$ is an n -dimensional vector bundle over $\mathbf{F}$. It is easy to check that ξ_f corresponds to f by the above isomorphism.

On the other hand, a vector bundle η which represents a generator of $\tilde{K}O(S^1) \cong \pi_0(\mathbf{O}) \cong \mathbf{Z}_2$ is constructed as follows. We regard S^1 as I with 0 and 1 identified. Let η be the one-dimensional vector bundle obtained by identifying $0 \times \mathbf{R}^1$ with $1 \times \mathbf{R}^1$ by $(0, v) \sim (1, -v)$ in $I \times \mathbf{R}^1$ (η is the Möbius band).

LEMMA 4.22. If $\tilde{K}G(S^{k+1})$ corresponds to $f: S^k \rightarrow G(n)$, then $\xi \wedge \eta \in \tilde{K}G(S^{k+1})$ corresponds to the map $g: S^{k+1} = S^k \wedge S^1 \rightarrow G(2n)$ constructed as follows: Let $w: I \rightarrow G(2n)$ be a path connecting $w(0) = I_{2n}$ and $w(1) = -I_{2n}$, then g is defined by the following equality:

$$g(x, t) = (f(x) \oplus I_n)w(t)(f(x) \oplus I_n)^{-1}w(t)^{-1},$$

where $x \in S^k$ and a point of $S^1 = I/\partial I$ is represented by $t \in I$.

PROOF. First we construct $\xi \hat{\otimes} \eta$. The map $\otimes: G(n) \times O(1) \rightarrow G(n)$ is nothing but the scalar multiplication by ± 1 . $\hat{\otimes}$ may be obtained by composition with a glueing function using this product. We consider $(\xi \hat{\otimes} \eta) \oplus (\varepsilon_{\mathbf{F}}^n \hat{\otimes} \eta)$. This (vector) bundle is obtained by introducing in $D^{n+1} \times \{0, 1\} \times I \times \mathbf{F}^{2n}$ the following relations

$$(x, 0, t, v) \sim (x, 1, t, (f(x) \oplus I_n)v) \quad \text{for } x \in S^k, t \in I, v \in \mathbf{F}^{2n};$$

$$(\bar{x}, \varepsilon, 0, v) \sim (\bar{x}, \varepsilon, 1, -v) \quad \text{for } \bar{x} \in D^{k+1}, \varepsilon = \pm 1, v \in \mathbf{F}^{2n}.$$

Here if we define a homeomorphism ψ of $D^{k+1} \times \{0, 1\} \times I \times \mathbf{F}^{2n}$ to itself by

$$\psi(\theta x, \varepsilon, t, v) = (\theta x, \varepsilon, t, w(\theta t)^{-1}v) \quad \text{for } x \in S^k, \theta, t \in I, v \in \mathbf{F}^{2n}$$

and rewrite the coordinate, then the relations become:

$$(x, 0, t, v) \sim (x, 1, t, w(t)(f(x) \oplus I_n)w(t)^{-1}v) \quad \text{for } x \in S^k,$$

$$(\bar{x}, \varepsilon, 0, v) \sim (\bar{x}, \varepsilon, 1, v).$$

For the base point e_1 , $f(v) = I_n$ implies that the bundle over $e_1 \times S^1$ is trivial. Therefore the bundle is considered to be defined on $S^{k+1} \times S^1 / e_1 \times S^1$. So it is obtained by glueing together two trivial fibre bundles over $D^{k+1} \times S^1 / e_1 \times S^1$ at $S^k \times S^1 / e_1 \times S^1$ by the above relations. Since $D^{k+1} \times S^1 / e_1 \times S^1$ is a cone over $S^k \times S^1 / e_1 \times S^1$, the bundle corresponds to

$$[g_1] \in [S^k \times S^1 / e_1 \times S^1, G(2n)]$$

where $g_1(x, t) = w(t)(f(x) \oplus I_n)w(t)^{-1}$. The map corresponding to

$$\xi \otimes \eta + e_R^n \otimes \eta - \xi \otimes e_R^1 \in \widetilde{KG}(S^{k+1} \times S^1)$$

can be given by $g_2(x, t) = (f(x) \oplus I_n)^{-1} g_1(x, t)$ using the multiplication in $G(2n)$. Since $g_2(S^k \times e_1) = I_{2n}$, the map g_2 is considered as a map from $S^k \wedge S^1 = S^k \times S^1 / S^k \vee S^1$, and so g_2 is seen to correspond to $\xi \wedge \eta$. Here we have $\xi \wedge \eta = (-\xi) \wedge \eta$, as $2\eta = 0$ in $\widetilde{KO}(S^1) \cong \pi_1(\mathbf{BO}) \cong \mathbb{Z}_2$. Thus the lemma is obtained by replacing $f(x)$ with $f(x)^{-1}$ in the definition of g_2 . Q.E.D.

§5. The periodicity of KF -groups

The Bott map

NOTATION. $R_t(n) = e^{\pi i t} I_n \oplus e^{-\pi i t} I_n \in SU(2n)$;

$$S_t(n) = r(e^{\pi i t} I_n) = \begin{pmatrix} \cos \pi t I_n & -\sin \pi t I_n \\ \sin \pi t I_n & \cos \pi t I_n \end{pmatrix} \in SO(2n).$$

DEFINITION. The maps defined below are called the Bott map:

$\beta(n): U(2n)/U(n) \times U(n) \rightarrow \Omega SU(2n)$ is defined by setting

$$\beta(n)\{A\}(t) = AR_t(n)A^{-1}R_t(n)^{-1} \quad \text{for } A \in U(2n), t \in I;$$

$\beta_{Sp}(n): Sp(n)/U(n) \rightarrow \Omega Sp(n)$ is defined by setting

$$\beta_{Sp}(n)\{A\}(t) = A(e^{\pi i t} I_n)A^{-1}(e^{\pi i t} I_n)^{-1} \quad \text{for } A \in Sp(n), t \in I;$$

$\beta_O(n): O(2n)/U(n) \rightarrow \Omega SO(2n)$ is defined by setting

$$\beta_O(n)\{A\}(t) = AS_t(n)A^{-1}S_t(n)^{-1} \quad \text{for } A \in O(2n), t \in I$$

(omitting $r: U(n) \hookrightarrow O(2n)$).

LEMMA 5.1. The maps

$$\beta: \mathbf{BU} \rightarrow \Omega \mathbf{SU}, \quad \beta_{Sp}: \mathbf{SP}/\mathbf{U} \rightarrow \Omega \mathbf{Sp}, \quad \beta_O: \mathbf{O}/\mathbf{U} \rightarrow \Omega \mathbf{SO}$$

given by putting $\beta|U(2n)/\tau_n(U(n) \times U(n)) = \Omega \tau_n \circ \beta(n) \circ \tau_n^{-1}$. $\beta_{Sp}|Sp(n)/U(n) = \beta_{Sp}(n)$, and $\beta_O|O(2n)/U(n) = \beta_O(n)$ are well defined and are H -maps.

PROOF. First we need to show that the definitions of $\beta(n)$, $\beta_{Sp}(n)$, and $\beta_O(n)$ are independent of the choice of representatives A . We have

$$(AB)R_t(n)(AB)^{-1}R_t(n)^{-1} = AR_t(n)BB^{-1}A^{-1}R_t(n)^{-1} = AR_t(n)A^{-1}R_t(n)^{-1},$$

since $B = B_1 \oplus B_2 \in U(n) \times U(n)$ commutes with $R_t(n)$. Thus $\beta(n)$ is determined independently of the choice of A . Since an element of $U(n)$ obviously commutes with $e^{\pi i t} I_n$, the map $\beta_{Sp}(n)$ is determined. So is $\beta_O(n)$, since an element of $U(n) = r(U(n))$ commutes with $S_t(n) = r(e^{\pi i t} I_n)$. Notice that these maps take values in loop spaces and map the base point $\{e\}$ to the base point (the constant path), since we have $R_0(n) = S_0(n) = I_{2n}$, $e^{\pi i 0} I_n = I_n$, $R_1(n) = S_1(n) = -I_{2n}$, $e^{\pi i 1} I_n = -I_n$.

The inclusion $Sp(n) \subset Sp(n+1)$ is given by $A \mapsto A \oplus 1$. Since we have $\beta_{Sp}(n+1)\{A \oplus 1\}(t) = (A \oplus 1)(e^{\pi i t} I_n \oplus e^{\pi i t}) (A \oplus 1)^{-1} (e^{\pi i t} I_n \oplus e^{\pi i t})^{-1} = \beta_{Sp}(n)\{A\}(t) \oplus 1$, we see that β_{Sp} can be defined. As for β , using (3.6) we have

$$\begin{aligned} & \tau_{n+1}[\beta(n+1)\{\tau_{n+1}^{-1}(A \oplus I_2)\}(t)] \\ &= (A \oplus I_2) \tau_{n+1}(R_t(n+1)) (A \oplus I_2)^{-1} \tau_{n+1}(R_t(n+1)^{-1}) \\ &= (A \oplus I_2) (\tau_n R_t(n) \oplus R_t(1)) (A^{-1} \oplus I_2) (\tau_n R_t(n)^{-1} \oplus R_t(1)^{-1}) \\ &= \tau_n(\beta(n)\{\tau_n^{-1} A\}(t)) \oplus I_2. \end{aligned}$$

This shows $\Omega \tau_{n+1} \circ \beta(n+1) \circ \tau_{n+1}^{-1} | U(2n)/U(n) \times U(n) = \Omega \tau_n \circ \beta(n) \circ \tau_n^{-1}$ and hence β is well defined. By using the second equality of (3.6), β_O is also shown to be well defined.

Define monomorphisms $\alpha, \beta: \mathbb{N} \rightarrow \mathbb{N}$ by $\alpha(i) = 2i - 1$, $\beta(i) = 2i$ and consider the multiplication $\psi_{\alpha, \beta}$ in $\mathbf{Sp}$. We have $\psi_{\alpha, \beta}(A, B) = \tau_n(A \oplus B)$ for $A, B \in Sp(n)$. Therefore we have

$$\begin{aligned} & \psi_{\alpha, \beta}(\beta_{Sp}(n)\{A\}(t), \beta_{Sp}(n)\{B\}(t)) \\ &= \tau_n[A e^{\pi i t} I_n A^{-1} e^{-\pi i t} I_n \oplus B e^{\pi i t} I_n B^{-1} e^{-\pi i t} I_n] \\ &= \tau_n[(A \oplus B) e^{\pi i t} I_{2n} (A \oplus B)^{-1} e^{-\pi i t} I_{2n}] \\ &= \psi_{\alpha, \beta}(A, B) e^{\pi i t} I_{2n} \psi_{\alpha, \beta}(A, B)^{-1} e^{-\pi i t} I_{2n} \\ &= \beta_{Sp}(2n)\{\psi_{\alpha, \beta}(A, B)\}(t). \end{aligned}$$

Thus β_{Sp} is an H -map: $\Omega \psi_{\alpha, \beta} \circ (\beta_{Sp} \times \beta_{Sp}) = \beta_{Sp} \circ \psi_{\alpha, \beta}$. Similarly by using $\psi_{\bar{\alpha}, \bar{\beta}}$ it follows that β_O is an H -map. Similarly for β . Q.E.D.

LEMMA 5.2. *The following diagrams are homotopy commutative:*

$$\begin{array}{ccc} \mathbf{Sp}/U & \xrightarrow{\beta_{Sp}} & \Omega \mathbf{Sp} \\ \downarrow j_H & & \downarrow \Omega j_C' \\ \mathbf{BU} & \xrightarrow{\beta} & \Omega \mathbf{SU} \end{array} \quad \begin{array}{ccc} \mathbf{O}/U & \xrightarrow{\beta_O} & \Omega \mathbf{SO} \\ \downarrow j_H & & \downarrow \Omega j_C \\ \mathbf{BU} & \xrightarrow{\beta} & \Omega \mathbf{SU}. \end{array}$$

PROOF. By $c(e^{\pi i t} I_n) = e^{\pi i t} I_n \oplus e^{-\pi i t} I_n = R_t(n)$ and the fact that c is a homomorphism we have

$$\begin{aligned}\beta(n)\{cA\}(t) &= (cA)R_t(n)(cA)^{-1}R_t(n)^{-1} \\ &= c(Ae^{\pi i t} I_n A^{-1} e^{-\pi i t} I_n) = c[\beta_{Sp}(n)\{A\}(t)],\end{aligned}$$

from which follows the commutativity of the first diagram $\beta \circ j_H = \Omega i_C \circ \beta_{Sp}$.

By using $\eta_n S_t(n) \eta_n^{-1} = \eta_n r(e^{\pi i t} I_n) \eta_n^{-1} = j_n(e^{\pi i t} I_n) = R_t(n)$, we have

$$\begin{aligned}\beta(n)\{j_n(A)\}(t) &= j_n(A)R_t(n)j_n(A)^{-1}R_t(n)^{-1} = j_n(AS_t(n)A^{-1}S_t(n)^{-1}) \\ &= j_n(\beta_O(n)\{A\}(t)) = \eta_n(\beta_O(n)\{A\}(t))\eta_n^{-1}.\end{aligned}$$

Since $U(2)$ is 0-connected, there is a path $w: I \rightarrow U(2)$ connecting $I_2 = w(0)$ and $\eta_1 = w(1)$. The path $w_n: I \rightarrow U(2n)$ defined by

$$w_n(s) = \tau_n^{-1}(w(s) \oplus \cdots \oplus w(s))$$

connects I_{2n} and $w_n(1) = \eta_n$. The homotopy $H(n): (O(2n)/U(n)) \times I \rightarrow \Omega SU(2n)$ defined by

$$H(n)(\{A\}, s)(t) = w_n(s)(\beta_O(n)\{A\}(t))w_n(s)^{-1}$$

shows that $\Omega i \circ \beta_O(n) \simeq \beta(n) \circ j_n$, where $\Omega i: \Omega SO(2n) \subset \Omega SU(2n)$. Now if the map $H: (O/U) \times I \rightarrow \Omega SU$ defined by

$$H|(O(2n)/\tau_n r U(n)) \times I = \Omega \tau_n \circ H(n) \circ (\tau_n^{-1} \times 1_I)$$

is determined independently of n , it is a homotopy which shows that $\Omega i_C \circ \beta_O \simeq \beta \circ j_R$. In order for H to be independent of n it suffices that

$$\tau_{n+1}[w_{n+1}(s)\tau_{n+1}^{-1}(B \oplus I_2)w_{n+1}(s)^{-1}] = \tau_n[w_n(s)(\tau_n^{-1}B)w_n(s)^{-1}] \oplus I_2$$

since we have $\Omega \tau_{n+1} \circ \beta_O(n+1) \circ \tau_{n+1}^{-1}|O(2n)/\tau_n r U(n) = \Omega \tau_n \circ \beta_O(n) \circ \tau_n^{-1}$. Here we have

$$\begin{aligned}\text{the left-hand side} &= \tau_{n+1}(w_{n+1}(s))(B \oplus I_2)\tau_{n+1}(w_{n+1}(s)^{-1}) \\ &= (w(s) + \cdots)(B \oplus I_2)(w(s)^{-1} \oplus \cdots) \\ &= [\tau_n(w_n(s))B\tau_n(w_n(s)^{-1})] \oplus I_2 \\ &= \text{the right-hand side},\end{aligned}$$

since τ_{n+1} and τ_n are homomorphisms. Q.E.D.

A cellular decomposition of $SU(n+1)$. We define a map $\bar{\beta}_{1,n}$ from the suspension $S(P_n(C))$ of the n -dimensional complex projective space $P_n(C) = U(n+1)/U(1) \times U(n)$ to $SU(n+1)$ by

$$\bar{\beta}_{1,n}(\{A\}, t) = AR_t(1, n)A^{-1}R_t(1, n)^{-1},$$

where $A \in U(n+1)$ and $R_t(1, n) = e^{\pi i t} \oplus e^{-\pi i t} I_n$. It is seen similarly to the case of $\beta(n)$ that the definition is independent of the choice of a representative A of $\{A\}$ and that $\bar{\beta}_{1,n}$ takes the value I_{n+1} at $t = 0, 1$ or for $A = I_n$. Let $p: SU(n+1) \rightarrow S^{2n+1} = SU(n+1)/SU(n)$ be the projection.

LEMMA 5.3. $\bar{\beta}_{1,n}|S(P_{n-1}(C)) = \bar{\beta}_{1,n-1}$ and

$$p \circ \bar{\beta}_{1,n}: (S(P_n(C)), S(P_{n-1}(C))) \rightarrow (S^{2n+1}, e_{n+1})$$

is a relative homeomorphism.

PROOF. The first equality is clear. For $A = (a_{ij})$, $A^{-1} = {}^t\bar{A} = (\bar{a}_{ji})$. Then, since

$$R_i(1, n)A^{-1}R_i(1, n)^{-1} = (\bar{a}_{ji}(1 + \delta_{ji}(e^{2\pi i t} - 1))(1 + \delta_{ji}(e^{-2\pi i t} - 1))),$$

we have

$$\begin{aligned}\bar{\beta}_{1,n}(\{A\}, t) &= \left(\sum_k a_{ik} \bar{a}_{jk} (1 + \delta_{ji}(e^{2\pi i t} - 1))(1 + \delta_{ik}(e^{-2\pi i t} - 1)) \right), \\ (p \circ \bar{\beta}_{1,n})(\{A\}, t) &= \left(\sum_k a_{ik} \bar{a}_{n+1,k} (1 + \delta_{n+1}(e^{2\pi i t} - 1))(1 + \delta_{ik}(e^{-2\pi i t} - 1)) \right) \\ &= \left(\left(\sum_k a_{ik} \bar{a}_{n+1,k} \right) + a_{i1} \bar{a}_{n+1,1} (e^{-2\pi i t} - 1) \right) \\ &= (\delta_{i,n+1} + a_{i1} \bar{a}_{n+1,1} (e^{-2\pi i t} - 1)).\end{aligned}$$

Here, denoting $\{A\} = [y_1 : \dots : y_{n+1}]$ where $y_i \doteq a_{i1}$, the map

$$h: (D^{2n}, S^{2n-1}) \rightarrow (P_n(C), P_{n-1}(C))$$

defined by

$$h(x) = h(x_1, \dots, x_n) = [x_1 : \dots : x_n : \sqrt{1 - \|x\|^2}]$$

is a relative homeomorphism. Then we have

$$(p \circ \bar{\beta}_{1,n} \circ Sh)((x_1, \dots, x_n), t) = (x_1 r \lambda, \dots, x_n r \lambda, 1 + r^2 \lambda),$$

where $r = \sqrt{1 - \|x\|^2}$ and $\lambda = e^{-2\pi i t} - 1$. It is easily verified that $p \circ \bar{\beta}_{1,n} \circ Sh$ is an injection onto $S^{2n+1} - e_{n+1}$ for $r > 0$, $0 < t < 1$ and hence $p \circ \bar{\beta}_{1,n}$ is also an injection onto $S^{2n+1} - e_{n+1}$, and it is a relative homeomorphism. Q.E.D.

THEOREM 5.4. $\bar{\beta}_{1,n}: S(P_n(C)) \rightarrow SU(n+1)$ is a homeomorphism into. If we put $e^{2n+1} = \bar{\beta}_{1,n}(SP_n(C) - SP_{n-1}(C))$ and define the product $e^{2a_1+1} \dots e^{2a_r+1}$ of e^{2a_i+1} by the multiplication of $SU(n+1)$, then the set of $e^0 = \{I_{n+1}\}$ and $e^{2a_1+1} \dots e^{2a_r+1}$ for $n \geq a_1 > a_2 > \dots > a_r \geq 1$ gives a cellular decomposition of $SU(n+1)$.

PROOF. It follows from Lemma 5.3 that $\bar{\beta}_{1,n}$ is one to one and hence a homeomorphism into. Since SD^{2n} is homeomorphic with D^{2n+1} , the relative homeomorphism $\bar{\beta}_{1,n} \circ Sh$ gives a characteristic map of e^{2n+1} , and

$\partial e^{2n+1} \subset \bar{\beta}_{1,n-1}(SP_{n-1}(C)) = e^0 \cup e^3 \cup \dots \cup e^{2n-1}$. If we consider the projection $p: SU(n+1) \rightarrow S^{2n+1} = SU(n+1)/SU(n)$ and the action of $SU(n)$, we see that the multiplication of $SU(n+1)$ gives a homeomorphism of $e^{2n+1} \times SU(n)$ onto $SU(n+1) - SU(n)$. Thus the cellular decomposition of the theorem is obtained by induction on n . Q.E.D.

Let $\beta_{1,n}: P_n(C) \rightarrow \Omega SU(n+1)$ be the adjoint map of $\bar{\beta}_{1,n}$.

LEMMA 5.5. (1) The homology ring $H_*(\Omega SU(n+1)) = \mathbb{Z}[e_2, \dots, e_{2n}]$ is generated by the image of $\beta_{1,n*}: H_*(P_n(C)) \rightarrow H_*(\Omega SU(n+1))$.

(2) There is a map $f: P_n(C) \rightarrow \mathbf{BU}$ such that the diagram

$$\begin{array}{ccc} P_n(C) & \xrightarrow{f} & \mathbf{BU} \\ \downarrow \beta_{1,n} & & \downarrow \beta \\ \Omega SU(n+1) & \xrightarrow{\Omega i} & \Omega \mathbf{SU} \end{array}$$

is homotopy commutative, where $i: SU(n+1) \rightarrow \mathbf{SU}$ is the inclusion.

PROOF. In the commutative diagram

$$\begin{array}{ccccc} H_{2n}(P_n(C)) & \xrightarrow{\beta_{1,n*}} & H_{2n}(\Omega SU(n+1)) & \xrightarrow{\Omega p_*} & H_{2n}(\Omega S^{2n+1}) \\ \cong \downarrow S & & \downarrow \Sigma & & \downarrow \Sigma \\ H_{2n+1}(S(P_n(C))) & \xrightarrow{\bar{\beta}_{1,n*}} & H_{2n+1}(SU(n+1)) & \xrightarrow{p_*} & H_{2n+1}(S^{2n+1}) \end{array}$$

S is an isomorphism and so is $p_* \circ \bar{\beta}_{1,n*}$ by Lemma 5.3.

We choose a generator e_{2n} so that $\Sigma \Omega p_* e_{2n}$ is a generator of $H_{2n+1}(S^{2n+1})$ (Theorem 2.8). Then e_{2n} is in the image of $\beta_{1,n*}$. The other generators $e_2, \dots, e_{2n-2}$ can be taken in the image from $H_*(\Omega SU(n))$. By Lemma 5.3 we have $\beta_{1,n-1} = \beta_{1,n}|_{\Omega SU(n)}$. By induction on n we obtain (1).

(2) If we define a monomorphism $\alpha: \mathbf{N} \rightarrow \mathbf{N}$ by $\alpha(1) = 1$ and $\alpha(i) = n + i - 1$ for $i > 1$ and consider $f_\alpha: U(n+1) \rightarrow U(2n)$, then f_α induces $f_\alpha: SU(n+1) \rightarrow SU(2n)$ and $f_\alpha: P_n(C) \rightarrow U(2n)/U(n) \times U(n)$ so that $\Omega f_\alpha \circ \beta_{1,n} = \beta(n) \circ f_\alpha: P_n(C) \rightarrow \Omega(SU(2n))$. Putting $f = \tau_n \circ f_\alpha$ we have $\beta \circ f = \Omega i \circ \Omega \tau_n \circ \beta_{1,n}$ by the definition of β . Using a path connecting a permutation matrix $P_\tau \in U(2n)$ such that $\tau_n(A) = P_\tau A P_\tau^{-1}$ and I_{2n} in $U(2n)$, $\tau_n \simeq 1$, which implies $\Omega \tau_n \simeq 1$ and hence $\beta \circ f \simeq \Omega i \circ \beta_{1,n}$. Q.E.D.

The periodicity of K-groups

THEOREM 5.6. $\beta: \mathbf{BU} \rightarrow \Omega \mathbf{SU}$ is a (weak) homotopy equivalence.

PROOF. By Corollary 1.6 it suffices to show $\beta_*: H_*(\mathbf{BU}) \cong H_*(\Omega \mathbf{SU})$. Since $\Omega i_*: H_*(\Omega SU(n+1)) \rightarrow H_*(\Omega \mathbf{SU})$ is an isomorphism for degrees

$\leq 2n$, $H_*(\Omega\mathbf{SU}) = \mathbf{Z}[e_2, e_4, \dots, e_{2n}, \dots]$ is multiplicatively generated by $\text{Im } \beta_*$ by Lemma 5.5. Since β is an H -map (Lemma 5.1), β_* is a homomorphism of rings and it follows that β_* is an epimorphism. On the other hand, $H_*(\mathbf{BU})$ and $H_*(\Omega\mathbf{SU})$ are free abelian groups isomorphic for each degree by Theorem 2.8 and 3.17 and hence β_* is an isomorphism. Q.E.D.

COROLLARY 5.7. (1) *There is an H -map $\mathbf{BU} \times \mathbf{Z} \rightarrow \Omega\mathbf{U}$ which is a (weak) homotopy equivalence.*

(2)

$$\tilde{K}(S^{2n}) \cong \pi_{2n}(\mathbf{BU}) \cong \pi_{2n-1}(\mathbf{U}) \cong \mathbf{Z} \quad \text{for } n \geq 1;$$

$$\tilde{K}(S^{2n-1}) = \pi_{2n-1}(\mathbf{BU}) = \pi_{2n-1}(\mathbf{U}) = 0 \quad \text{for } n \geq 1.$$

PROOF. (1) $\pi_1(\mathbf{U}) \cong \pi_1(U(1)) \cong \mathbf{Z}$, hence $\Omega\mathbf{U}$ is homeomorphic to $\Omega_0\mathbf{U} \times \mathbf{Z}$ as an H -space. $\pi_i(\mathbf{SU}) \cong \pi_i(\mathbf{U})$ for $i \geq 1$, hence $\pi_i(\Omega\mathbf{SU}) \cong \pi_i(\Omega\mathbf{U})$ for $i \geq 0$. Thus $\Omega\mathbf{SU} \rightarrow \Omega_0\mathbf{U}$ is a weak homotopy equivalence, and (1) is obtained.

(2) is obvious for $n = 1$. For $i > 0$, $\pi_i(\mathbf{BU}) \cong \pi_i(\mathbf{BU} \times \mathbf{Z}) \cong \pi_i(\Omega\mathbf{U}) \cong \pi_{i+1}(\mathbf{U}) \cong \pi_{i+2}(\mathbf{BU})$. Thus $\pi_i(\mathbf{BU}) \cong \pi_{i+2k}(\mathbf{BU})$, and (2) is obtained. Q.E.D.

Let $\bar{\beta}: \mathbf{SBU} \rightarrow \mathbf{SU}$ be the adjoint map of the Bott map $\beta: \mathbf{BU} \rightarrow \Omega\mathbf{SU}$ and

$$\Sigma\bar{\beta} = \sigma_{\mathbf{SU}} \circ S\bar{\beta}: S^2\mathbf{BU} \rightarrow \mathbf{SSU} \rightarrow \mathbf{BSU}$$

be the adjoint map of $\bar{\beta}$.

LEMMA 5.8.

$$(1) (\Sigma\bar{\beta})^*(c_{k+1}) = \pm S^2 s_k(c_1, \dots, c_k), \quad \beta^* \sigma^2(c_{k+1}) = \pm s_k(c_1, \dots, c_k)$$

for $k \geq 1$, where $s_k(c_1, \dots, c_k) = k \cdot ch_k = c_1^k + \dots + (-1)^{k-1} k c_k$.

(2) *The image of $ch_n: \tilde{K}(S^{2n}) \rightarrow H^{2n}(S^{2n}; \mathbf{Q}) \cong \mathbf{Q}$ coincides with $H^*(S^{2n}) \cong \mathbf{Z}$.*

(3) $g_n^*(c_n) = \pm(n-1)! u_{2n}$ for a generator $[g_n]$ of $\pi_{2n}(\mathbf{BU})$ and a generator u_{2n} of $H^{2n}(S^{2n})$.

PROOF. (1) The images $\sigma^2(c_{k+1})$ of c_{k+1} under the suspension homomorphisms

$$\sigma: H^{2k+2}(\mathbf{BSU}) \rightarrow H^{2k+1}(\mathbf{SU}) \quad \text{and} \quad \sigma: H^{2k+1}(\mathbf{SU}) \rightarrow H^{2k}(\Omega\mathbf{SU})$$

are primitive (Lemma 2.16). Since $\beta^*: H^*(\Omega\mathbf{SU}) \cong H^*(\mathbf{BU})$ is an isomorphism of Hopf algebras, $\beta^* \sigma^2(c_{k+1}) \in H^{2k}(\mathbf{BU})$ is primitive. By Theorem 2.18 there is an $a \in \mathbf{Z}$ such that $\beta^* \sigma^2(c_{k+1}) = a s_k(c_1, \dots, c_k)$. To determine a we consider $f_\alpha: P_n(\mathbf{C}) = U(n+1)/U(1) \times U(n) \rightarrow U(2n)/U(n) \times U(n)$ used in the proof of Lemma 5.5(2). Using the same α we define $f_\alpha: U(m+1)/U(1) \times U(m) \rightarrow U(n+m)/U(n) \times U(m)$, and obtain $p_\alpha: \mathbf{BU}(1) \rightarrow \mathbf{BU}(n)$ as the limit, as m becomes large. The maps f_α and p_α induce

equivalent homomorphisms of cohomology groups for degrees $\leq 2n$. Since $p_\alpha^*(c_1) = c_1$ and $p_\alpha^*(c_i) = 0$ for $i > 1$, the same holds for f_α^* . Therefore $f^*(c_1) = t$ is a generator of $H^2(P_n(C))$ and $f^*(c_i) = 0$ ($i > 1$) for the map $f: P_n(C) \rightarrow \mathbf{BU}$ ($n \geq k$) in Lemma 5.5, hence $f^* \beta^* \sigma^2(c_{k+1}) = af^* s_k(c_1, \dots, c_k) = af^*(c_1^k) = at^k$.

On the other hand, $H^*(SU(n+1)) = \Lambda(\sigma(c_2), \dots, \sigma(c_{n+1}))$. By Theorem 5.4, $\bar{\beta}_{1,n}: H_*(S(P_n(C))) \rightarrow H_*(SU(n+1))$ is a monomorphism so $\bar{\beta}_{1,n}^*: H^*(SU(n+1)) \rightarrow H^*(S(P_n(C)))$ is an epimorphism.

Since $\alpha, \beta \in H^+(SX)$ implies $\alpha\beta = 0$ in general, $\bar{\beta}_{1,n}^* \sigma(c_{k+1})$ must be a generator of $H^{2k+1}(S(P_n(C)))$. By the commutativity of the diagram we obtain $\beta_{1,n}(c_{k+1}) = \pm t^k$.

$$\begin{array}{ccc} H^{2k+1}(SU(n+1)) & \xrightarrow{\bar{\beta}_{1,n}^*} & H^{2k+1}(S(P_n(C))) \\ \downarrow \sigma & & \cong \downarrow S^{-1} \\ H^{2k}(\Omega SU(n+1)) & \xrightarrow{\beta_{1,n}^*} & H^{2k}(P_n(C)) \end{array}$$

By Lemma 2.5(2) we have $\pm t^k = \beta_{1,n}^*(\sigma^2(c_{k+1})) = f^* \beta^* \sigma^2(c_{k+1}) = at^k$. Thus $a = \pm 1$, that is, $\beta^* \sigma^2(c_{k+1}) = \pm s_k(c_1, \dots, c_k)$.

Since $\Sigma \bar{\beta}$ is the adjoint map of β , we have $(\Sigma \bar{\beta})^*(c_{k+1}) = S^2 \beta^*(c_{k+1})$.

(2) Let $g_n: S^{2n} \rightarrow \mathbf{BU}$ be a map representing a generator of $\tilde{K}(S^{2n}) \cong [S^{2n}, \mathbf{BU}]_0 \cong \pi_{2n}(\mathbf{BU}) \cong \mathbf{Z}$. Then it is easy to see that $g_{n+1} = \Sigma \bar{\beta} \circ S^2 g_n: S^{2n+2} \rightarrow S^2 \mathbf{BU} \rightarrow \mathbf{BSU} \rightarrow \mathbf{BU}$ represents a generator of $\tilde{K}(S^{2n+2})$. Thus we suppose that g_n is constructed successively from $g_1: S^2 = P_1(C) \xrightarrow{f} \mathbf{BU}$ in this manner. Easily by Lemma 5.5 $g_1^*(c_1) = u_2$ is a generator of $H^2(S^2)$. If we put $g_n^*(c_n) = b_n u_{2n}$ for $b_n \in \mathbf{Z}$ where $u_{2n} = S^2(u_{2n-2})$ is a generator of $H^{2n}(S^{2n})$, by (1) and $g_n^*(c_i) = 0$ for $i \neq 0, n$ we have

$$\begin{aligned} b_{n+1} \cdot u_{2n+2} &= g_{n+1}^*(c_{n+1}) = (S^2 g_n)^*(\Sigma \bar{\beta})^*(c_{n+1}) = \pm (S^2 g_n)^* S^2 s_k \\ &= \pm (S^2 g_n)^*(c_1^n + \dots + (-1)^{n-1} n c_n) \\ &= \pm (-1)^{n-1} n (S^2 g_n)^*(c_n) = \pm (-1)^{n-1} n b_n u_{2n+2}. \end{aligned}$$

So inductively $b_n = \pm(n-1)!$. The formula

$$ch_n = \frac{1}{n!} s_n = \frac{1}{n!} c_1^n + \dots + (-1)^{n-1} \frac{1}{(n-1)!} c_n$$

implies

$$\begin{aligned} ch_n(g_n) &= g_n^* \left\{ \frac{1}{n!} c_1^n + \cdots + (-1)^{n-1} \frac{1}{(n-1)!} c_n \right\} \\ &= (-1)^{n-1} \frac{1}{(n-1)!} g_n^*(c_n) = \pm u_{2n}. \end{aligned}$$

(3) is obvious from the above considerations. Q.E.D.

THEOREM 5.9 (The periodicity of K -groups). *Let X be a CW -complex of finite dimension.*

(1) $\otimes : K(X) \otimes K(S^2) \cong K(X \times S^2)$ (an isomorphism).

(2) $\wedge : \tilde{K}(X) \otimes \tilde{K}(S^2) \cong \tilde{K}(X \wedge S^2) = \tilde{K}(S^2 X)$ (an isomorphism).

In other words, $\wedge g : \tilde{K}(X) \cong \tilde{K}(S^2 X)$ for a generator g of $K(S^2)$.

PROOF. By considering the direct sum decompositions $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$ and $K(X \times S^2) \cong \tilde{K}(X \wedge S^2) \oplus \tilde{K}(X) \oplus \tilde{K}(S^2) \oplus \mathbb{Z}$ one sees that (1) and (2) are equivalent. If X is nonconnected, then (1) is reduced to the case where X is connected by dividing it into each component. Thus it suffices to prove (2) for X connected. Further, since a connected X is homotopy equivalent to a CW -complex with base point just one vertex such that the attaching map of each cell preserves the base point, one may suppose that X satisfies such a condition.

By Lemma 5.8, $ch(g) = s$ is a generator of $H^2(S^2)$ for a generator g of $\tilde{K}(S^2)$. Since $ch(g^n) = s \wedge \cdots \wedge s$ is a generator of $H^{2n}(S^{2n})$ for $g^n = g \wedge \cdots \wedge g$ (n copies); again by Lemma 5.8, g^n is a generator of $\tilde{K}(S^{2n})$. Thus (2) holds for $X = S^{2n}$ ($n \geq 1$). Also (2) is clear for $X = S^{2n-1}$ since $\tilde{K}(X) = \tilde{K}(S^2 X) = 0$. Using $\tilde{K}(\vee S_\alpha^m) = \sum_\alpha \tilde{K}(S_\alpha^m)$ for a wedge $\vee S_\alpha^m$, (2) holds for $X = \vee S_\alpha^m$. In particular, if $\dim Y \leq 1$ then $Y =$ a point or $Y = \vee S_\alpha^1$ so (1) holds for $X = Y$ and $X = SY$.

We will show inductively that $\wedge g$ is an isomorphism for $\dim X < N$ and an epimorphism for $\dim X = N$. If $\dim X = m + 1$, there is a map $f : \vee S_\alpha^m \rightarrow X^m$ the m -skeleton X^m of X such that $X = C_f$ is the mapping cone of f . By 4.1 we have the following exact and commutative diagram:

$$\begin{array}{ccccccccc} \tilde{K}(SX^m) & \xrightarrow{sf^*} & \tilde{K}(S(\vee S_\alpha^m)) & \xrightarrow{\pi^*} & \tilde{K}(X) & \xrightarrow{i^*} & \tilde{K}(X^m) & \xrightarrow{f^*} & \tilde{K}(\vee S_\alpha^m) \\ \downarrow \wedge g & & \downarrow \wedge g & & \downarrow \wedge g & & \downarrow \wedge g & & \downarrow \wedge g \\ \tilde{K}(S^3 X^m) & \xrightarrow{s^3 f^*} & \tilde{K}(S^3(\vee S_\alpha^m)) & \xrightarrow{s^2 \pi^*} & \tilde{K}(S^2 X) & \xrightarrow{s^2 i^*} & \tilde{K}(S^2 X^m) & \xrightarrow{s^2 f^*} & \tilde{K}(S^2(\vee S_\alpha^m)). \end{array}$$

Applying the 5-lemma for $m = N$ and $N + 1$ we obtain that $\wedge g$ is an isomorphism for $\dim X \leq N$ and an epimorphism $\dim X = N + 1$. Thus (2) is proved by induction. Q.E.D.

Periodicity of KO-groups

THEOREM 5.10. $\beta_O : O/U \rightarrow SO$ is a (weak) homotopy equivalence.

PROOF. Consider $\beta_O(1) : O(2)/U(1) = \text{two points} \rightarrow \Omega SO(2)$. Since $J S_i(1) J^{-1} = S_{-i}(1) = S_i(1)^{-1}$ for $J = 1 \oplus (-1)$, we have

$$\beta_O(1)\{J\}(t) = S_{-i}(1)^2 = S_{-2i}(1).$$

Thus $\beta_O(1)\{J\}$ represents a generator $(-i_1)$ of $\pi_0(\Omega SO(2)) \cong \mathbb{Z}$. It follows that β_O maps two components of O/U to those two of ΩSO respectively. Since β_O is an H -map, it suffices to prove the theorem for β_O restricted on one component: $SO/U \rightarrow \Omega_0 SO$. By Lemma 5.2 the commutative diagram is obtained:

$$\begin{array}{ccc} H^*(\Omega SU) & \xrightarrow{\beta^*} & H^*(BU) \\ \downarrow \Omega^* & & \downarrow V \\ H^*(\Omega_0 SO) & \xrightarrow{\beta_O^*} & H^*(SO/U). \end{array}$$

By 3.18, $2e_{2n}$ is in the image of j_R^* for the generator e_{2n} of $H^*(SO/U)$. Since β^* is an isomorphism by Theorem 5.6, there is $x \in H^{2n}(\Omega SU)$ such that $2e_{2n} = \beta_O^*(\Omega_C^*(x))$. There is $y \in H^{2n}(\Omega_0 SO)$ such that $2y = \Omega_C^*(x)$ by Lemma 3.15. Since $H^*(SO/U)$ is a free abelian group, $\beta_O^*(2y) = 2e_{2n}$ implies $\beta_O^*(y) = e_{2n}$ and hence β_O^* is an epimorphism. Since $H^*(\Omega_0 SO)$ and $H^*(SO/U)$ are free abelian groups of the same rank for each degree by 3.12, 3.13*, and 3.18, β_O^* is an isomorphism. Then β_O is a (weak) homotopy equivalence by Corollary 1.6. Q.E.D.

LEMMA 5.11. Let η be a generator of $\widetilde{KO}(S^1) \cong \mathbb{Z}_2$ and g a generator of $\widetilde{K}(S^2) \cong \mathbb{Z}$, then the following sequence is exact for $n \geq 1$:

$$\widetilde{KO}(S^{n+1}) \xrightarrow{\wedge \eta} \widetilde{KO}(S^{n+2}) \xrightarrow{(\wedge g)^{-1} \circ c} \widetilde{K}(S^n) \xrightarrow{r} \widetilde{KO}(S^n) \xrightarrow{\wedge \eta} \widetilde{KO}(S^{n+1}),$$

PROOF. The sequence $\pi_{n-1}(U) \xrightarrow{i_*} \pi_{n-1}(O) \xrightarrow{\pi_*} \pi_{n-1}(O/U)$ induced from $U \xrightarrow{i} O \xrightarrow{\pi} O/U$ is exact, since it is equivalent to $\pi_{n-1}(U(N)) \xrightarrow{i_*} \pi_{n-1}(O(2N)) \xrightarrow{\pi_*} \pi_{n-1}(O(2N)/U(N))$ for sufficiently large N . Using the isomorphisms

$$\pi_{n-1}(G) \cong \pi_n(BG) \cong \widetilde{KG}(S^n)$$

and

$$\pi_{n-1}(SO/U) \cong \pi_{n-1}(\Omega_0 SO) \cong \pi_n(SO) \cong \widetilde{KO}(S^{n+1}),$$

the sequence

$$\widetilde{K}(S^n) \xrightarrow{r} \widetilde{KO}(S^n) \xrightarrow{h} \widetilde{KO}(S^{n+1})$$

is exact, where h corresponds to π_* . Similarly we obtain homotopy exact sequences associated with $O \xrightarrow{\pi} O/U \xrightarrow{j_R} BU$ and $O/U \xrightarrow{j_R} BU \xrightarrow{\rho_R} BO$ where j_R corresponds to $c: \widetilde{KO}(S^{n+2}) \rightarrow \widetilde{K}(S^{n+2})$ using the isomorphisms $\widetilde{K}(S^{n+2}) \cong \pi_n(\Omega SU) \xrightarrow{\beta_*^{-1}} \pi_n(BU) \cong \widetilde{K}(S^n)$ by Lemma 5.2 and Theorems 5.6 and 5.10. Theorem 5.9 shows that β_*^{-1} can be replaced by $(\wedge g)^{-1}$. After all it suffices to show $h = \wedge \eta$. Observe that h is equivalent to the map induced by $\beta_O \circ \pi: O \rightarrow O/U \rightarrow \Omega SO$, i.e., $\beta_O(N) \circ \pi: O(2N) \rightarrow O(2N)/U(N) \rightarrow \Omega SO(2N)$. Since $(\beta_O(N) \circ \pi)\{A\}(t) = AS_t(N)A^{-1}S_t(N)^{-1}$, the map induces $\wedge \eta$ by Lemma 4.22. Q.E.D.

THEOREM 5.12. *Let $n \geq 1$. $\widetilde{KO}(S^n) \cong \pi_n(BO) \cong \pi_{n-1}(O)$ is isomorphic to $Z, Z_2, Z_2, 0, Z, 0, 0, 0$ according as $n \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$.*

If $n \equiv 0 \pmod{8}$ then $c: \widetilde{KO}(S^n) \cong \widetilde{K}(S^n)$.

If $n \equiv 1, 2 \pmod{8}$ then $\wedge \eta: \widetilde{KO}(S^{n-1}) \cong \widetilde{KO}(S^n)$ is an isomorphism.

If $n \equiv 4 \pmod{8}$ then $c: \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^n)$ is an isomorphism onto $2(\widetilde{K}(S^n))$.

PROOF. The case $n = 1$ is well known. A generator of $\widetilde{KO}(S^2) \cong \pi_1(O) \cong \pi_1(SO) \cong Z_2$ is represented by $\beta_O(1)(J)$ as is seen in the first part of Theorem 5.10 and the generator is $\eta \wedge \eta$ by Lemma 4.22. We have $\widetilde{KO}(S^3) \cong \pi_2(SO) \cong \pi_2(SO(4)) \cong \pi_2(\text{Spin}(4)) \cong \pi_2(Sp(1) \times Sp(1))$, where we have $\widetilde{KO}(S^3) = 0$ as $Sp(1)$ is 2-connected. For $n \geq 4$ we apply induction on n to the sequence in Lemma 5.11.

If $n \equiv 4 \pmod{8}$ then in the exact sequence $0 = \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{KO}(S^n) \xrightarrow{(\wedge g)^{-1} \circ c} \widetilde{K}(S^{n-2}) \xrightarrow{r} \widetilde{KO}(S^{n-2}) \xrightarrow{\wedge \eta} \widetilde{KO}(S^{n-1}) = 0$ we have $\widetilde{KO}(S^{n-2}) \cong Z_2$ and $\widetilde{K}(S^{n-2}) \cong Z$ and hence $c: \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^n)$ is an isomorphism onto $2(\widetilde{K}(S^n))$ and so $\widetilde{KO}(S^n) \cong Z$.

If $n \equiv 5 \pmod{8}$ then in the exact sequence $\widetilde{K}(S^{n-1}) \xrightarrow{r} \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^{n-2}) = 0$, $r \circ c = 2$ and $c: \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{K}(S^{n-1})$ is a homomorphism multiplying by 2 and so r is an isomorphism and hence $\widetilde{KO}(S^n) = 0$.

If $n \equiv 6 \pmod{8}$ then in the exact sequence $0 = \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^{n-2}) \xrightarrow{r} \widetilde{KO}(S^{n-2})$, r is an isomorphism as above and so $\widetilde{KO}(S^n) = 0$.

If $n \equiv 7 \pmod{8}$ then $0 = \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^{n-2}) = 0$ is exact and so $\widetilde{KO}(S^n) = 0$.

If $n \equiv 0 \pmod{8}$ then $0 = \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{KO}(S^n) \xrightarrow{(\wedge g)^{-1} \circ c} \widetilde{K}(S^{n-2}) \rightarrow \widetilde{KO}(S^{n-1}) = 0$ is exact. Thus $c: \widetilde{KO}(S^n) \cong \widetilde{K}(S^n) \cong Z$. It follows from $r \circ c = 2$ that $\text{Cok} r \cong Z_2$ for $r: \widetilde{K}(S^n) \rightarrow \widetilde{KO}(S^n)$.

If $n \equiv 1 \pmod{8}$ then $0 \rightarrow \text{Cokerr}(\cong \mathbb{Z}_2) \xrightarrow{\wedge \eta} \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^{n-2}) = 0$ is exact so $\widetilde{KO}(S^n) \cong \mathbb{Z}_2$ and $\wedge \eta: \widetilde{KO}(S^{n-1}) \rightarrow \widetilde{KO}(S^n)$ is an epimorphism.

If $n \equiv 2 \pmod{8}$ then $0 \rightarrow \widetilde{K}(S^{n-1}) \rightarrow \widetilde{KO}(S^{n-1}) \xrightarrow{\wedge \eta} \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^{n-2}) \cong \mathbb{Z}$ is exact so $\wedge \eta: \widetilde{KO}(S^{n-1}) \cong \widetilde{KO}(S^n) (\cong \mathbb{Z}_2)$.

If $n \equiv 3 \pmod{8}$ then $\widetilde{KO}(S^{n-1}) \xrightarrow{\wedge \eta} \widetilde{KO}(S^n) \rightarrow \widetilde{K}(S^{n-2}) = 0$ is exact. By the above consideration $\wedge \eta^3: \widetilde{KO}(S^{n-3}) \rightarrow \widetilde{KO}(S^n)$ is an epimorphism where $\wedge \eta^3 = \eta \wedge \eta \wedge \eta$. On the other hand, $\eta^3 \in \widetilde{KO}(S^3) = 0$ implies $\widetilde{KO}(S^n) = 0$. Q.E.D.

THEOREM 5.13 (Periodicity of KO -groups). *Let X be a CW-complex of finite dimension.*

(1) $\otimes: KO(X) \otimes KO(S^8) \cong KO(X \times S^8)$ (an isomorphism).

(2) $\wedge: \widetilde{KO}(X) \otimes \widetilde{KO}(S^8) \cong \widetilde{KO}(X \wedge S^8) = \widetilde{KO}(S^8 X)$ (an isomorphism).

In other words, $\wedge \sigma: \widetilde{KO}(X) \cong \widetilde{KO}(S^8 X)$ for a generator σ of $\widetilde{KO}(S^8)$.

PROOF. Arguing as in the proof of Theorem 5.9 one can see that it suffices to show $\wedge \sigma: \widetilde{KO}(S^n) \cong \widetilde{KO}(S^{n+8})$ for $n \geq 1$. This is obvious for $n \equiv 3, 5, 6, 7 \pmod{8}$. By Theorem 5.12 one can choose a generator σ so that $c(\sigma) = g^4$. Put $\sigma^k = \sigma \wedge \dots \wedge \sigma$ (k copies). c is a homomorphism of rings. Thus $c(\sigma^k) = g^{4k}$, so σ^k is a generator of $\widetilde{KO}(S^{8k})$. It follows that $\sigma^k \wedge \eta = \eta \wedge \sigma^k$ is a generator of $\widetilde{KO}(S^{8k+1})$. Thus $\sigma^k \wedge \eta^2 = \eta^2 \wedge \sigma^k$ is a generator of $\widetilde{KO}(S^{8k+2})$. $\widetilde{KO}(S^4)$ is generated by an element ν satisfying $c(\nu) = 2g^2$. $c(\nu \wedge \sigma^k) = 2g^2 \wedge g^{4k} = 2g^{4k+2}$ implies $\nu \wedge \sigma^k$ is a generator of $\widetilde{KO}(S^{8k+4})$. It follows that $\wedge \sigma: \widetilde{KO}(S^n) \cong \widetilde{KO}(S^{n+8})$ for $n \geq 1$. Q.E.D.

Periodicity of KSp -groups

THEOREM 5.14. $\beta_{Sp}: Sp/U \rightarrow \Omega Sp$ is a (weak) homotopy equivalence.

PROOF. The spaces under consideration are both 1-connected. In addition $j_C: H^*(BU) \rightarrow H^*(Sp/U)$ is an epimorphism by 3.18 and hence the theorem is proved by Lemma 5.2, 3.13, and 3.18. Q.E.D.

LEMMA 5.15. *Let η be a generator of $\widetilde{KO}(S^1)$ and g a generator of $\widetilde{K}(S^2)$. Then the following sequence is exact for $n \geq 1$:*

$$\widetilde{KSp}(S^{n+1}) \xrightarrow{\wedge \eta} \widetilde{KSp}(S^{n+2}) \xrightarrow{(\wedge g)^{-1} \circ c'} \widetilde{K}(S^n) \xrightarrow{\mathfrak{q}} \widetilde{KSp}(S^n) \xrightarrow{\wedge \eta} \widetilde{KSp}(S^{n-1}).$$

PROOF. Similar to Lemma 5.11. Q.E.D.

THEOREM 5.16. *Let $n \geq 1$. $\widetilde{KSp}(S^n) \cong \pi_n(\mathbf{BSp}) \cong \pi_{n-1}(\mathbf{Sp})$ is isomorphic to $\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0$ accordingly as $n \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$.*

If $n \equiv 0 \pmod{8}$ then $c': \widetilde{KSp}(S^n) \rightarrow \widetilde{K}(S^n)$ is an epimorphism onto $2(\widetilde{K}(S^n))$.

If $n \equiv 4 \pmod{8}$ then $c': \widetilde{KSp}(S^n) \cong \widetilde{K}(S^n)$.

If $n \equiv 5, 6 \pmod{8}$ then $\wedge\eta: \widetilde{KSp}(S^{n-1}) \rightarrow \widetilde{KSp}(S^n)$ is an epimorphism.

PROOF. The theorem is obvious for $n \leq 3$, since $\mathbf{Sp}$ is 2-connected. The case $n \geq 4$ is proved similarly to Theorem 5.12, using Lemma 5.15, $c'_*\mathbf{q} = 2$, and $\eta^3 = 0$. Q.E.D.

THEOREM 5.17 (Periodicity of KSp -groups). *Let X be a CW-complex of finite dimension.*

(1) $\otimes: KSp(X) \otimes KO(S^8) \cong KSp(X \times S^8)$ (an isomorphism).

(2) $\wedge: \widetilde{KSp}(X) \otimes \widetilde{KO}(S^8) \cong \widetilde{KSp}(X \wedge S^8) \cong \widetilde{KSp}(S^8 X)$ (an isomorphism).

In other words, $\wedge\sigma: \widetilde{KSp}(X) \cong \widetilde{KSp}(S^8 X)$ for a generator σ of $\widetilde{KO}(S^8)$.

PROOF. If we choose $\nu \in \widetilde{KSp}(S^4)$ and $\mu \in \widetilde{KSp}(S^8)$ so that $c'(\nu) = g^2$ and $c'(\mu) = 2g^2$, then $\nu \wedge \sigma^k, \mu \wedge \sigma^k, \mu \wedge \sigma^k \wedge \eta, \mu \wedge \sigma^k \wedge \eta^2$ are generators of $\widetilde{KSp}(S^{8k+j})$ for $j = 4, 8, 9$, and 10 respectively. The remainder is similar to Theorem 5.13. Q.E.D.

§6. Homotopy groups of classical groups and homogeneous spaces

Homotopy groups of symmetric spaces of infinite dimension. The homotopy groups of $\mathbf{BG}$ and $\mathbf{G}$ have already been determined in §5 for $\mathbf{G} = \mathbf{U}, \mathbf{O}$, and $\mathbf{Sp}$. The homotopy groups of $\mathbf{G} = \mathbf{SU}, \mathbf{SO}$ are isomorphic to those of $\mathbf{G} = \mathbf{U}, \mathbf{O}$ in high dimensions respectively, and the only different cases are the following:

$$\pi_1(\mathbf{BSO}) = \pi_0(\mathbf{SO}) = 0, \quad \pi_2(\mathbf{BSU}) = \pi_1(\mathbf{U}) = 0.$$

In the remaining cases, once $\mathbf{U/O}, \mathbf{O/U}, \mathbf{U/Sp}$, and $\mathbf{SO/U}$ are dealt with, the other cases such as $\mathbf{SU/SO}$ can easily be obtained with only slight differences in low dimensions.

THEOREM 6.1. (1) $r: \widetilde{K}(S^n) \rightarrow \widetilde{KO}(S^n)$ for $n > 0$ is

an isomorphism onto $2(\widetilde{KO}(S^n))$ if $n \equiv 0 \pmod{8}$.

an epimorphism if $n \equiv 2 \pmod{8}$,

an isomorphism if $n \equiv 4 \pmod{8}$.

(2) $q: \widetilde{K}(S^n) \rightarrow \widetilde{KSp}(S^n)$ for $n > 0$ is

an isomorphism if $n \equiv 0 \pmod{8}$,

an isomorphism onto $2(\widetilde{KSp}(S^n))$ if $n \equiv 4 \pmod{8}$,

an epimorphism if $n \equiv 6 \pmod{8}$.

(3) $t = (-1)^n: \widetilde{K}(S^{2n}) \rightarrow \widetilde{K}(S^{2n})$ for $n > 0$.

TABLE 4.1.

M :	U	BU	O	BO	Sp	BSp	O/U	Sp/U	U/SO	U/Sp
$n \equiv 0$	0	Z	Z_2	Z	0	Z	Z_2	0	0	0
$n \equiv 1$	Z	0	Z_2	Z_2	0	0	0	0	Z	Z
$n \equiv 2$	0	Z	0	Z_2	0	0	Z	Z	Z_2	0
$n \equiv 3$	Z	0	Z	0	Z	0	0	Z_2	Z_2	0
$n \equiv 4$	0	Z	0	Z	Z_2	Z	0	Z_2	0	0
$n \equiv 5$	Z	0	0	0	Z_2	Z_2	0	0	Z	Z
$n \equiv 6$	0	Z	0	0	0	Z_2	Z	Z	0	Z_2
$n \equiv 7$	Z	0	Z	0	Z	0	Z_2	0	0	Z_2

PROOF. (1) is contained in the proof of Theorem 5.12. (2) is similarly obtained from Lemma 5.15. One obtains (3) from $cr = 1 + t = c'q$ and $rc = qc' = 2$ in 4.6(1). See Table 4.1. Q.E.D.

THEOREM 6.2. *The n -dimensional homotopy groups $\pi_n(M)$ of the symmetric space M of infinite dimension for $n > 0$ are given for all values of n modulo 8.*

PROOF. For the first six spaces, the results are already known in Corollary 5.7, Theorems 5.12 and 5.16. From Theorem 5.10, $\pi_n(O/U) \cong \pi_n(\Omega SO) \cong \pi_{n+1}(SO) \cong \pi_{n+1}(O)$ for $n \geq 0$. From Theorem 5.14, $\pi_n(Sp/U) \cong \pi_n(\Omega Sp) \cong \pi_{n+1}(Sp)$. Thus we have obtained the results for O/U and Sp/U .

Since the homotopy fibres of the maps $j_H, \Omega_{i_C}, j_R, \Omega_{i_C}$ in Lemma 5.2 are weakly homotopy equivalent to $Sp, \Omega^2(SU/Sp), O, \Omega^2(SU/SO)$ respectively, the maps between their homotopy fibres are obtained from the homotopy commutativity of the diagrams in Lemma 5.2. By applying the 5-lemma to the homomorphisms between the homotopy exact sequences of the fibrations induced by these maps and β_{Sp}, β and β_O, β , we obtain the following:

6.2'. *There are (weak) homotopy equivalence $Sp \rightarrow \Omega^2(SU/Sp)$ and $O \rightarrow \Omega^2(SU/SO)$.*

The results on U/Sp and U/O are easily deduced from (6.2'). Q.E.D.

As can be conjectured from Theorem 6.2, we have the following four homotopy equivalences other than $\beta, \beta_O, \beta_{Sp}$ and $\sigma_G: G \rightarrow \Omega BG$:

$$\begin{aligned} \beta_{O/U}: U/Sp &\rightarrow \Omega(SO/U), & \beta_{U/O}: BO &\rightarrow \Omega(SU/Sp), \\ \beta_{Sp/U}: U/SO &\rightarrow \Omega(Sp/U), & \beta_{U/Sp}: BSp &\rightarrow \Omega(SU/Sp). \end{aligned}$$

These are induced by the following Bott maps:

$\beta_{O/U}(n) : U(2n)/Sp(n) \rightarrow \Omega(SO(4n)/U(2n))$ defined by

$$\beta_{O/U}(n)\{A\}(t) \equiv AT_t(n)A^{-1}T_t(n)^{-1};$$

$\beta_{U/O}(n) : O(2n)/O(n) \otimes O(n) \rightarrow \Omega(SU(2n)/SO(2n))$ defined by

$$\beta_{U/O}(n)\{A\}(t) \equiv AR_t(n)A^{-1}R_t(n)^{-1};$$

$\beta_{Sp/U}(n) : U(n)/O(n) \rightarrow \Omega(Sp(n)/U(n))$ defined by

$$\beta_{Sp/U}(n)\{A\}(t) \equiv A(e^{\pi i t/2}I_n)A^{-1}(e^{-\pi i t/2}I_n);$$

$\beta_{U/Sp}(n) : Sp(2n)/Sp(n) \times Sp(n) \rightarrow \Omega(SU(4n)/Sp(2n))$ defined by

$$\beta_{U/Sp}(n)\{A\}(t) \equiv AR_t^2(n)A^{-1}R_t^2(n)^{-1},$$

where $T_t(n) = S_{t/2}(n) \oplus S_{-t/2}(n)$, $R_t^2(n) = R_t(n) \oplus R_t(n)$.

THEOREM 6.3. $\beta_{O/U}$, $\beta_{U/O}$, $\beta_{Sp/U}$, and $\beta_{U/Sp}$ are (weak) homotopy equivalences.

Bott's proof uses Morse theory (Chapter VI). A proof using homology groups and cohomology groups was given by Dyer-Lashof [26]. Also there is a method of using Clifford algebras to deduce the periodicity of KF -groups from these results [28].

Hopf fiberings

DEFINITION. An algebra obtained from H^2 by defining the product of two elements $x = (p_1, p_2)$, $y = (q_1, q_2)$ by $xy = (p_1q_1 - p_2\bar{q}_2, p_1q_2 + \bar{q}_1p_2)$ is called the *Cayley numbers (field)*, denoted by $\mathfrak{C}$.

The product in $\mathfrak{C}$ is not associative, but the following properties can be checked by a direct calculation:

6.4. If one puts $\bar{x} = (\bar{p}_1, -p_2)$ and $|x| = \sqrt{|p_1|^2 + |p_2|^2}$ for $x = (p_1, p_2)$, then

$$(1) \quad x\bar{x} = \bar{x}x = |x|^2, \text{ i.e., } x^{-1} = \bar{x}/|x|, \text{ for } x \neq 0,$$

$$(2) \quad |xy| = |x| \cdot |y|.$$

THEOREM 6.5. (1) If $n = 2, 4, 8$, the principal bundle $(SO(n), p, S^{n-1}, SO(n-1))$ has a cross section.

(2) If the principal bundle $(SO(n), p, S^{n-1}, SO(n-1))$ has a cross section, then $S^{n-1} \times SO(n-1) \cong SO(n)$ (a homeomorphism), and there exists a fibre bundle $S^{n-1} \xrightarrow{i} S^{2n-1} \xrightarrow{h} S^n$ such that $(\alpha, \beta) \mapsto S\alpha + h_*\beta$ gives an isomorphism $\pi_i(S^{n-1}) \oplus \pi_{i+1}(S^{2n-1}) \cong \pi_{i+1}(S^n)$.

PROOF. (1) Consider $\mathbf{R}^n = \mathbf{C}, \mathbf{H}, \mathfrak{C}$ accordingly as $n = 2, 4, 8$, and put $s(x)y = xy$ by using the multiplication. Since $|s(x)y| = |y|$ for $x \in S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$, $s(x): \mathbf{R}^n \rightarrow \mathbf{R}^n$ is an orthogonal transformation. Thus

we have defined $s: S^{n-1} \rightarrow SO(n)$, since S^{n-1} is connected. The map s is a cross section with respect to $p: SO(n) \rightarrow S^{n-1}$ defined by $p(r) = r(e_1)$, since $y = e_1$ is the identity element of the product.

(2) For a cross section $s: S^{n-1} \rightarrow SO(n)$ we define

$h_+: D^n \otimes S^{n-1} \rightarrow E_+^n$ (the upper hemisphere) by

$$h_+(tx, y) = \left(\sqrt{1-t^2} s(x)(y), t \right),$$

$h_-: S^{n-1} \otimes D^n \rightarrow E_-^n$ (the lower hemisphere) by

$$h_-(x, ty) = \left(\sqrt{1-t^2} s(x)(y), -t \right),$$

for $t \in I$ and $x, y \in S^{n-1}$. The maps h_+ and h_- define $h: S^{2n-1} \rightarrow S^n$ as $D^n \times S^{n-1} \cup S^{n-1} \times D^n = \partial(D^n \times D^n)$ is homeomorphic with S^{2n-1} . It is easy to see that (S^{2n-1}, h, S^n) is a fibre bundle with fibre S^{n-1} and structure group $SO(n)$. The isomorphism in the theorem is obtained by verifying that $\Delta \circ S = \pm \text{id}$ for $\Delta: \pi_{i+1}(S^n) \rightarrow \pi_i(S^{n-1})$ in the homotopy exact sequence of the fibre bundle. Q.E.D.

DEFINITION. The map $h: S^{2n-1} \rightarrow S^n$ for $n = 2, 4, 8$ constructed by (2) from the cross section defined in the proof of Theorem 6.5(1) is called a *Hopf fibering*.

THEOREM 6.6. If $n \neq 2, 4, 8$, then the principal bundle $(SO(n), p, S^{n-1}, SO(n-1))$ does not have a cross section, and $p_*: \pi_{n-1}(SO(n)) \rightarrow \pi_{n-1}(S^{n-1})$ is not an epimorphism.

PROOF. We quote from Mimura-Toda [14] the following: If $n \neq 2, 4, 8$, then the Hopf invariant $H: \pi_{2n-1}(S^n) \rightarrow \mathbb{Z}$ is not an epimorphism, [14, Chapter VII, Theorems 7.1 and 7.2], and so $S: \pi_{2n-3}(S^{n-1}) \rightarrow \pi_{2n-2}(S^n)$ is not a monomorphism [14, Chapter II, Corollary 2.5]. The first half follows from Theorem 6.5(2). If p_* is an epimorphism, there is $f: S^{n-1} \rightarrow SO(n)$ such that $p \circ f \simeq 1: S^{n-1} \rightarrow S^{n-1}$. By the covering homotopy property there is s such that $f \simeq s$ and $p \circ s = 1$, i.e., s is a cross section. Thus if $n \neq 2, 4, 8$, then p_* is not an epimorphism. Q.E.D.

Now we calculate some homotopy groups of spheres. By Corollary 1.4,

6.7. The suspension homomorphism $S: \pi_k(S^n) \rightarrow \pi_{k+1}(S^{n+1})$ is an epimorphism for $k \leq 2n-1$ and an isomorphism for $k < 2n-1$.

By Theorem 6.5(2),

6.7'. For $n = 1, 3, 7$, $S: \pi_k(S^n) \rightarrow \pi_{k+1}(S^{n+1})$ is a monomorphism, in particular,

$$S: \pi_{2n-1}(S^n) \cong \pi_{2n}(S^{n+1}).$$

Since in the covering space $(\mathbf{R}, p, S^1)$ $\mathbf{R}$ is contractible, one has

$$\pi_k(S^1) = 0 \quad \text{for } k > 1, \quad \pi_1(S^1) \cong \mathbf{Z}.$$

Therefore by Theorem 6.5(2),

$$h_* : \pi_k(S^3) \cong \pi_k(S^2) \quad \text{for } k \geq 3. \quad (6.8)$$

Recall that $S^3 \cong Sp(1)$. By Corollary 3.17 of Chapter II,

$$\pi_k(S^3) \cong \pi_k(Sp(1)) \cong \pi_k(\mathbf{Sp}) \quad \text{for } k < 6.$$

This is already known by Theorem 6.2 and so the following results are obtained:

- 6.9. (1) $\pi_k(S^n) = 0$ for $k < n$;
 (2) $\pi_n(S^n) \cong \mathbf{Z}$ for $n \geq 1$;
 (3) $\pi_2(S^1) = 0$; $\pi_3(S^2) \cong \mathbf{Z}$; and $\pi_{n+1}(S^n) \cong \mathbf{Z}_2$ for $n \geq 3$;
 (4) $\pi_3(S^1) = 0$; and $\pi_{n+2}(S^n) \cong \mathbf{Z}_2$ for $n \geq 2$.

Here we note the following:

- 6.9'. (1) The generator of $\pi_n(S^n)$ is the class $\iota_n = [1_{S^n}]$ of the identity map.
 (2) The generator of $\pi_3(S^2)$ is the class $\eta_2 = [h]$ of the Hopf fibering $h: S^3 \rightarrow S^2$. The generator of $\pi_{n+1}(S^n)$ is $\eta_n = S^{n-2}\eta_2$ for $n \geq 2$.
 (3) The generator of $\pi_{n+2}(S^n)$ is $\eta_n^2 = \eta_n \circ \eta_{n+1}$ for $n \geq 2$.

THEOREM 6.10 (Serre). (1) If n is odd then $\pi_k(S^n)$ is finite for $k \neq n$.

(2) If n is even then $\pi_k(S^n)$ is finite for $k \neq n, 2n-1$; $\pi_k(S^n)$ is a direct sum of an infinite cyclic group and a finite group for $k = 2n-1$.

PROOF. Let K_n denote the Eilenberg-Mac Lane space $K(\mathbf{Z}, n)$. We see by induction on $n \geq 1$ that

$$H^*(K_n; \mathbf{Q}) = \begin{cases} \Lambda(u_n), & \text{for odd } n, \\ \mathbf{Q}[u_n], & \text{for even } n, \end{cases} \quad (6.10')$$

beginning with $S^1 = K(\mathbf{Z}, 1)$ and using the spectral sequence of a fibration $K_n = \Omega K_{n+1} \rightarrow PK_{n+1} \rightarrow K_{n+1}$. By the Serre $\mathcal{S}$ -theory, $H_*(S^n)$ finitely generated implies that $\pi_*(S^n)$ is finitely generated. Suppose that a generator of $\pi_n(K_n) \cong \mathbf{Z}$ is represented by the inclusion $S^n \subset K_n$. If n is odd, then $H_*(K_n, S^n; \mathbf{Q}) = 0$. Thus $\pi_*(K_n, S^n)$ is finite and we have proved (1). If n is even, then $H^*(\Omega(K_n, S^n)) = \Lambda(u_{2n-1})$ by using the spectral sequence of the fibration $\Omega(K_n, S^n) \rightarrow P(K_n, S^n) \rightarrow K_n$. Therefore $\pi_*(\Omega(K_n, S^n))$ is $\mathcal{S}$ -isomorphic with $\pi_*(S^{2n-1})$ for the class $\mathcal{S}$ of finite abelian groups. Combining this with (1), we obtain (2). Q.E.D.

Semistable homotopy groups of classical groups. Consider homotopy groups of the classical group $G(n)$ for $G = U, SU, Sp$, and SO . Since we have

$$\pi_k(SU(n)) \cong \pi_k(U(n)) \quad \text{for } k > 1$$

it suffices to study either one of $SU(n)$ and $U(n)$.

From Corollary 3.17 of Chapter II we obtain the following isomorphism:

$$\begin{aligned} \text{If } k < d(n+1) - 2 \text{ then } i_* : \pi_k(G(n)) &\cong \pi_k(G) \\ \text{for } d = 1(G = SO), \quad d = 2(G = SU, U), \quad d = 4(G = Sp). \end{aligned} \quad (6.11)$$

$\pi_k(G(n))$, in the range $k < d(n+1) - 2$, is said to be *stable* and is called a *stable homotopy group*.

For sufficiently large n , $\pi_k(G(n)) \cong \pi_k(G)$ and hence we obtain a homotopy exact sequence

$$\cdots \rightarrow \pi_{k+1}(G(N)/G(n)) \xrightarrow{\Delta} \pi_k(G(n)) \xrightarrow{i_*} \pi_k(G) \xrightarrow{p_*} \pi_k(G(N)/G(n)) \rightarrow \cdots \quad (6.12)$$

where $G(N)/G(n)$ is $(d(n+1) - 2)$ -connected. Hence by Corollary 1.14 we obtain the following isomorphism:

$$\begin{aligned} \text{For } k+1 \leq 2d(n+1) - 4, \\ S^M : \pi_{k+1}(G(N)/G(n)) \cong \pi_{k+M+1}(S^M(G(N)/G(n))). \end{aligned} \quad (6.12')$$

In other words, $\pi_{k+1}(G(N)/G(n))$ is a stable homotopy group under the suspension, and this is the range where various methods in the homotopy theory are easy to apply. $\pi_k(G(n))$ in the range $k < 2d(n+1) - 6$ is called a *semistable homotopy group*.

We will determine the image of the homomorphism

$$p_* : \pi_{d(n+1)-1}(G(n+1)) \rightarrow \pi_{d(n+1)-1}(S^{d(n+1)-1}) \cong \mathbb{Z}$$

induced from the projection $p : G(n+1) \rightarrow G(n+1)/G(n) = S^{d(n+1)-1}$.

THEOREM 6.13. (1) $p_* \pi_{2n}(SO(2n+1)) = 0$,

$$p_* \pi_{2n-1}(SO(2n)) = \begin{cases} \pi_{2n-1}(S^{2n-1}) & \text{for } n = 1, 3, 7, \\ 2\pi_{2n-1}(S^{2n-1}) & \text{for } n \neq 1, 3, 7. \end{cases}$$

$$(2) \quad p_* \pi_{2n+1}(U(n+1)) = p_* \pi_{2n+1}(SU(n+1)) = n! \pi_{2n+1}(S^{2n+1}).$$

(3)

$$p_* \pi_{4n+3}(Sp(n+1)) = \begin{cases} (2n+1)! \pi_{4n+3}(S^{4n+3}) & \text{for } n \text{ even,} \\ 2(2n-1)! \pi_{4n+3}(S^{4n+3}) & \text{for } n \text{ odd.} \end{cases}$$

PROOF. (1) Consider the homotopy exact sequence

$$\begin{aligned} \pi_{2n}(S^{2n}) &\xrightarrow{\Delta} \pi_{2n-1}(S^{2n-1}) \\ &\xrightarrow{i_*} \pi_{2n-1}(SO(2n+1)/SO(2n-1)) \rightarrow \pi_{2n-1}(S^{2n}) = 0 \end{aligned}$$

associated with the fibre bundle $S^{2n-1} \rightarrow SO(2n+1)/SO(2n-1) \rightarrow S^{2n}$. By Corollary 3.13 of Chapter III, $H_{2n-1}(SO(2n+1)/SO(2n-1)) \cong \mathbb{Z}_2$, whence by Theorem 1.3, $\pi_{2n-1}(SO(2n+1)/SO(2n-1)) \cong \mathbb{Z}_2$ and so $\text{Im } \Delta = 2\pi_{2n-1}(S^{2n-1})$. Since Δ is decomposed as $\pi_{2n}(S^{2n}) \xrightarrow{\Delta'} \pi_{2n-1}(SO(2n)) \xrightarrow{p_*} \pi_{2n-1}(S^{2n-1})$, we have $\text{Im } p_* \supset \text{Im } \Delta = 2\pi_{2n-1}(S^{2n-1})$. Thus the latter half of (1) is deduced from Theorems 6.5 and 6.6.

We have $p_*\pi_{2n}(SO(2n+1)) = 0$, since the sequence

$$\pi_{2n}(SO(2n+1)) \xrightarrow{p_*} \pi_{2n}(S^{2n}) \xrightarrow{\Delta'} \pi_{2n-1}(SO(2n))$$

is exact and Δ' is a monomorphism.

(2) By (6.11), $\pi_{2n+1}(U(n+1)) \cong \pi_{2n+1}(\mathbb{U})$, whence $\pi_{2n+2}(BU(n+1)) \cong \pi_{2n+2}(\mathbb{B}\mathbb{U})$. By Lemma 5.8(3) we have $g^*(c_{n+1}) = \pm n! u_{2n+2}$ for a map $g: S^{2n+2} \rightarrow BU(n+1)$ representing a generator of $\pi_{2n+2}(BU(n+1))$. We easily see that $f^*(\sigma(c_{n+1})) = \pm n! u_{2n+1}$ for the adjoint map $f: S^{2n+1} \rightarrow U(n+1)$ of g . On the other hand, since $p^*(u_{2n+1}) = \sigma(c_{n+1}) + (\text{decomposables})$ for $p: U(n+1) \rightarrow S^{2n+1}$, we have $(p \circ f)^*(u_{2n+1}) = \pm n! u_{2n+1}$. A similar fact holds for the corresponding homology group and hence for the homotopy group by the Hurewicz homomorphism: $(p \circ f)_* i_{2n+1} = \pm n! i_{2n+1} \in \pi_{2n+1}(S^{2n+1})$. This proves (2).

(3) By the commutative diagram

$$\begin{array}{ccc} Sp(n+1) & \xrightarrow{c'} & SU(2n+2) \\ \downarrow p & & \downarrow p \end{array}$$

$$Sp(n+1)/Sp(n) \cong SU(2n+2)/SU(2n+1)$$

it suffices to study the behavior of

$$c'_*: \pi_{4n+3}(Sp(n+1)) \rightarrow \pi_{4n+3}(SU(2n+2)).$$

By using (6.10') it is $c'_*: \pi_{4n+3}(\mathbf{Sp}) \rightarrow \pi_{4n+3}(\mathbb{U})$ and is equivalent to

$$c': \widetilde{KSp}(S^{4n+4}) = \pi_{4n+4}(\mathbf{BSp}) \rightarrow \widetilde{K}(S^{4n+4}) = \pi_{4n+4}(\mathbb{B}\mathbb{U}).$$

By Theorem 5.16, c' is an isomorphism if $n+1$ is odd, and is an isomorphism onto $2\widetilde{K}(S^{4n+4})$ if $n+1$ is even. Thus (3) is obtained. Q.E.D.

COROLLARY 6.14.

(1) Let $n \neq 1, 2, 3, 7$.

$$\pi_{n-1}(SO(n)) \cong \mathbb{Z} \oplus \mathbb{Z} \quad (n \equiv 0, 4 \pmod{8}),$$

$$\pi_{n-1}(SO(n)) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad (n \equiv 2 \pmod{8}),$$

$$\pi_{n-1}(SO(n)) \cong \mathbb{Z} \quad (n \equiv 6 \pmod{8}),$$

$$\pi_{n-1}(SO(n)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad (n \equiv 1 \pmod{8}),$$

$$\pi_{n-1}(SO(n)) \cong \mathbb{Z}_2 \quad (n \equiv 3, 5, 7 \pmod{8}),$$

where $\pi_2(SO(3)) = \pi_6(SO(7)) = 0$.

- (2) $\pi_{2n}(U(n)) \cong \mathbb{Z}_{n!}$.
 (3) $\pi_{4n-2}(Sp(n)) \cong \mathbb{Z}_{2(2n+1)!}$ for odd n ,
 $\pi_{4n-2}(Sp(n)) \cong \mathbb{Z}_{(2n+1)!}$ for even n .

PROOF. The corollary is obtained by applying Theorems 6.13 and 6.2 to the exact sequence

$$\pi_{d(n+1)-1}(G(n+1)) \xrightarrow{p_*} \pi_{d(n+1)-1}(S^{d(n+1)-1}) \xrightarrow{\Delta} \pi_{d(n+1)-2}(G(n)) \\ \rightarrow \pi_{d(n+1)-2}(G) \rightarrow 0$$

obtained from (6.12). Here the group extension problem occurs at $\pi_{n-1}(SO(n))$ for $n \equiv 1, 2 \pmod{8}$. To determine the extension it suffices to show the following:

6.15. $\wedge \eta : \widetilde{KO}(S^n) \rightarrow \widetilde{KO}(S^{n+1})$ is given by the composition $\eta_n^* : [S^n, \mathbf{BO}] \rightarrow [S^{n+1}, \mathbf{BO}]$ of the generator η_n of $\pi_{n+1}(S^n)$, for $n \geq 2$. (The proof is omitted.)

Using this one can obtain an element of order 2 of $\pi_{n-1}(SO(n))$ for $n \equiv 1, 2 \pmod{8}$ such that its image by i_* is a generator of $\pi_{n-1}(\mathbf{SO})$. Thus the group extension problem is trivial. Q.E.D.

REMARK 6.16. To carry out the calculation further, we need various facts from homotopy theory, for which we have no space in this book. We just mention the following examples:

$$\begin{aligned} \pi_{2n+1}(U(n)) &\cong \mathbb{Z}_2 \quad \text{for } n \text{ even,} \\ &= 0 \quad \text{for } n \text{ odd;} \\ \pi_{2n+2}(U(n)) &\cong \mathbb{Z}_{(n+1)!} \oplus \mathbb{Z}_2 \quad \text{for } n \text{ even,} \\ &\cong \mathbb{Z}_{(n+1)!/2} \quad \text{for } n \text{ odd;} \\ \pi_{2n+3}(U(n)) &\cong \mathbb{Z}_{d(n)} \quad \text{for } n \text{ even,} \\ &\cong \mathbb{Z}_{d(n)/2} \quad \text{for } n \text{ odd,} \end{aligned}$$

where $n \geq 3$ and $d(n) = (n(3n+1), 24)$ is the greatest common divisor.

It was shown in §5 of Chapter II that $\text{Spin}(n)$ is isomorphic to a classical group for $n = 3, 4, 5, 6$. In addition, using the isomorphism $\pi_k(\text{Spin}(n)) \cong \pi_k(SO(n))$ for $k > 1$ by the covering map, one can obtain the following isomorphisms:

6.17. Let $k > 1$.

$$\begin{aligned} \pi_k(Sp(1)) &\cong \pi_k(SU(2)) \cong \pi_k(SO(3)), \\ \pi_k(Sp(1) \times Sp(1)) &\cong \pi_k(SO(4)), \\ \pi_k(Sp(2)) &\cong \pi_k(SO(5)), \\ \pi_k(SU(4)) &\cong \pi_k(SO(6)). \end{aligned}$$

TABLE 4.2.

	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
$\pi_k(Sp(1))$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
$\pi_k(Sp(2)), n \geq 2$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}$	0
$\pi_k(U(3))$	$\mathbf{Z}$	0	$\mathbf{Z}$	$\mathbf{Z}_6$	0	$\mathbf{Z}_{12}$
$\pi_k(U(4))$	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	$\mathbf{Z}_{24}$
$\pi_k(U(n)), n \geq 5$	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0
$\pi_k(O(7))$	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\pi_k(O(8))$	$\mathbf{Z}$	0	0	0	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\pi_k(O(9))$	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\pi_k(O(n)), n \geq 10$	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$

Excluding these isomorphisms, $\pi_k(SU(n)) \cong \pi_k(U(n))$ for $k > 1$ and $\pi_k(SO(n)) \cong \pi_k(O(n))$ for $k > 0$, the homotopy groups of classical groups for dimensions ≤ 8 are given in Table 4.2, where $\pi_2(G)$ is omitted as it is always 0. We use also the following:

REMARK 6.18. Using the Cayley numbers one can construct an inclusion of $\text{Spin}(7)$ in $\text{Spin}(9)$ as a suitable subgroup such that $\text{Spin}(9)/\text{Spin}(7) \cong S^{15}$ (cf. I. Yokota [8]). See Table 4.2.

Mod p decomposition of classical groups by homogeneous spaces

DEFINITION. Let p be a prime or 0.

A map $f: X \rightarrow Y$ is called a p -equivalence if f induces a bijection of each arcwise component and $f_*: \pi_*(X_\alpha) \rightarrow \pi_*(Y_\alpha)$ is a $\mathcal{G}_p$ -isomorphism on each component, where $\mathcal{G}_0 = \mathcal{F}$.

NOTATION. In this situation we write $f: X \simeq_p Y$.

According to Theorem 1.12,

6.19. If X and Y are 1-connected and of finite type, then $f: X \simeq_p Y$ is equivalent to $f_*: H_*(X; \mathbf{Z}_p) \cong H_*(Y; \mathbf{Z}_p)$, and to $f^*: H^*(Y; \mathbf{Z}_p) \cong H^*(X; \mathbf{Z}_p)$, where $\mathbf{Z}_0 = \mathbf{Z}$.

DEFINITION. A map $s: B \rightarrow E$ of a fibration (E, π, B) is called a mod p cross section if $\pi \circ s: B \rightarrow B$ is a p -equivalence.

6.20. If a principal fibration (E, π, B, F) admits a mod p cross section then there is a p -equivalence $f: B \times F \simeq_p E$, and $\pi_i(E)$ is $\mathcal{G}_p$ -isomorphic to $\pi_i(B) \oplus \pi_i(F)$.

One can put $f(b, y) = \mu(s(b), y)$ for the action μ of F on E .

COROLLARY 6.21. If p is an odd prime then

$$SO(2n-1) \times S^{2n-1} \simeq_p SO(2n).$$

PROOF. By Theorem 6.13(1) there is $s: S^{2n-1} \rightarrow SO(2n)$ such that $p \circ s$ has mapping degree 2 and hence s is a mod p cross section. Thus we obtain the corollary from 6.20. Q.E.D.

LEMMA 6.22. Let p be an odd prime. Then we have

$$S^{4n-1} \simeq_p SO(2n+1)/SO(2n-1).$$

$i_*: \pi_i(SO(2n+1)) \rightarrow \pi_i(\mathbf{SO})$ and

$$i_*: \pi_i(SU(2n+1)/SO(2n+1)) \rightarrow \pi_i(\mathbf{SU}/\mathbf{SO})$$

are $\mathcal{E}_p$ -isomorphisms for $i < 4n+2$ and $\mathcal{E}_p$ -epimorphisms for $i = 4n+2$.

PROOF. The double covering $S^3 = Sp(1) \rightarrow SO(3)$ is a p -equivalence. $V = SO(2n+1)/SO(2n-1)$ is 4-connected for $n > 1$. We have a map $f: S^{4n-1} \rightarrow V$ such that $\Xi[f] \not\equiv 0 \pmod{p}$ by Theorem 1.11, since

$$H_*(V; \mathbb{Z}_p) \cong H_*(S^{4n-1}; \mathbb{Z}_p).$$

The map f is a p -equivalence. The latter half is obtained by replacing the isomorphisms and so on with $\mathcal{E}_p$ -isomorphisms and so on in the proof of Lemma 3.11. Q.E.D.

THEOREM 6.23 (The Harris decomposition). For p an odd prime, there are (mod p) cross sections $s_{n+1}: SU(2n+1)/SO(2n+1) \rightarrow SU(2n+1)$ and $s'_{2n}: SU(2n)/Sp(n) \rightarrow SU(2n)$ so that

$$\begin{aligned} (SU(2n+1)/SO(2n+1)) \times SO(2n+1) &\simeq_p SU(2n+1), \\ (SU(2n)/Sp(n)) \times Sp(n) &\simeq_p SU(2n). \end{aligned} \quad \text{AS}$$

PROOF. We define $s_m: SU(m)/SO(m) \rightarrow SU(m)$ by $s_m\{A\} = A\bar{A}^{-1}$. For $B \in SO(m)$, $\bar{B} = B$ and $s_m\{AB\} = AB\bar{B}^{-1}\bar{A}^{-1} = A\bar{A}^{-1}$. Thus, s_m is well defined and naturally extended to $s: \mathbf{SU}/\mathbf{SO} \rightarrow \mathbf{SU}$. If we put $f = s \circ \pi: \mathbf{SU} \rightarrow \mathbf{SU}/\mathbf{SO} \rightarrow \mathbf{SU}$ the composition with the projection, we have $f(A) = A\bar{A}^{-1}$ and hence the morphism in $\tilde{K}(S^{2j})$ equivalent to $f_*: \pi_{2j-1}(\mathbf{SU}) \rightarrow \pi_{2j-1}(\mathbf{SU})$ is $1 - t: \tilde{K}(S^{2j}) \rightarrow \tilde{K}(S^{2j})$. By Theorem 6.1(3), $f_* = 1 + (-1)^{j-1}$ is a $\mathcal{E}_p$ -isomorphism for j odd. On the other hand, by considering Theorem 6.2 and the homotopy exact sequence associated with $\mathbf{SO} \xrightarrow{1} \mathbf{SU} \xrightarrow{\pi} \mathbf{SU}/\mathbf{SO}$, we have $\pi_*: \pi_{4k+1}(\mathbf{SU}) \rightarrow \pi_{4k+1}(\mathbf{SU}/\mathbf{SO})$ is a $\mathcal{E}_p$ -isomorphism and hence

$$(\pi \circ s)_* = \pi_* \circ s_*: \pi_i(\mathbf{SU}/\mathbf{SO}) \rightarrow \pi_i(\mathbf{SU}/\mathbf{SO})$$

is a $\mathcal{E}_p$ -isomorphism. Putting $W_n = SU(2n+1)/SO(2n+1)$, by Lemma 6.22 $(p \circ s)_*: \pi_i(W_n) \rightarrow \pi_i(W_n)$ is a $\mathcal{E}_p$ -isomorphism for $i < 4n+2$. Here, as we have $H^*(W_n; \mathbb{Z}_p) = \Lambda(e_5, e_9, \dots, e_{4n+1})$ (cf. Theorem 6.7(2) of Chapter III), $(p \circ s)^*: H^*(W_n; \mathbb{Z}_p) \cong H^*(W_n; \mathbb{Z}_p)$ implies that $p \circ s$ is a p -equivalence ($s = s_{2n+1}$) and hence s_{2n+1} is a mod p cross section.

By using 2.14(2) of Chapter I we can define $s'_{2n}: SU(2n)/Sp(n) \rightarrow SU(2n)$ by $s'_{2n}\{A\} = AJ_n\bar{A}^{-1}J_n$. We have $(s'_{2n} \circ p')(A) = AJ_n\bar{A}^{-1}J_n$. Using a path connecting J_n and I_n in $SU(2n)$, we obtain $s'_{2n} \circ p' \simeq s_{2n} \circ p: SU(2n) \rightarrow SU(2n)$. The remainder is similar to the above, using Lemma 3.11(3) instead of Lemma 6.22. Q.E.D.

DEFINITION. A Lie group G is called p -regular if there is a p -equivalence $f: \prod S^{n_i} \simeq_p G$ from the direct product of spheres S^{n_i} .

6.24. If G is p -regular then $H^*(G; \mathbb{Z}_p) = \Lambda(x_{n_1}, \dots, x_{n_i})$, where $x_{n_i} \in H^{n_i}$, $\pi_*(G)$ and $\sum_i \pi_*(S^{n_i})$ are $\mathbb{Z}_p$ -isomorphic.

THEOREM 6.25 (Serre). (1) $U(n)$ and $SU(n)$ are p -regular iff $n \leq p$.

(2) $Sp(n)$ is p -regular iff $2n \leq p$.

(3) $O(n)$ and $SO(n)$ are p -regular iff $n-1 \leq p$.

PROOF. (1) We will show this for $SU(n)$, since the multiplication gives a homeomorphism $S^1 \times SU(n) \cong U(n)$ (where $S^1 = U(1)$). If $n \leq p$, then by Theorem 6.13(2), there is $s: S^{2n-1} \rightarrow SU(n)$ such that the mapping degree of $p \circ s$ is $(n-1)! \not\equiv 0 \pmod{p}$. By 6.20, $SU(n-1) \times S^{2n-1} \simeq_p SU(n)$. By induction on n , $n \leq p$ implies $SU(n)$ is p -regular.

Conversely, suppose that $SU(n)$ is p -regular for $n > p$. Since

$$H^*(SU(n)) = \Lambda(e_3, e_5, \dots, e_{2n-1}),$$

there is a p -equivalence $f: S^3 \times S^5 \times \dots \times S^{2n-1} \rightarrow SU(n)$. If we take f_n to be the restriction of f on S^{2n-1} , then $f_n^*(e_{2n-1}) \not\equiv 0 \pmod{p}$. Since we can take $e_{2n-1} = p^* u_{2n-1}$ for a generator u_{2n-1} of $H^{2n-1}(S^{2n-1})$, we have $(p \circ f_n)^*(u_{2n-1}) \not\equiv 0 \pmod{p}$. On the other hand, by Theorem 6.13(2) $[p \circ f_n] \in p_* \pi_{2n-1}(SU(n)) = (n-1)! \pi_{2n-1}(S^{2n-1})$ implies $(p \circ f_n)^*(u_{2n-1}) \equiv 0 \pmod{(n-1)!}$. This is a contradiction, as p is a factor of $(n-1)!$. Thus if $SU(n)$ is p -regular we must have $n \leq p$.

(2) is proved similarly to (1), by using Theorem 6.13(3).

(3) It suffices to show the result for $SO(n)$, since there is a homeomorphism $S^0 \times SO(n) \cong O(n)$. Also by Corollary 6.14, it suffices to show the result for odd $n = 2m+1$. When $m = 1$, the double covering $p: Sp(1) \rightarrow SO(3)$ is a p -equivalence for $p > 2$. $SO(n)$ is not 2-regular, since $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$. Using the above facts, we prove the result for $n = 2m+1 \geq 5$ and $p \geq 3$. Then we can use Theorem 6.23.

Let $n-1 = 2m \leq p$, then $2m+1 \leq p$. A p -equivalence $f: S^3 \times S^5 \times \dots \times S^{4m+1} \rightarrow SU(2m+1)$ in (1) can be constructed from maps $f_i: S^{2i+1} \rightarrow SU(2m+1)$ such that $[f_i]$ is an a_i -multiple of a generator of

$$\pi_{2i+1}(SU(2m+1))$$

with $a_i \not\equiv 0 \pmod{p}$ by using the multiplication of $SU(2m+1)$. Here, taking Theorem 6.23 into account, one sees that f_i can be chosen so that

$f_{2j+1}(S^{4j+3}) \subset SO(2m+1)$ for odd $i = 2j+1$. One constructs a map $\bar{f}: Y \rightarrow SO(2m+1)$ by using the maps f_{2j+1} and the multiplication of $SO(2m+1)$, where $Y = S^3 \times S^7 \times \cdots \times S^{4m-1}$. One also constructs $g': Y' \rightarrow SU(2m+1)$ for $Y' = S^5 \times S^9 \times \cdots \times S^{4m+1}$ using f_{2j} and puts $\bar{g} = p \circ g': Y' \rightarrow SU(2m+1)/SO(2m+1)$. For the p -equivalence

$$h(= s_{2n+1} \circ c): (SU(2m+1)/SO(2m+1)) \times SO(2m+1) \rightarrow SU(2m+1)$$

in Theorem 6.23, the composition $h \circ (\bar{f} \times \bar{g}): Y \times Y' \rightarrow SU(2m+1)$ is a p -equivalence. Therefore, $\bar{f}$ is a p -equivalence and hence $SO(2m+1)$ is p -regular.

If we suppose that $SO(2m+1)$ is p -regular for $n-1 = 2m > p$, then there is a p -equivalence $g: S^3 \times \cdots \times S^{4m-1} \rightarrow SO(2m+1)$. Since we have $p_m^* = (-1)^m \rho_c^*(c_{2m})$ in $H^*(SO(2m+1); \mathbb{Z}_p) = \Lambda(e_3, \dots, e_{4m-1})$ with $e_{4m-1} = \sigma(p_m)$ and $H^*(SU(2m+1); \mathbb{Z}_p) = \Lambda(e_3, \dots, e_{4m+1})$ with $e_{2j-1} = \sigma(c_j)$, we have $c^*(e_{4m-1}) = (-1)^m e_{4m-1}$. Putting $f_{2m} = c \circ (g|S^{4m-1}): S^{4m-1} \rightarrow SO(2m+1) \subset SU(2m+1)$, we have $f_{2m}^*(e_{4m-1}) \not\equiv 0 \pmod{p}$ and hence we would get a contradiction as in the proof of the latter half of (1). Q.E.D.

Mod 0 decomposition of compact Lie groups. The following is the classical Hopf theorem from which originated the theory of H -spaces and Hopf algebras.

THEOREM 6.26 (Hopf). *Let k be a field of characteristic 0. For a compact connected Lie group G , there exist elements $x_{2n_i+1} \in H^{2n_i+1}(G; k)$ of odd degree for $i = 1, \dots, l$, such that*

$$H^*(G; k) = \Lambda(x_{2n_1+1}, \dots, x_{2n_l+1}).$$

The results also hold for an arcwise connected H -space G with $\dim_k H^*(G; k) < \infty$. Moreover, one can replace $H^*(G; k)$ by a connected, commutative Hopf algebra A^* of finite dimension over k .

PROOF. It suffices to show the theorem for A^* . We choose a minimal number of multiplicative generators $\{a_i\}$ of A^* , where $a_i \in A^{m_i}$ for $i = 1, 2, \dots$ proceeding successively from the one of the lowest degree. First we show that the degree m_i of a_i is odd. Suppose m_k even, we will deduce a contradiction. The quotient ring $B^* = A^*/(a_i \text{ for } i \neq k)$ is generated by a_k , but there is $t > 0$ such that $a_k^{t-1} \neq 0$ and $a_k^t = 0$, since $\dim_k A^* < \infty$. Let $\pi: A^* \rightarrow B^*$ be the projection and consider a homomorphism

$$\phi' = (\pi \otimes 1) \circ \phi: A^* \rightarrow A^* \otimes A^* \rightarrow B^* \otimes A^*$$

of rings. Since $\phi'(a_k) = (\pi \otimes 1)(a_k \otimes 1 + \cdots + 1 \otimes a_k)$, we have

$$0 = \phi'(a_k^t) = (\phi'(a_k))^t = (a_k \otimes 1 + 1 \otimes a_k)^t = \sum_{0 < j < t} \binom{t}{j} a_k^j \otimes a_k^{t-j}.$$

This is a contradiction, since the right-hand side is not 0. Thus the degree of every generator is odd. So $a_i^2 = -a_i^2$ implies $a_i^2 = 0$.

Obviously, the number of generators is finite (say l) and hence the natural homomorphism

$$\Lambda(a_1, \dots, a_l) \rightarrow A^*$$

is surjective. We will show that if $\Lambda(a_1, \dots, a_{k-1}) \rightarrow A^*$ is a monomorphism, then so is $\Lambda(a_1, \dots, a_k) \rightarrow A^*$. For a_k , consider ϕ' in the above. Since $\phi'(a_i) = 1 \otimes a_i$ for $i < k$, $y \in \Lambda(a_1, \dots, a_{k-1})$ implies $\phi'(y) = 1 \otimes y$. It suffices to show that $x = 0$, supposing that $x \in \Lambda(a_1, \dots, a_k)$ vanishes in A^* . Since we can write $x = a_k y + y'$ for $y, y' \in \Lambda(a_1, \dots, a_{k-1})$, we have in $B^* \otimes A^*$

$$\begin{aligned} 0 &= \phi'(x) = \phi'(a_k y + y') = (a_k \otimes 1 + 1 \otimes a_k)(1 \otimes y) + 1 \otimes y' \\ &= a_k \otimes y + 1 \otimes x = a_k \otimes y. \end{aligned}$$

Since $\Lambda(a_k) \otimes \Lambda(a_1, \dots, a_{k-1}) \rightarrow B^* \otimes A^*$ is a monomorphism, $y = 0$, which implies $y' = x = 0$. Therefore, by induction on k , $\Lambda(a_1, \dots, a_l) \cong A^*$ is proved. Q.E.D.

DEFINITION. A sequence $(2n_1 + 1, \dots, 2n_l + 1)$ of odd degrees, in the theorem above, written in the order of the size is called the *type* of G and the number of the generators the *rank* of G , denoted by $l = \text{rank } G$.

THEOREM 6.27. Let G be a 1-connected Lie group of type $(2n_1 + 1, \dots, 2n_l + 1)$. Then there exists a 0-equivalence

$$f: S^{2n_1+1} \times \dots \times S^{2n_l+1} \simeq_0 G$$

that is, G is 0-regular.

PROOF. We have $H^*(G; \mathbf{k}) = \Lambda(x_{2n_1+1}, \dots, x_{2n_l+1})$. By induction on i , we will show the existence of maps f_i such that

$$f_i: S^{2n_i+1} \rightarrow G \quad \text{and} \quad f_i^*(x_{2n_i+1}) \neq 0.$$

We may consider this trivial for $i = 0$. Suppose that there exists such $f_1, \dots, f_i$ and put $Y_i = S^{2n_1+1} \times \dots \times S^{2n_i+1}$. By using the multiplication of G we define $F = f_1 \cdots f_i: Y \rightarrow G$ by

$$F(x_1, \dots, x_i) = f_1(x_1) \cdots f_i(x_i).$$

For simplicity we put $k = 2n_i + 1$. Since $F^*: H^*(G; \mathbf{k}) \rightarrow H^*(Y_i; \mathbf{k}) \cong \Lambda(x_{2n_1+1}, \dots, x_{2n_i+1})$ is an epimorphism, we have

$$\text{Ker } F^* = (\text{the ideal generated by } x_k, \dots, x_{2n_i+1})$$

rechoosing the generators $x_k, \dots, x_{2n_i+1}$ if necessary. Using the mapping cylinder, we may suppose $Y_i \subset G$. We have $H_j(G, Y_i; \mathbf{k}) = H^j(G, Y_i; \mathbf{k}) = 0$ for $0 < j < k$ and so $H_j(G, Y_i) \in \mathcal{F}$ (the class of finite groups).

By using Theorem 1.11(2) one sees that $\Xi: \pi_k(G, Y_i) \rightarrow H_k(G, Y_i)$ is an $\mathcal{F}$ -isomorphism. In the exact sequence

$$\pi_k(G) \xrightarrow{j_*} \pi_k(G, Y_i) \xrightarrow{\partial} \pi_{k-1}(Y_i),$$

$\pi_{k-1}(Y_i) \cong \sum_{j < i} \pi_{k-1}(S^{2n_j+1})$ is a finite group by Theorem 6.10. Therefore $\langle x_k, \Xi j_*(\alpha) \rangle \neq 0$ for $\alpha \in \pi_k(G)$, that is, we can take a map $f_{i+1}: S^k \rightarrow G$ representing α such that $f_{i+1}^*(x_k) \neq 0$.

By induction we see that there exist $f_1, \dots, f_l$ so that $f = f_1 \cdots f_l$ is the required 0-equivalence. Q.E.D.

COROLLARY 6.28. $H_*(\Omega G; \mathbf{k}) \cong H^*(\Omega G; \mathbf{k}) = \mathbf{k}[y_{2n_1}, \dots, y_{2n_l}]$.

PROOF. By Corollary 2.6, $H_*(\Omega S^{2n+1}; \mathbf{k}) = \mathbf{k}[e_{2n}]$; and hence by Theorem 2.18, $H^*(\Omega S^{2n+1}; \mathbf{k}) = \mathbf{k}[y_{2n}]$. Theorem 6.27 implies that Ωf is a 0-equivalence. The corollary is obtained by using the Künneth formula. Q.E.D.

COROLLARY 6.29. For the Poincaré series $P(\ , t)$ with respect to a coefficient field of characteristic 0, we have

$$P(G, t) = (1 + t^{2n_1+1}) \cdots (1 + t^{2n_l+1}), \quad P(\Omega G, t) = (1 - t^{2n_1})^{-1} \cdots (1 - t^{2n_l})^{-1}.$$

Postscript

In writing the first volume we made every possible effort so that the reader would not need any preliminary knowledge but we did not have enough space to write in detail on topology, in particular on (co)homology theory and manifolds. We list references and give some supplementary explanation below.

References

First of all we refer to the following for the general theory of algebraic topology:

1. A. Komatsu, M. Nakaoka, and M. Sugawara, *Topology*. I, Iwanami-Shoten, 1967 (in Japanese).
2. E. Spanier, *Algebraic topology*, McGraw-Hill Book Co., 1966.
3. R. M. Switzer, *Algebraic topology—homotopy and homology*, Springer, 1977.

We list below references related to the facts that are studied in each chapter. Of course we are sure that there are many other good books dealing with these topics.

Chapter I. We think that this chapter requires little preliminary knowledge but there are several places where we use terminology and basic facts from general topology, for which we refer for example to:

4. J. L. Kelley, *General topology*, Princeton, 1955.

As for other facts, e.g., topological groups, linear algebras, and classical groups etc. in this chapter there are many good books, for example:

5. L. S. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N. J., 1939.
6. S. Murakami, *The foundation on the theory of topological groups*, The Basic Mathematics Series 25, Asakura-Shoten, 1973 (in Japanese).
7. I. Satake, *Linear algebras*, Shōka-Bō, 1976 (in Japanese).
8. I. Yokata, *Groups and topology*, Shōka-Bō, 1971 (in Japanese).
- 8'. —, *Groups and representation*, Shōka-Bō, 1973 (in Japanese).
9. C. Chevalley, *Theory of Lie groups*. I, Princeton Univ. Press, Princeton, N. J., 1946.

For differentiable manifolds and Lie groups see the following:

10. S. Takizawa, *Manifolds*, Chikuma-Shobô, 1971 (in Japanese).
11. Y. Matsushima, *Differentiable manifolds*, Decker, New York, 1972.
12. J. F. Adams, *Lie groups*, Benjamin, 1965.

Chapter II. Fibre bundles are presented also in [1–3] but

13. N. E. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1951,

is now a classic which is worth reading. As for the classifying space, the reader is recommended to read Chapter 5 of

14. M. Mimura and H. Toda, *Homotopy theory*, Kinokuniya Math. Ser. 3, Kinokuniya-Shoten, 1975 (in Japanese)

and the part related it.

As for the relation between Lie groups and Lie algebras stated in this chapter, see [6], [11] as well as

15. N. Iwahori, *Theory of Lie groups*. I, II, Iwanami-Kôza Modern Appl. Math., Iwanami-Shoten, 1957 (in Japanese),

16. M. Ise, *Lie groups*. I, Iwanami-Kôza Basic Mathematics, Iwanami-Shoten, 1977 (in Japanese).

Chapter III. The Serre spectral sequences are stated in [1], [2] but the reader is recommended to read the original paper:

17. J.-P. Serre, *Homologie singulière des espaces fibrés*, Ann. Math. **54** (1951), 425–505.

As for the cohomology of classical groups and their homogeneous spaces see the following:

18. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. Math. **57** (1953), 115–207.

19. —, *Sur l'homologie et la cohomologie de groupes de Lie compacts connexes*, Amer. J. Math. **76** (1954), 273–342.

20. —, *Topics in the homology theory of fibre bundles*, Lecture Notes in Math., vol. 36, Springer-Verlag, 1967.

21. N. E. Steenrod, *Cohomology operations*, Ann. of Math. Stud., vol. 50, Princeton Univ. Press, Princeton, N. J., 1962.

Also the following is interesting as there are applications to algebraic geometry.

22. F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, 1956.

Chapter IV. For Serre $\mathcal{E}$ -theory the reader is recommended to read the original paper :

23. J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. Math. **58** (1953), 258–294,

and [20] for the relation with H -spaces.

Algebraic proofs of Bott periodicity are given in

25. Séminaire H. Cartan, *Périodicité des groupes d'homotopie stables des*

groupes classiques, d'après Bott, (1959/60).

26. E. Dyer and R. Lashof, *A topological proof of the Bott periodicity theorems*, Ann. Math. Pure Appl. **54** (1961), 231–254.

27. H. Toda, *A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups*, Mem. Coll. Sci. Univ. Kyoto **32** (1959), 103–119.

28. R. Wood, *Banach algebras and Bott periodicity*, Topology **4** (1965/66), 371–379.

For proofs of the Bott periodicity using Morse theory see the following:

29. R. Bott, *The stable homotopy of the classical groups*, Ann. Math. **70** (1959), 313–337.

30. J. Milnor, *Morse theory*, Ann. Math. Stud., vol. 51, Princeton Univ. Press, Princeton, N. J., 1963.

For K -theory see the following:

31. S. Araki, *The generalized cohomology*, Kinokuniya Math. Ser. 4, Kinokuniya-Shoten, 1975 (in Japanese).

32. M. F. Atiyah, *K-theory*, Benjamin, 1967.

33. R. Bott, *Lectures on $K(X)$* , Harvard Univ. Press, 1963.

34. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., 1961, pp. 7–38.

For p -regularity and its generalization see the following:

35. M. Mimura and H. Toda, *Cohomology operations and the homotopy of compact Lie groups*. I, Topology **9** (1970), 317–336.

36. S. Oka, *On the homotopy groups of sphere bundles over spheres*, J. Sci. Hiroshima Univ. **33** (1969), 161–195.

37. M. Mimura, G. Nishida, and H. Toda, *Mod p decomposition of compact Lie groups*, Publ. RIMS of Kyoto Univ., **13** (1977), 627–680.

The de Rham cohomology

Let M be an n -dimensional differentiable manifold and T_x^*M the dual space of the tangent space T_xM at $x \in M$. Put $C_{DR}^i(M) = \{\omega: M \rightarrow \bigcup_{x \in M} \Lambda^i(T_x^*M) \text{ is an exterior differential form of degree } i\}$, where $\Lambda^i(T_x^*M)$ is the part of degree i of the exterior algebra over T_x^*M (the multiplication is denoted by $\wedge$), and $C_{DR}^*(M) = \sum_{0 \leq i \leq n} C_{DR}^i(M)$ (an infinite-dimensional algebra over $\mathbf{R}$). Then we have

2.1. *There exists a unique linear map $d: C_{DR}^*(M) \rightarrow C_{DR}^*(M)$ satisfying the following properties (d is called the exterior derivative):*

(1) *df is a derivative (an exterior differential form of degree 1) of f for $f \in C_{DR}^0$;*

(2) *$d(C_{DR}^i(M)) \subset C_{DR}^{i+1}(M)$ for $0 \leq i \leq n$;*

(3) *$d \circ d = 0$;*

$$(4) \quad d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (d\beta) \text{ for } \alpha, \beta \in C_{DR}^*(M).$$

If we put

$$Z_{DR}^i = \{\omega \in C_{DR}^i(M) | d\omega = 0\}, \quad B_{DR}^i = \{d\omega \in C_{DR}^i(M) | \omega \in C_{DR}^{i-1}(M)\},$$

then we have $B_{DR}^i \subset Z_{DR}^i$ by 2.1(3) and so we can define algebraically the cohomology

$$H_{DR}^i(M) = Z_{DR}^i / B_{DR}^i.$$

We call it the *de Rham cohomology* of the manifold M .

2.2. For a paracompact M we have the following isomorphism:

$$\varphi : H_{DR}^*(M) \cong H^*(M; \mathbf{R}).$$

We define the correspondence

$$\varphi : H_{DR}^i(M) \rightarrow H^i(M; \mathbf{R}) = \text{Hom}(H_i(M), \mathbf{R})$$

by

$$\varphi(\omega)(C) = \int_C \omega \quad (\text{the integral of } \omega \text{ on } C)$$

for any exterior differential form ω and a singular cycle C of dimension i . For details see the following:

38. Y. Kawada et al, *Topology*, Modern Math. Exercise Series, Iwanami-Shoten, 1965 (in Japanese).

39. S. T. Hu, *Differentiable manifolds*, Holt, Rinehart, and Winston Inc., 1969.

The duality theorem

For a topological space X let $X \supset A \supset B$. Since the set $\{(U, V)\}$ of a neighborhood (U, V) of (A, B) (i.e., U, V are neighborhoods of A, B respectively) forms an inductive system under inclusions, we can consider the inductive limit $\varinjlim H^i(U, V; R)$ of the set $\{H^i(U, V; R) | (U, V) \text{ is a neighborhood of } (A, B)\}$. So we define

$$H_N^i(A, B; R) = \varinjlim H^i(U, V; R),$$

where R is a coefficient ring. Under the definition we have

3.1 (The Alexander duality theorem). If $\mathbf{R}^n \supset A$ is a compact set, then

$$\tilde{H}_q(\mathbf{R}^n - A; G) \cong H_N^{n-q-1}(A; G),$$

where G is an R -module.

Further in general, for an orientable manifold M^n of dimension n we have

3.2. If (A, B) is a compact pair of M^n , then

$$H_q(M^n - B, M^n - A; G) \cong H_N^{n-q}(A, B; G).$$

If M^n is compact, then $H_N^q(M^n; R) \cong H^q(M^n; G)$ and so by applying 3.2 to $(M^n, \emptyset)$ we have (cf. §3 of Chapter VII):

3.3 (The Poincaré duality theorem). *If M^n is compact, then*

$$H_q(M^n; G) \cong H^{n-q}(M^n; G).$$

Let (M^n, A) be a relative n -dimensional manifold (i.e., M^n is a Hausdorff space with A a closed subset, and $M^n - A$ is an n -dimensional manifold). Then

3.4 (The Lefschetz duality theorem). *If (M^n, A) is a compact, relative n -dimensional manifold such that $M^n - A$ is orientable, then*

$$H_q(M^n - A; G) \cong H_N^{n-q}(M^n, A; G).$$

Let M^n be an n -dimensional manifold with boundary ∂M^n (i.e., M^n is a paracompact and Hausdorff space such that $(M^n, \partial M^n)$ is a relative n -dimensional manifold and such that any $x \in M^n$ has a neighborhood V satisfying $(V, V \cap \partial M^n) \cong (\mathbb{R}^{n-1} \times (I, 0))$). We define an n -dimensional manifold M^n with boundary ∂M^n to be orientable if $M^n - \partial M^n$ is orientable. Then

3.5 (The Lefschetz duality theorem). *If a compact n -manifold M^n with boundary ∂M^n is orientable, then*

$$H_q(M^n; G) \cong H^{n-q}(M^n, \partial M^n; G), \quad H_q(M^n, \partial M^n; G) \cong H^{n-q}(M^n; G).$$

Note that the case where $\partial M^n = \emptyset$ is the Poincaré duality theorem 3.3. These duality theorems are thoroughly described in Spanier [2].

Characteristic classes

Suppose that an n -dimensional differentiable manifold M^n is immersed in $\mathbb{R}^{n+k}$. For the Grassmannian manifold $G_{n,k}(\mathbb{R}) = \{n\text{-dimensional subspaces of } \mathbb{R}^{n+k}\}$ we define the (generalized) Gauss map

$$\varphi: M^n \rightarrow G_{n,k}(\mathbb{R}),$$

by $\varphi(x) = (n\text{-dimensional subspace through the origin of } \mathbb{R}^{n+k} \text{ parallel to the tangent plane at } x \in M^n)$.

Denote by $\gamma_k^n = (E(\gamma_k^n), p, G_{n,k}(\mathbb{R}), \mathbb{R}^n)$ the n -dimensional vector bundle over $G_{n,k}(\mathbb{R})$, where $E(\gamma_k^n) = \{(n\text{-dimensional subspace through the origin, vector in that space})\}$.

If we denote by τ_M the tangent bundle of M^n , then

4.1. *There exists a bundle map $f: \tau_M \rightarrow \gamma_k^n$ such that the map between the base spaces is the Gauss map $\varphi: M^n \rightarrow G_{n,k}(\mathbb{R})$.*

If we put

$$G_n(\mathbf{R}) = G_{n,\infty}(\mathbf{R}) = \{n\text{-dimensional subspaces in } \mathbf{R}^\infty\},$$

$$\gamma^n = \gamma_\infty^n = (E(\gamma^n), p, G_n(\mathbf{R}), \mathbf{R}^n),$$

$$E(\gamma^n) = \{(n\text{-dimensional subspace in } \mathbf{R}^\infty, \text{ vector in that space})\},$$

then we can take $G_n(\mathbf{R}) = BGL(n, \mathbf{R})$ (or $= BO(n)$) and hence

4.2. *For an arbitrary n -dimensional real vector bundle ζ over a paracompact space X there exists a bundle map $f: \zeta \rightarrow \gamma^n$ such that the map between the base spaces is a Gauss map.*

Here we have $H^*(G_n(\mathbf{R}); \mathbf{Z}_2) = H^*(BO(n); \mathbf{Z}_2) = \mathbf{Z}_2[w_1, \dots, w_n]$. If we put $\bar{f}^*(w_i) = w_i(\zeta)$ for $0 \leq i \leq n$, $w(\zeta) = 1 + w_1(\zeta) + \dots + w_n(\zeta)$ and $w_0(\zeta) = 1$ for the homomorphism

$$\bar{f}^*: H^*(G_n(\mathbf{R}); \mathbf{Z}_2) \rightarrow H^*(X; \mathbf{Z}_2)$$

induced by the map $\bar{f}$ between the base spaces of a bundle map f in 4.2, then the following properties are satisfied:

4.3.1. $h^*(w(\zeta)) = w(h^*\zeta)$ for $h: X \rightarrow Y$,

4.3.2. $w(\zeta \oplus \xi) = w(\zeta)w(\xi)$ for the Whitney sum $\zeta \oplus \xi$, that is,

$$w_k(\zeta \oplus \xi) = \sum_{i+j=k} w_i(\zeta)w_j(\xi),$$

which corresponds to Theorem 5.11(3) of Chapter III.

4.3.3. $w_1(\xi) \neq 0$ for a nontrivial one dimensional vector bundle ξ over $S^1 = P^1(\mathbf{R})$.

We call $w_i(\zeta)$ and $w(\zeta)$ the i th Stiefel-Whitney class and the total Stiefel-Whitney class of ζ respectively. Conversely one can characterize the Stiefel-Whitney class by (4.3.1) ~ (4.3.3). Also putting

$$w_i(M^n) = w_i(\tau_M) \quad \text{and} \quad w(M^n) = w(\tau_M)$$

for the tangent bundle τ_M of a differentiable manifold M^n , we call them the i th Stiefel-Whitney class and the total Stiefel-Whitney class of M^n respectively.

Similar facts hold also for the complex case. That is, we denote by

$$\gamma^n(\mathbf{C}) = (E(\gamma^n(\mathbf{C})), p, G_n(\mathbf{C}), \mathbf{C}^n)$$

the n -dimensional complex vector bundle over the complex Grassmannian manifold $G_n(\mathbf{C}) = \{n\text{-dimensional subspace of } \mathbf{C}^\infty\}$, where $E(\gamma^n(\mathbf{C})) = \{(n\text{-dimensional subspace, vector in that space})\}$. Then we have $G_n(\mathbf{C}) = BGL(n, \mathbf{C})$ (or $= BU(n)$) and

4.2'. *For arbitrary n -dimensional complex vector bundle ζ over a paracompact space X there exists a bundle map $f: \zeta \rightarrow \gamma^n(\mathbf{C})$ such that the map between the base spaces is a Gauss map.*

Here we have $H^*(G_n(\mathbb{C}); \mathbb{Z}) = H^*(BU(n); \mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n]$. If we put $\bar{f}^*(c_i) = c_i(\zeta)$ for $0 \leq i \leq n$, $c(\zeta) = 1 + c_1(\zeta) + \dots + c_n(\zeta)$ and $c_0(\zeta) = 1$ for the homomorphism

$$\bar{f}^*: H^*(G_n(\mathbb{C}); \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$$

induced by the map $\bar{f}$ between the base spaces of a bundle map f in 4.2', then the following properties are satisfied:

4.3.1'. $h^*(c(\zeta)) = c(h^*\zeta)$ for $h: X \rightarrow Y$,

4.3.2'. $c(\zeta \oplus \xi) = c(\zeta)c(\xi)$ for the Whitney sum $\zeta \oplus \xi$, that is,

$$c_k(\zeta \oplus \xi) = \sum_{i+j=k} c_i(\zeta)c_j(\xi),$$

which corresponds to Theorem 5.8(3) of Chapter III.

4.3.3'. $c_1(\xi) \neq 0$ for a nontrivial line bundle ξ over $S^2 = P^1(\mathbb{C})$.

We call $c_i(\zeta)$ and $c(\zeta)$ the i th Chern class and the total Chern class of ζ respectively. Conversely one can characterize the Chern class by 4.3.1' ~ 4.3.3'.

We consider the complexification $\zeta_{\mathbb{C}}$ of an n -dimensional real vector bundle ζ . Putting

$$\begin{aligned} p_i(\zeta) &= (-1)^i c_{2i}(\zeta_{\mathbb{C}}), & p_0(\zeta) &= 1, \\ p(\zeta) &= 1 + p_1(\zeta) + \dots + p_m(\zeta), & m &= [n/2], \end{aligned}$$

we call them the i th Pontrjagin class and the total Pontrjagin class of ζ respectively. Then

4.4. $p(\zeta \oplus \eta) - p(\zeta)p(\eta)$ is a sum of elements of order 2.

For these facts we recommend to the reader the following:

40. J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. of Math. Stud., vol. 76, Princeton Univ. Press, Princeton, N. J., 1974.

Obstruction theory

We refer the reader to [1], [2] for general obstruction theory. Here we briefly state the relation between the characteristic classes above and the obstruction cocycles to the existence of a cross section of a fibre bundle $\xi = (E, p, B, F)$. For simplicity, we suppose that B is a 1-connected finite polyhedron and that F is simple.

We consider construction of a cross section $s: B \rightarrow E$ (i.e., $p \circ s = 1_B$) of the bundle ξ by induction on the q -skeleton $B^{(q)}$ of B . Now suppose given a cross section on $B^{(q)}$, and we wish to extend it over $B^{(q+1)}$. To that end it suffices to extend $s|_{\partial e}$ over e for each $(q+1)$ -cell e of $B^{(q+1)}$. Since e

is contractible, $\xi|_e$ is a product bundle and hence there exist a bundle map $\phi_e: E \times F \rightarrow E_e$ (the section of ξ over e) and a map $p_e: E_e \rightarrow F$ such that $\phi_e(p(x), p_e(x)) = x$ for $x \in E$. Then

5.1. A cross section $s|_{\partial e}$ is extendable to a cross section over e iff $s|_{\partial e}$ is extendable to a map $e \rightarrow E_e$.

In fact, if $s': e \rightarrow E_e$ is an extension of $s|_{\partial e}$, then the map s'' defined by $s''(x) = \phi_e(x, p_e s'(x))$ is a cross section, which is an extension of $s|_{\partial e}$.

Next, consider a fibre F_x over $x \in e$. Since the inclusion $F_x \subset E_e$ induces an isomorphism $\pi_i(F_x) \cong \pi_i(E_e)$, $s|_{\partial e}$ is homotopic to a map $\bar{s}: \partial e \rightarrow F_x$ in E_e and the homotopy class of $\bar{s}$ in F_x depends only on the homotopy class of $s|_{\partial e}$ in E_e ; thus one sees that

$\bar{s}$ is extendable to $\bar{s}: e \rightarrow F_x$ iff $s|_{\partial e}$ is extendable to a cross section over e .

Now for a cross section s over $B^{(q)}$ we denote by $c(s, e)$ the element represented by $s|_{\partial e}$ in $\pi_q(F_x) \cong \pi_q(F)$. Then

5.2. A cross section s over $B^{(q)}$ is extendable over $B^{(q+1)}$ iff $c(s, e) = 0$ for any $(q+1)$ -cell e .

Further, we define $c(s)$ by $c(s)(e) = c(s, e)$ and call it an obstruction cocycle of s .

5.3. Let s be a cross section over $\xi|_{B^{(q-1)}}$ and suppose that s is extendable over $B^{(q)}$. Then the set $\{c(s')\}$ of the q -dimensional obstruction cocycles of all the extensions s' of s forms one cohomology class $\bar{c}(s) \in H^{q+1}(B; \pi_q(F))$ such that

s is extendable over $B^{(q+1)}$ iff $\bar{c}(s) = 0$.

Now let ζ be an n -dimensional vector bundle and

$$\zeta(q) = (E, p, B, V_{n, n-q}(\mathbf{R}))$$

be the associated bundle of ζ ($q < n$). More concretely a point in the fiber over $b \in B$ is given by

$$(b, p^{-1}(b)) \cong \text{a frame } (\mathbf{v}_1, \dots, \mathbf{v}_{n-q}) \text{ in } \mathbf{R}^n.$$

Then the primary obstruction to a cross section of $\zeta(q)$ is an element $o_{q+1}(\zeta) \in H^{q+1}(B; \pi_q(V_{n, n-q}(\mathbf{R})))$, where $\pi_q(V_{n, n-q}(\mathbf{R})) \cong \mathbf{Z}$ or $\mathbf{Z}_2$. For the mod 2 reduction $o_{q+1}(\zeta)_2$ of the element $o_{q+1}(\zeta)$ we have the following:

$$o_{q+1}(\zeta)_2 = w_{q+1}(\zeta). \quad (5.4)$$

Moreover if ζ is orientable we have the following for the Euler class $\chi(\zeta)$ of §3 of Chapter III:

$$o_n(\zeta) = \chi(\zeta). \quad (5.5)$$

Similarly, if ζ is an n -dimensional complex vector bundle, we have $\pi_{2q+1}(V_{n,n-q}(C)) \cong \mathbb{Z}$ and the primary obstruction

$$o_{2q+2}(\zeta) \in H^{2q+2}(B; \pi_{2q+1}(V_{n,n-q}(C)))$$

satisfies the following:

$$o_{2q+2}(\zeta) = \pm c_{q+1}(\zeta). \quad (5.4')$$

The obstruction theory of a cross section is thoroughly stated in Chapter III of [13]. The reader is referred to [40] for proofs of (5.4) and (5.5).

Topology of Lie Groups, II

From the theory of compact Lie groups
to exceptional groups

Preface

Lie groups are very general objects of mathematical study that appear in numerous fields of mathematics such as topology, functional analysis, and algebra as well as differential geometry and differential topology. The purpose of this book is to provide a guide to the topology of Lie groups and homogeneous spaces by putting together all the known results relating to them.

It is not the case, however, that the authors have a thorough knowledge of Lie groups but rather that one cannot find good references that give laymen the necessary information smoothly, possibly because too many years have passed since the work of E. Cartan. As matters stand in the theory of compact Lie groups, which is what topologists need, the facts that look obvious turn out to be very difficult as is the case in Euclidean geometry. Likewise, if one goes too far into the theory of Lie algebras, the readers as well as the authors lose an intuitive viewpoint.

Fortunately, by virtue of the convenient Iwasawa decomposition, whose proof is beyond the scope of this book, a Lie group is homeomorphic to a direct product of a maximal compact subgroup and a Euclidean space. Thus, for Lie groups, it suffices to study compact Lie groups from the topological viewpoint.

Among Lie groups, the classical groups are indispensable for practical use. And as far as the classical groups are concerned, there is no need for such a thing as the theory of Lie groups. So, in this series of books, we study in detail the classical groups in the first volume. The theory of compact Lie groups is studied in the second volume. It is our goal to determine the cohomology of exceptional Lie groups.

In the first volume we thoroughly discussed topological properties of the classical groups as typical examples of Lie groups. In the second volume we study general properties of compact Lie groups, in particular, the exceptional groups.

In Chapter V, we first deal with the representation of compact Lie groups and among other results we show that a compact Lie group can be regarded as a closed subgroup of an appropriate classical group. Then we describe the theory of maximal tori and roots associated with them, which is a special

feature of the theory of compact Lie groups. The classification of subgroups of maximal rank of compact Lie groups is also discussed here, assuming results about Lie algebras as given.

In Chapter VI we describe Morse theory and the subsequent theory of Bott-Samelson. As applications of these ideas we obtain the result that the loop space of a compact Lie group has only even-dimensional homology and we determine the type and the dimension as well as the homotopy groups of the exceptional groups.

In Chapter VII we begin with a supplement on the theory of Hopf algebras and a preparatory section on the comparison and the transgression theorems; we then state general results on the cohomology of classifying spaces and Weyl groups. Next, after introducing cohomology operations, we determine the mod p cohomology of exceptional groups by our own methods. We owe much to a suggestion of Akira Kono for the latter half of this chapter.

We briefly describe representation rings and K -theory of Lie groups and homogeneous spaces in the postscript, since there is not sufficient space elsewhere for this purpose.

In these books we consider Lie groups from the topological viewpoint, and we hope for the tolerance of the reader for a point of view that is fairly dogmatic and biased about the theory of Lie groups itself.

Last, but not least, we wish to express our gratitude to A. Kozłowski and J. McCleary for reading the manuscript in the English version and for suggesting many improvements.

Spring 1990
The authors

CHAPTER V

Compact Lie Groups

The object of this chapter is to develop the general theory of compact (connected) Lie groups. After introducing the Haar integral on compact Lie groups and the kernel operator in function spaces in §1, we use them in §2 to show the complete reducibility of a real (or complex) representation of finite dimension and its equivalence with an orthogonal (or unitary) representation. Further we prove that G is isomorphic to a closed subgroup of an orthogonal group. In §3 we prove that the properties of maximal tori and Weyl groups stated for the classical groups in Chapter I, as a matter of fact, hold also for a general G . For example, the conjugacy of maximal tori will be proved by using "the fixed point theorem," which is well known in topology. In §4 we define the notion of root by considering the restriction of the adjoint representation on a maximal torus and also study the action of the Weyl group. In §5 we express the fundamental group of G in terms of the integer lattice $\Gamma(G) = \exp^{-1}(e)$ and its subgroups and give a decomposition of G by making use of the decomposition of the Lie algebra. We assign a Dynkin diagram to the simple root system of a simple group G in §6, and by classifying them we classify (the local isomorphism classes of) G . By studying the simple roots and the dominant roots in §7 we describe all the type of subgroups of the same rank as G .

§1. Integration on compact groups

Invariant integrals. Throughout this chapter a compact topological group is simply called a *compact group*.

DEFINITION. An *invariant integral* or a *Haar integral*

$$\int f dx = \int f(x) dx = \int_G f(x) dx = \int_{x \in G} f(x) dx$$

on a compact group G assigns a real value $\int f(x) dx \in \mathbb{R}$ to a real-valued continuous function $f: G \rightarrow \mathbb{R}$ on G and satisfies the following conditions (1) ~ (4):

$$\begin{aligned} \int (f(x) + g(x)) dx &= \int f(x) dx + \int g(x) dx, \\ \int cf(x) dx &= c \int f(x) dx \quad (c \in \mathbb{R}); \end{aligned} \tag{1}$$

$$f(x) \geq 0 \quad \text{for all } x \in G \text{ implies } \int f(x) dx \geq 0; \quad (2)$$

$$f(x) = 1 \quad \text{for all } x \in G \text{ implies } \int f(x) dx = 1; \quad (3)$$

$$\int f(xy) dx = \int f(x) dx \quad \text{for all } y \in G. \quad (4)$$

THEOREM 1.1. *If a compact group G satisfies the second axiom of countability, then there exists uniquely an invariant integral on G , and in addition to (1) $\sim$ (4), it satisfies the following:*

$$f(x) \geq 0 \quad \text{for all } x \in G \text{ and } f \neq 0 \text{ imply } \int f(x) dx > 0; \quad (2')$$

$$\int f(yx) dx = \int f(x) dx \quad \text{for all } y \in G; \quad (5)$$

$$\int f(x^{-1}) dx = \int f(x) dx. \quad (6)$$

Let $A = \{a_1, \dots, a_n\}$ be a finite subset of G . One obtains $\int f(x) dx$ as the limit, in a certain sense, of the mean

$$M(A, f(x)) = (f(xa_1) + \dots + f(xa_n))/n$$

of $f(x)$ with respect to A . That is, $\int f(x) dx$ can be characterized as the real number such that for arbitrary $\varepsilon > 0$ there exists a finite set A satisfying

$$\left| M(A, f(y)) - \int f(x) dx \right| < \varepsilon \quad \text{for all } y \in G.$$

We refer for the details of the proof of the theorem to Chapter 4 in Pontrjagin [37].

Properties (1), (2), and (2') are those of ordinary integration. From them it follows easily that

$$\begin{aligned} f(x) \geq g(x) & \text{ implies } \int f(x) dx \geq \int g(x) dx, \\ \left| \int f(x) dx \right| & \leq \int |f(x)| dx \leq \text{Sup}\{|f(x)| | x \in G\}. \end{aligned} \quad (1.2)$$

THEOREM 1.3. *Let G be a compact group satisfying the second axiom of countability. If Y is a topological space and $f: G \times Y \rightarrow \mathbf{R}$ is continuous, then the map $F: Y \rightarrow \mathbf{R}$ defined by*

$$F(y) = \int f(x, y) dx$$

is continuous.

Furthermore, if $Y = \mathbf{R}$ and if the differential $df/dt: G \times \mathbf{R} \rightarrow \mathbf{R}$ exists and is continuous, then

$$\frac{d}{dt} F(t) = \int \frac{d}{dt} f(x, t) dx.$$

PROOF. Let $\varepsilon > 0$ and $y_0 \in Y$. Using the compactness of G , one can see the existence of a neighborhood U of y_0 such that if $y \in U$ then

$$|f(x, y) - f(x, y_0)| < \varepsilon \quad \text{for all } x \in G.$$

Then by (1.2)

$$|F(y) - F(y_0)| = \left| \int (f(x, y) - f(x, y_0)) dx \right| < \varepsilon.$$

This shows the continuity of F .

For the latter half, fix $t \in \mathbf{R}$ and consider $0 \neq s \in \mathbf{R}$. Then there is $\theta = \theta(x)$ with $0 < \theta < 1$ such that $(1/s)(f(x, t+s) - f(x, t)) = (d/dt)f(x, t+\theta s)$. Since df/dt uniformly converges on a compact subset of $G \times \mathbf{R}$, for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |s| < \delta$ then

$$\left| \frac{f(x, t+s) - f(x, t)}{s} - \frac{d}{dt}f(x, t) \right| < \varepsilon \quad \text{for all } x \in G.$$

Integrating it we have

$$\left| \frac{F(t+s) - F(t)}{s} - \int \frac{d}{dt}f(x, t) dx \right| < \varepsilon$$

which implies $\frac{d}{dt}F(t) = \int \frac{d}{dt}f(x, t) dx$. Q.E.D.

COROLLARY 1.4. If $f: G \times \mathbf{R}^n \rightarrow \mathbf{R}$ has all (higher order) partial derivatives with respect to the variables $t_1, \dots, t_n$ of $\mathbf{R}^n$ that are continuous on $G \times \mathbf{R}^n$, then $F(t_1, \dots, t_n) = \int f(x, t_1, \dots, t_n) dx$ is a real-valued C^∞ -function on $\mathbf{R}^n$.

THEOREM 1.5. Let G and H be compact Lie groups satisfying the second axiom of countability. If $f: G \times H \rightarrow \mathbf{R}$ is a continuous function, then

$$\int \left(\int f(x, y) dy \right) dx = \int \left(\int f(x, y) dx \right) dy$$

(which is simply denoted by $\iint f(x, y) dx dy$).

PROOF. By Theorem 1.3 both sides can be shown to satisfy the conditions for the invariant integral on $G \times H$. So both sides are equal by the uniqueness of the invariant integral in Theorem 1.1. Q.E.D.

When f takes values in the vector space $\mathbf{R}^n$, one can define

$$\int f(x) dx \in \mathbf{R}^n$$

for $f: G \rightarrow \mathbf{R}^n$ by componentwise integration and literally apply the above arguments. By the linearity of integration, one has

1.6. For a linear map $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$,

$$\int T(f(x)) dx = T \left(\int f(x) dx \right).$$

Similarly for a complex-valued function $f: G \rightarrow \mathbf{C}$ (or $\mathbf{C}^n$).

Function spaces on a compact group

NOTATION. $V = V(G) = \{f: G \rightarrow \mathbb{R}^n \text{ continuous}\}$.

The structure of a vector space in $\mathbb{R}^n$ makes V into a vector space (of infinite dimension). The values of each f are bounded, since G is compact.

DEFINITION. Let A be a subset of V . A is called *uniformly bounded* if there exists $M > 0$ such that $f \in A$ and $x \in G$ imply $\|f(x)\| < M$.

A is called a set of *equicontinuous functions* if for arbitrary $\varepsilon > 0$, there is a neighborhood U of the identity e of G such that $f \in A$, $x \in G$, $y \in U$ imply $\|f(xy) - f(x)\| < \varepsilon$.

LEMMA 1.7. *If a sequence of functions $\{f_n\}$ in V belongs to a uniformly bounded set A of equicontinuous functions, then one can choose a subsequence $\{f_{n_k}\}$, which converges uniformly on G .*

PROOF. First, for a positive integer k consider a subset A_k of A such that if $f, g \in A_k$ satisfy $\|f(x) - g(x)\| < 1/k$ for all $x \in G$ then $f = g$. We show that A_k is a finite set. By the equicontinuity of A there is a neighborhood U of e such that if $f \in A$, $x \in G$, and $y \in U$ then $\|f(xy) - f(x)\| < 1/6k$. Since G is compact, there are $x_1, \dots, x_N \in G$ such that $G = \bigcup_{i=1}^N x_i U$.

Put $D = \{(f, g) | f, g \in A_k, f \neq g\}$, then for each $(f, g) \in D$ there is $z \in G$ such that $\|f(z) - g(z)\| \geq 1/k$. We have $D = \bigcup D_i$, where D_i is the set of (f, g) satisfying $z \in x_i U$ for the above z . Since $\{f(x_i) | f \in A\}$ is contained in a compact subset of $\mathbb{R}^n$, we can take a finite subset B_i of A so that for arbitrary $f \in A$ there is $h_f \in B_i$ satisfying

$$\|f(x_i) - h_f(x_i)\| < 1/6k.$$

Here if we suppose that D_i contains more than or the same number of elements as the square of the number of B_i , then there is $(f, g) \in D_i$ such that $h_f = h_g = h \in B_i$. Since z in the above is expressed as $z = x_i y$ with $y \in U$, we have

$$\begin{aligned} \|f(z) - h(z)\| &\leq \|f(x_i y) - f(x_i)\| + \|f(x_i) - h(x_i)\| \\ &\quad + \|h(x_i y) - h(x_i)\| < 1/2k. \end{aligned}$$

Similarly, $\|g(z) - h(z)\| < 1/2k$. Thus we obtain the contradictory conclusion that $\|f(z) - g(z)\| < 1/k$. Therefore, D_i is finite, so $D = \bigcup_{i=1}^N D_i$ is finite, and A_k is finite.

Now let us denote afresh by A_k the maximal one of such A_k with respect to the inclusion relation. Then A_k has the following properties: for arbitrary $f \in A$ there exists $g \in A_k$ satisfying $\|f(x) - g(x)\| < 1/k$ for any $x \in G$. For a given arbitrary sequence of functions $\{h_n\}$ in A , there are infinitely many n such that $\|h_n(x) - g_k(x)\| < 1/k$ (for any $x \in G$) for at least one $g_k \in A_k$. Since A_k is finite, we can choose a subsequence $\{h'_m\}$ of $\{h_n\}$ with such n . Applying this process successively to $A_1, A_2, \dots$, for

a given sequence of functions $\{f_n\}$ we can choose a sequence $\{f_n^{(k)}\}$ such that $\|f_n^{(k)}(x) - g_k(x)\| < 1/k$.

If we put $f_n^* = f_n^{(n)}$, then $\{f_n^*\}$ is a subsequence of $\{f_n\}$ such that if $m, n \geq k$ then

$$\|f_m^*(x) - f_n^*(x)\| \leq \|f_m^*(x) - g_k(x)\| + \|f_n^*(x) - g_k(x)\| < 2/k.$$

Thus $\{f_n^*\}$ is a uniformly convergent subsequence. Q.E.D.

We define an inner product $\langle f, g \rangle: V \times V \rightarrow \mathbb{R}$ on V by

$$\langle f, g \rangle = \int (f(x), g(x)) dx \quad (f, g: G \rightarrow \mathbb{R}^n),$$

where $(f(x), g(x)) = f(x) \cdot g(x)$ is the inner product in $\mathbb{R}^n$. Obviously V is a metric vector space and the following hold:

1.8. (1) $\langle a_1 f_1 + a_2 f_2, g \rangle = a_1 \langle f_1, g \rangle + a_2 \langle f_2, g \rangle$ for $a_1, a_2 \in \mathbb{R}$;

(2) $\langle g, f \rangle = \langle f, g \rangle$;

(3) $\langle f, f \rangle \geq 0$, and $\langle f, f \rangle > 0$ for $f \neq 0$.

We call $\|f\| = \sqrt{\langle f, f \rangle} \geq 0$ the *norm* of f . By the usual method one shows that

(4) $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$, $\|f + g\| \leq \|f\| + \|g\|$;

(5) $\int \|f(x)\| dx \leq \|f\|$.

Observe that the above arguments also hold for the case $\mathbb{C}$ with $V = \{f: G \rightarrow \mathbb{C}^n \text{ continuous}\}$, where 1.8(2) is replaced by the following form:

(2') $\langle g, f \rangle = \overline{\langle f, g \rangle}$.

Operators and characteristic functions

DEFINITION. $K: V \rightarrow V$ is called an *operator* if K is linear and continuous.

We call $\varphi \in V$ a *characteristic function* of K and λ a *characteristic value* of φ , if $K\varphi = \lambda\varphi$ for $\lambda \in \mathbb{R}$ with $\varphi \neq 0$; $\{f \in V | Kf = \lambda f\}$ is called the *eigenspace* with respect to λ .

$K: V \rightarrow V$ is called a *special operator* if K is an operator satisfying the following:

1.9. (1) K maps the set $\{f \in V | \|f\| \leq 1\}$ to the set of uniformly bounded, equicontinuous functions in V .

(2) $\langle Kf, g \rangle = \langle f, Kg \rangle$ for $f, g \in V$.

DEFINITION. For a symmetric, continuous function $k: G \times G \rightarrow \mathbb{R}$, $k(x, y) = k(y, x)$, the operator $K: V \rightarrow V$ defined by

$$(Kf)(x) = \int k(x, y)f(y) dy$$

is called the *kernel operator* with the *kernel function* k .

Using Theorems 1.3 and 1.1, the compactness of G and (1.2), Theorem 1.5 and the symmetry of k , one can obtain the following.

1.10. A kernel operator is a special operator.

LEMMA 1.11. A special operator K has a characteristic function. In fact, $c = \sup\{\|Kf\| \mid \|f\| = 1\}$ or $-c$ is a characteristic value and any characteristic value λ satisfies $|\lambda| \leq c$.

PROOF. Putting $c' = \sup\{\langle Kf, f \rangle \mid \|f\| = 1\}$, any characteristic value λ satisfies $|\lambda| \leq c'$. We have $c' \leq c$, since $\|f\| = 1$ implies $|\langle Kf, f \rangle| \leq \|Kf\| \cdot \|f\| \leq c$.

Obviously, $|\langle Kf, f \rangle| \leq c'\|f\|^2$ for $f \in V$, and hence

$$c'\|f + g\|^2 \geq \langle K(f + g), f + g \rangle = \langle Kf, f \rangle + 2\langle Kf, g \rangle + \langle Kg, g \rangle,$$

$$-c'\|f - g\|^2 \leq \langle K(f - g), f - g \rangle = \langle Kf, f \rangle - 2\langle Kf, g \rangle + \langle Kg, g \rangle.$$

Thus putting $\|f\| = \|g\| = 1$, we have $4\langle Kf, g \rangle \leq c'\|f + g\|^2 + c'\|f - g\|^2 = 2c'(\|f\|^2 + \|g\|^2) = 4c'$. Thus $\langle Kf, g \rangle \leq c'$. If $Kf \neq 0$, then we have $\|Kf\| \leq c'$ by putting $g = Kf/\|Kf\|$. If $Kf = 0$, then $\|Kf\| = 0 \leq c'$. It follows that $c \leq c'$. Thus we see that $c = c'$.

When $c = c' = 0$, we always have $Kf = 0$, and 0 is a characteristic value.

Let $c = c' \neq 0$. By the definition of c' , there is a sequence of functions $\{f_i \mid \|f_i\| = 1\}$ such that $\langle Kf_i, f_i \rangle$ converges to $\lambda = \pm c$. By Lemma 1.7 we can select $\{f_i\}$ so that $\{Kf_i\}$ uniformly converges to $\varphi \in V$. We have

$$0 \leq \|Kf_i - \lambda f_i\|^2 = \|Kf_i\|^2 + \lambda^2\|f_i\|^2 - 2\lambda\langle Kf_i, f_i \rangle \rightarrow \|\varphi\|^2 - \lambda^2,$$

and so $\|\varphi\|^2 \geq \lambda^2 > 0$, which implies $\varphi \neq 0$. On the other hand, since $\|Kf_i\| \leq c' = c = |\lambda|$, we have

$$\|Kf_i - \lambda f_i\|^2 \leq 2\lambda^2 - 2\lambda\langle Kf_i, f_i \rangle$$

so $\lim \|Kf_i - \lambda f_i\| = 0$. It follows that

$$\lim \|K(Kf_i) - \lambda(Kf_i)\| = \lim \|K(Kf_i - \lambda f_i)\| = 0.$$

Thus $\|K\varphi - \lambda\varphi\| = 0$ so $K\varphi = \lambda\varphi$. Q.E.D.

THEOREM 1.12. For characteristic functions and characteristic values of a special (in particular, kernel) operator K , the following hold:

(1) Characteristic functions φ and φ' with different characteristic values are orthogonal: $\langle \varphi, \varphi' \rangle = 0$.

(2) For any $c > 0$, the number of characteristic values λ with $|\lambda| \geq c$ is finite.

(3) For $\lambda \neq 0$, the eigenspace with respect to λ is a vector space of finite dimension.

(4) Let $\{\varphi_i\}$ be a maximal, orthonormal system ($\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$) of characteristic functions with nonzero characteristic values. Then the set $\{\varphi_i\}$ is countable, and the (Fourier) series $\sum_i \langle Kf, \varphi_i \rangle \varphi_i$ uniformly converges to Kf for any $f \in V$.

PROOF. (1) Let λ and λ' be the characteristic values of characteristic functions φ and φ' respectively. Then by 1.9(2),

$$\lambda\langle\varphi, \varphi'\rangle = \langle\lambda\varphi, \varphi'\rangle = \langle K\varphi, \varphi'\rangle = \langle\varphi, K\varphi'\rangle = \langle\varphi, \lambda'\varphi'\rangle = \lambda'\langle\varphi, \varphi'\rangle.$$

Thus $\langle\varphi, \varphi'\rangle = 0$ if $\lambda \neq \lambda'$.

(2) Consider distinct characteristic values $\{\lambda_i\}$ satisfying $|\lambda_i| \geq c$ and choose one by one a characteristic function φ_i with λ_i as its characteristic value satisfying $\|\varphi_i\| = 1$. By (1) we have

$$\|K\varphi_i - K\varphi_j\|^2 = \langle\lambda_i\varphi_i - \lambda_j\varphi_j, \lambda_i\varphi_i - \lambda_j\varphi_j\rangle = \lambda_i^2 + \lambda_j^2 \geq 2c^2$$

for $i \neq j$. Thus there is $x \in G$ such that $\|K\varphi_i(x) - K\varphi_j(x)\| \geq \sqrt{2}c$. Consider $A_k = \{K\varphi_i\}$ for $\sqrt{2}c > 1/k$. Since A_k is uniformly bounded, equicontinuous by 1.9(1), the set A_k is a finite set by the first half of the proof of Lemma 1.7. So (2) is obtained from (1).

(3) The set of characteristic functions with characteristic value λ together with 0 obviously forms a vector subspace of V , from which we choose an orthonormal system $\{\varphi_i\}$ and apply the arguments in (2). Then $\{\varphi_i\}$ must be finite. This proves (3).

(4) (2) and (3) imply that $\{\varphi_i\}$ is countable. We will prove the following Bessel inequality:

$$\sum_i \langle f, \varphi_i \rangle^2 \text{ converges for any } f \in V \text{ and the sum } \leq \|f\|^2. \quad (*)$$

Put $f_k = f - \sum_{i \leq k} \langle f, \varphi_i \rangle \varphi_i$. Obviously, if $j \leq k$ then $\langle f_k, \varphi_j \rangle = 0$. Therefore

$$\|f\|^2 = \left\| f_k + \sum_{i \leq k} \langle f, \varphi_i \rangle \varphi_i \right\|^2 = \|f_k\|^2 + \sum_{i \leq k} \langle f, \varphi_i \rangle^2.$$

So $\sum_i \langle f, \varphi_i \rangle^2$ is bounded and converges to a value $\leq \|f\|^2$.

Let $\lambda_i \neq 0$ be the characteristic value of φ_i . Then

$$\langle Kf, \varphi_i \rangle \varphi_i = \langle f, K\varphi_i \rangle \varphi_i = \lambda_i \langle f, \varphi_i \rangle \varphi_i = K(\langle f, \varphi_i \rangle \varphi_i).$$

By 1.9(1) there exists $M > 0$ such that if $\|f\| = 1$ and $x \in G$ then $\|Kf(x)\| < M$. Thus $\|Kf(x)\| \leq M\|f\|$ for any $f \in V$ and $x \in G$.

Consider the partial sum

$$\sum_{i=n}^m \langle f, \varphi_i \rangle \varphi_i \in V.$$

Since $\|\sum c_i \varphi_i\| = (\sum |c_i|^2)^{1/2}$, we have

$$\left\| \sum_{i=n}^m \langle Kf, \varphi_i \rangle \varphi_i(x) \right\| \leq M \left\| \sum_{i=n}^m \langle f, \varphi_i \rangle \varphi_i \right\| = M \left(\sum_{i=n}^m \langle f, \varphi_i \rangle^2 \right)^{1/2}.$$

By (*) the right-hand side converges to 0 as $n, m \rightarrow \infty$. Thus $\sum_i \langle Kf, \varphi_i \rangle \cdot \varphi_i$ uniformly converges.

It suffices to show that the sum is Kf . If one puts

$$K_m f = Kf - \sum_{i=1}^m \lambda_i \langle f, \varphi_i \rangle \varphi_i = Kf - \sum_{i=1}^m \langle Kf, \varphi_i \rangle \varphi_i,$$

it is easy to see that K_m is a special operator. We have

$$\langle K_m f, \varphi_j \rangle = \langle Kf, \varphi_j \rangle - \lambda_j \langle f, \varphi_j \rangle = \langle f, K\varphi_j - \lambda_j \varphi_j \rangle = 0,$$

for $1 \leq j \leq m$. Now if we denote by φ a characteristic function of K_m with a characteristic value $\lambda \neq 0$, we have by the above equality

$$\langle \varphi, \varphi_j \rangle = 0 \quad \text{for } 1 \leq j \leq m \quad \text{and} \quad \lambda \varphi = K_m \varphi = K\varphi,$$

and φ is a characteristic function of K and hence φ is a linear combination of φ_i with characteristic value λ :

$$\varphi = \sum_{\lambda_i = \lambda} \langle \varphi, \varphi_i \rangle \varphi_i = \sum_{\lambda_i = \lambda, i > m} \langle \varphi, \varphi_i \rangle \varphi_i.$$

For $\varepsilon > 0$ the number of i such that $|\lambda_i| \geq \varepsilon$ is finite by (2). So, taking m sufficiently large, one sees by the above arguments that the characteristic value λ of K_m satisfies $|\lambda| < \varepsilon$. According to Lemma 1.11 we have $\|K_m f\| < \varepsilon \|f\|$ for any $f \in V$, i.e., $\lim \|K_m f\| = 0$. On the other hand $Kf = K_m f + \sum_{i=1}^m \langle Kf, \varphi_i \rangle \varphi_i$ and we have

$$Kf = \lim \sum_{i=1}^m \langle Kf, \varphi_i \rangle \varphi_i = \sum_i \langle Kf, \varphi_i \rangle \varphi_i. \quad \text{Q.E.D.}$$

The above arguments also hold for the case **C** in place of **R** by suitably reinterpreting the expressions.

Invariant Riemannian metrics on manifolds

DEFINITION. A manifold M has a *Riemannian metric* g if g gives an inner product (a positive definite, real symmetric bilinear form) in $T_x M$ and is a C^∞ -function with respect to x .

In a local coordinate system $(x^1, \dots, x^m)$ of M this is expressed as

$$g(\partial x^i, \partial x^j) = g_{ij}(x), \quad \text{where } \partial x^i, \partial x^j \in T_x M$$

such that $(g_{ij}(x))$ is a positive definite symmetric matrix and g_{ij} is a C^∞ -function with respect to x .

THEOREM 1.13. (1) Any (paracompact, differentiable) manifold has a Riemannian metric.

(2) If a compact (Lie) group K acts on a manifold from left (or right), then M has a Riemannian metric invariant under the action of K .

(3) Let K and H be compact (Lie) groups. If K acts from left and H from right on a manifold M so that $(k \cdot x) \cdot h = k \cdot (x \cdot h)$ always holds for

$k \in K$, $h \in H$, $x \in M$, then there is a Riemannian metric on M invariant under the actions of K and H . In particular, a compact Lie group G has a Riemannian metric invariant under its left and right actions.

PROOF. (1) Let $m = \dim M$ and $F' = \{A \in M(m, \mathbf{R}) | A = {}^t A\} \supset F = \{A \in F' | \text{positive definite}\}$. $P \in GL(m, \mathbf{R})$ acts on F' and F by $P \cdot A = PA^tP$.

Now, let $\zeta = (E, p, M, F, GL(m, \mathbf{R}))$ be the associated fibre bundle of the tangent bundle TM of M . Then the following is clear:

1.14. A Riemannian metric g of M can be identified with a cross section $g: M \rightarrow E$.

F' is a vector space of dimension $m(m+1)/2$ over $\mathbf{R}$. Let $A, B \in F$. Since $vA^t v > 0$ and $vB^t v > 0$ for any nonzero $v \in \mathbf{R}^m$, we have $v(\theta A + (1-\theta)B)^t v > 0$ for $0 \leq \theta \leq 1$, which shows that $\theta A + (1-\theta)B \in F$, and hence F is a convex set in F' . Obviously F is also an open set of F' . It follows that F is homeomorphic with $\mathbf{R}^{m(m+1)/2}$. By Lemma 2.7 of Chapter II, ζ has a cross section, and hence by (1.14) M has a Riemannian metric.

(2) If $g_0(x)$ is a Riemannian metric on M , then so is $g_0(k \cdot x)$ with $k \in K$. If we consider the invariant integral $g(x) = \int_K g_0(k \cdot x) dk$ on K , then $g(x)$ is a Riemannian metric invariant under the action of K . (Here note that $g_0(k \cdot x) \in F$ implies $g(x) \in F$, since F is convex.) Similarly for the right action.

(3) Apply the process of (2) to a Riemannian metric invariant under the right action of H . Q.E.D.

DEFINITION. A manifold assigned a Riemannian metric is called a *Riemannian manifold*.

A transformation of a Riemannian manifold that preserves the Riemannian metric is called an *isometric transformation*.

A (Riemannian metric of a) Riemannian manifold such as in (2) or (3) of Theorem 1.13 is said to be *K-invariant* or *K-H-invariant*, respectively. In particular, a Riemannian metric on a Lie group G is simply called an *invariant Riemannian metric* if it is G - G -invariant under the action by the multiplication of G .

A Riemannian metric of M is *K-invariant* iff the action L_k (or R_k) of K is an isometric transformation for all $k \in K$.

§2. Representations of compact groups

Finite-dimensional representations of compact groups

THEOREM 2.1. Let G be a compact group. A representation of G over a finite-dimensional real vector space is equivalent to an orthogonal representation. A representation of G over a finite-dimensional complex vector space is equivalent to a unitary representation.

PROOF. Let V be a representation space. We suppose that $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$. Let $V = \mathbb{R}^n$ and denote the action of G by $(x, v) \mapsto xv$. When an inner product $u \cdot v$ is given in $\mathbb{R}^n$, we define a new one by

$$(u, v) = \int_G (xu) \cdot (xv) dx.$$

The inner product (u, v) is obviously linear and symmetric. Since $u \neq 0$ implies $xu \neq 0$ and hence $(x, u) \cdot (xu) > 0$, we have

$$(u, u) = \int_G (xu) \cdot (xu) dx > 0 \quad \text{for } u \neq 0,$$

that is, it is positive definite. In addition, by the properties of invariant integration, we have

$$(yu, yv) = \int_G (xyu) \cdot (xyv) dx = \int_G (xu) \cdot (xv) dx = (u, v) \quad \text{for } y \in G.$$

If we replace a basis of V by an orthonormal basis with respect to the inner product $(\cdot, \cdot)$, then G acts by orthogonal transformations. This shows that the action of G is equivalent to an orthogonal representation.

The case $V = \mathbb{C}^n$ can be similarly proved with the inner product $u \cdot v = uv^*$ of $\mathbb{C}^n$. Q.E.D.

In an orthogonal (or unitary) representation, an orthogonal complement of a G -invariant subspace is also G -invariant (Theorem 2.11 of Chapter I, and hence we obtain the following:

COROLLARY 2.2. *A finite-dimensional real (or complex) representation of a compact group is completely reducible.*

Representations of compact groups by function spaces. We shall use the notation $V = \{f: G \rightarrow \mathbb{R}^n\}$ in the previous section.

DEFINITION. For $f \in V$, $g \in G$ and a linear transformation $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$R_y: V \rightarrow V \quad \text{is defined by} \quad (R_y f)(x) = f(xy),$$

$$L_y: V \rightarrow V \quad \text{is defined by} \quad (L_y f)(x) = f(y^{-1}x),$$

$$A: V \rightarrow V \quad \text{is defined by} \quad (Af)(x) = A(f(x)).$$

Obviously,

2.3. (1) R_y, L_y , and A are linear transformations.

$$R_y \circ L_y = L_y \circ R_y, \quad A \circ R_y = R_y \circ A, \quad A \circ L_y = L_y \circ A.$$

(2) $R_y \circ R_x = R_{xy}$, $L_y \circ L_x = L_{xy}$ for $x, y \in G$.

Thus R and L are representations of G over V .

Suppose that an operator K is constructed from a kernel function k satisfying the following condition.

2.4. There is a function $h: G \rightarrow \mathbb{R}$ satisfying $h(x) = h(x^{-1})$ such that

$$k(x, y) = h(xy^{-1}).$$

LEMMA 2.5. (1) $R_y \circ K = K \circ R_y$ for any $y \in G$.

(2) If h satisfies $h(yxy^{-1}) = h(x)$ for any $x \in G$, then $L_y \circ K = K \circ L_y$.

(3) $A \circ K = K \circ A$ for any linear transformation A .

PROOF. (1)

$$\begin{aligned} K(R_y f)(x) &= \int h(xz^{-1})f(zy) dz = \int h(xy(zy)^{-1})f(zy) dz \\ &= \int h((xy)z^{-1})f(z) dz = (Kf)(xy) = R_y(Kf)(x). \end{aligned}$$

(2)

$$\begin{aligned} K(L_y f)(x) &= \int h(xz^{-1})f(y^{-1}z) dz \\ &= \int h(yy^{-1}x(y^{-1}z)^{-1}y^{-1})f(y^{-1}z) dz \\ &= \int h(yy^{-1}xz^{-1}y^{-1})f(z) dz = \int h(y^{-1}xz^{-1})f(z) dz \\ &= (Kf)(y^{-1}x) = L_y(Kf)(x). \end{aligned}$$

(3)

$$K(Af)(x) = \int h(xz^{-1})A(f(z)) dz = A \int h(xz^{-1})f(z) dz = A(Kf)(x).$$

Q.E.D.

COROLLARY 2.6. R_y and A map an eigenspace with respect to one characteristic value to itself. So does L_y if h satisfies $h(yxy^{-1}) = h(x)$.

PROOF. $K\phi = \lambda\phi$ implies $K(R_y\phi) = R_y(K\phi) = R_y(\lambda\phi) = \lambda(R_y\phi)$.

The proofs of the other assertions are similar. Q.E.D.

NOTATION. For $c > 0$,

$V_c =$ (the subspace spanned by characteristic functions of K ,
which belong to a characteristic value λ satisfying $|\lambda| \geq c$)

$= \sum$ (the eigenspace with respect to λ satisfying $|\lambda| \geq c$);

$$V_c^\perp = \{g \in V \mid (g, f) = 0, \forall f \in V_c\}.$$

By Theorem 1.12, V_c is of finite dimension and has an orthonormal basis $\{\phi_1, \dots, \phi_r\}$ consisting of characteristic functions. If we define

$$P_c: V \rightarrow V_c \quad \text{by} \quad P_c(f) = \sum_{i=1}^r (f, \phi_i) \phi_i,$$

then P_c is the projection satisfying

$$P_c(V_c^\perp) = 0, \quad P_c|_{V_c} = \text{id}, \quad P_c \circ P_c = P_c,$$

and we obtain a direct sum decomposition

$$V = V_c \oplus V_c^\perp$$

by $f = P_c f \oplus (f - P_c f)$. *1

LEMMA 2.7. *The above projection P_c commutes with R_y and A for $c > 0$. Further, P_c commutes with L_y if h in 2.4 satisfies $h(yxy^{-1}) = h(x)$:*

$$P_c \circ R_y = R_y \circ P_c, \quad P_c \circ A = A \circ P_c, \quad P_c \circ L_y = L_y \circ P_c.$$

These morphisms R_y , A , and L_y map V_c to V_c and $V_c^\perp$ to $V_c^\perp$.

PROOF. By Corollary 2.6 we have $R_y(V_c) \subset V_c$. Now if we suppose $R_y(V_c^\perp) \subset V_c^\perp$, we have $R_y P_c f, P_c R_y f \in V_c$ and $R_y(f - P_c f), R_y f - P_c R_y f \in V_c^\perp$ in

$$R_y P_c f + R_y(f - P_c f) = R_y f = P_c R_y f + (R_y f - P_c R_y f).$$

By the uniqueness of the direct sum decomposition, we have $R_y P_c f = P_c R_y f$. We obtain $R_y(V_c^\perp) \subset V_c^\perp$ from the first of the equalities of (2.7)' below. The lemma for A and L_y follows from the remaining equalities of (2.7') below. The lemma for A and L_y follows from the remaining equalities of (2.7') and Corollary 2.6.

$$\begin{aligned} \langle R_y g, f \rangle &= \langle g, R_{y^{-1}} f \rangle, & \langle L_y g, f \rangle &= \langle g, L_{y^{-1}} f \rangle, \\ \langle A g, f \rangle &= \langle g, {}^t A f \rangle, \end{aligned} \quad (2.7')$$

where ${}^t A$ is the transpose of A (which corresponds to the transpose of a matrix).

The first equality is

$$\begin{aligned} \langle R_y g, f \rangle &= \int (g(xy), f(x)) dx = \int (g(xy), f((xy)y^{-1})) dx \\ &= \int (g(x), f(xy^{-1})) dx = \langle g, R_{y^{-1}} f \rangle. \end{aligned}$$

The rest is similar. Q.E.D.

Extension of representations of closed subgroups

LEMMA 2.8. *For an arbitrary neighborhood U of the identity element e of a compact group G , there exists a continuous function $h: G \rightarrow \mathbb{R}$ satisfying the following conditions:*

$$\begin{aligned} h(x) &\geq 0 \text{ for } x \in G, & h(e) &= 1, & x \notin U \text{ implies } h(x) &= 0, \\ h(x) &= h(x^{-1}) \text{ for } x \in G, & h(yxy^{-1}) &= h(x) \text{ for } x, y \in G. \end{aligned}$$

PROOF. One can replace U with a neighborhood contained in it. By the compactness of G and the continuity of $A(g, u) = gug^{-1}$, we see that $\bigcap gUg^{-1}$ is also a neighborhood of e . Replacing U with $\bigcap gUg^{-1}$, we have $gUg^{-1} = U$ for any $g \in G$. In addition, we see that we may suppose that $U = U^{-1}$, by considering $U \cap U^{-1}$.

Since G is a normal space, there is a continuous function $r: G \rightarrow \mathbb{R}$ such that

$$r(x) \geq 0 \text{ for } x \in G, \quad r(e) = 1, \quad x \notin U \text{ implies } r(x) = 0.$$

Then putting

$$h(x) = \frac{1}{2} \int_G (r(xz) + r(zx)) dz$$

the properties of the lemma are easily verified. Q.E.D.

In the arguments below we suppose that the operator K arises from a continuous function h in Lemma 2.8 for a suitable small neighborhood U :

$$(Kf)(x) = \int_G h(xy^{-1})f(y) dy.$$

The projection $P_c: V_c \rightarrow V_c$ is the one corresponding to this K and some $c > 0$.

LEMMA 2.9. Suppose given a representation $\rho: H \rightarrow GL(n, \mathbb{R})$ of a closed subgroup H of a compact group G . Then there exist $c > 0$ and a continuous function $f: G \rightarrow \mathbb{R}^n$ such that

$$f \in V_c, \quad f(e) \neq 0, \quad f(yx) = \rho(y)f(x) \text{ for } y \in H, x \in G.$$

PROOF. Since the map

$$g_1: H \rightarrow \mathbb{R}^n \text{ defined by } g_1(y) = \rho(y) \cdot e_1 \text{ for } y \in H,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ is the unit vector, is continuous, and since G is a compact Hausdorff space and H is a closed subset, we can extend g_1 to $g_2: G \rightarrow \mathbb{R}^n$, $g_2|_H = g_1$, by the Tietze theorem. The map $g: G \rightarrow \mathbb{R}^n$ defined by $g(x) = \int_H \rho(y^{-1})g_2(yx) dy$ for $x \in G$ is a continuous map. We have

$$g(z) = \int_H \rho(y^{-1})g_1(yz) dy = \int_H g_1(z) dy = g_1(z) \text{ for } z \in H$$

since g_1 satisfies $g_1(yy') = \rho(y)g_1(y')$ for $y, y' \in H$, that is, g is also an extension of g_1 . Furthermore, we have

$$\begin{aligned} g(zx) &= \int_H \rho(y^{-1})g_2(yzx) dy = \int_H \rho(z(yz)^{-1})g_2((yz)x) dy \\ &= \int_H \rho(zy^{-1})g_2(yx) dy = \rho(z) \int_H \rho(y^{-1})g_2(yz) dy = \rho(z)g(x) \end{aligned}$$

for any $z \in H$ and $x \in G$.

Since $g(e) = g_1(e) = \rho(e) \cdot e_1 = e_1 = (1, 0, \dots, 0)$, the first coordinate of $g(x)$ can be taken to be positive on a neighborhood U of e in G . Then we have $Kg(e) \neq 0$, since the first coordinate of $Kg(e) = \int_G h(ey^{-1})g(y)dy$ is positive for h in Lemma 2.8.

Put $f = P_c(Kg) \in V_c$. If we put $A = \rho(y)$, then A is a linear transformation and the equality $g(yx) = \rho(y)g(x)$ in the above is equivalent to $L_{y^{-1}}(g) = A(g)$. By Lemmas 2.5 and 2.7 we have

$$L_{y^{-1}}(f) = L_{y^{-1}}P_c(Kg) = P_cKL_{y^{-1}}(g) = P_cKA(g) = AP_c(Kg) = A(f),$$

that is, $f(yx) = \rho(y)f(x)$ is satisfied for $y \in H$, $x \in G$.

If we state Theorem 1.12(4) in terms of P_c , it asserts that $P_c(Kg)$ uniformly converges to Kg as $c \rightarrow 0$. Since $Kg(e) \neq 0$, taking c sufficiently small we have $f(e) = P_c(Kg)(e) \neq 0$. Q.E.D.

THEOREM 2.10. *Let G be a compact group and H a closed subgroup. For an arbitrary representation $\rho: H \rightarrow GL(n, \mathbb{R})$ of H over $\mathbb{R}^n$, there exist representations $\eta: G \rightarrow GL(m, \mathbb{R})$ of G and $\mu: H \rightarrow GL(m-n, \mathbb{R})$ of H such that the restriction $\eta|_H$ of η on H is equivalent to a direct sum $\rho \oplus \mu$.*

PROOF. It suffices to show the theorem for irreducible ρ , since a representation of a compact group is completely reducible. We put

$$\begin{aligned} W &= \{f \in V_c | f(yx) = \rho(y) \cdot f(x), y \in H, x \in G\} \\ &= \{f \in V_c | L_{y^{-1}}f = \rho \circ f, y \in H\}. \end{aligned}$$

Then W is a linear subspace of V_c and so is of finite dimension.

We shall show $R_z(W) \subset W$ for $z \in G$. If for $f \in V_c$ we take $f_0 \in V$ so that $f = P_c(f_0)$, we have $R_z f = R_z P_c(f_0) = P_c R_z(f_0) \in V_c$ by Lemma 2.7. Also we have $R_z f \in W$, since by 2.3(1)

$$L_{y^{-1}}(R_z f) = R_z(L_{y^{-1}}f) = R_z(\rho(y) \circ f) = \rho(y) \circ R_z f$$

for $f \in W$ and $y \in H$. In addition, by 2.3(2) the assignment $z \mapsto R_z|_W$ determines a representation $\eta: G \rightarrow GL(W)$ of G over W .

If we define $J: W \rightarrow \mathbb{R}^n$ by $J(f) = f(e)$, then the diagram is commutative

$$\begin{array}{ccc} W & \xrightarrow{J} & \mathbb{R}^n \\ \downarrow R_y & & \downarrow \rho(y) \\ W & \xrightarrow{J} & \mathbb{R}^n \end{array}$$

for arbitrary y :

$$JR_y(f) = R_y f(e) = f(y) = \rho(y)f(e) = \rho(y)J(f).$$

In particular, $\text{Ker } J$ is an $(\eta|_H)$ -invariant subspace of W , since $R_y(\text{Ker } J) \subset \text{Ker } J$, where $\eta|_H: H \rightarrow GL(W)$. By the complete reducibility

of a representation of a compact group H , there is an $(\eta|H)$ -invariant subspace W_0 of W such that $W = W_0 \oplus \text{Ker } J$ (a direct sum). If we denote by ρ_0 and μ the restrictions of $\eta|H$ on W_0 and $\text{Ker } J$ respectively, we have

$$\rho|H = \rho_0 \oplus \mu.$$

Since $W_0 \cong W/\text{Ker } J$, J induces a monomorphism $J_0: W_0 \rightarrow \mathbb{R}^n$. Regarding W_0 as a subspace of $\mathbb{R}^n$, W_0 is ρ -invariant in $\mathbb{R}^n$ by the previous commutative diagram. Then $W_0 = 0$ or $J_0: W_0 \cong \mathbb{R}^n$, since ρ is irreducible.

The map f in Lemma 2.9 belongs to W and $J(f) = f(e) \neq 0$. That is, $J_0: W_0 \cong \mathbb{R}^n$ and ρ_0 and ρ are equivalent through J_0 . Thus $\eta|H$ is equivalent to $\rho \oplus \mu$. Q.E.D.

Compact groups isomorphic to subgroups of orthogonal groups

THEOREM 2.11. *Let G be a compact group, H its closed subgroup, and U a neighborhood of the identity e in G . Then for some n there exist a representation $\rho: G \rightarrow GL(n, \mathbb{R})$ and a vector $v \in \mathbb{R}^n$ such that $H \subset G_v \subset UH$, where G_v is the isotropy group at v .*

PROOF. Take a neighborhood W of e so that $W = W^{-1}$ and $W^2 \subset U$, and choose a continuous function $h: G \rightarrow \mathbb{R}$ such that $h(e) = 1$, $h(x) = h(x^{-1}) \geq 0$, $h(x) = 0$ for $x \notin W$.

Since G/H is also a compact Hausdorff space, we can choose a continuous function $f_0: G/H \rightarrow \mathbb{R}$ such that $f_0(eH) = 1$, $f_0(xH) \geq 0$, and $f_0(xH) = 0$ for $x \notin WH$. If we define $f: G \rightarrow \mathbb{R}$ by

$$f(x) = f_0(xH),$$

then $R_y f = f$ for $y \in H$, $f(e) = 1$, and $f(x) = 0$ for $x \notin WH$. Consider

$$Kf(x) = \int h(xy^{-1})f(y)dy.$$

If $Kf(x) \neq 0$, then we have $xy^{-1} \in W$ and $y \in WH$ for some $y \in G$. Therefore we must have $x \in W^2H \subset UH$. Since $h(xy^{-1})f(y) \geq 0$ and $h(e)f(e) = 1$, we see the following:

$$Kf(x) \geq 0, \quad Kf(e) \geq 0, \quad Kf(x) = 0 \quad \text{for } x \notin UH.$$

By Theorem 1.12(4), $P_c(Kf)$ uniformly converges to Kf as $c \rightarrow 0$, and so by taking c sufficiently small, it follows that

$$P_c(Kf)(e) > P_c(Kf)(x) = R_x P_c(Kf)(e) \quad \text{for } x \notin UH.$$

Consider $V_c \subset V = \{G \rightarrow \mathbb{R}\}$ for such c and put $v = P_c(Kf) \in V_c$. We may suppose $V_c = \mathbb{R}^n$, as V_c is of finite dimension.

Let ρ be the representation of G determined by the assignment

$$x(\in G) \mapsto R_x|V_c: V_c \rightarrow V_c \quad (\text{Lemma 2.7}).$$

As is shown above, if $x \notin UH$, then $v(e) = P_c(Kf)(e) > R_x P_c(Kf)(e) = R_x(v)(e)$ so $v \neq R_x(v) = \rho(x)(v)$ and $x \notin G_v$, that is, $G_v \subset UH$.

Next, recall that $R_y f = f$ for $y \in H$. Since R_y commutes with K and P_c , we have

$$R_y(v) = R_y P_c(Kf) = P_c K(R_y f) = P_c(Kf) = v,$$

and hence $H \subset G_v$. Q.E.D.

COROLLARY 2.12. *Let G be a compact group and U a neighborhood of the identity e of G . Then there exists a homomorphism $\rho: G \rightarrow O(n)$ such that $\text{Ker } \rho \subset U$.*

PROOF. If we put $H = \{e\}$ in Theorem 2.11, we have $\text{Ker } \rho \subset G_v \subset U$. In addition, by Theorem 2.1 we can replace the inner product of $\mathbb{R}^n$ so that $\rho(G) \subset O(n)$. Q.E.D.

DEFINITION. A topological group G is said to contain no *small group* if there is a neighborhood U of the identity e of G containing no subgroup of G other than $\{e\}$.

LEMMA 2.13. *A Lie group contains no small group.*

PROOF. Let G be a Lie group of dimension n and consider the exponential map $\exp: \mathbb{R}^n = L(G) \rightarrow G$. Since $\exp$ is a local homeomorphism, we choose a bounded convex neighborhood V of 0 in $\mathbb{R}^n$ so that $\exp$ is a homeomorphism of V to a neighborhood of the identity e of G . Put $\frac{1}{2}V = \{\frac{1}{2}A | A \in V\}$ and $U = \exp(\frac{1}{2}V)$. Now, suppose that a subgroup H of G satisfies $\{e\} \neq H \subset U$. If $e \neq h \in H$, then there is $v \in \frac{1}{2}V$ such that $\exp(v) = h$ and $v \neq 0$. Since V is bounded and convex, there is an integer $r > 1$ such that $2^r v \in V - \frac{1}{2}V$. Then we have $h^{2^r} = \exp(2^r v) \in \exp(V) - U$, as $\exp|_V$ is a monomorphism. This contradicts the assumption that H is a subgroup which implies $h^{2^r} \in H \subset U$. Thus G contains no small group. Q.E.D.

THEOREM 2.14. *The following three conditions are equivalent for a compact group G .*

- (1) G contains no small group.
- (2) G is isomorphic to a (closed) subgroup of $O(n)$ for some n .
- (3) G is (or has a structure of) a Lie group.

PROOF. (1) implies (2). We can take $\text{Ker } \rho = \{e\}$ in Corollary 2.12. G is homeomorphic to $\rho(G)$, since $\rho: G \rightarrow O(n)$ is a monomorphism and since G and $O(n)$ are compact Hausdorff spaces. Therefore G is isomorphic to a (closed) subgroup of $\rho(G)$.

(2) $\implies$ (3) is clear from Theorem 5.13 of Chapter I.

(3) $\implies$ (1) is clear from Lemma 2.13. Q.E.D.

§3. Maximal tori

Tori. A topological group isomorphic to a direct product of n copies of $U(1)$ is called an n -dimensional torus, denoted by

$$T = T^n \cong U(1) \times \cdots \times U(1), \quad n = \dim T \quad (\text{the dimension of } T).$$

Obviously, a torus is a compact, connected, abelian Lie group. If we define $p: \mathbb{R}^n \rightarrow T^n$ by

$$p(t_1, \dots, t_n) = (\exp(2\pi i t_1), \dots, \exp(2\pi i t_n)),$$

then p is a universal covering homomorphism of Lie groups, and hence we can take

$$T^n = \mathbb{R}^n / \mathbb{Z}^n \quad (3.1)$$

since $\text{Ker } p = \mathbb{Z}^n = \{(t_1, \dots, t_n) | t_i \in \mathbb{Z}\}$; a point $p(t_1, \dots, t_n)$ of T^n is simply denoted by $(t_1, \dots, t_n)$.

DEFINITION. A topological group G is called *monogenic* if there is an element g of G such that any closed subgroup containing g coincides with G . g is called a *generator* of G .

Consider a subgroup H generated by g ; then the closure $\bar{H}$ is a closed subgroup. The existence of g such that $\bar{H} = G$ is equivalent to G being monogenic with generator g . Since H in the above is abelian, then so is $\bar{H}$. Thus

3.2. If G is monogenic, then it is abelian.

THEOREM 3.3. A torus T^n is monogenic and generators are dense in T^n .

PROOF. (In fact, the linear independence of $\{t_1, \dots, t_n | t_i \in \mathbb{R}\}$ over the field of rationals is equivalent to $(t_1, \dots, t_n)$ being a generator of T^n .) If we denote a point of T^n by its coordinates $(t_1, \dots, t_n) \in \mathbb{R}^n$, then the group multiplication is expressed as addition. A cube with side of length a (> 0) in T^n is $C = \{(x_1, \dots, x_n) | \xi_i \leq x_i \leq \xi_i + a\}$ for some point $(\xi_1, \dots, \xi_n)$. Observe that $C = T^n$ if $a \geq 1$. Now, it suffices to show that there exists a generator in an arbitrary cube C .

Let $U_1, U_2, \dots$ be a countable open basis of T . Starting with $C = C_0$, we will construct inductively a decreasing sequence of cubes $C_0 \supset C_1 \supset C_2 \supset \dots$ as follows: if $\varepsilon > 0$ is the length of a side of C_{m-1} , there exists an integer $N(m)$ such that $N(m)\varepsilon \geq 1$. We can take a cube C_m contained in C_{m-1} such that $N(m)C_m \subset U_m$, since $N(m)C_{m-1} = T^n$. Then we have $\bigcap C_m \neq \emptyset$, since $\{C_m\}$ thus constructed is a decreasing sequence of nonempty closed sets in the compact set T^n . If $g \in \bigcap C_m$, then $N(m)g \in U_m$. Thus the subgroup generated by g is dense in T^n and so g is a generator of T^n . Q.E.D.

LEMMA 3.4. If an abelian topological group G contains a torus T^n as a subgroup such that G/T^n is a cyclic subgroup, then G is monogenic.

PROOF. Let t be a generator of T^n and $u \in G$ an element such that its projection is a generator of G/T^n . Then $mu \in T^n$ for the order m of

G/T^n . Expressing everything in terms of coordinates of $\mathbf{R}^n$, it is clear that for $t - mu \in T^n$ there exists $s \in T^n$ such that $ms = t - mu$. Putting $g = u + s$, the subgroup H generated by $mg = m(u + t) = t$ is dense in T^n and $ig \cdot H$ is dense in $ig \cdot T^n = iu \cdot T^n$ for $i = 1, 2, \dots, m$. Since $G = \bigcup_i iu \cdot T^n$, g is a generator of G . Q.E.D.

THEOREM 3.5. *Let G be a connected, abelian Lie group. Then there are integers n and k such that*

$$G \cong \mathbf{R}^{n-k} \times T^k \quad (\text{an isomorphism of Lie groups}).$$

PROOF. The Lie algebra $\mathfrak{g} = L(G)$ of G is isomorphic with $\mathbf{R}^n$, where $n = \dim G$. Now observe that

3.5'. *If G is an abelian Lie group then the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a homomorphism of Lie groups.*

In fact, by the commutativity, $\exp(tX)\exp(tY)$ is a one parameter subgroup with the initial vector $X + Y$, and so $\exp(tX + tY) = \exp(tX)\exp(tY)$. Putting $t = 1$, we have $\exp(X + Y) = \exp(X)\exp(Y)$. Since $\exp$ is a local homeomorphism, $\exp^{-1}(e)$ is a discrete subgroup of $\mathfrak{g}$. Let $x_1, \dots, x_k$ be independent elements of the maximal number of $\exp^{-1}(e)$ in the vector space $\mathfrak{g}$ over $\mathbf{R}$ and construct an $\mathbf{R}$ -basis of $\mathfrak{g}$ by adding $x_{k+1}, \dots, x_n$ to them. Let $\mathbf{R}^k$ and $\mathbf{R}^{n-k}$ be the subspaces spanned by the former and the latter respectively: $\mathfrak{g} = \mathbf{R}^k \oplus \mathbf{R}^{n-k}$. Obviously $\exp^{-1}(e) \subset \mathbf{R}^k$, and hence $G \cong \mathbf{R}^{n-k} \times (\mathbf{R}^k / \exp^{-1}(e))$. Let F be the subgroup of $\exp^{-1}(e)$ generated by $x_1, \dots, x_k$; $\exp^{-1}(e)/F$ is finite, since it is a discrete subgroup of $\mathbf{R}^k/F$ and since $\mathbf{R}^k/F \cong T^k$ is compact. F is finitely generated so $\exp^{-1}(e)$ is finitely generated. Since there is no element of finite order in $\mathbf{R}^n$ and hence in $\exp^{-1}(e)$, by the fundamental theorem of abelian groups, $\exp^{-1}(e)$ is a free abelian group of k generators. Thus $\mathbf{R}^n / \exp^{-1}(e) \cong T^k$ and hence $G \cong \mathbf{R}^{n-k} \times T^k$. Q.E.D.

COROLLARY 3.6. *A compact, connected, abelian Lie group is a torus.*

THEOREM 3.7. *If T and T' are tori such that T' is a subgroup of T , then there is a torus T'' which is a subgroup of T such that $T = T' \times T''$.*

PROOF. Put $K = \exp^{-1}(e) \subset L(T)$ and $K' = K \cap L(T')$. $\exp$ induces isomorphisms $L(T)/K \cong T$ and $L(T')/K' \cong T'$. K/K' is a finitely-generated abelian group so K/K' is a direct sum of a free abelian group F and a finite group S . Let F' be the kernel of the composition of the projections $K \rightarrow K/K' \rightarrow F$ and F'' be the lift of F ; then $K = F' \oplus F''$ (a direct sum), $F' \supset K'$ and $F'/K' \cong S$. If we denote by V' and V'' the subspaces over $\mathbf{R}$ spanned by F' and F'' respectively, then $L(T) = V' \oplus V''$. $L(T')$ is spanned by K' so $L(T') \subset V'$. Besides, $V'/L(T') \cong F'/K' \cong S$ is finite.

So $V' = L(T')$. It follows that

$$T' \cong L(T')/K' \cong V'/K', \quad T \cong L(T)/K = (V'/F') \times (V''/F'').$$

An inclusion $i: T' \rightarrow T$ is given by the natural map $V'/K' \rightarrow V'/F' \subset T$, and $S \cong K'/T' = \text{Ker } i = 0$ implies $K' = F'$. Putting $V''/F'' = T''$, we have $T = T' \times T''$. Q.E.D.

COROLLARY 3.8. *If T and T' are tori, $T \supset T'$ and $\dim T = \dim T'$ then $T = T'$.*

Maximal tori. Hereafter in this section we assume that G is a compact, connected Lie group. By Theorem 2.14 we may suppose $G \subset O(n)$. By regarding the tangent space $T_x G$ of G as a subspace of $T_x O(n)$, we introduce a natural inner product, and the actions of G

$$dL_g, dR_g, dA_g = dL_g \circ dR_{g^{-1}}: TG \rightarrow TG \text{ preserve the inner product.} \quad (3.9)$$

In other words, G has a Riemannian metric invariant under the multiplication of G (Theorem 1.13(3)). In particular, the adjoint action of G on $\mathfrak{g} = L(G) = T_e G$

$$Ad(g) = dA_g|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathfrak{g} \text{ is an orthogonal transformation.} \quad (3.9')$$

A subgroup T of G isomorphic to a torus is called simply a *torus* of G . Since T is compact, the torus T is a closed subgroup of G and hence is a closed Lie subgroup; by Corollary 3.6 it is characterized as a connected, abelian, closed subgroup of G .

DEFINITION. T is called a *maximal torus* of G if T is a torus of G and if a torus T' of G containing T satisfies $T = T'$.

THEOREM 3.10. (1) *Any torus of G is contained in a maximal torus of G .*

(2) *A connected, abelian subgroup containing a maximal torus T of G coincides with T .*

PROOF. (1) If there is no maximal torus containing a torus T of G , we can take an infinite, strictly increasing sequence of tori $T \subsetneq T_1 \subsetneq T_2 \subsetneq \dots$. By Corollary 3.8, $\dim T < \dim T_1 < \dim T_2 < \dots \leq \dim G$. This is a contradiction, and so (1) is obtained.

(2) If A is a connected, abelian subgroup such that $T \subset A \subset G$, then the closure $\bar{A}$ of A is a connected abelian closed subgroup. G is compact so $\bar{A}$ is compact, and by Corollary 3.6, $\bar{A}$ is a torus. Therefore $T = \bar{A}$ implies $T = A$. Q.E.D.

LEMMA 3.11. *The automorphism group $\text{Aut } T^n$ of a torus T^n is discrete with respect to the compact-open topology.*

PROOF. The covering $p: \mathbb{R}^n \rightarrow T^n = \mathbb{R}^n/\mathbb{Z}^n$ is the universal covering group. For an isomorphism $f: T^n \rightarrow T^n$, we can take uniquely a lifting

$\tilde{f}: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of $f \circ p$ such that $\tilde{f}(0) = 0$ and $\tilde{f}$ is an isomorphism. Consider a map $\varphi: \text{Aut } T^n \rightarrow \text{Aut } \mathbf{R}^n = GL(n, \mathbf{R})$ given by $f \mapsto \tilde{f}$. Since T^n is a compact metric space, the compact-open topology in $\text{Aut } T^n$ is equivalent to the one defined by the distance metric of maps. It follows from this that φ is a continuous injection. Since $\text{Im } \varphi$ consists of matrices with integer entries such that the determinant $= \pm 1$, it is a discrete subgroup of $GL(n, \mathbf{R})$. By the continuity of φ , $\text{Aut } T^n$ is a discrete set. Q.E.D.

NOTATION. (revision)

$N(S) = N_G(S) = \{g \in G | gS = Sg\}$ denotes the *normalizer* of (a subset) S ;

$Z(S) = Z_G(S) = \{g \in G | ga = ag\}$ for all $a \in S$ denotes the *centralizer* of S ;

H_0 = the connected component containing the identity of (a subgroup) H ;

$A: G \times G \rightarrow G$ denotes the map $A(g, x) = gxg^{-1}$ (an adjoint action);

$A_g: G \rightarrow G$ for $g \in G$ denotes the map $A_g(x) = gxg^{-1}$ (an inner automorphism);

$N(S)$ acts on S by A , that is, $A|N(S): N(S) \times S \rightarrow S$.

LEMMA 3.12. If T is a maximal torus of G , then $N(T)_0 = Z(T)_0 = T$, and $N(T)/T$ is a finite group.

PROOF. $N(T)$ is a Lie group, since it is a closed subgroup of a Lie group G . The action of $N(T)$ on T induces a continuous map $N(T) \rightarrow \text{Aut } T$. By Lemma 3.11, the connected component $N(T)_0$ is mapped to the identity. That is, $N(T)_0$ acts trivially on T and $N(T)_0 \subset Z(T)$. $T \cup \{\exp tX | t \in \mathbf{R}\}$ generates a connected abelian subgroup H for $X \in L(Z(T))$. By Theorem 3.10(2), $T = H$ and hence $Z(T)_0 = N(T)_0 = T$. $N(T)/T = N(T)/N(T)_0$ is a finite group since it has discrete topology and is compact. Q.E.D.

The conjugacy of maximal tori. Now, we quote without proof a fixed point theorem.

THEOREM 3.13 (The Lefschetz fixed point theorem). If there is a fixed point free self-map f of a polyhedron X (i.e., $f(x) \neq x$ for any $x \in X$) such that $f \simeq 1_X: X \rightarrow X$, then $\chi(X) = 0$.

Here $\chi(X)$ is the Euler characteristic

$$\chi(X) = \sum_i (-1)^i \dim_k H_i(X; k) = \sum_i (-1)^i \# \{\text{the } i\text{-dimensional cells } \subset X\}.$$

Recall that $\chi(X)$ is defined by the first equality for a general topological space and that the second equality holds for a polyhedron or for a finite CW-complex (cf. §3 of Chapter VII).

THEOREM 3.14. *Let T be a maximal torus of G . Then $\dim(G/N(T))$ is even, and*

$$\chi(G/N(T)) = 1, \quad \chi(G/T) = |N(T)/T| \quad (\text{the order of } N(T)/T).$$

PROOF. Consider the action of T on $G/N(T)$ by left translation. Since $T \subset N(T)$, the point $N(T) = p(e)$ of $G/N(T)$ is a fixed point of the action. If we identify $T_{p(e)}(G/N(T))$ with $T_e(N(T))^\perp$ (the orthogonal complement in $T_e G$ by $T_e G/T_e(N(T)) \cong T_{p(e)}(G/N(T))$), the action of each element is an orthonormal transformation by (3.9). By considering $\exp|T_e(N(T))^\perp|$ we see that there exists a disk $D^m \subset G/N(T)$ with center $p(e)$, where $m = \dim(G/N(T))$, such that

$$D^m, \quad S^{m-1} = \partial D^m, \quad M = G/N(T) - \text{Int } D^m$$

are mapped by the action of T to themselves.

Let t be a generator of T and suppose that $p(g) = gN(T)$ is invariant under the action of t , then the following statements are equivalent $t \cdot p(t) = p(g)$; $t g N(T) = g N(T)$; $g^{-1} t g \in N(T)$; $g^{-1} t^n g \in N(T)$ $n = 1, 2, \dots$

This is equivalent to $g^{-1} T g \subset N(T)$, since $\{t^n\}$ is dense in T . This, in turn, is equivalent to $g^{-1} T g = T$, i.e., $g \in N(T)$, since $g^{-1} T g$ is connected and $N(T)_0 = T$ (Lemma 3.12). Thus the fixed point of the left translation by T is the point $p(e)$. As there is a path connecting t and e in T , we have $t \simeq 1_M: (M, S^{m-1}) \rightarrow (M, S^{m-1})$. By using Theorem 3.13 we have

$$\chi(M) = \chi(S^{m-1}) = 0.$$

In particular, m is even, since $\chi(S^{m-1}) = 1 + (-1)^{m-1}$. By using a cellular decomposition or the excision axiom, the following equality holds (in general):

$$\chi(G/N(T)) = \chi(M) + \chi(D^m) - \chi(S^{m-1}).$$

Thus $\chi(G/N(T)) = \chi(D^m) = 1$.

G/T is a covering space of $G/N(T)$ with covering degree $|N(T)/T|$. Since we can obtain a cellular decomposition of G/T by lifting each cell of a cellular decomposition of $G/N(T)$ to G/T , we have

$$\chi(G/T) = |N(T)/T| \cdot \chi(G/N(T)) = |N(T)/T|.$$

Q.E.D.

The following theorem is a generalization of Theorem 3.11 of Chapter I.

THEOREM 3.15. *If T is a maximal torus of G , then for any element $x \in G$, there exists $g \in G$ such that $g^{-1} x g \in T$, that is,*

$$G = \bigcup_{g \in G} g T g^{-1}.$$

PROOF. Consider a map $f: G/T \rightarrow G/T$ defined by $hT \mapsto x h T$. Using a path connecting x and e , we have $f \simeq 1: G/T \rightarrow G/T$. On the other

hand, by Theorem 3.14, $\chi(G/T) \neq 0$. So by Theorem 3.13 f must have a fixed point gT . Then $gT = f(gT) = xgT$, which implies $g^{-1}xgT = T$ and hence $g^{-1}xg \in T$. Q.E.D.

COROLLARY 3.16. *Maximal tori are conjugate to each other. That is, if T is a maximal torus, then T' is a maximal torus of G iff there is $g \in G$ such that $T' = gTg^{-1}$.*

PROOF. For a generator x of T' there is $g \in G$ such that $x \in gTg^{-1}$. This implies $T' \subset gTg^{-1}$ and hence $T' = gTg^{-1}$. Conversely, gTg^{-1} is a maximal torus, since it is a conjugate of T . Q.E.D.

COROLLARY 3.17. *$\exp: \mathfrak{g} \rightarrow G$ is an epimorphism for G compact and connected.*

PROOF. Any $x \in G$ is written as $x = gtg^{-1}$ with $t \in T$, $g \in G$. $s(\alpha) = g(\exp(\alpha X))g^{-1}$ is a one parameter subgroup with $x = s(1)$. We have $x = \exp Y$ for the initial vector $Y \in \mathfrak{g}$ of s . Q.E.D.

THEOREM 3.18. *Let T be a maximal torus of G and A an arbitrary subset of T . If $g \in G$ satisfies $gAg^{-1} \subset T$, then there exists $k \in N(T)$ such that*

$$kak^{-1} = gag^{-1} \text{ for all } a \in A.$$

PROOF. $Z(A) = \{h \in G | ha = ah \text{ for any } a \in A\}$ is a closed subgroup of G . Thus the identity component $H = Z(A)_0$ of $Z(A)$ is a compact connected Lie group. Obviously $T \subset H$. $A \subset g^{-1}Tg$ and $g^{-1}Tg$ is abelian, hence $g^{-1}Tg \subset H$. Since T as well as $g^{-1}Tg$ are maximal tori of H , there exists $h \in H$ such that $g^{-1}Tg = h^{-1}Th$. Then, putting $k = gh^{-1}$, $kTk^{-1} = T$, which $k \in N(T)$. Also for any $a \in A$, we have $kak^{-1} = gh^{-1}ahg^{-1} = gag^{-1}$. Q.E.D.

LEMMA 3.19. *If S is a connected abelian subgroup of G and if $g \in G$ commutes with all the elements of S , then there exists a torus containing g and S .*

PROOF. Let H be a subgroup generated by g and S . Since H is abelian, $\overline{H}$ is compact and abelian. It follows that the identity component $\overline{H}_0$ of $\overline{H}$ is a torus. $\overline{H}/\overline{H}_0$ is a finite cyclic group, since it is finite and is generated by g . By Lemma 3.4, $\overline{H}$ has a generator h . Then we have $\overline{H} \subset T$, since there exists a maximal torus T containing h by Theorem 3.15. Thus $S \cup g \subset H \subset \overline{H} \subset T$. Q.E.D.

COROLLARY 3.20. *If S is a connected, abelian subgroup then $Z(S) = \bigcup (\text{maximal tori containing } S)$. In particular, if T is a maximal torus then $Z(T) = T$.*

THEOREM 3.21. *Let T be a maximal torus of G . If an abelian subgroup A of G contains T , then $A = T$. That is, a maximal torus is a maximal abelian subgroup.*

PROOF. Take arbitrary $g \in A$. There is a torus U such that $g \cup T \subset U$ by Lemma 3.19. Since T is a maximal torus, $g \in U = T$, and hence $A = T$. Q.E.D.

THEOREM 3.22. *If T' is an arbitrary torus of G , the centralizer $Z(T')$ of T' is connected and coincides with the identity component $N(T')_0$ of the normalizer $N(T')$.*

PROOF. Obviously, $Z(T') \subset N(T')$. By Corollary 3.20, $Z(T') = \bigcup$ (a maximal torus containing T') is connected. Thus $Z(T') \subset N(T')_0$.

On the other hand, the first half of the proof of Lemma 3.12 holds for a torus T' , and $N(T')_0 \subset Z(T')$, since $N(T')_0$ acts on T' trivially. Thus $Z(T') = N(T')_0$. Q.E.D.

By Corollary 3.16 the dimension of a maximal torus of a compact, connected Lie group G is constant independently of the choice of T . Thus we have the following:

DEFINITION. The dimension of a maximal torus T of (a compact, connected Lie group) G is called the *rank* of G , denoted by $l = \text{rank } G$ ($= \dim T$).

It will be shown in Theorem 5.5 of Chapter VI that the above-defined rank l is equal to the one defined by using the cohomology ring in §6 of Chapter IV.

The Weyl group

DEFINITION. Let T be a torus of G . The set of all the conjugate transformations of T to itself

$$W(G : T) = \{\varphi: T \rightarrow T \mid \varphi(t) = gtg^{-1} \text{ for some } g \in G\}$$

is called the *Weyl group* with respect to T . In particular, if T is a maximal torus, $W(G : T)$ is simply called the *Weyl group* of G , and denoted by $W(G)$.

For $\varphi(t) = gtg^{-1}$, the condition that $\varphi \in W(G : T)$ is equivalent to $gTg^{-1} = T$, which is the same as $g \in N(T)$. Also, $\varphi = 1_T$ is equivalent to $gtg^{-1} = t$ for all $t \in T$, that is, $g \in Z(T)$. Combining these with Theorem 3.22, we obtain

$$W(G : T) \cong N(T)/Z(T) = N(T)/N(T)_0 \quad (\text{a finite group}). \quad (3.23)$$

If T is a maximal torus, $Z(T) = T$ by Corollary 3.20. Further, by using Corollary 3.16 one obtains

3.24. $W(G) = N(T)/Z(T) = N(T)/T$, which is uniquely determined up to isomorphism, independent of the choice of a maximal torus T .

LEMMA 3.25. *Suppose that T is a maximal torus of G and H is a normal closed subgroup of G contained in T . Then the following hold:*

$$(1) \quad N(T/H) = N(T)/H.$$

(2) T/H is a maximal torus in G/H .

(3) $W(G/H) = W(G)$.

PROOF. Let $\pi: G \rightarrow G/H$ be the projection.

(1) $n \in N(T)$ implies $\pi(n) \in N(T/H)$, since π is a homomorphism. Conversely, if $\pi(n) \in N(T/H)$, we have $n(tH)n^{-1} \subset T$ for any $t \in T$. Since H is normal, $ntHn^{-1} = ntn^{-1}H \subset T$. Thus $ntn^{-1} \in T$ and so $t \in N(T)$. Hence $N(T/H) = N(T)/H$.

(2) Certainly T/H is a torus, since it is abelian, connected, and compact. If a torus U/H of G/H contains T/H , then $U/H \subset N(T/H) = N(T)/H$. This implies $T \subset U \subset N(T)$ so $\dim T = \dim U$. Thus $\dim T/H = \dim U/H$ so $T/H = U/H$. Thus T/H is a maximal torus.

(3) $W(G/H) = N(T/H)/(T/H) = (N(T)/H)/(T/H) \cong N(T)/T = W(G)$. Q.E.D.

LEMMA 3.26. If $\text{rank } G = 1$, then (1) $\dim G = 1$, $W(G) = 0$ or (2) $\dim G = 3$, $W(G) \cong \mathbb{Z}_2$.

PROOF. $\dim G = 1$ implies $G = T$ so $W(G) = 0$.

Let $\dim G > 1$ and consider a vector X with length 1 of $L(T)$. The map $f: G/T \rightarrow S^{n-1} \subset L(G)$ given by

$$f(gT) = \text{Ad}(g) \cdot X$$

is well defined. In fact, as $\text{Ad}(t)X = X$ for $t \in T$, we have

$$f(gtT) = \text{Ad}(g)\text{Ad}(t)X = \text{Ad}(g)X = f(gT).$$

If $\text{Ad}(g)X = \text{Ad}(g')X$, then $\text{Ad}(g^{-1}g')X = X$ so $g^{-1}g' \in Z(T) = T$ (Corollary 3.20), and hence $gT = g'T$. This shows that f is a monomorphism. Since f is a C^∞ -map and an injection between the manifolds of the same dimension, it induces a monomorphism and hence an isomorphism of the tangent space at a point x of G/T . Namely, f maps a neighborhood U of x to a neighborhood V of $f(x)$ homeomorphically, and $f^{-1}(V) = U$. The diagram

$$\begin{array}{ccccc} H_{n-1}(G/T) & \xrightarrow{i_*} & H_{n-1}(G/T, G/T-x) & \xleftarrow{j_*} & H_{n-1}(U, U-x) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_{n-1}(S^{n-1}) & \xrightarrow{i_*} & H_{n-1}(S^{n-1}, S^{n-1}-f(x)) & \xleftarrow{j_*} & H_{n-1}(V, V-f(x)) \end{array}$$

is commutative, where the two morphisms j_* are excision isomorphisms; the two morphisms i_* are seen to be isomorphisms, since G/T and S^{n-1} are orientable (if this is hard to see, consider it mod 2). Obviously, f_* on the right-hand side is an isomorphism, so the maps f_* in the diagram are all isomorphisms. Now if we suppose that f is not surjective, then $f \simeq 0$, as

S^{n-1} —(a point) is contractible, which implies $f_* = 0$; this contradicts the property that f_* is an isomorphism. Thus f is surjective.

Now, since f is surjective, there is $h \in G$ such that $Ad(h)X = -X$. Then $Ad(h)(L(T)) \subset L(T)$ implies $A_h(T) = hTh^{-1} \subset T$ so $h \in N(T)$. In addition, A_h maps $t \in T$ to t^{-1} . Since $\dim T = 1$, we have $\text{Aut}(T) \cong \mathbf{Z}_2$, which is generated by A_h . Thus we have shown $W(G) \cong \mathbf{Z}_2$.

Since f is a bijection of compact manifolds, it is a homeomorphism. Hence one obtains the following exact sequence:

$$\pi_2(S^{n-1}) \xrightarrow{\partial} \pi_1(T) \xrightarrow{i_*} \pi_1(G) \rightarrow \pi_1(S^{n-1}),$$

where $\pi_1(T) \cong \mathbf{Z}$ is generated by the class $\iota = [1_T]$ of the identity map. If h_θ denotes a path connecting the identity and h , the homotopy $t \in T \mapsto h_\theta t h_\theta^{-1}$ shows $i_*(\iota) = i_*(-\iota)$, i.e., $i_*(2\iota) = 0$. Thus, $\partial(\pi_2(S^{n-1})) \supset 2\pi_1(T)$ implies $\pi_2(S^{n-1}) \neq 0$ so $n = 3$. Q.E.D.

COROLLARY 3.27. *If $\text{rank } G = 1$ and $\dim G = 3$, then G is an S^1 -bundle over S^2 , and $\pi_1(G)$ is isomorphic to 0 or $\mathbf{Z}_2$.*

REMARK 3.28. In Corollary 3.27, if $\pi_1(G) = 0$ then $G \cong Sp(1)$; if $\pi_1(G) \cong \mathbf{Z}_2$ then $G \cong SO(3)$.

§4. Roots and the Weyl groups

The definition of a root. Throughout this section let G be a compact, connected Lie group and let an inner product be given in $L(G) = \mathfrak{g} = T_e G$ so that the adjoint action (representation) $Ad(g) = dA_g: L(G) \rightarrow L(G)$ is an orthogonal transformation for $g \in G$ as in (3.9').

DEFINITION. The kernel $\exp^{-1}(e)$ of the exponential map $\exp: L(T) \rightarrow T$ of a torus is called the *integer lattice*, denoted by $\Gamma = \Gamma(G)$.

The integer lattice is a free abelian group with $k = \dim T$ generators $t_1, \dots, t_k$, and $L(T)$ can be identified with $\mathbf{R}^k$ with them as basis. When T is a maximal torus of G , the choice of these generators $t_1, \dots, t_k$ is suitably determined, depending on G .

THEOREM 4.1. (1) *Let T be a torus of G . Then there is an orthogonal decomposition of $L(G)$:*

$$L(G) = V_0 \oplus \sum_{r=1}^m V_r \quad \text{with } \dim V_r = 2,$$

where V_0 and V_r ($r = 1, 2, \dots, m$) are invariant subspaces under the restriction of the adjoint action Ad to T so that T acts on V_0 trivially and the action on each V_r is given by a nontrivial (surjective) homomorphism $\bar{\theta}_r: T \rightarrow \mathbf{R}/\mathbf{Z}$ with respect to an orthogonal basis of V_r as follows:

$$Ad(t)|_{V_r} = \mathbf{r}(e^{2\pi i \bar{\theta}_r(t)}) = \begin{pmatrix} \cos 2\pi \bar{\theta}_r(t) & -\sin 2\pi \bar{\theta}_r(t) \\ \sin 2\pi \bar{\theta}_r(t) & \cos 2\pi \bar{\theta}_r(t) \end{pmatrix}$$

(2) Each $\bar{\theta}_r$ above is unique up to sign and order, that is, the set $\{\pm\bar{\theta}_1, \dots, \pm\bar{\theta}_m\}$ is determined by G and T independently of the orthogonal decomposition in (1) and of the choice of an orthonormal basis.

PROOF. Let $t \in T$ be a generator. Since $Ad(t): L(t) \rightarrow L(G)$ is an orthogonal transformation, it is represented by an orthogonal matrix $A \in O(n) \subset M(n, \mathbb{R})$ with $n = \dim G$. By Theorem 3.5 of Chapter I, A can be transformed into the form $\lambda_1 \oplus \dots \oplus \lambda_{n-2m} \oplus r(\alpha_1) \oplus \dots \oplus r(\alpha_m)$, $\lambda_j \in \mathbb{R}$, $\alpha_r (\neq \pm 1) \in \mathbb{C}$ by an orthogonal matrix. Namely, by a choice of an orthogonal basis one has a decomposition $L(G) = \sum_{j=1}^{n-2m} V'_j \oplus \sum_{r=1}^m V_r$ with $\dim V'_j = 1$ and $\dim V_r = 2$ so that $Ad(t)$ acts on each V'_j and V_r as an orthogonal transformation. Similarly for $Ad(t^k) = Ad(t)^k$ ($k = 1, 2, \dots$). Since $\{t^k\}$ is dense in T and $Ad: T \rightarrow \text{Aut}(L(G))$ is continuous, T acts on each V'_j and V_r so that it gives rise to homomorphisms $\bar{\theta}'_j: T \rightarrow SO(1)$ and $\bar{\theta}_r: T \rightarrow SO(2)$ respectively. Clearly $\bar{\theta}'_j = 0$. And, $\alpha_r \neq \pm 1$ implies that $\bar{\theta}_r$ is not trivial. Thus (1) is proved.

The set $\{\alpha_r, \alpha_r^{-1}\}$ is unique (Theorem 3.5 of Chapter I) and (2) follows. Q.E.D.

Consider the covering maps $\exp: L(T) \rightarrow T$ and $p: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$.

Since $L(T)$ is 1-connected, the composition $\bar{\theta}_r \circ \exp: L(T) \rightarrow \mathbb{R}/\mathbb{Z}$ is uniquely lifted to a map $\theta_r: L(T) \rightarrow \mathbb{R}$ preserving the identities so that θ_r is a homomorphism of the integer lattice Γ to $\mathbb{Z}$. Conversely, for such a homomorphism θ_r , there is a uniquely determined homomorphism $\bar{\theta}_r$ such that the diagram

$$\begin{array}{ccc} L(T) & \xrightarrow{\theta_r} & \mathbb{R} \\ \downarrow \exp & & \downarrow p \\ T & \xrightarrow{\bar{\theta}_r} & \mathbb{R}/\mathbb{Z} \end{array}$$

is commutative. In this manner we obtain a linear form

$$\theta_r: L(T) \rightarrow \mathbb{R}, \quad \text{where } \theta_r(\text{integer lattice}) \subset \mathbb{Z},$$

which corresponds bijectively with $\bar{\theta}_r$.

DEFINITION. If T is a maximal torus, then the linear forms $\pm\theta_r: L(T) \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, m$ are called the *roots* of G . Also $\pm\bar{\theta}_r: L(T) \rightarrow \mathbb{R}/\mathbb{Z}$ are called the *roots*.

THEOREM 4.2. T is a maximal torus if and only if $V_0 = L(T)$.

PROOF. Sufficiency. Suppose $V_0 = L(T)$ and $T \subset T'$ (a torus). Since T acts trivially on $L(T')$, we have $L(T') \subset V_0$. $L(T) \subset L(T') \subset V_0 = L(T)$ implies $\dim T = \dim T'$ and hence $T = T'$ (Corollary 3.8). Thus T is a maximal torus.

Necessity. Obviously $L(T) \subset V_0$. If we suppose $X \in V_0$ and $X \notin L(T)$, then $\exp(\alpha X)$ commutes with T elementwise for $\alpha \in \mathbb{R}$. By Lemma 3.19 there is a torus T_α containing T and $\exp(\alpha X)$ for all α . On the other hand, if we take $\alpha > 0$ sufficiently small, $\exp(\alpha X) \notin T$, which contradicts the maximality of T . Thus if T is a maximal torus, then $V_0 = L(T)$. Q.E.D.

COROLLARY 4.3. $\dim G = l + 2m$, where $l = \text{rank } G$ and $2m$ is the number of roots.

The roots of compact, connected, classical groups and spinor groups. We will illustrate the roots of $G = U(l)$, $SU(l+1)$, $Sp(l)$, $SO(n)$, and $\text{Spin}(n)$ where $[n/2] = l$ as examples. We take the standard maximal torus in §3 of Chapter I as the maximal torus of each of the above groups.

NOTATION. E_{rs} = (the square matrix with (r, s) -element 1 and the other elements 0).

$G = U(l)$. We choose coordinates $(x_1, \dots, x_l)$ of $L(T)$ so that

$$\exp(x_1, \dots, x_l) = e^{2\pi i x_1} \oplus \dots \oplus e^{2\pi i x_l}.$$

$L(U(l)) = \{A \in M(l, \mathbb{C}) | A^* + A = O\}$ is decomposed as a direct sum of

$$V_0 = L(T), \text{ where } (x_1, \dots, x_l) = ix_1 \oplus \dots \oplus ix_l,$$

$$V_{rs} = \{zE_{rs} - \bar{z}E_{sr} | z \in \mathbb{C}\} \quad (1 \leq r < s \leq l),$$

where V_{rs} is spanned by $\{E_{rs} - E_{sr}, iE_{rs} + iE_{sr}\}$.

Since the action of $\exp(x_1, \dots, x_l)$ on V_{rs} is given by

$$zE_{rs} - \bar{z}E_{sr} \mapsto wE_{rs} - \bar{w}E_{sr}, \quad w = e^{2\pi i(x_r - x_s)} z,$$

we have

4.4. The roots of $U(l)$ are $\{\pm(x_r - x_s) | 1 \leq r < s \leq l\}$.

$G = SU(l+1)$. Using the coordinate of the maximal torus T^{l+1} of $U(l+1)$, we express the coordinate of $L(T)$ by $(x_1, \dots, x_{l+1})$ with $x_1 + \dots + x_{l+1} = 0$. Then similarly to the above

4.5. The roots of $SU(l+1)$ are $\{\pm(x_r - x_s) | 1 \leq r < s \leq l+1\}$.

$G = Sp(l)$. $V_0 = L(T)$ is similar to $U(l)$.

$L(Sp(l)) = \{A \in M(l, \mathbb{H}) | A^* + A = O\}$ is decomposed as a direct sum of V_0 ,

$$V_r^* = \{zjE_{rr} | z \in \mathbb{C}\} \quad (1 \leq r \leq l),$$

$$V_{rs} = \{zE_{rs} - \bar{z}E_{rs} | z \in \mathbb{C}\} \quad (1 \leq r < s \leq l),$$

$$V_{rs}^* = \{zjE_{rs} + \bar{z}jE_{sr} | z \in \mathbb{C}\} \quad (1 \leq r < s \leq l),$$

and the action of $\exp(x_1, \dots, x_l)$ is given as follows:

$$zjE_{rr} \mapsto e^{2\pi i x_r} zj e^{-2\pi i x_r} E_{rr} = e^{4\pi i x_r} zj E_{rr},$$

$$zE_{rs} - \bar{z}E_{sr} \mapsto wE_{rs} - \bar{w}E_{sr}, \quad w = e^{2\pi i(x_r - x_s)} z,$$

$$zj(E_{rs} + E_{sr}) \mapsto e^{2\pi i(x_r + x_s)} zj(E_{rs} + E_{sr}).$$

Therefore

4.6. *The roots of $Sp(l)$ are*

$$\{\pm 2x_r \text{ for } 1 \leq r \leq l; \pm(x_r - x_s), \pm(x_r + x_s) \text{ for } 1 \leq r < s \leq l\}.$$

$G = SO(2l)$. Since we take the maximal torus T of $SO(2l)$ so that $T = r(T')$ for T' a maximal torus of $U(l)$, we can use the coordinates of $L(T')$ as those of $L(T)$ via r . $L(SO(2l)) = \{A \in M(2l, \mathbb{R}) | A + A^T = 0\}$ has

$$S(r, s) = E_{rs} - E_{sr} \quad (1 \leq r < s \leq l)$$

as a basis. $L(SO(2l))$ is decomposed as a direct sum of

$$V_0 = L(T) \quad (\text{which has } \{S(r, r+l) \text{ for } 1 \leq r \leq l\} \text{ as a basis}),$$

$$V_{rs} = \{x(S(r, s) + S(r+l, s+l)) + y(S(r, s+l) - S(s, r+l)) |$$

$$x, y \in \mathbb{R}\},$$

$$V_{rs}^* = \{x(S(r, s) - S(r+l, s+l)) + y(S(r, s+l) + S(s, r+l)) |$$

$$x, y \in \mathbb{R}\},$$

and the action of T is similar to the case of $Sp(l)$, with $z = x + iy$.

Therefore

4.7. *The roots of $SO(2l)$ are $\{\pm(x_r - x_s), \pm(x_r + x_s) | 1 \leq r < s \leq l\}$.*

$G = SO(2l+1)$. By considerations similar to the case $SO(2l)$, with $SO(2l) \times 1 \subset SO(2l+1)$, we see that $L(SO(2l+1))$ has a direct summand

$$V_r = \{xS(r, 2l+1) + yS(r+l, 2l+1) | x, y \in \mathbb{R}\}$$

in addition to V_0, V_{rs}, V_{rs}^* and that the action of T on V_r is given by $S'(r, z) \mapsto S'(r, e^{2\pi i x} z)$, regarded as

$$S'(r, s) = xS(r, 2l+1) + yS(r+l, 2l+1) \quad \text{for } z = x + iy.$$

Therefore

4.8. *The roots of $SO(2l+1)$ are*

$$\{\pm x_r \text{ for } 1 \leq r \leq l; \pm(x_r - x_s), \pm(x_r + x_s) \text{ for } 1 \leq r < s \leq l\}.$$

$G = \text{Spin}(n)$ for $n = 2l$ and $n = 2l+1$.

THEOREM 4.9. *Let $\tilde{G}$ and G be compact Lie groups in a covering group $(\tilde{G}, p, G)$. Maximal tori $\tilde{T}$ of $\tilde{G}$ and maximal tori T of G correspond bijectively under $p(\tilde{T}) = T$ and $p^{-1}(T) = \tilde{T}$, and $dp: L(\tilde{T}) \rightarrow L(T)$ is an isomorphism.*

If we identify $L(\tilde{T}) = L(T)$ through dp , the roots of $\tilde{G}$ and G coincide and we have $\Gamma(\tilde{G}) \subset \Gamma(G)$ and $\Gamma(G)/\Gamma(\tilde{G}) \cong \text{Ker } p$.

PROOF. First, $dp: L(\tilde{G}) \rightarrow L(G)$, since p is a local homeomorphism. For a torus $\tilde{T}$ of $\tilde{G}$, $T = p(\tilde{T})$ is a torus of G and, identifying $L(\tilde{T}) = L(T)$,

the two sets of linear forms θ_i obtained thus coincide. By Theorem 4.2, $\tilde{T}$ is maximal iff T is maximal, and their roots coincide. Using Theorem 3.21 we see that $p(\tilde{T}) = T$ being maximal implies $\tilde{T} = p^{-1}(T)$. The rest is easy. Q.E.D.

As in the proof of Theorem 4.14 of Chapter II, the image by $\exp$ of the segment joining the origin of $L(T)$ and $(a_1, \dots, a_l) \in \mathbb{Z}^l = \Gamma(SO(n))$ represents a $\sum a_i$ multiple of the generator of $\pi_1(SO(n)) \cong \mathbb{Z}_2$, and

4.10. $\Gamma(\text{Spin}(n))$ consists of the points of $\Gamma(SO(n))$ such that the sum of the coordinates is even.

Regular elements and singular elements. For an element g of G we have $N(g) = Z(g)$, which is a closed subgroup of G and hence a compact Lie group. We make the following definition related to this.

DEFINITION. An element g of G is said to be *regular* if $\dim Z(g) = \text{rank } G$, and *singular* otherwise.

Any element of G is contained in at least one maximal torus (Theorem 3.15) and so we always have $\dim Z(g) \geq \text{rank } G$. Therefore

4.11. $G \ni g$ is singular iff $\dim Z(g) > \text{rank } G$.

LEMMA 4.12. The identity component $Z(g)_0$ of the centralizer of an element g of G is the union of maximal tori of G containing g .

PROOF. Obviously $Z(g)_0$ contains all the maximal tori of G containing g . Conversely, since $Z(g)_0$ is a compact, connected Lie group, there is a maximal torus S of $Z(g)_0$ containing an arbitrary element x . Applying Lemma 3.19 to S and g , we see the existence of a maximal torus of G containing x . Q.E.D.

THEOREM 4.13. g is regular iff g is contained in just one maximal torus. g is singular iff g is contained in more than one maximal torus.

PROOF. If g is contained in only one maximal torus T , then $Z(g)_0 = T$ implies $\dim Z(g) = \dim Z(g)_0 = \dim T = \text{rank } G$ so g is regular.

Suppose that g is contained in maximal tori T_1 and T_2 such that $T_1 \neq T_2$. If $L(T_1) = L(T_2)$, then we have $T_1 = T_2$; hence $L(T_1) \neq L(T_2)$. Since $T_1 \cup T_2 \subset Z(g)$ by Lemma 4.12, $L(Z(g)) \supset L(T_1) + L(T_2)$. Thus $\dim Z(g) > \dim L(T_1) = \text{rank } G$ so g is singular. Q.E.D.

Hereafter, T is a maximal torus of G and $l = \text{rank } G = \dim T$ unless stated otherwise.

NOTATION. $U_r = \text{Ker}(\bar{\theta}_r: T \rightarrow \mathbb{R}/\mathbb{Z})$, where $\bar{\theta}_r$ is a root of G for $i = 1, 2, \dots, m$.

As $\bar{\theta}_r \neq 0$, we have $\dim U_r = l - 1$ and $(U_r)_0$ is a torus. $U_r/(U_r)_0$ is a cyclic group, since it is a finite subgroup of a one-dimensional torus $T/(U_r)_0$. By Lemma 3.4 we have

4.14. U_r is a monogenic $(l - 1)$ -dimensional closed subgroup of T .

U_r is not necessarily connected. (Example: $U_1 = \{0, 1/2 \pmod{1}\}$ in $Sp(1)$.)

THEOREM 4.15. *If an element t of T belongs to exactly μ pieces of U_r ,*

$$\dim Z(t) = l + 2\mu.$$

PROOF. Let $V = \{X \in L(G) | Ad(t)X = X\}$. By Theorem 4.1 we have

$$V = V_0 \oplus \sum_{i \in U_r} V_i, \text{ from which follows } \dim V = l + 2\mu.$$

It suffices to show $V = L(Z(t))$. Since t commutes with an element of $Z(t)$, A_t acts on $Z(t)$ and so $Ad(t)$ acts on $L(Z(t))$ as the identity map. Hence $L(Z(t)) \subset V$.

If $X \in V$, then $t \exp(\alpha X) t^{-1} = \exp(\alpha Ad(t)X) = \exp(\alpha X)$ for $\alpha \in \mathbb{R}$. Therefore $\exp(\alpha X) \in Z(t)$, hence $X \in L(Z(t))$. Q.E.D.

COROLLARY 4.16. *An element t of T is regular iff t does not belong to any U_r .*

An element t of T is singular iff t belongs to some U_r .

Consider the adjoint representation of a Lie group G :

$$Ad: G \rightarrow GL(L(G)) \text{ given by } Ad(g) = d_e A_g, A_g(t) = gxg^{-1}.$$

$g \exp(\alpha X) g^{-1}$ is a one-parameter subgroup with the initial vector $Ad(g)X$ for $g \in G$ and for a one-parameter subgroup $\exp(\alpha X)$ with $\alpha \in \mathbb{R}$. A one parameter subgroup is determined by its initial vector and hence

$$g(\exp X)g^{-1} = \exp(Ad(g)X), \quad X \in L(G), \quad g \in G. \quad (4.17)$$

THEOREM 4.18. *The center $Z(G)$ of a compact, connected Lie group G is given by:*

(1) $Z(G) = \text{Ker } Ad = \{g \in G | Ad(g) = 1\}$. (This is true for a connected Lie group G .)

(2) $Z(G) = \bigcap_{r=1}^m U_r = \{t \in T | \bar{\theta}_r(t) = e \text{ (the unit) } (r = 1, 2, \dots, m)\}$.

PROOF. (1) $g \in Z(G)$ implies $A_g = 1_G$. Thus $Ad(g) = 1$ which is equivalent to $g \in \text{Ker } Ad$. Conversely, if $Ad(g) = 1$, then $g(\exp X) = (\exp X)g$ for all $X \in L(G)$ by (4.17). Since $\exp$ is a local homeomorphism and since a connected G is generated by the neighborhood of the identity, we have $gx = xg$ for any $x \in G$. Thus $g \in Z(G)$.

(2) First, $T \subset G$ and hence $Z(G) \subset Z(T) = T$. Therefore $Ad(t)$ is given by Theorem 4.1, $Ad(t) = 1$ is equivalent to $\bar{\theta}_r(t) = e$ for $r = 1, 2, \dots, m$. It follows from (1) that $Z(G) = \bigcap_{r=1}^m U_r$. Q.E.D.

Weyl groups, roots, and weights

DEFINITION. A linear form (a homomorphism) $\lambda: L(T) \rightarrow \mathbf{R}$, which maps the integer lattice $\Gamma(G)$ to $\mathbf{Z}$, is called a *weight* (of G).

Of course roots $\pm\theta_r: L(T) \rightarrow \mathbf{R}$ are weights.

Since $L(T)$ is spanned by $\Gamma = \Gamma(G)$ over $\mathbf{R}$, $\lambda|_{\Gamma}$ determines λ and $\bar{\lambda}: T (= L(T)/\Gamma) \rightarrow \mathbf{R}/\mathbf{Z}$ induced by λ . Also λ is identified with $\bar{\lambda}_*: H_1(T) \rightarrow H_1(\mathbf{R}/\mathbf{Z}) \cong \mathbf{Z}$ through the isomorphisms $H_1(T) \cong \pi_1(T) \xrightarrow{\partial} \pi_0(\Gamma) \cong \Gamma$. Furthermore, it is considered as $\bar{\lambda}_* \in H^1(T)$ by the isomorphism $H^1(T) \cong \text{Hom}(H_1(T), \mathbf{Z})$. From these we have

$$\begin{aligned} \{\text{the weights of } G\} &= \text{Hom}(\Gamma, \mathbf{Z}) \\ &= \text{Hom}(T, \mathbf{R}/\mathbf{Z}) \quad (\text{identification}) \\ &= H^1(T). \end{aligned} \quad (4.19)$$

Let $\varphi: T \rightarrow T$ be an element of the Weyl group $W(G) = N(T)/T$. The action of φ on a weight λ is defined by the following:

$$\varphi(\lambda) = \lambda \circ d\varphi^{-1}: L(T) \rightarrow L(T) \rightarrow \mathbf{R}. \quad (4.20)$$

Obviously, $(\varphi\varphi')(\lambda) = \varphi(\varphi'(\lambda))$, and

4.20'. *The Weyl group induces permutations of weights.*

THEOREM 4.21. *The action of the Weyl group maps a root to a root.*

PROOF. Let $\varphi = nT$ with $n \in N(T)$. If one puts $\varphi^*(f) = \text{Ad}(n)^{-1} \circ f \circ \text{Ad}(n)$ in the diagram

$$\begin{array}{ccc} T & \xrightarrow{\varphi^{-1}} & T \\ \downarrow \text{Ad} & & \downarrow \text{Ad} \\ \text{Aut } L(G) & \xrightarrow{\varphi^*} & \text{Aut } L(G) \end{array}$$

one easily sees that it is commutative. For the orthogonal decomposition $L(G) = V_0 \oplus \sum_r V_r$ in Theorem 4.1(1), $\text{Ad}(n)^{-1}V_0 = V_0$ and $\text{Ad}(n)^{-1}V_r$ give an orthogonal decomposition of $L(G)$, which is invariant under the action of $\varphi^* \circ \text{Ad} = \text{Ad} \circ \varphi^{-1} \varphi(\theta_r) = \theta_r \circ \varphi^{-1}$ is a root by Theorem 4.1(2). Q.E.D.

LEMMA 4.22. *For each U_r , there exists an element φ_r of $W(G)$ that fixes every element of U_r and is not the identity on T .*

PROOF. Let u be a generator of U_r . Then $U_r \subset T \subset Z(u)_0$. Since an element of $Z(u)_0$ commutes with u and hence with any element of U_r , U_r is a closed normal subgroup of $Z(u)_0$. Applying Lemma 3.25 to $G = Z(u)_0$ and $H = U_r$, one has that T/U_r is a maximal torus of $Z(u)_0/U_r$ and $W(Z(u)_0/U_r) \cong W(Z(u)_0)$.

Since T/U_r is one-dimensional, the dimension of $Z(u)_0/U_r$ is 1 or 3 by Lemma 3.26, and if it is 3, then $W(Z(u)_0/U_r) \cong \mathbf{Z}_2$. By Theorem 4.15, $u \in U_r$ implies $\dim Z(u)_0 \geq l+2$, whence $\dim Z(u)_0 = l+2$ and $W(Z(u)_0) \cong \mathbf{Z}_2$. If we denote the generator of $W(Z(u)_0)$ by φ_r , it is easily seen to satisfy the properties of the lemma. Q.E.D.

COROLLARY 4.23. φ_r is represented by an element of $N(T) \cap Z(U_r)_0$ which does not lie in T .

COROLLARY 4.24. The number of components of U_r is at most 2.

PROOF. The action of φ_r on $T/(U_r)_0 \cong U(1)$ is not trivial. Thus $\varphi_r(t) = t^{-1}$, which implies that the number of fixed points is 2. Since $U_r/(U_r)_0$ is invariant elementwise under φ_r , it consists of at most 2 points. Q.E.D.

THEOREM 4.25. If $s \neq r$, then θ_s and θ_r are linearly independent.

PROOF. If one replaces U_r with $(U_r)_0$ and u with a generator v of $(U_r)_0$ in the proof of Lemma 4.22, one obtains $\dim Z(v) = l+2$ quite similarly. Then Theorem 4.15 shows that $v \in U_r$ and $v \notin U_s$ for $s \neq r$. $\bar{\theta}_r(U_r) = e$ implies $\theta_r(L(U_r)) \in \mathbf{Z}$ and hence $\theta_r(L(U_r)) = 0$. On the other hand, $\theta_s(\exp^{-1}(v)) \neq 0$ for $\exp^{-1}(v) \in U_r$. Therefore θ_s is not a scalar multiple of θ_r . Similarly, θ_r is not a scalar multiple of θ_s , and the theorem is proved. Q.E.D.

COROLLARY 4.26. If $s \neq r$, then $\dim(U_r \cap U_s) = l-2$.

The infinitesimal diagram and the Weyl group

DEFINITION. $D'(G) = \bigcup_{r=1}^m L(U_r)$ is called an *infinitesimal diagram* and each $L(U_r)$ a *hyperplane* of $D'(G)$. Each connected component $\mathfrak{F}$ of $L(T) - D'(G)$ is called a *Weyl chamber*. If the intersection of the closure of a Weyl chamber with a hyperplane $L(U_r)$ has dimension $l-1$, it is called a *wall* of the Weyl chamber.

A Weyl chamber, its closure, and a wall are convex sets that are not bounded, and if a point x lies in one of them, so does tx for $t > 0$. The union of the walls of a Weyl chamber is the boundary of the Weyl chamber. A Weyl chamber $\mathfrak{F}$ is expressed as

$$\mathfrak{F} = \{t \in L(T) \mid \varepsilon_r \theta_r(t) > 0; \quad r = 1, 2, \dots, m\}$$

with suitable $\varepsilon_r = \pm 1$. Note that with some ε_r the set can be empty.

We consider the action of $\varphi_r \in W(G)$ in Lemma 4.22 on $L(T)$. It is an orthogonal transformation that fixes a hyperplane $L(U_r)$ and is not the identity transformation. Hence

4.27. The action of φ_r is the reflection in the hyperplane $L(U_r)$.

By Theorem 4.21

4.28. *An element of the Weyl group induces a permutation of Weyl chambers.*

LEMMA 4.29. (1) *If $X \in L(T)$ and $\psi \in W(G)$ satisfy $\psi(X) = X$ and $\psi \neq 1$ (the identity), X lies in some hyperplane $L(U_r)$.*

(2) *If an element ψ of $W(G)$ satisfies $\psi\mathfrak{F} = \mathfrak{F}$ for some Weyl chamber $\mathfrak{F}$, then $\psi = 1$.*

PROOF. (1) If $\psi = nT$ with $n \in N(T)$, $\exp(\alpha X) = \psi(\exp(\alpha X)) = n(\exp(\alpha X))n^{-1}$ then n commutes with every element of the one parameter subgroup $H = \{\exp(\alpha X) | \alpha \in \mathbb{R}\}$. By Lemma 3.19 there is a maximal torus T' that contains n and H . $\psi \neq 1$ implies $n \notin T$ so $T \neq T'$. Since $\exp X \in H \subset T \cap T'$, by Theorem 4.13 and Corollary 4.16 $\exp X \in U_r$ for some r so $X \in L(U_r)$.

(2) We have $\psi^q = 1$ for some positive integer q , since $W(G)$ is a finite group. Since $\mathfrak{F}$ is a convex set, $X \in \mathfrak{F}$ implies $\psi^i X \in \mathfrak{F}$ and $\frac{1}{q} \sum_{i=1}^q \psi^i X \in \mathfrak{F}$. We have $\psi = 1$ by (1), since $\frac{1}{q} \sum \psi^i X$ is invariant under ψ and does not lie in any $L(U_r)$. Q.E.D.

THEOREM 4.30. (1) *$W(G)$ permutes the Weyl chambers simply transitively (transitively and effectively).*

(2) *For any Weyl chamber $\mathfrak{F}$, $W(G)$ contains the reflections in the walls of $\mathfrak{F}$ and these reflections generate $W(G)$.*

(3) *The stabilizer $W_X = \{\varphi \in W(G) | \varphi X = X\}$ of $X \in L(T)$ permutes simply transitively the Weyl chambers whose closures contain X .*

(4) *If $X \in \mathfrak{F} - \mathfrak{F}'$ for a Weyl chamber $\mathfrak{F}$, then the above W_X contains the reflections in the walls containing X of $\mathfrak{F}$, and these reflections generate W_X .*

PROOF. (1) and (2) are obtained by putting $X = 0$ in (3) and (4) respectively.

(3) Effectiveness follows from Theorem 4.29(2). Let $X \in \mathfrak{F}_0 \cap \overline{\mathfrak{F}'}$ for two Weyl chambers $\mathfrak{F}_0$ and $\mathfrak{F}'$ and consider $X_0 \in \mathfrak{F}_0$ and $X' \in \mathfrak{F}'$. Since $L(U_r) \cap L(U_s)$ is of dimension $l-2$ for $r \neq s$, by moving slightly a line joining X_0 and X' we can join X_0 and X' by a polygonal line that satisfies the following conditions:

(a) it does not meet any $L(U_r) \cap L(U_s)$,

(b) it does not meet any $L(U_r)$ that does not contain X ,

(c) it is not tangential to any $L(U_r)$.

If the polygonal line crosses $L(U_{r_1}), \dots, L(U_{r_k})$ successively to get from $\mathfrak{F}_0$ to $\mathfrak{F}_1, \dots$, to $\mathfrak{F}_k = \mathfrak{F}'$, then $\varphi_{r_k} \cdots \varphi_{r_2} \varphi_{r_1}$ maps $\mathfrak{F}_0$ via $\mathfrak{F}_1, \dots, \mathfrak{F}_{k-1}$ to $\mathfrak{F}'$. Thus W_X acts transitively.

(4) The first half is clear from 4.27. If we put $\psi(\mathfrak{F}) = \mathfrak{F}'$ for $\psi \in W_X$ and take $\varphi_{r_k} \cdots \varphi_{r_1}$, which maps $\mathfrak{F}$ to $\mathfrak{F}'$ as in (3), then $\psi^{-1} \varphi_{r_k} \cdots \varphi_{r_1}$ maps $\mathfrak{F}$ to $\mathfrak{F}$ and hence we see that $\psi = \varphi_{r_k} \cdots \varphi_{r_1}$ by Lemma 4.29(2). Observe

that $L(U_{r_i})$ is not necessarily a wall of $\mathfrak{F}$, although it contains X . We show by induction on s that $\varphi_{r_s} \cdots \varphi_{r_1}$ ($1 \leq s \leq k$) is expressed as a product of reflections in the walls containing X of $\mathfrak{F}$. If $s = 1$, this is obvious. Suppose that it holds for $s < k$. In (3), $L(U_{r_{s+1}})$ is a wall of $\mathfrak{F}_s$ and $(\varphi_{r_s} \cdots \varphi_{r_1})^{-1}$ maps $\mathfrak{F}_s$ to $\mathfrak{F}$ and $L(U_{r_{s+1}})$ to a wall $L(U_m)$ containing X of $\mathfrak{F}$. Then $\varphi_{r_{s+1}}$ and $\varphi_{r_s} \cdots \varphi_{r_1} \varphi_m (\varphi_{r_s} \cdots \varphi_{r_1})^{-1}$ coincide, since both of them map $\mathfrak{F}_s$ to $\mathfrak{F}_{s+1}$. Therefore $\varphi_{r_{s+1}} \varphi_{r_s} \cdots \varphi_{r_1} = \varphi_{r_s} \cdots \varphi_{r_1} \varphi_m$ is expressed in terms of reflections in the walls containing X of $\mathfrak{F}$. Thus (4) is proved by the induction. Q.E.D.

COROLLARY 4.31. *The intersection $W(G) \cdot X \cap \overline{\mathfrak{F}}$ of the orbit by $W(G)$ of a point X of $L(T)$ with the closure of each Weyl chamber $\mathfrak{F}$ consists of exactly one point.*

PROOF. If $X \in \overline{\mathfrak{F}}_0$, by taking ψ such that $\psi \mathfrak{F}_0 = \mathfrak{F}$, we have $\psi(X) \in W(G) \cdot X \cap \overline{\mathfrak{F}}$. Let X and $\varphi(X) \in \overline{\mathfrak{F}}$. $X \in \varphi^{-1} \overline{\mathfrak{F}}$ and W_X acts transitively on these $\mathfrak{F}$ and $\varphi^{-1} \mathfrak{F}$, and hence there is $\sigma \in W_X$ such that $\varphi^{-1} \mathfrak{F} = \sigma^{-1} \mathfrak{F}$. $\varphi^{-1} \mathfrak{F} = \sigma^{-1} \mathfrak{F}$ implies $\varphi^{-1} = \sigma^{-1}$ and hence $\varphi = \sigma$. Further, since $\sigma(X) = X$, we have $\varphi(X) = \sigma(X) = X$, that is, $W(G) \cdot X \cap \overline{\mathfrak{F}}$ is a point. Q.E.D.

§5. The fundamental group and the decomposition of a compact connected Lie group

The Stiefel diagram and the extended Weyl group. We will use the following notation concerning a root $\theta_r: L(T) \rightarrow \mathbf{R}$ of a compact connected Lie group G .

NOTATION. $L_r = \theta_r^{-1}(\mathbf{Z}) = \exp^{-1}(U_r)$,

$L_{r,n} = \theta_r^{-1}(n)$, $n \in \mathbf{Z}$,

$\varphi_{r,n}: L(T) \rightarrow L(T)$ is the reflection in $L_{r,n}$.

In particular, $L_{r,0} = L(U_r)$ and $\varphi_{r,0} = \varphi_r \in W(G)$.

DEFINITION. $D(G) = \bigcup_{r=1}^m L_r = \bigcup_{r=1}^m \bigcup_{n \in \mathbf{Z}} L_{r,n}$ is called the *Stiefel diagram*, each $L_{r,n}$ is the *hyperplane* of the Stiefel diagram, and each connected component Δ of $L(T) - D(G)$ is the *Weyl cell*.

The Weyl cell is an open convex set in $L(T)$ and its boundary is a part of $D(G)$. The intersection $\overline{\Delta} \cap L_{r,n}$ of the closure of a Weyl cell Δ with a hyperplane $L_{r,n}$ of $D(G)$ is called a *face* of the Weyl cell Δ , if it is of dimension $l-1$ (i.e., it contains an interior point), and $L_{r,n}$ is called a *wall* of Δ . The boundary of a Weyl cell is the union of its faces.

DEFINITION. The congruent transformation group generated by the reflections $\varphi_{r,n}$ in the hyperplanes $L_{r,n}$ of the Stiefel diagram $D(G)$ is called the *extended Weyl group*, denoted by $\overline{W}(G)$.

THEOREM 5.1. *An element w of the extended Weyl group $\overline{W}(G)$ is a congruent transformation that maps the Stiefel diagram $D(G)$ to $D(G)$. Therefore, $w \in \overline{W}(G)$ maps a hyperplane of $D(G)$ to that of $D(G)$.*

PROOF. The diagram

$$\begin{array}{ccc} L(T) & \xrightarrow{\varphi_{r,n}} & L(T) \\ \downarrow \exp & & \downarrow \exp \\ T & \xrightarrow{\varphi} & T \end{array}$$

is commutative for $\varphi: T \rightarrow T$ induced by $\varphi_{r,n}$. Since φ is given by an inner automorphism, it maps a singular element to some singular element by Theorem 4.13, that is, $\varphi(\bigcup U_i) = \bigcup U_i$. We have $\varphi_{r,n}(D(G)) = D(G)$, since $\bigcup L_i = \exp^{-1}(\bigcup U_i)$. Therefore a congruent transformation $\varphi_{r,n}$ maps a hyperplane of $D(G)$ to that of $D(G)$. Thus we obtain the theorem, since $\varphi_{r,n}$ generate $\overline{W}(G)$. Q.E.D.

COROLLARY 5.2. An element of $\overline{W}(G)$ gives a congruent transformation of the set of Weyl cells.

LEMMA 5.3. Let L' be the orthogonal complement of $\mathfrak{z} = L(Z(G))$ in $L(T)$ so that $L(T) = \mathfrak{z} \times L'$. Then a Weyl cell Δ of G satisfies $\Delta = \mathfrak{z} \times (\Delta \cap L')$ and $\Delta \cap L'$ is the interior of a bounded convex polyhedron.

PROOF. $\theta_r(\mathfrak{z}) = 0$ implies $\mathfrak{z} \subset L_{r,n}$ so $L_{r,n} = \mathfrak{z} \times (L_{r,n} \cap L')$. The connectedness of $\mathfrak{z}$ implies $\Delta = \mathfrak{z} \times (\Delta \cap L')$. By Theorem 4.18(2) one easily obtains

$$\mathfrak{z} = L(Z(G)) = \{X \in L(T) | \theta_r(X) = 0\}. \quad (5.3')$$

If $\Delta \cap L'$ is not bounded, then there is a half-line A contained in it such that $\theta_r(A) = 0$ for any r , or that there is some r such that $\theta_r|_A$ is not identically 0. In the former case we have $A \subset \mathfrak{z}$, which is a contradiction. Also in the latter case, $\theta_r|_A$ takes an integer value at some point, since the values of it are unbounded, and so the point lies in U_r , which is a contradiction. Thus $\Delta \cap L'$ is bounded. Q.E.D.

THEOREM 5.4. The extended Weyl group $\overline{W}(G)$ permutes the Weyl cells of $D(G)$ simply transitively.

PROOF. Two Weyl cells, which have a face in common, are mapped to each other by the reflection in the hyperplane containing the face. Let U be the union of a Weyl cell and the interiors of the faces, then U is arcwise connected, since $L(T) - U$ is of dimension $\leq l-2$. Thus for two given Weyl cells Δ and Δ' , there is a sequence of Weyl cells $\Delta = \Delta_0, \Delta_1, \dots, \Delta_n = \Delta'$ such that Δ_i and Δ_{i+1} have a face in common. Therefore Δ is mapped to Δ' by a product of the reflections in the hyperplanes containing these faces, and the action of $\overline{W}(G)$ is transitive.

Consider an element $\varphi \in \overline{W}(G)$ that maps every Weyl cell in itself. Since $\varphi|_{\mathfrak{z}} = 1$, we have $\varphi|_{\Delta} = 1_{\mathfrak{z}} \times (\varphi|_{\Delta \cap L'})$. Since every $\Delta \cap L'$ is congruent, its diameter $d > 0$ is constant, and for any $X \in \Delta$ and hence for any $X \in L(T)$, the distance between X and $\varphi(X)$ is less than or equal to d .

Such a congruent transformation φ can be nothing but a translation. On the other hand, if Δ and Δ' have a face in common, the face is mapped to itself. For, Δ and Δ' are mapped to themselves respectively. It follows that the vertices of $\Delta \cap \Delta'$ are mapped to themselves. Thus $\varphi = 1$ and so the action of $\overline{W}(G)$ is effective. Q.E.D.

The inverse root and translations. For simplicity, we denote the dual space of $L(T)$ by

$$L(T)^* = \text{Hom}(L(T), \mathbb{R}).$$

Roots and weights are elements of $L(T)^*$. Since $L(T)$ is of finite dimension,

$$L(T) \cong \text{Hom}(L(T)^*, \mathbb{R}).$$

DEFINITION. We put $\theta_r^* = \varphi_{r,1}(\mathbf{0}) \in L(T)$ the image of the origin $\mathbf{0}$ by $\varphi_{r,1}$ (the reflection in the hyperplane $L_{r,1}$) for a root θ_r of G and call it the *inverse root* of θ_r .

As is easily seen, for $\varphi_r \in W(G)$ we have

5.5. θ_r^* is uniquely determined by the following:

$$\varphi_r(\theta_r^*) = -\theta_r^* \quad \text{and} \quad \theta_r(\theta_r^*) = 2.$$

THEOREM 5.6. (1) $\varphi_r(X) = X - \theta_r(X)\theta_r^*$ for any $X \in L(T)$.

(2) $\varphi_r(\theta_i) = \theta_i - \theta_i(\theta_r^*)\theta_r$ (which is also a root) for a root $\theta_i \in L(T)^*$.

PROOF. (1) X is decomposed as $X = a\theta_r^* + Y$ with $a \in \mathbb{R}$ and $Y \in L_{r,0}$, and we have $\varphi_r(X) = -a\theta_r^* + Y = X - 2a\theta_r^*$, since $\varphi_r(Y) = Y$. On the other hand $\theta_r(X) = a\theta_r(\theta_r^*) = 2a$ as $\theta_r(Y) = 0$. Thus $\varphi_r(X) = X - \theta_r(X)\theta_r^*$.

(2) Since $\varphi_r = \varphi_r^{-1}$, we have $(\varphi_r(\theta_i))X = \theta_i(\varphi_r(X)) = \theta_i(X - \theta_r(X)\theta_r^*) = (\theta_i - \theta_i(\theta_r^*)\theta_r)X$ by (4.20). Therefore $\varphi_r(\theta_i) = \theta_i - \theta_i(\theta_r^*)\theta_r$. Q.E.D.

NOTATION. $\Gamma_0 = \Gamma_0(G)$ = (the subgroup generated by inverse roots $\theta_r^* \in L(T)$).

From the commutative diagram in the proof of Theorem 5.1 we have

$$\exp(\theta_r^*) = \exp(\varphi_{r,1}(\mathbf{0})) = \varphi_r(\exp \mathbf{0}) = \varphi_r(e) = e.$$

Hence

$$\Gamma_0 \subset \Gamma = \Gamma(G) = \exp^{-1} e. \quad (5.7)$$

An element Y of Γ or Γ_0 acts on $L(T)$ by the translation

$$X \mapsto X + Y \quad \text{for } X \in L(T)$$

and hence Γ or Γ_0 is regarded as the congruent transformation group of $L(T)$.

NOTATION. $\overline{W}(G)$ = (the congruent transformation group generated by Γ and $\overline{W}(G)$).

THEOREM 5.8. (1) An element of $\overline{W}(G)$ maps (hyperplanes of) $D(G)$ to (hyperplanes of) $D(G)$.

(2) By the commutative diagram

$$\begin{array}{ccc} L(T) & \xrightarrow{w} & L(T) \\ \downarrow \exp & & \downarrow \exp \\ T & \xrightarrow{\varphi} & T \end{array}$$

a homomorphism

$$\pi: \overline{W}(G) \rightarrow W(G) \quad \text{with} \quad \pi(w) = \varphi$$

is defined, with $\Gamma = \text{Ker } \pi$.

(3) $\overline{W}(G) = \Gamma \cdot W(G)$ (a semidirect product).

(4) $w(Y) \equiv Y \pmod{\Gamma_0}$ for $w \in \overline{W}(G)$ and $Y \in \Gamma \subset L(T)$.

(5) $\overline{W}(G) = \Gamma_0 \cdot W(G)$ (a semidirect product).

PROOF. (1) Since $\exp(Y) = e$ for $Y \in \Gamma$, $X \in L_r$ implies $\exp(X) \in U_r$. Thus $\exp(X + Y) = \exp(X) \in U_r$ and $X + Y \in L_r$. Therefore an element of Γ maps $D(G)$ to $D(G)$. This is already known by Theorem 5.1 for an element of $\overline{W}(G)$.

(2) The unique existence of φ is known for any element of $\overline{W}(G)$. Obviously $\pi(w) = 1_T$ for $w \in \Gamma$, and $\Gamma \subset \text{Ker } \pi$. It is also obvious that π is defined to be a homomorphism for any element of $\overline{W}(G)$, which is generated by Γ and $\overline{W}(G)$. If $w \in \text{Ker } \pi$, then $\pi(w) = 1_T$. $\exp(w(0)) = \exp(0) = e$ implies $Y = w(0) \in \Gamma$. If one regards Y as a translation, we have $\pi(Y) = 1_T$. We have $w = Y \in \Gamma$, since w and Y are both covering transformations and have an identical value at 0 (§1 of Chapter II). Thus $\Gamma = \text{Ker } \pi$.

(3) By assigning $d\varphi: L(T) \rightarrow L(T)$ to $\varphi: T \rightarrow T$ ($\varphi \in W(G)$) we obtain a cross section of π in the exact sequence $1 \rightarrow \Gamma \rightarrow \overline{W}(G) \xrightarrow{\pi} W(G)$, and hence $\overline{W}(G)$ is a semidirect product of a normal subgroup $\Gamma = \text{Ker } \pi$ and $W(G)$.

(4) Composition of the reflections in the two hyperplanes that are parallel is a translation, and $\varphi_{r,n}\varphi_r(0) = \varphi_{r,n}(0) = n\theta_r^*$, whence

$$n\theta_r^* = \varphi_{r,n}\varphi_r \in \overline{W}(G) \quad (n \in \mathbf{Z}; r = 1, 2, \dots, m). \quad (5.9)$$

By applying Theorem 5.6(1) to $\varphi_{r,n} = (n\theta_r^*)\varphi_r^{-1} = (n\theta_r^*)\varphi_r$, one has

$$\varphi_{r,n}(X) = X + (n - \theta_r(X))\theta_r^* \quad (X \in L(T)). \quad (5.10)$$

$Y \in \Gamma$ implies $\theta_r(Y) \in \mathbf{Z}$ and hence $\varphi_{r,n}(Y) \equiv Y \pmod{\Gamma_0}$. Since $\varphi_{r,n}$ generates $\overline{W}(G)$, (4) is obtained.

(5) Since $\text{Ker } \pi = \text{Ker } \pi \cap \overline{W}(G) = \Gamma \cap \overline{W}(G)$ for $\pi = \pi|_{\overline{W}(G)}$, we have $\overline{W}(G) = (\Gamma \cap \overline{W}(G)) \cdot W(G)$ (a semidirect product) similarly to (3). Thus it suffices to show

$$\Gamma_0 = \Gamma \cap \overline{W}(G).$$

By (5.9), $\theta_r^* = \varphi_{r,1} \varphi_r \in \overline{W}(G)$ and so $\Gamma_0 \subset \overline{W}(G)$. Using (5.7), $\Gamma_0 \subset \Gamma \cap \overline{W}(G)$. By (5.10) an element w of $\overline{W}(G)$ maps Γ_0 to Γ_0 . Furthermore, if $w \in \Gamma$, then it is a translation and so $w \in \Gamma_0$. Thus $\Gamma \cap \overline{W}(G) \subset \Gamma_0$. Q.E.D.

COROLLARY 5.11. $\Gamma/\Gamma_0 \cong \overline{W}(G)/\overline{W}(G)$ (a bijection).

The fundamental group of a compact, connected Lie group

NOTATION. $G_R = \{\text{regular elements in } G\}$.

$T_R = G_R \cap T = \{\text{regular elements in } T\}$,

$L(T)_R = \exp^{-1}(T_R) = L(T) - D(G)$.

LEMMA 5.12. $G - G_R$ is the union of the images under C^∞ -maps of a finite number of differentiable manifolds of dimension $\leq \dim G - 3$.

PROOF. If u is a generator of U_r , then $\dim Z(u) \geq l + 2$ by Theorem 4.15. Every element of $Z(u)$ commutes with a power of u and hence with all elements of U_r , and there is a C^∞ -map $f_r: (G/Z(u)) \times U_r \rightarrow G$ defined by

$$f_r(g \cdot Z(u), t) = gtg^{-1}$$

so that $\text{Im } f_r = \bigcup_{g \in G} gU_r g^{-1}$. We see that $G - G_R = \bigcup_{r=1}^m \text{Im } f_r$ by Theorem 4.13 and the conjugacy of maximal tori. We have

$$\dim((G/Z(u)) \times U_r) = \dim G - \dim Z(u) + (l - 1) \leq \dim G - 3. \quad \text{Q.E.D.}$$

COROLLARY 5.13. $\pi_1(G_R) \cong \pi_1(G)$.

PROOF. If the image of an arbitrary C^∞ -map $h: I \rightarrow G$ (or $H: I \times I \rightarrow G$) meets $G_S = G - G_R$ at a point x , then for dimensional reasons, there is a direction along which we can move h (or H) slightly so that it does not meet G_S in a neighborhood of x . Using this we can take a loop representing an element of $\pi_1(G)$ and its homotopy inside G_R . (The idea is used in the proof of Theorem 4.20 and Lemma 5.19.) Q.E.D.

Consider the following two maps:

$$\bar{f}: (G/T) \times T \rightarrow G \quad \text{defined by} \quad \bar{f}(gT, t) = gtg^{-1},$$

$$f: (G/T) \times L(T) \rightarrow G \quad \text{defined by} \quad f = \bar{f} \circ (1 \times \exp).$$

Define the action of the Weyl group $W(G)$ on $(G/T) \times T$ by

$$\varphi(gT, t) = (gn^{-1}T, \varphi(t)) \quad \text{for } \varphi = [n] \in W(G) \text{ and } n \in N(T)$$

and the action of the group $\overline{W}(G)$ on $(G/T) \times L(T)$ by

$$w(gT, X) = (gn^{-1}T, w(X)) \quad \text{for } w \in \overline{W}(G) \text{ and } \pi(w) = [n].$$

Then we easily obtain the following identities:

$$\bar{f}(\varphi(gT, t)) = \bar{f}(gT, t) \quad \text{and} \quad f(w(gT, X)) = f(gT, X). \quad (5.14)$$

LEMMA 5.15. (1) $\bar{f}$ gives the projection of a $W(G)$ -bundle $\bar{f}_R: (G/T) \times T_R \rightarrow G_R$.

(2) f gives the projection of a $\overline{W}(G)$ -bundle $f_R: (G/T) \times L(T)_R \rightarrow G_R$.

PROOF. (1) $G = \bigcup gTg^{-1}$ and $gG_Rg^{-1} = G_R$ imply $G_R = \bigcup gT_Rg^{-1}$. Thus $\bar{f}_R$ is an epimorphism. If $\varphi(gT, t) = (gT, t)$, then $tn = nt$. Since t is a regular element, $n \in T$ implies $\varphi = 1$. Therefore $W(G)$ acts without fixed points, and it is seen by (5.14) that $\bar{f}_R$ induces a surjection of the quotient space $(G/T) \times_{W(G)} T_R$ onto G_R .

If $\bar{f}(gT, t) = \bar{f}(g_1T, t_1)$, then $g_1^{-1}gtg^{-1}g_1 = t_1 \in T$. By Theorem 3.18 there is $n \in N(T)$ such that $t_1 = ntn^{-1}$, and hence $(g^{-1}g_1n)t = t_1(g^{-1}g_1n)$. t is regular so $g^{-1}g_1n \in T$. If one puts $\varphi = [n]$, then $\varphi(gT, t) = (g_1T, t_1)$. As a result, we obtain a surjection $(G/T) \times_{W(G)} T_R \rightarrow G_R$ of manifolds, which is a homeomorphism. Since $W(G)$ is finite, $\bar{f}_R$ is the projection of a $W(G)$ -bundle.

(2) By (1) and the fact that $1 \times \exp: (G/T) \times L(T)_R \rightarrow (G/T) \times T_R$ is the projection of a $\Gamma(G)$ -bundle, we see that $f_R = \bar{f}_R \circ (1 \times \exp)$ is the projection of a $\overline{W}(G) = \Gamma(G) \cdot W(G)$ -bundle. Q.E.D.

THEOREM 5.16. G/T is 1-connected.

PROOF. $\pi_0(G) = 0$ implies $\pi_0(G/T) = 0$. Take a point X_0 of $L(T)_R$ and consider the composition

$$h: G/T \xrightarrow{\times X_0} G/T \times L(T)_R \xrightarrow{f_R} G_R \xrightarrow{i} G.$$

Then $h(gT) = gt_0g^{-1}$ with $t_0 = \exp X_0$. We obtain $h \simeq 0$ by considering a path connecting t_0 and the identity in T . On the other hand, $(\times X_0)_*$ is the inclusion into a direct summand, f_{R*} is a monomorphism on π_1 as f_R is a covering map and i_* is a monomorphism by Corollary 5.13; since all of them induce monomorphisms on π_1 , $h_*: \pi_1(G/T) \rightarrow \pi_1(G)$ is also a monomorphism. $h \simeq 0$ implies $h_* = 0$, whence $\pi_1(G/T) = 0$. Q.E.D.

COROLLARY 5.17. $i_*: \pi_1(T) \rightarrow \pi_1(G)$ induced by the inclusion i is an epimorphism.

LEMMA 5.18. $i_*(\partial^{-1}\Gamma_0) = 0$ for the isomorphism $\partial: \pi_1(T) \cong \Gamma$.

PROOF. Consider a path $\tilde{l}: I \rightarrow L(T)$ (given by $\tilde{l}(t) = t\theta_r^*$) connecting the origin and a generator θ_r^* of Γ_0 and consider its image $l: (I, \partial I) \rightarrow (G, e)$ (given by $l(t) = \exp(t\theta_r^*)$) by $\exp$. Then it suffices to show $l \simeq 0$, since $i_*\partial^{-1}(\theta_r^*) = [l]$.

Since $\varphi_{r,1}$ acts on the line segment $\tilde{l}$ as a point symmetry with respect to the middle point $\tilde{l}(1/2) \in L_{r,1}$, we have

$$\varphi_r(l(t)) = l(1-t) \quad \text{and} \quad l(1/2) \in U_r.$$

There is an $n \in N(T) \cap Z(U_r)_0$ by Corollary 4.23 such that $\varphi_r = [n]$. If $h: I \rightarrow Z(U_r)$ is a path connecting $h(0) = n$ and $h(1) = e$, then we have $h(\theta)l(1/2)h(\theta)^{-1} = l(1/2)$ for $0 \leq \theta \leq 1$. Therefore one can define a homotopy $l_\theta: (I, \partial I) \rightarrow (G, e)$ by

$$l_\theta(t) = \begin{cases} l(t) & (0 \leq t \leq 1/2), \\ h(\theta)l(t)h(\theta)^{-1} & (1/2 \leq t \leq 1), \end{cases}$$

so that $l = l_0 \simeq l_1$ and $l_1(t) = l_1(1-t)$, and hence $l_{2-\theta}(t) = l_{2-\theta}(1-t) = l_1(\theta t)$ for $0 \leq t \leq 1/2$ shows $0 = l_2 \simeq l_1$, whence $l \simeq 0$. Q.E.D.

THEOREM 5.19. $i_* \circ \partial^{-1}: \Gamma \cong \pi_1(T) \rightarrow \pi_1(G)$ induces the following isomorphism:

$$\Gamma/\Gamma_0 \cong \pi_1(G).$$

PROOF. By Lemma 5.18, $i_* \circ \partial^{-1}$ induces a homomorphism $\lambda: \Gamma/\Gamma_0 \rightarrow \pi_1(G)$, which is an epimorphism by Corollary 5.17. Thus it suffices to show that $i_*(\partial^{-1}(\gamma)) = 0$ implies $\gamma \in \Gamma_0$ for any $\gamma \in \Gamma$. We first show

5.20. *There is a Weyl cell Δ such that the closure $\bar{\Delta}$ contains the origin and a representative γ of $\gamma \cdot \Gamma_0$.*

In fact, we take a representative γ , which is the closest to the origin. $\theta_r(\gamma)$ is an integer. If $\theta_r(\gamma) > 1$, then $\varphi_{r,1}(\gamma)$ is nearer to the origin than γ . By Theorem 5.8(4), $\varphi_{r,1}(\gamma)$ is also a representative of $\gamma \cdot \Gamma_0$, which contradicts the choice of γ , and hence $\theta_r(\gamma) \leq 1$. Similarly, $\theta_r(\gamma) \geq -1$. Thus $\theta_r(\gamma) = 1, 0$, or -1 . Since the set $\bar{\Delta}$ covers $L(T)$, there is a $\bar{\Delta}$ that meets an interior point of the line segment (or a point) $\overline{0\gamma}$ connecting the origin 0 and γ . Then we have $\overline{0\gamma} \subset \bar{\Delta}$, since the only hyperplane which passes through an interior point of $\overline{0\gamma}$ is the one passing through the origin. Thus (5.20) is proved.

If we take a point P of Δ sufficiently close to 0 and a point Q of Δ sufficiently close to γ , the paths represented by $\overline{0\gamma}$ and $\overline{0P} + \overline{PQ} + \overline{Q\gamma}$ are homotopic; if l_0, l , and l_1 are the images under $\exp$ of $\overline{0P}$, $\overline{PQ}$, and $\overline{Q\gamma}$ respectively, we have $[l_0 l l_1] = i_*(\partial^{-1}\gamma) = 0$, and l_0 and l_1 lie in a contractible neighborhood U_ϵ of e and l lies in G_R . If we put $x_0 = \exp P$ and $y_0 = \exp Q$, then there is a path k connecting x_0 and y_0 in $U_\epsilon \cap G_R$ (Lemma 5.12). U_ϵ is contractible so $k \simeq l_0^{-1} l_1^{-1} \simeq l_0^{-1} (l_0 l l_1) l_1^{-1} \simeq l$, and hence $k \simeq l$ in G_R by $\pi_1(G_R, x_0) \cong \pi_1(G, x_0)$ (Corollary 5.13). Consider the projection

$$f_r: (G/T) \times L(T)_R \rightarrow G_R$$

of the bundle in Lemma 5.15(2). Regard as $L(T)_R \subset (G/T) \times L(T)_R$ by $X \mapsto (T, X)$. As the fibre $\overline{W}(G)$ of f_r has the discrete topology, the liftings $\tilde{k}$ and $\tilde{l}$ of paths k and l with initial point P are unique and the terminal points of $\tilde{k}$ and $\tilde{l} = \overline{PQ}$ coincide by the fact that $k \simeq l$. (In fact,

$\tilde{k}$ and $\tilde{l}$ are paths in $(G/T) \times \Delta$ and $f_R: (G/T) \times \Delta \rightarrow G_R$ is a (universal) covering space.) Taking ε sufficiently small, $\tilde{k}$ must lie in a sufficiently small region and hence the distance between P and Q must be arbitrarily small. If $0 \neq \gamma$, then this cannot happen; so $\gamma = 0 \in \Gamma_0$. Thus the theorem is proved. Q.E.D.

The Lie algebra of a compact Lie group. Consider the adjoint representations

$$Ad: G \rightarrow GL(\mathfrak{g}), \quad ad: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$$

of a connected Lie group G and its Lie algebra $\mathfrak{g} = L(G)$. Here observe that the following hold (§5 of Chapter II):

$$Ad(\exp X) = \exp(adX), \quad [X, Y] = \lim_{t \rightarrow 0} \frac{1}{t} \{Ad(\exp tX)Y - Y\}. \quad (5.21)$$

LEMMA 5.22. $[X, Y] = 0$ for $X, Y \in \mathfrak{g}$ iff $\exp tX \exp sY = \exp sY \exp tX$, for any $s, t \in \mathbb{R}$.

PROOF. First by using (5.21) we have

$$\begin{aligned} \exp tX \exp sY (\exp tX)^{-1} &= \exp(s \cdot Ad(\exp tX)Y) \\ &= \exp(s \cdot \exp(t \cdot adX)Y). \end{aligned} \quad (*)$$

If $[X, Y] = 0$, then we have $\exp(t \cdot adX)Y = \sum (t^i/i!)(adX)^i Y = Y$ in $\mathfrak{gl}(\mathfrak{g})$ and hence the right-hand side of (*) is $\exp sY$ and so one obtains $\exp tX \exp sY = \exp sY \exp tX$.

Conversely, if the last equality holds, we have $\exp(s \cdot \exp(t \cdot adX)Y) = \exp sY$ by (*). Therefore we have $\exp(t \cdot adX)Y = Y$ considering the initial vector. We have $[X, Y] = 0$ by the second equality of (5.21). Q.E.D.

THEOREM 5.23. The Lie algebra of $Z(G)$ is given by $\mathfrak{z} = \text{Ker } ad = \{X \in \mathfrak{g} | [X, Y] = 0, \forall Y \in \mathfrak{g}\}$.

PROOF. $L(Z(G)) = \{X \in \mathfrak{g} | \exp X \in Z(G)\} = \{X \in \mathfrak{g} | \exp tX \cdot g = g \cdot \exp tX, \forall t \in \mathbb{R}, \forall g \in G\}$. Considering $g = \exp sY$, we have $L(Z(G)) \subset \text{Ker } ad$ by Lemma 5.22.

The converse inclusion is obtained from Lemma 5.22 by the same reason as in the second half of the proof of Theorem 4.18(1) Q.E.D.

DEFINITION. We denote by $Ad(\mathfrak{g})$ the image $\exp(ad(\mathfrak{g}))$ of the Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ by $\exp: \mathfrak{gl}(\mathfrak{g}) \rightarrow GL(\dim G, \mathbb{R})$ and call it the *adjoint group* of the Lie algebra $\mathfrak{g}$. We call the image $Ad(G)$ of $Ad: G \rightarrow GL(\mathfrak{g})$ the *adjoint group* of the Lie group G .

By Theorem 5.23, (5.21), and Theorem 4.18 we have

$$ad(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}, \quad Ad(\mathfrak{g}) \cong Ad(G) \cong G/Z(G). \quad (5.24)$$

5.24'. $Ad(G)$ is determined by the Lie algebra $L(G)$ and $Z(Ad(G)) = 0$.

The adjoint representation $Ad: G \rightarrow GL(\mathfrak{g})$ is completely reducible by Corollary 2.2, and so we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{i=1}^k \mathfrak{g}_i, \quad (5.25)$$

where $Ad|_{\mathfrak{g}_i}$ for $1 \leq i \leq k$ is an irreducible representation and $\mathfrak{g}_0$ consists of all the vectors invariant under Ad . Thus $\mathfrak{g}_i$ is a simple Lie algebra.

Here the differential of the adjoint representation Ad is the adjoint representation $ad: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{g}$, which has the same irreducible decomposition. Every $\mathfrak{g}_i$ is invariant under ad and $\mathfrak{g}_0$ consists of all the elements fixed by ad :

$$[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_i, \quad \mathfrak{g}_0 = \mathfrak{z} = \{X \in \mathfrak{g} | [X, \mathfrak{g}] = 0\}.$$

In particular, $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_i$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ ($i \neq j$).

LEMMA 5.26. $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ ($1 \leq i \leq k$).

PROOF. By the Jacobi identity we see that $[\mathfrak{g}_i, \mathfrak{g}_i]$ is an ideal of $\mathfrak{g}$ and hence of $\mathfrak{g}_i$. Since $ad|_{\mathfrak{g}_i}$ is an irreducible representation, $[\mathfrak{g}_i, \mathfrak{g}_i] \neq \mathfrak{g}_i$ implies $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$, whence $[\mathfrak{g}, \mathfrak{g}_i] = 0$ implies $\mathfrak{g}_i \subset \mathfrak{z} = \mathfrak{g}_0$, which is a contradiction. Thus $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$. Q.E.D.

For the Lie algebra $\mathfrak{g} = L(G)$ of a compact Lie group G , we now have:

5.27. The Lie algebra $\mathfrak{g} = L(G)$ is decomposed into a direct sum of the center $\mathfrak{z}$ and simple ideals $\mathfrak{g}_i$:

$$\mathfrak{g} = \mathfrak{z} \oplus \sum_{i=1}^k \mathfrak{g}_i.$$

DEFINITION. G is called *simple* if the Lie algebra $\mathfrak{g} = L(G)$ of G is simple.

G is called *semisimple* if the Lie algebra $\mathfrak{g} = L(G)$ of G is semisimple (i.e., a direct sum of simple Lie algebras).

The Lie subgroup $G_i = \exp \mathfrak{g}_i$ of G for a simple ideal $\mathfrak{g}_i$ in (5.25) is called a *simple factor*.

By (5.24) and 5.27 we have

5.28. $Ad(G)$ is semisimple.

The decomposition of a compact Lie group. We begin with the following characterization of a semisimple Lie group.

THEOREM 5.29. The following conditions are equivalent for a connected compact Lie group G :

- (1) G is semisimple.
- (2) $\mathfrak{z} = L(Z(G)) = 0$.
- (3) $Z(G)$ is finite.

- (4) $Ad: G \rightarrow Ad(G)$ is a covering group.
- (5) The roots $\{\theta_r\}$ of G span $L(T)^*$ over $\mathbf{R}$.
- (6) The index of the subgroup of $H^1(T)$ generated by the roots $\{\theta_r\}$ is finite.
- (7) The inverse roots $\{\theta_r^*\}$ of G span $L(T)$ over $\mathbf{R}$.
- (8) Every Weyl cell is bounded (an open, convex polyhedron).
- (9) $\pi_1(G) (\cong H_1(G))$ is finite.
- (10) $H^1(G) = 0$.
- (11) The universal covering group $\tilde{G}$ of G is compact.

PROOF. (1) $\Leftrightarrow$ (2) follows from 5.27, (2) $\Leftrightarrow$ (3) is trivial and (3) $\Leftrightarrow$ (4) is obtained by (5.24). Using Theorem 4.18(2) one obtains (3) $\Leftrightarrow$ (5). Using 4.19 one sees (5) $\Leftrightarrow$ (6). (5) $\Leftrightarrow$ (7) is trivial. By Lemma 5.3 one obtains (2) $\Leftrightarrow$ (8). (7) is equivalent to $L(T)/\Gamma_0$ being compact (and hence a torus) where Γ_0 is generated by θ_r^* . Since Γ/Γ_0 is a discrete subgroup in $L(T)/\Gamma_0$, using Theorem 5.14, (7) implies that $\Gamma/\Gamma_0 \cong \pi_1(G)$ is finite, that is, (9). Conversely, since Γ spans $L(T)$, (9) implies (7). (9) $\Leftrightarrow$ (10) follows from the universal coefficient theorem. (11) holds iff the covering degree is finite; and this holds iff (9) holds. Q.E.D.

THEOREM 5.30. If G is a compact, conned Lie group, and each simple factor $G_i = \exp \mathfrak{g}_i$ is a normal closed (Lie) subgroup of G . Then the map

$$f: Z(G)_0 \times G_1 \times \cdots \times G_k \rightarrow G$$

obtained by the multiplication of G is a covering group with finite covering degree and its kernel is a finite group contained in the center.

PROOF. Since $Ad(G)$ is semisimple, its universal covering group $\tilde{G}$ is compact. The Lie algebra of $\tilde{G}$ is isomorphic with $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. $\tilde{G}_i = \exp \mathfrak{g}_i$ is a connected, normal subgroup of $\tilde{G}$ and a closed subgroup by 5.14 of Chapter II and hence is compact. Since $\tilde{G}_i$ is (semi)simple, the local homeomorphism corresponding to $\mathfrak{g}_i \hookrightarrow \mathfrak{g}$ extends to $\tilde{G}_i \rightarrow G$ and the image of $\tilde{G}_i$ is G_i . Therefore the simple factor G_i of G is compact, and hence a closed subgroup. $\mathfrak{g}_i$ is an ideal so G_i is normal.

We have $ghg^{-1}h^{-1} \in G_i \cap G_j$ for $g \in G_i$ and $h \in G_j$ ($i \neq j$). If g and h are elements in a neighborhood of the identity, then we have $ghg^{-1}h^{-1} = e$, that is, $gh = hg$. Since G_i and G_j are connected and generated by their neighborhoods, their elements commute each other. Therefore a homomorphism $f: Z(G)_0 \times G_1 \times \cdots \times G_k \rightarrow G$ of Lie groups is defined by the multiplication and it is a local isomorphism, since it induces an isomorphism of Lie algebras. Since $\exp: \mathfrak{g} \rightarrow G$ is an epimorphism (Corollary 3.17), f is an epimorphism and a covering map. Since $Z(G)$ is a commutative, closed subgroup, $Z(G)_0$ is a torus. Thus $Z(G)_0 \times G_1 \times \cdots \times G_k$ is compact and so the theorem is proved. Q.E.D.

COROLLARY 5.31. *For an arbitrary compact connected Lie group G , there exist a torus T , compact simple 1-connected Lie groups $\tilde{G}_1, \dots, \tilde{G}_k$, and a finite group K contained in the center $T \times Z(\tilde{G}_1) \times \dots \times Z(\tilde{G}_k)$ of $T \times \tilde{G}_1 \times \dots \times \tilde{G}_k$ such that*

$$(T \times \tilde{G}_1 \times \dots \times \tilde{G}_k)/K \cong G \quad (\text{an isomorphism}).$$

Simple Lie groups and roots. Consider the orthogonal decomposition

$$L(G) = L(T) \oplus V_1 \oplus \dots \oplus V_m \quad (\dim V_r = 2)$$

of the Lie algebra $L(G)$ by the restriction $Ad|_T$ of the adjoint representation to a maximal torus T of G and the roots $\theta_r: L(T) \rightarrow \mathbb{R}$. Computing the differential of

$$Ad(t) = \begin{pmatrix} \cos 2\pi\theta_r(t) & -\sin 2\pi\theta_r(t) \\ \sin 2\pi\theta_r(t) & \cos 2\pi\theta_r(t) \end{pmatrix}$$

by using the second equality of (5.21) one obtains the following result:

$$[Z, Y] = 2\pi \begin{pmatrix} 0 & -\theta_r(Z) \\ \theta_r(Z) & 0 \end{pmatrix} (Y) \quad \text{for } Z \in L(T) \text{ and } Y \in V_r. \quad (5.32)$$

We express the decomposition of $\mathfrak{g} = L(G)$ above as

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \quad \text{with } \mathfrak{t} = L(T) \text{ and } \mathfrak{m} = \sum_{r=1}^m V_r.$$

We have for $Z \in \mathfrak{t} = L(T)$:

LEMMA 5.33. $Z \in L(T) - D'(G)$ iff $\mathfrak{m} = (ad Z)(\mathfrak{m})$. In particular, $\mathfrak{m} = [\mathfrak{t}, \mathfrak{m}]$.

PROOF. If $Z \in L(T) - D'(G)$, then $\theta_r(Z) \neq 0$ and so one can deduce from (5.32) the existence of $Y_r \in V_r$ such that $[Z, Y_r] = X_r$ for some $X_r \in V_r$. Taking the sum with respect to r , $\mathfrak{m} \subset (ad Z)(\mathfrak{m}) \subset [\mathfrak{t}, \mathfrak{m}]$. Also by (5.32), $[\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m}$ and hence $\mathfrak{m} = (ad Z)(\mathfrak{m})$. Conversely, if $Z \in D'(G)$, then we have $V_r \not\subset [Z, \mathfrak{m}]$ for r such that $\theta_r(Z) = 0$. Q.E.D.

Since the inner product is invariant under $Ad g$ with $g \in G$, we have

$$(Ad(\exp tX)Y, Z) = (Y, Ad(\exp(-tX))Z).$$

By applying it to the second equality of (5.21) we obtain the following relation:

$$([X, Y], Z) + (Y, [X, Z]) = 0 \quad \text{for } X, Y, Z \in \mathfrak{g}. \quad (5.34)$$

THEOREM 5.35. *A noncommutative G is simple if and only if $Ad|_T$ is not decomposable, that is, there is no orthogonal decomposition $L(\mathfrak{t}) = \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ such that the roots satisfy $\theta_i|_{\mathfrak{t}_2} = 0$ for $1 \leq i \leq k$ and $\theta_j|_{\mathfrak{t}_1} = 0$ for $k \leq j \leq m$ (after suitably changing the order).*

PROOF. *Sufficiency.* If the group is not simple, obviously there is such a decomposition.

Necessity. Assume that there is such a decomposition and put

$$\mathfrak{m}_1 = V_1 \oplus \cdots \oplus V_k \quad \text{and} \quad \mathfrak{m}_2 = V_{k+1} \oplus \cdots \oplus V_m$$

in $\mathfrak{g} = \mathfrak{t} \oplus V_1 \oplus \cdots \oplus V_k \oplus V_{k+1} \oplus \cdots \oplus V_m$. Obviously $[t_i, t_j] = 0$. Using (5.32) one sees the following:

$$[t_i, m_j] \subset m_j \text{ for } i = 1, 2 \quad \text{and} \quad [t_i, m_j] = 0 \text{ for } i \neq j.$$

For any $X_1 \in m_1$, there are $Z_i \in t_i$ and $Y_i \in m_i$ such that $X_1 = [Z_1 + Z_2, Y_1 + Y_2]$ (Lemma 5.33). By the above properties we have $X_1 = [Z_1, Y_1] + [Z_2, Y_2]$ with $[Z_i, Y_i] \in m_i$, whence $X_1 = [Z_1, Y_1]$. For any $X_2 \in m_2$ we have the following by the Jacobi identity

$$\begin{aligned} [X_1, X_2] &= [[Z_1, Y_1], X_2] = [Z_1, [Y_1, X_2]] + [[Z_1, X_2], Y_1] \\ &= [Z_1, [Y_1, X_2]]. \end{aligned}$$

By decomposing $[Y_1, X_2] \in \mathfrak{g} = \mathfrak{t} \oplus m_1 \oplus m_2$, we see that $[Z_1, [Y_1, X_2]] \in [t_1, m_1] \subset m_1$. Similarly, interchanging X_1 with X_2 , we have $[X_1, X_2] \in m_2$. Thus $[X_1, X_2] = 0$, i.e., $[m_1, m_2] = 0$.

By using (5.34) we have

$$([X_1, Y_1], X_2) = -(Y_1, [X_1, X_2]) = -(Y_1, 0) = 0$$

for any $X_1, Y_1 \in m_1$ and $X_2 \in m_2$. Since X_2 is arbitrary, we have $[X_1, Y_1] \in t_1 \oplus m_1$, that is, $[m_1, m_2] \subset t_1 \oplus m_1$.

Putting $g_i = t_i \oplus m_i$, we have $[g_1, g_1] = [t_1, t_1] + [t_1, m_1] + [m_1, m_1] \subset t_1 \oplus m_1 = g_1$, and $[g_1, g_2] = [t_1, t_2] + [t_1, m_2] + [t_2, m_1] + [m_1, m_2] = 0$. Also $[g_2, g_2] \subset g_2$. Thus $\mathfrak{g}$ is a direct sum of the ideals g_1 and g_2 and so it is not simple. Q.E.D.

§6. Simple roots and the Dynkin diagram

The inner product of roots. Roots and weights are homomorphisms (linear forms) of $L(T)$ to $\mathbf{R}$ but it is convenient to regard them as vectors of the dual space $L(T)^* = \text{Hom}_{\mathbf{R}}(L(T), \mathbf{R})$ and represent them over $L(T)$. That is, the map

$$\iota: L(T) \rightarrow L(T)^* \quad \text{given by} \quad \iota(X)Y = (X, Y) \quad \text{for } X, Y \in L(T),$$

which is defined by using the $W(G)$ -invariant inner product $(\ , \)$ in $L(T)$, is an isomorphism. Via ι one can naturally define an inner product on $L(T)^*$. That is, one defines

$$(h, h') = (\iota^{-1}(h), \iota^{-1}(h')) = h(\iota^{-1}(h'))$$

for $h, h' \in L(T)^*$.

Furthermore, the action of $W(G)$ given by (4.20) preserves the inner product.

6.1. If $v \in L(T)$ corresponds to $\theta = \iota(v) \in L(T)^*$, then we have $v = \theta(e)e$ for the vector e of length 1 in the direction of the half space $\theta > 0$ and orthogonal to the hyperplane $\theta = 0$.

6.1'. $\theta_r = \frac{1}{2}(\theta_r, \theta_r)\iota(\theta_r^*)$ for an inverse root θ_r^* .

An element $\varphi_r \in W(G)$ corresponding to a root θ_r acts on $L(T)$ as the reflection in $\theta_r = 0$. Since $\varphi_r = \varphi_r^{-1}$, it acts on $L(T)^*$ by $h \mapsto h \cdot \varphi_r$. By 6.1 we have

6.2. $\varphi_r: L(T)^* \rightarrow L(T)^*$ is the reflection satisfying $\varphi_r(\theta_r) = -\theta_r$.

THEOREM 6.3. (1) $\varphi_r(h) = h - (2(\theta_r, h)/(\theta_r, \theta_r))\theta_r$ for $h \in L(T)^*$.

(2) If h is a weight (or a root), so is $\varphi_r(h)$, and $2(\theta_r, h)/(\theta_r, \theta_r)$ is an integer.

PROOF. (1) is the formula for the map in 6.2. The first half of (2) is the same as 4.20' and Theorem 4.21. If we choose $X \in L(T)$ satisfying $\theta_r(X) = 1$, then one has $\exp X \in U_r$. Since φ_r fixed U_r , $X - \varphi_r(X)$ lies in the integer lattice and hence $h(X - \varphi_r(X))$ is an integer. Thus $2(\theta_r, h)/(\theta_r, \theta_r) = h(X) - h(X) + (2(\theta_r, h)/(\theta_r, \theta_r))\theta_r(X) = h(X) - \varphi_r(h(X))$ is also an integer. Q.E.D.

THEOREM 6.4. Between two roots α and β with $\alpha \neq \pm\beta$ one of the following relations holds:

- (0) α is perpendicular to β .
- (1) α and β make an angle of 60° or 120° and their lengths are equal.
- (2) α and β make an angle of 45° or 135° and the ratio of the lengths is $\sqrt{2}$ or $1/\sqrt{2}$.
- (3) α and β make an angle of 30° or 150° and the ratio of the lengths is $\sqrt{3}$ or $1/\sqrt{3}$.

We will prove this together with the following theorem.

THEOREM 6.5. If α and β are roots satisfying $\alpha \neq \pm\beta$, then $\beta + k\alpha$ is also a root for an integer k satisfying $0 \leq k \leq -2(\alpha, \beta)/(\alpha, \alpha)$.

PROOF OF THEOREMS 6.4 AND 6.5. Let $a_\alpha = -2(\alpha, \beta)/(\alpha, \alpha)$ and $a_\beta = -2(\alpha, \beta)/(\beta, \beta)$; by Theorem 6.3(2) these are integers. We have $\cos^2 \omega = (1/2)^2 a_\alpha a_\beta$ for the angle ω between α and β . $\alpha \neq \pm\beta$ implies $0 < \omega < 180^\circ$ so $0 \leq 4 \cos^2 \omega = a_\alpha a_\beta < 4$.

If $(\alpha, \beta) = 0$, we have the case of Theorem 6.4(0) and Theorem 6.5 is trivial.

If $(\alpha, \beta) \neq 0$, $a_\alpha a_\beta$ is 1, 2, or 3. Replacing α by $-\alpha$ if necessary, we have $(\alpha, \beta) < 0$, i.e., we may suppose that a_α and a_β are positive.

Let $a_\alpha = a_\beta = 1$, then $\varphi_\alpha(\beta) = \beta + a_\alpha \alpha = \beta + \alpha$ is also a root, and Theorem 6.5 is true; we have $|\alpha|^2 = a_\alpha(\alpha, \alpha) = -2(\alpha, \beta) = a_\beta(\beta, \beta) = |\beta|^2$.

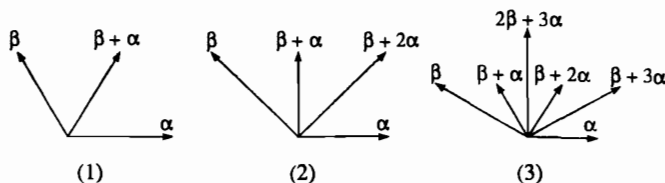


FIGURE 5.1

Thus the lengths of α and β are equal, and $\cos \omega = (\pm 1/2) \sqrt{a_\alpha a_\beta} = \pm 1/2$ which implies $\omega = 60^\circ$ or 120° and we have Theorem 6.4(1).

Let $a_\alpha = 2$; then $a_\beta = 1$, whence $2|\alpha|^2 = a_\alpha(\alpha, \alpha) = -2(\alpha, \beta) = |\beta|^2$, and hence $|\beta| = \sqrt{2}|\alpha|$ and $\cos \omega = \pm 1/\sqrt{2}$. Thus $\omega = 45^\circ$ or 135° and we have Theorem 6.4(2). Both $\varphi_\beta(\alpha) = \alpha + a_\beta \beta = \alpha + \beta$ and $\varphi_\alpha(\beta) = \beta + a_\alpha \alpha = \beta + 2\alpha$ are roots and Theorem 6.5 is proved.

Let $a_\alpha = 3$; then $a_\beta = 1$ and hence $3|\alpha|^2 = -2(\alpha, \beta) = |\beta|^2$, which implies $|\beta| = \sqrt{3}|\alpha|$ and $\cos \omega = \pm \sqrt{3}/2$ which gives Theorem 6.4(3). $\varphi_\beta(\alpha) = \alpha + \beta$, $\varphi_\alpha(\beta) = \beta + a_\alpha \alpha = \beta + 3\alpha$, and $\varphi_\alpha(\beta + \alpha) = \varphi_\alpha(\beta) + \varphi_\alpha(\alpha) = \beta + 3\alpha - \alpha = \beta + 2\alpha$ are all roots, and Theorem 6.5 is proved.

The case $a_\alpha = 1$ and $a_\beta = 2, 3$ is reduced to the case $a_\alpha = 2, 3$, since Theorem 6.5 is unchanged by switching α and β . Q.E.D.

LEMMA 6.6. *For the case of rank 2, examples giving each case of Theorem 6.4 are (0) $SO(4)$, (1) $SU(3)$, (2) $Sp(2)$ and $SO(5)$, (3) G_2 (the automorphism group of the Cayley numbers as an algebra over $\mathbb{R}$), respectively, and we can illustrate one of the roots $\pm \theta_r$ in (1), (2), and (3) as in Figure 5.1.*

Simple roots

DEFINITION. We choose a Weyl chamber $\mathcal{F}$ of $L(T)$ and call it the *fundamental Weyl chamber*.

Alternating the signs of the roots $\pm \theta_1, \dots, \pm \theta_m$ appropriately, we make an agreement:

6.7. The roots $\theta_1, \dots, \theta_m$ are chosen so that the fundamental chamber is given by

$$\mathcal{F} = \{X \in L(T) \mid \theta_r(X) > 0 \text{ for } r = 1, \dots, m\}.$$

DEFINITION. Under the above agreement, $\theta_1, \dots, \theta_m$ are called *positive roots* and $-\theta_1, \dots, -\theta_m$ are *negative roots*.

We determine the fundamental Weyl chamber for the case of classical groups by the following condition (cf. 6.26):

6.8. (1) $G = U(l), SU(l)$: $x_1 > x_2 > \dots > x_l$ (positive roots: $x_r - x_s$ ($r < s$)).

(2) $G = Sp(l), SO(2l+1)$: $x_1 > x_2 > \cdots > x_l > 0$.

(3) $G = SO(2l)$: $x_1 > x_2 > \cdots > x_{l-1} > x_l > -x_{l-1}$.

LEMMA 6.9. For positive root θ_r , if $\sum_r \lambda_r \theta_r = 0$ with $\lambda_r \geq 0$ for all r , then $\lambda_r = 0$ for all r .

PROOF. $\sum \lambda_r \theta_r(X) = 0$ for $X \in \mathcal{F}$ with $\theta_r(X) > 0$ implies $\lambda_r = 0$. Q.E.D.

DEFINITION. α is called a *simple root* if α is a positive root and is not a sum of two positive roots.

THEOREM 6.10. Any positive root is a linear combination of simple roots with nonnegative integer coefficients.

PROOF. If one keeps representing a positive root as a sum of positive roots, then the process terminates after a finite number of steps. Q.E.D.

COROLLARY 6.11. If $\alpha_1, \dots, \alpha_s$ are all the simple roots, then the fundamental Weyl chamber is given by $\alpha_1 > 0, \dots, \alpha_s > 0$.

(It is seen by Theorem 6.14 that one cannot reduce the number of simple roots any further.)

THEOREM 6.12. The reflections φ_α corresponding to simple roots α generate the Weyl group $W(G)$.

PROOF. If a root γ is a sum of roots α and β , we have $\gamma = \varphi_\alpha(\beta)$ or $\gamma = \varphi_\beta(\alpha)$ as in the proof of Theorems 6.4 and 6.5. Since $\gamma = \varphi_\alpha(\beta)$ implies $\varphi_\gamma = \varphi_\alpha \varphi_\beta \varphi_\alpha$, we obtain the theorem from Theorems 6.10 and 4.30(2). Q.E.D.

LEMMA 6.13. If α and β are distinct simple roots, then $(\alpha, \beta) \leq 0$.

PROOF. Suppose (α, β) is positive; then $-2(\alpha, \beta)/(\alpha, \alpha)$ is a positive integer, which implies that $\beta - \alpha$ is a root by Theorem 6.5, hence $\beta - \alpha$ or $\alpha - \beta$ is a positive root. Thus $\beta = (\beta - \alpha) + \alpha$ or $\alpha = (\alpha - \beta) + \beta$ is not a simple root, which contradicts the assumption. Q.E.D.

THEOREM 6.14. The simple roots are linearly independent.

PROOF. Suppose that a linear combination of simple roots is 0. If we transpose the terms with negative coefficients, we have $\sum \mu_i \alpha_i = \sum \nu_j \beta_j$, where $\mu_i \geq 0$, $\nu_j \geq 0$, and α_i, β_j are distinct simple roots. Denote this element by γ and then $(\gamma, \gamma) = \sum \mu_i \nu_j (\alpha_i, \beta_j) \leq 0$ so $\gamma = 0$. By Lemma 6.9 we have $\mu_i = \nu_j = 0$. That is, the coefficients of the first linear combination are all 0. Q.E.D.

THEOREM 6.15. If θ_r is a simple root, then φ_r permutes the positive roots except θ_r , which maps to $-\theta_r$.

PROOF. Let $\alpha_1, \dots, \alpha_s$ be simple roots and let θ_i be a positive root. We write $\theta_i = n_1 \alpha_1 + \cdots + n_s \alpha_s$ with $n_i \geq 0$ (Theorem 6.10). If $\theta_r = \alpha_j$, then

$\varphi_r(\theta_i) = \theta_i - 2((\theta_i, \alpha_j)/(\alpha_j, \alpha_j))\alpha_j$ differs from θ_i only in the coefficient of α_j . Thus, if $\theta_i \neq \theta_r$, then there is $i (\neq j)$ such that $n_i > 0$. $\varphi_r(\theta_i)$ is a root, which is either a positive root or a negative root, but it must be a positive root by Theorem 6.10 if there is $i (\neq j)$ such that $n_i > 0$. Q.E.D.

THEOREM 6.16.

- (1) Any root can be obtained from a simple root by the actions of the Weyl group.
- (2) The minimal set M satisfying the following three conditions coincides with the set of roots:
 - (a) A simple root belongs to M .
 - (b) If α belongs to M , then so does $-\alpha$.
- (3) If α and β belong to M and if $\beta \neq \pm\alpha$, then $\beta + k\alpha$ belongs to M for an integer k satisfying $0 \leq k \leq -2(\alpha, \beta)/(\alpha, \alpha)$.

PROOF. (1) Consider the Weyl chamber with a wall $L(U_r)$ for a root θ_r . There is an element φ of the Weyl group that maps the Weyl chamber to the fundamental Weyl chamber by Theorem 4.30(1). Since $\varphi(L(U_r))$ is a wall of the fundamental Weyl chamber, denoting it by $L(U_s)$, we have that θ_s is a simple root such that $\varphi^{-1}\theta_r = \pm\theta_s$. If $\varphi^{-1}\theta_r = -\theta_s$, then $(\varphi^{-1}\varphi_r)\theta_r = \theta_s$.

(2) We see by Theorem 6.5 that M consists of roots. It suffices to show by (1) that the actions of the Weyl group are given by (b) and (c), but by Theorem 4.30(2) it suffices to show this for a reflection φ_r corresponding to a simple root $\alpha = \theta_r$.

First, $\varphi_r(\pm\alpha) = \mp\alpha$ is obtained by (b). When $\beta \neq \pm\alpha$, $\varphi_r(\beta) = \beta - 2(\alpha, \beta)/(\alpha, \alpha)\alpha$ is obtained by (c). Q.E.D.

The admissible system of vectors

DEFINITION. A system of s linearly independent vectors $\{\alpha_1, \dots, \alpha_s\}$ in the Euclidean space $\mathbf{R}^l$ is called an *admissible system of vectors* if it satisfies the condition

$$a_{ij} = -\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \text{ is an integer, and } a_{ij} \geq 0 \text{ for } i \neq j. \quad (*)$$

Theorems 6.3(2) and 6.14 show that this is an abstraction of the concept of a system of simple roots. The definition is independent of the order of the vectors.

DEFINITION. Two admissible systems are defined to be *equivalent* if there is a one to one correspondence between their vectors such that the integers in (*) coincide for any two corresponding pairs (i, j) .

The definition of a simple root of G depends on the choices of a maximal torus T and the fundamental Weyl chamber, but the maximal tori are mapped to each other by inner automorphisms and so are Weyl chambers by the action of the Weyl group. Also these actions do not alter the inner product, and the action of the Weyl group maps positive roots with respect

to a fundamental Weyl chamber $\mathcal{F}$ to positive roots, which have the image of $\mathcal{F}$ as the fundamental Weyl chamber.

It follows from these that

6.17. *The simple roots of G form an admissible system of vectors in $L(T) \cong \mathbf{R}^l$ and there is a uniquely determined correspondence, which assigns to G an equivalence class of admissible systems of simple roots.*

In the proof of Theorem 6.4 we have only used the fact that roots form an admissible system. So, Theorem 6.4 also holds for two vectors of an admissible system, and besides we have $a_{ij} \geq 0$ for $i \neq j$ and so from the proof of Theorem 6.4 we see the following:

6.18. *For two vectors α_i and α_j ($i \neq j$) of an admissible system of vectors and for the integers in (*) we always have:*

- (1) a_{ij} and a_{ji} are simultaneously 0, or one is 1 and the other is 1, 2, or 3.
- (2) If $a_{ji} = 1$ then $|\alpha_i| = \sqrt{a_{ij}}|\alpha_j|$.
- (3) $a_{ij}|\alpha_j|/|\alpha_j| = a_{ji}|\alpha_i|/|\alpha_j| = \sqrt{a_{ij}a_{ji}}$.

Various restrictions are imposed on an admissible system of vectors in addition to 6.18. The following lemma is used to show them.

LEMMA 6.19. *Let $\{\alpha_1, \dots, \alpha_s\}$ be an admissible system of vectors. The following inequality holds for real numbers $t_1, t_2, \dots, t_s$ all of which are not 0:*

$$\sum_{i=1}^s t_i^2 > \sum_{i < j} t_i t_j \sqrt{a_{ij} a_{ji}}.$$

PROOF. $\alpha_1, \dots, \alpha_s$ are linearly independent so $\alpha = \sum_{i=1}^s t_i \alpha_i / |\alpha_i| \neq 0$. By applying 6.18(3) to

$$0 < |\alpha|^2 = (\alpha, \alpha) = \sum_{i,j=1}^s \frac{t_i t_j (\alpha_i, \alpha_j)}{|\alpha_i| \cdot |\alpha_j|} = \sum_{i=1}^s t_i^2 - \frac{1}{2} \sum_{i < j} t_i t_j \left(\frac{a_{ij} |\alpha_j|}{|\alpha_i|} + \frac{a_{ji} |\alpha_i|}{|\alpha_j|} \right)$$

we obtain the inequality of the lemma. Q.E.D.

COROLLARY 6.20. *The number of ordered pairs (i, j) for which $a_{ij} \geq 1$ ($i < j$) in the admissible system $\{\alpha_1, \dots, \alpha_s\}$ is at most $s - 1$.*

PROOF. If we put $t_i = 1$ in Lemma 6.19, we have $s > \sum_{i < j} \sqrt{a_{ij} a_{ji}}$. If $a_{ij} \geq 1$ then $\sqrt{a_{ij} a_{ji}} \geq 1$ by 6.18(1) and hence the corollary is obtained. Q.E.D.

LEMMA 6.21. $\sum_{i \neq r} a_{ir} a_{ri} \leq 3$ for every r .

PROOF. First we see

$$a_{ir} \geq 1, a_{jr} \geq 1 \ (i \neq j) \text{ implies } a_{ij} = 0 \quad (6.22)$$

by applying Corollary 6.20 to the admissible system $\{\alpha_i, \alpha_j, \alpha_r\}$. We put $t_r = t$, $t_i = \sqrt{a_{ir}a_{ri}}$ if $a_{ir} \geq 1$ and $t_i = 0$ if $a_{ir} = 0$ in Lemma 6.19. Since $t_i t_j \neq 0$ ($i \neq j \neq r \neq i$) implies $a_{ij} = 0$ by (6.22), we have, for $a = \sum_{i \neq r} a_{ir} a_{ri} = \sum_{i \neq r} t_i^2$,

$$t^2 + a = \sum t_i^2 > \sum_{i < j} t_i t_j \sqrt{a_{ij} a_{ji}} = \sum_{i \neq r} t \left(\sqrt{a_{ir} a_{ri}} \right)^2 = at. \quad 6.23$$

Putting $t = a/2$, $a^2/4 + a > a^2/2$ so $4a > a^2$. Since a is an integer ≥ 0 , we have $a \leq 3$. Q.E.D.

The Dynkin diagram. The one-dimensional complex, which is constructed from an admissible system $\{\alpha_1, \dots, \alpha_s\}$ of vectors as described below, is called the *Dynkin diagram* (or the *Schläfli diagram*) or the *Coxeter diagram* of the admissible system. If the admissible system is a system of simple roots of a compact connected Lie group G , it is called the *Dynkin diagram* of G .

First we assign to each vector α_i a vertex (denoted by $\circ$).

If $a_{ij} \neq 0$, we assign $a_{ij}a_{ji}$ edges (one-dimensional simplexes) joining α_i and α_j , where we add an arrow directed from the larger one to the smaller one to indicate the size explicitly when the lengths of α_i and α_j are distinct. By 6.18 there are the following possibilities:

$\alpha_i \quad \alpha_j$ $\circ \quad \circ$	$\alpha_i \quad \alpha_j$ $\circ \text{---} \circ$	$\alpha_i \quad \alpha_j$ $\circ \Rightarrow \circ$	$\alpha_i \quad \alpha_j$ $\circ \Rightarrow \Rightarrow \circ$
$a_{ij} = 0 (= a_{ji})$	$a_{ij} = a_{ji} = 1$	$a_{ij} = 2 (a_{ji} = 1)$	$a_{ij} = 3 (a_{ji} = 1)$

DEFINITION. A root system $\{\pm\theta_1, \dots, \pm\theta_m\}$ is said to be *irreducible* if it spans $L(T)^* \cong \mathbf{R}^l$ and if it cannot be divided into two nonempty subsets so that any two vectors from each subset are orthogonal to each other. *Irreducibility* is also defined for the simple root system as well as an admissible system of vectors.

The following is obvious:

6.23. An admissible system of vectors $\{\alpha_1, \dots, \alpha_l\}$ is irreducible iff its Dynkin diagram is connected (as a one-dimensional polyhedron).

Now, we quote the following theorem from the theory of semisimple Lie algebras.

THEOREM 6.24. There is a one-to-one correspondence between the local isomorphism classes of compact Lie groups G (the isomorphism classes of Lie algebra $L(G)$) and admissible systems of vectors.

The reader is referred to Chapter 9 of Y. Matsushima, *The Theory of Lie Algebras*, Kyōritsu-Shuppan (in Japanese), for a proof of the theorem, since we have no space to give it here. This book is mainly concerned with semisimple Lie algebras but we remark that the root system of $L(G) \otimes \mathbb{C}$ is obtained as a $2\pi i$ multiple of the root system of G given here. Also we observe that in order to put back a Lie algebra over the complex number field $\mathbb{C}$ to a Lie algebra over the real number field $\mathbb{R}$ we have only to take its compact real form.

THEOREM 6.25. *There is a one-to-one correspondence between the local isomorphism classes of compact connected simple Lie groups G and isomorphism classes of connected Dynkin diagrams viewed as one-dimensional complexes (including the arrows).*

PROOF. By Theorems 5.29, 6.24, and 5.35, we see that semisimplicity corresponds to $L(T)^*$ being generated by the roots and simplicity to irreducibility of the root system. Also by using Theorems 6.10 and 6.16, we see that root systems correspond bijectively with simple root systems together with their irreducibility. Thus Theorem 6.24 and 6.23 imply the theorem. Q.E.D.

Classification of simple Lie groups. We will list below the Dynkin diagrams of simple Lie groups classified by E. Cartan and the root systems realizing them (cf. 4.5 ~ 4.8 and 6.8).

6.26. *The orthonormal basis $e_i \in \text{Hom}(\mathbb{R}^l, \mathbb{R})$ dual to $\mathbb{R}^l = \{(x_1, \dots, x_l)\}$ are given by $e_i(x_j) = \delta_{ij}$.*

Type A_l : $l \geq 1$. On $\mathbb{R}^l \cong \mathbb{R}^{l+1} | x_1 + \dots + x_{l+1} = 0$.



simple roots: $\alpha_i = e_i - e_{i+1}$ ($1 \leq i \leq l$);

positive roots: $e_i - e_j$ ($1 \leq i < j \leq l+1$).

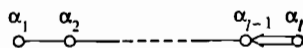
Type B_l : $l \geq 2$.



simple roots: $\alpha_i = e_i - e_{i+1}$ ($1 \leq i < l$), $\alpha_l = e_l$;

positive roots: e_i ($1 \leq i \leq l$), $e_i + e_j$, $e_i - e_j$ ($1 \leq i < j \leq l$).

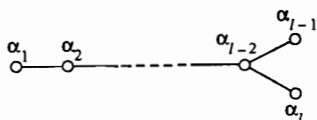
Type C_l : $l \geq 2$.



simple roots: $\alpha_i = e_i - e_{i+1}$ ($1 \leq i < l$), $\alpha_l = 2e_l$;

positive roots: $2e_i$ ($1 \leq i \leq l$), $e_i + e_j$, $e_i - e_j$ ($1 \leq i < j \leq l$).

Type D_l : $l \geq 3$.



simple roots: $\alpha_i = e_i - e_{i+1}$ ($1 \leq i < l$), $\alpha_l = e_{l-1} + e_l$;

positive roots: $e_i + e_j$, $e_i - e_j$ ($1 \leq i < j \leq l$).

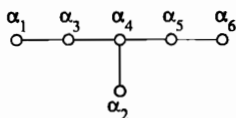
Type E_l : $l = 6, 7, 8$. On $\mathbf{R}^6 \cong \mathbf{R}^8 | x_6 = x_7 = -x_8$, $\mathbf{R}^7 \cong \mathbf{R}^8 | x_7 = -x_8$, $\mathbf{R}^8$.

simple roots: $\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7)$,

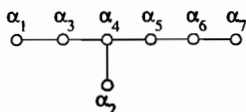
$\alpha_2 = e_1 + e_2$, $\alpha_i = e_{i-1} - e_{i-2}$ ($3 \leq i \leq l$);

positive roots:

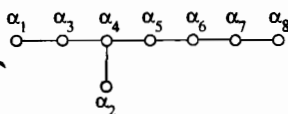
$l = 6$. $\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^5 (-1)^{\mu(i)} e_i)$, where $\sum_{i=1}^5 \mu(i)$ is even;
 $e_i + e_j$, $-e_i + e_j$ ($1 \leq i < j \leq 5$).



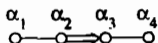
$l = 7$. $\frac{1}{2}(e_8 - e_7 + \sum_{i=1}^6 (-1)^{\mu(i)} e_i)$, where $\sum_{i=1}^6 \mu(i)$ is odd;
 $e_i + e_j$, $-e_i + e_j$ ($1 \leq i < j \leq 6$), $e_8 - e_7$.



$l = 8$. $\frac{1}{2}(e_8 + \sum_{i=1}^7 (-1)^{\mu(i)} e_i)$, where $\sum_{i=1}^7 \mu(i)$ is even;
 $e_i + e_j$, $-e_i + e_j$ ($1 \leq i < j \leq 8$).



Type F_4 .



simple roots: $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$,

$$\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4);$$

positive roots: e_i ($1 \leq i \leq 4$), $e_i + e_j$, $e_i - e_j$ ($1 \leq i < j \leq 4$),

$$\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4).$$

Type G_2 : On $\mathbf{R}^2 \cong \mathbf{R}^3 | x_1 + x_2 + x_3 = 0$.

$$\begin{array}{c} \alpha_1 \quad \alpha_2 \\ \circ \rightleftharpoons \circ \end{array}$$

simple roots: $\alpha_1 = e_1 - e_2$, $\alpha_2 = -2e_1 + e_2 + e_3$;

positive roots: $e_1 - e_2$, $-e_1 + e_3$, $-e_2 + e_3$,

$$-2e_1 + e_2 + e_3, e_1 - 2e_2 + e_3, -e_1 - e_2 + 2e_3.$$

For each case listed above we extend the given simple roots by Theorem 6.16 to obtain a root system with positive roots obtained correspondingly. Thus we have

6.26'. There exist compact connected simple Lie groups with root systems isomorphic to the root systems (A_l) , (B_l) , (C_l) , (D_l) , (E_l) , (F_4) , and (G_2) listed above, and they are not isomorphic to each other.

THEOREM 6.27. There are no local isomorphism classes of simple Lie groups other than the ones listed in 6.26', that is, there are no connected Dynkin diagrams other than the ones listed in 6.26.

We prepare to prove the theorem with some lemmas.

LEMMA 6.28. (1) If a connected Dynkin diagram contains $\circ \Rightarrow \circ$, it contains nothing else.

(2) If one replaces $\circ \Rightarrow \circ$ and $\circ \Rightarrow \circ$ with $\circ - \circ$ in a Dynkin diagram, then the diagram does not contain a loop (that is, it is a tree).

PROOF. (1) Lemma 6.21 shows that the number of the edges which issue from a vertex is at most 3 and hence (1) is obvious.

(2) The part containing closed paths is also an admissible system but then we have the number of edges $\geq$ the number of vertices, which contradicts Corollary 6.20. Q.E.D.

LEMMA 6.29. A Dynkin diagram does not contain, as a part of it, any of the diagrams in Figure 5.2 where the arrow in $\circ \Rightarrow \circ$, etc. may be reversed and the number of edges in $\circ - \dots - \circ$ may be 0. (The numerical values and $\bullet$ are given for later use.)

PROOF. If we substitute the numerical values given at each vertex of the lemma for the values of $t_1, t_2, \dots, t_s$ in Lemma 6.19, we have the equality $\sum t_i^2 = \sum_{i < j} t_i t_j \sqrt{a_{ij} a_{ji}}$ whose value is the numerical value given in

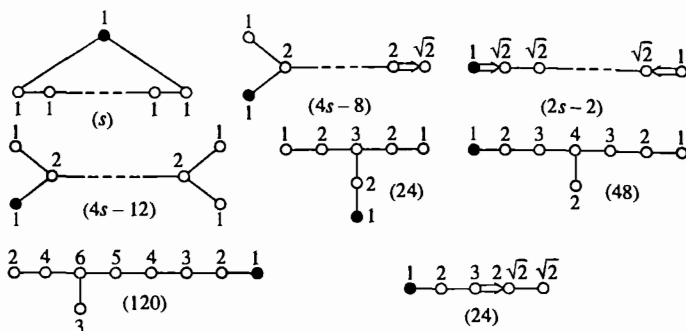


FIGURE 5.2

(), which contradicts Lemma 6.19. Therefore each diagram above does not correspond to an admissible system and hence it is not a part of a Dynkin diagram. Q.E.D.

PROOF OF THEOREM 6.27. By Lemma 6.28(2) a Dynkin diagram does not have a closed path but is a tree; we see, however, by Lemma 6.21 that it has only the form $\circ \circ \circ$ at a ramified vertex. If it contains $\circ \Rightarrow \circ$, it is of G_2 -type by Lemma 6.28(1). If it contains $\circ \Rightarrow \circ$, it is seen by Lemma 6.29 to be either of B_l , C_l , or F_4 type. If it does not contain $\circ \Rightarrow \circ$ or $\circ \Rightarrow \circ$, it must be of A_l , D_l , or E_l type by Lemma 6.29. Q.E.D.

REMARK 6.30. A_l , B_l , C_l , and D_l are the root systems of the classical groups $SU(l+1)$, $SO(2l+1)$, $Sp(l)$, and $SO(2l)$ respectively (4.4 ~ 4.8). Lie groups that have root systems of the remaining E_l ($l = 6, 7, 8$), F_4 , or G_2 types are called *Lie groups of the exceptional type* or simply *exceptional groups*.

Dominant roots and the extended Dynkin diagram

DEFINITION. The root $\tilde{\alpha}$ given below for each of the classification types 6.26 of simple Lie groups G is called the *dominant root* or the *maximal root*:

- 6.31. $(A_l) \tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_l = e_1 - e_{l+1}$;
 $(B_l) \tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l = e_1 + e_2$;
 $(C_l) \tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l = 2e_1$;
 $(D_l) \tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l = e_1 + e_2$;
 $(E_6) \tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \frac{1}{2}(e_1 + e_2 + \cdots + e_5 - e_6 - e_7 + e_8)$;
 $(E_7) \tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = -e_7 + e_8$;
 $(E_8) \tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = e_7 + e_8$;
 $(F_4) \tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = e_1 + e_2$;
 $(G_2) \tilde{\alpha} = 3\alpha_1 + 2\alpha_2 = -e_1 - e_2 + 2e_3$.

The following can be verified from the table of positive roots:

6.32. If $\tilde{\alpha} = m_1\alpha_1 + \cdots + m_l\alpha_l$ is a dominant root, we have $\mu_i \leq m_i$ ($i = 1, 2, \dots, l$) for any positive root $\theta = \mu_1\alpha_1 + \cdots + \mu_l\alpha_l$.

DEFINITION. The diagram constructed, similarly to the Dynkin diagram, from the vector system of simple roots $\alpha_1, \dots, \alpha_l$ of G with added $-\tilde{\alpha}$, the dominant root with its sign changed, is called the *extended Dynkin diagram*.

6.33. *Extended Dynkin diagrams are given by the diagram in Lemma 6.29 with the following exceptions, where $\bullet$ is the dominant root.*

$$(A_1) \begin{array}{c} \alpha_1 \\ \circ \end{array} \begin{array}{c} -\tilde{\alpha} \\ \longleftarrow \end{array} \bullet \quad -\alpha_1$$

$$(B_2) \begin{array}{c} \alpha_1 \\ \circ \end{array} \begin{array}{c} \alpha_2 \\ \longrightarrow \end{array} \begin{array}{c} -\tilde{\alpha} \\ \longrightarrow \end{array} \bullet$$

$$(G_2) \begin{array}{c} \alpha_1 \\ \circ \end{array} \begin{array}{c} \alpha_2 \\ \longrightarrow \end{array} \begin{array}{c} -\tilde{\alpha} \\ \longrightarrow \end{array} \bullet$$

DEFINITION. The Weyl cell Δ contained in the fundamental Weyl chamber $\mathcal{F} = \{X \in L(T) | \alpha_i(X) > 0 \text{ for } i = 1, 2, \dots, l\}$ such that $\bar{\Delta}$ contains the origin is called the *fundamental Weyl cell*.

THEOREM 6.34. (1) *G is (semi)simple if and only if the Weyl cells are the interiors of (a direct product of) bounded simplexes.*

(2) *If G is simple, the fundamental Weyl cell Δ is given by $\Delta = \{X \in L(T) | \alpha_i(X) > 0 \text{ for } i = 1, 2, \dots, l \text{ and } \tilde{\alpha}(X) < 1\}$.*

PROOF. The Weyl cells of G are congruent to each other under the action of the extended Weyl group. If G is not semisimple, Weyl cells are not bounded. If G is semisimple and has simple factors $G_1, \dots, G_k$, then Δ is congruent to $\Delta_1 \times \dots \times \Delta_k$ for each Weyl cell Δ_i of G_i . Now (1) is seen to follow from (2). One sees by 6.32 that Δ given by (2) is the Weyl cell satisfying

$$\Delta = \{X \in L(T) | 0 < \alpha(X) < 1 \text{ for all positive roots } \alpha\}.$$

Furthermore this Δ is in the only Weyl cell in the fundamental Weyl chamber $\mathcal{F}$ such that the closure $\bar{\Delta}$ contains the origin. Q.E.D.

The fundamental weight and the center of a 1-connected Lie group

THEOREM 6.35. *If G is a 1-connected, compact Lie group, the inverse roots $\alpha_1^*, \dots, \alpha_l^*$ of the simple roots $\alpha_1, \dots, \alpha_l$ form a basis of the free abelian group $\Gamma (\cong \pi_1(T))$.*

PROOF. Since $\Gamma = \Gamma_0$ is a free abelian group of rank l generated by the inverse roots θ_r^* by the assumption and Theorem 5.19, it suffices to show that any θ_r^* is a linear combination with integer coefficients of $\alpha_1^*, \dots, \alpha_l^*$. There are α_i and $\varphi \in W(G)$ by Theorem 6.16(1) such that $\varphi(\alpha_r) = \theta_r$. Also by Theorem 6.12, $W(G)$ is generated by the reflections φ_α in the simple root α . Thus we may suppose that $\theta_r = \varphi_\alpha(\theta_s)$ and that θ_s^* is a linear combination with integer coefficients of $\alpha_1^*, \dots, \alpha_l^*$. It follows from Theorem 5.6(1) that $\varphi_\alpha(\theta_s^*) = \theta_s^* - \alpha(\theta_s^*)\alpha^*$, where $\alpha(\theta_s^*)$ is an integer, since $\theta_r^* \in \Gamma_0 = \Gamma$. Since $\varphi_{r,1} = \varphi_\alpha \varphi_{s,1} \varphi_\alpha$ holds, we have that $\varphi_r^* = \varphi_{r,1}(0) = \varphi_\alpha \varphi_{s,1} \varphi_\alpha(0) = \varphi_\alpha \varphi_{s,1}(0) = \varphi_\alpha(\theta_s^*) = \theta_s^* - \alpha(\theta_s^*)\alpha^*$ is a linear combination with integer coefficients of $\alpha_1^*, \dots, \alpha_l^*$. Q.E.D.

DEFINITION. The vectors $w_1, \dots, w_l$ determined by the relation

$$2(w_i, \alpha_j) / (\alpha_j, \alpha_j) = \delta_{ij},$$

where $\alpha_1, \dots, \alpha_l$ are the simple roots are called the *fundamental weights*.

THEOREM 6.36. If G is a 1-connected, compact Lie group, the fundamental weights are weights; namely, $w_i(\Gamma) \subset \mathbb{Z}$. If we regard it as $w_i \in H^1(T) = \text{Hom}(\Gamma, \mathbb{Z})$ by (4.19), the fundamental weights $w_1, \dots, w_l$ form a basis over $\mathbb{Z}$ of $H^1(T)$.

PROOF. Since $(w_i, \alpha_j) = 0$ for $i \neq j$, the hyperplane $w_i = 0$ is perpendicular to the hyperplane $\alpha_j = 0$. Therefore, α_j^* lies in the hyperplane $w_i = 0$, and $w_i(\alpha_j^*) = 0$ for $i \neq j$. $2(w_i, \alpha_i) = (\alpha_i, \alpha_i)$ implies $(\beta, \alpha_i) = 0$ for $\beta = 2w_i - \alpha_i$. Similarly to the above, $\beta(\alpha_i^*) = 0$ and hence $2w_i(\alpha_i^*) = \beta(\alpha_i^*) + \alpha_i(\alpha_i^*) = 2$. It follows that $w_i(\alpha_j^*) = \delta_{ij}$. By Theorem 6.35 this shows that $\{w_i\}$ forms a $\mathbb{Z}$ -basis of $\text{Hom}(\Gamma, \mathbb{Z})$. Q.E.D.

By a simple calculation one can verify the following:

6.37. The fundamental weights $\{w_i\}$ of a 1-connected, compact, simple Lie group are given as follows and they are linear combinations of the simple roots $\{\alpha_i\}$ with rational coefficients:

$$w_i = \sum_{j=1}^i e_j - \frac{i}{l+1} \sum_{k=1}^{l+1} e_k; \quad (A_l)$$

$$w_i = \sum_{j=1}^i e_j \quad (1 \leq i < l), \quad w_l = \frac{1}{2} \sum_{j=1}^l e_j; \quad (B_l)$$

$$w_i = \sum_{j=1}^i e_j; \quad (C_l)$$

$$w_i = \sum_{j=1}^i e_j \quad (1 \leq i < l-1), \quad w_{l-1} = \frac{1}{2} \sum_{j=1}^{l-1} e_j - \frac{1}{2} e_l, \quad (D_l)$$

$$w_l = \frac{1}{2} \sum_{j=1}^l e_j; \quad (E_l)$$

$$w_1 = \frac{2}{3}(e_8 - e_7 - e_6), \quad w_2 = \frac{1}{2} \sum_{j=1}^5 e_j + \frac{1}{2}(e_8 - e_7 - e_6), \quad (E_6)$$

$$w_3 = -\frac{1}{2}e_1 + \frac{1}{2} \sum_{j=2}^5 e_j + \frac{5}{6}(e_8 - e_7 - e_6),$$

$$w_i = \sum_{j=i-1}^5 e_j + \frac{7-i}{3}(e_8 - e_7 - e_6) \quad (i = 4, 5, 6), \quad (w_2 = \tilde{\alpha});$$

$$w_1 = e_8 - e_7, \quad (E_7)$$

$$w_2 = \frac{1}{2} \sum_{j=1}^6 e_j + (e_8 - e_7), \quad w_3 = -\frac{1}{2} e_1 + \frac{1}{2} \sum_{j=2}^6 e_j + \frac{3}{2} (e_8 - e_7),$$

$$w_i = \sum_{j=i-1}^6 e_j + \frac{8-i}{2} (e_8 - e_7) \quad (i = 4, 5, 6, 7), \quad (w_1 = \tilde{\alpha});$$

$$w_1 = 2e_8, \quad w_2 = \frac{1}{2} \sum_{j=1}^7 e_j + \frac{5}{2} e_8, \quad w_3 = -\frac{1}{2} + \frac{1}{2} \sum_{j=2}^7 e_j + \frac{7}{2} e_8, \quad (E_8)$$

$$w_i = \sum_{j=i-1}^7 e_j + (9-i)e_8 \quad (i = 4, 5, 6, 7, 8), \quad (w_8 = \tilde{\alpha});$$

$$w_1 = e_1 + e_2 = \tilde{\alpha}, \quad w_2 = 2e_1 + e_2 + e_3, \quad (F_4)$$

$$w_3 = \frac{1}{2} (3e_1 + e_2 + e_3 + e_4), \quad w_4 = e_1;$$

$$w_1 = -e_2 + e_3 = 2\alpha_1 + \alpha_2, \quad w_2 = -e_1 - e_2 + 2e_3 = 3\alpha_1 + 2\alpha_2 = \tilde{\alpha}. \quad (G_2)$$

THEOREM 6.38. Let G be a 1-connected, compact Lie group. If A is a subgroup of $H^1(T) = \mathbb{Z} \cdot w_1 + \dots + \mathbb{Z} \cdot w_l$ generated by the simple roots $\alpha_1, \dots, \alpha_l$, the center of G is given by $Z(G) \cong H^1(T)/A$. In particular, if G is an exceptional group, it is given as follows:

$$Z(E_6) \cong \mathbb{Z}_3, \quad Z(E_7) \cong \mathbb{Z}_2, \quad Z(E_8) = Z(F_4) = Z(G_2) = 0.$$

PROOF. We have by Theorems 4.18(2) and 6.10

$$\begin{aligned} \exp^{-1}(Z(G)) &= \{X \in L(T) \mid \theta_r(X) \equiv 0 \pmod{1}, \quad r = 1, 2, \dots, m\} \\ &= \{X \in L(T) \mid \alpha_i(X) \equiv 0 \pmod{1}, \quad i = 1, 2, \dots, l\} \end{aligned}$$

for $\exp: L(T) \rightarrow T$. Therefore, putting $\tilde{\Gamma} = \exp^{-1}(Z(G))$ we obtain an isomorphism $\text{Hom}(\tilde{\Gamma}, \mathbb{Z}) \cong A$, and $i^*: \text{Hom}(\tilde{\Gamma}, \mathbb{Z}) \rightarrow \text{Hom}(\Gamma, \mathbb{Z})$, induced by the inclusion $i: \Gamma = \Gamma_0 = \exp^{-1}(e) \rightarrow \tilde{\Gamma}$ is equivalent to the inclusion $A \hookrightarrow H^1(T)$. Since $Z(G) \cong \tilde{\Gamma}/\Gamma$ is finite, an exact sequence $0 \rightarrow \Gamma \rightarrow \tilde{\Gamma} \rightarrow Z(G) \rightarrow 0$ induces an exact sequence

$$0 \rightarrow \text{Hom}(\tilde{\Gamma}, \mathbb{Z}) \xrightarrow{i^*} \text{Hom}(\Gamma, \mathbb{Z}) \rightarrow \text{Ext}(Z(G), \mathbb{Z}) \rightarrow 0,$$

where we have $H^1(T)/A \cong \text{Coker } i^* \cong \text{Ext}(Z(G), \mathbb{Z}) \cong Z(G)$. The second half can be verified by writing α_i as a linear combination of w_j by 6.37 and 6.36. Q.E.D.

§7. Subgroups of maximal rank

Subgroups of maximal rank and the direct product

DEFINITION. A closed subgroup G' of a compact, connected Lie group G ($G' \subsetneq G$) is called a *subgroup of maximal rank* if $\text{rank } G = \text{rank } G'$.

The object of this section is to study what kind of groups G' can be subgroups of maximal rank contained in a given compact connected Lie group G . Since G' can be transferred to another subgroup by an automorphism, we take no notice of how G' is contained in G .

The following is clear from the conjugacy of maximal tori.

7.1. Let G' be a subgroup of maximal rank of G , and T a maximal torus of G . Then we can choose G' in its conjugacy class so that T is a maximal torus of G' :

$$T \subset G' \subset G.$$

A finite degree covering group $\tilde{G}$ of G contains the inverse image of G' as a subgroup of maximal rank, and we see that we can replace G by a suitable $\tilde{G}$ if we are only concerned with the local isomorphism class of G' (= the isomorphism class of the Lie algebra). In particular we see by the following theorem that it suffices to study one-dimensional, simple groups, since $\tilde{G}$ can be taken to be a direct product of a torus and some 1-connected, simple groups (Corollary 5.31).

THEOREM 7.2. A subgroup G' of maximal rank of the direct product $G_1 \times \cdots \times G_k$ of compact, connected Lie groups G_i is isomorphic to $G'_1 \times \cdots \times G'_k$, where G'_i is either G_i or a subgroup of maximal rank of G_i .

PROOF. We may suppose that $k = 2$. Also we may suppose $T_1 \times T_2 \subset G'$ as in 7.1 for a maximal torus T_i of G_i . $G'_i = \{a_i | (a_1, a_2) \in G'\}$ is a subgroup of G_i containing T_i ($i = 1, 2$). For $a = (a_1, a_2) \in G'$ there exists $b = (b_1, b_2) \in G'$ such that $bab^{-1} \in T_1 \times T_2$ (Theorem 3.15). Putting $t = (b_1 a_1 b_1^{-1}, e_2) \in T_1 \times 1$ for the identity e_2 of G_2 , we have $(a_1, e_2) = b^{-1} t b \in G'$, that is, $G'_1 \times 1 \subset G'$. Similarly, $1 \times G'_2 \subset G'$. Obviously, $G' \subset G'_1 \times G'_2$ and hence $G' = G'_1 \times G'_2$. Q.E.D.

The roots of a subgroup of maximal rank. Hereafter in this section, let G be a compact, connected Lie group, G' a subgroup of maximal rank, and T a maximal torus of G contained in G' . The roots of G and G' are considered with respect to a common maximal torus T .

THEOREM 7.3. A root of G' is a root of G .

PROOF. We have an orthogonal decomposition $L(G') = L(T) \oplus \sum_{j=1}^{m'} V'_j$ by the invariant subspaces V'_j of $\text{Ad}|T$ corresponding to the roots of G' . On the other hand, the orthogonal complement $L(G')^\perp$ of $L(G')$ in $L(G)$ is invariant under $\text{Ad}|T$, and we can choose an orthogonal decomposition

$L(G) = L(T) \oplus \sum_{i=1}^m V_i$ so that $V'_j = V_j$ for $j \leq m'$. Thus the set of the roots of G' is a part of the set of roots of G . Q.E.D.

The restriction of the adjoint action Ad of G on $L(G)$ to G' is closed in $L(G')$ and hence in $L(G)^\perp$. Let $\pi: G \rightarrow G/G'$ be the projection and $\bar{e} = \pi(e)$. Then the differential $d\pi$ of π gives an isomorphism $L(G')^\perp \cong T_{\bar{e}}(G/G') \cong \mathbb{R}^{2m-2m'}$, where $\dim G = l + 2m$ and $\dim G' = l + 2m'$, and $Ad|_{G'}$ defines a homomorphism

$$h: G' \rightarrow O(2m - 2m').$$

LEMMA 7.4. If G is simple, then $Z(G) = \text{Ker } h$.

PROOF. We have $Z(G) \subset T \subset G'$, since $Z(G)$ is contained in an arbitrary maximal torus. We also have $Z(G) \subset \text{Ker } h$, since Ad is trivial on $Z(G)$.

For $a \in G'$, L_a and A_a induce the same map $G/G' \rightarrow G/G'$ and $dL_a: T_{\bar{e}}(G/G') \rightarrow T_{\bar{e}}(G/G')$ is equivalent to $Ad|_{L(G)^\perp}$. Therefore $\text{Ker } h = \{a \in G' | dL_a = 1\} = \{a \in G' | a \cdot \exp tX \in \exp tX \cdot G', \forall X \in L(G)\} = \{a \in G' | ag \in aG', \forall g \in G\} = \{a \in G' | g^{-1}ag \in G', \forall g \in G\}$. Thus, $a \in \text{Ker } h$ implies $g^{-1}ag \in \text{Ker } h$ for any g . That is, $\text{Ker } h$ is a normal, closed subgroup of G . If G is simple then $\dim \text{Ker } h = 0$ so $\text{Ker } h$ is a discrete group and $\text{Ker } h \subset Z(G)$ (Theorem 1.12(3) of Chapter I). Thus $Z(G) = \text{Ker } h$. Q.E.D.

DEFINITION. In the Stiefel diagram $D(G) = \bigcup_{r=1}^m \theta_r^{-1}(Z)$ of G we define

$$(\text{the center of } D(G)) = \exp^{-1}(Z(G))$$

$$= \{X \in L(T) | \theta_r^{-1}(X) \equiv 0 \pmod{1} \text{ for } r = 1, \dots, m\}.$$

When we choose the first half of the set of roots $\pm\theta_1, \dots, \pm\theta_{m'}$ of G as the roots of a subgroup of maximal rank G' , we put

$$D(G) - D(G') = \bigcup_{r=m'+1}^m \theta_r^{-1}(Z),$$

$$(\text{the center of } D(G) - D(G'))$$

$$= \{X \in L(G) | \theta_r(X) \equiv 0 \pmod{1} \text{ for } r = m' + 1, \dots, m\}.$$

THEOREM 7.5. If G is simple, then

$$(\text{the center of } D(G) - D(G')) = (\text{the center of } D(G)).$$

PROOF. If X is an element of the center of $D(G) - D(G')$, then $\exp X$ is an element of $\text{Ker } h$, since it acts on $L(G')^\perp = \sum_{i=m'+1}^m V_i$ trivially. Conversely, if $\exp X \in \text{Ker } h$, then X lies in the center of $D(G) - D(G')$ and hence $D(G) - D(G') = \exp^{-1}(\text{Ker } h)$. Thus the theorem follows immediately from Lemma 7.4. Q.E.D.

COROLLARY 7.6. $\dim G' \leq \dim G - 2 \text{rank } G$ for G simple.

PROOF. The center of $D(G) - D(G')$ is of dimension 0 and so $m - m' \geq l$. Q.E.D.

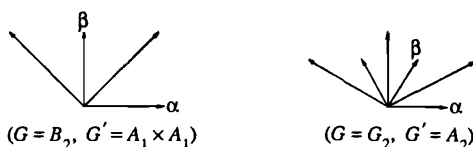


FIGURE 5.3

LEMMA 7.7. *Let α and β be roots of G' . If $\alpha + \beta$ is a root of G , then $\alpha + \beta$ is also a root of G' .*

PROOF. If $\beta = \pm\alpha$, then $\alpha + \beta$ is not a root, and hence $\text{rank } G \geq 2$.

First consider the case of $\text{rank } G = 2$. Only $A_2, B_2 = C_2$, and G_2 could be the type of G in the lemma. When it is of A_2 type, α and β of the lemma make an angle 60° or 120° and either $\alpha - \beta$ or $\alpha + \beta$ must be a root of G' . This shows $G' = G$, which is a contradiction, and hence the case in the lemma does not occur. For the cases of B_2 and G_2 , the only possibilities for α and β are the ones in Figure 5.3 contradicting the lemma.

If we take the coordinate in $\alpha = x_1$ and $\beta = x_2$ in the left upper diagram we have

$$(\text{the center of } D(G)) = \{(n_1, n_2) | n_1, n_2 \in \mathbb{Z}\}.$$

Since $D(G) - D(G')$ is given by the hyperplane $x_1 \pm x_2 \equiv 0 \pmod{1}$, $(\frac{1}{2}, \frac{1}{2})$ lies in the center but not in the center of $D(G)$. Thus this case does not occur.

If we take the coordinate in $\alpha = x_1$ and $\beta = x_2$ in the right upper diagram, $D(G)$ is given by the hyperplanes

$$x_1 \equiv 0, \quad x_2 \equiv 0, \quad x_1 - x_2 \equiv 0 \pmod{1}$$

$$x_1 + x_2 \equiv 0, \quad x_1 - 2x_2 \equiv 0, \quad 2x_1 - x_2 \equiv 0 \pmod{1}$$

and the equalities in the lower give the hyperplanes of $D(G) - D(G')$. Thus $(\frac{1}{3}, -\frac{1}{3})$ lies in the center of $D(G) - D(G')$ but not in the center of $D(G)$, and we see that this case does not occur either. Thus the lemma is true for $\text{rank } G = 2$.

Next let $\text{rank } G > 2$. Let U be the subspace of $L(T)$ given by $\alpha = 0$ and $\beta = 0$ and H the centralizer of $\exp U$ in G . We see that $D(H)$ consists of all the hyperplanes of $D(G)$ parallel to U and that the roots of H consist of all the roots of G contained in the two-dimensional plane V spanned by α and β . Locally H is a direct product of an $(l-2)$ -dimensional torus (which has U as a covering group) and a semisimple group H_2 of rank 2 which has the same roots as H . Similarly for G' , the roots of G' contained in V can be identified with the roots of $H'_2 = H_2 \cap G'$. Therefore, by the result for the case of rank 2, $\alpha + \beta$ is a root of H'_2 and hence of G' , and we obtain the lemma. Q.E.D.

The existence condition for a subgroup of maximal rank. We begin with the following lemma.

LEMMA 7.8. *Suppose that the roots $\theta, \theta_1, \dots, \theta_k$ of G satisfy the following conditions:*

- (1) $\theta_1, \theta_2, \dots, \theta_k$ are linearly independent, and $\theta \neq \pm\theta_j$ for $1 \leq j \leq k$.
- (2) There are positive numbers a_i such that

$$\theta = a_1\theta_1 + \dots + a_p\theta_p - a_{p+1}\theta_{p+1} - \dots - a_k\theta_k.$$

Then there exists at least one index j such that $\theta - \theta_j$ is a root of G if $j \leq p$ and $\theta + \theta_j$ is a root of G if $j > p$.

PROOF. Suppose that there does not exist such j as claimed in the lemma. If $1 \leq i \leq p$, then $\theta - \theta_i$ is not a root and hence by Theorems 6.5 and 6.3(2), $2(\theta, \theta_i)/(\theta_i, \theta_i) \leq 0$ implies $(\theta, \theta_i) \leq 0$. Similarly, if $p < i \leq k$, $\theta + \theta_i$ is not a root and hence $(\theta, \theta_i) \geq 0$. Thus $(\theta, \theta) = \sum_{i \leq p} a_i(\theta, \theta_i) - \sum_{i > p} a_i(\theta, \theta_i) \leq 0$ implies $\theta = 0$ so θ is not a root. This is a contradiction. Q.E.D.

COROLLARY 7.9. *If $a_1, \dots, a_k$ are integers in Lemma 7.9, then one can reach $\pm\theta$ by a finite sequence of roots $\theta_i, \theta_i \pm \theta_j, \dots, \pm\theta$ starting from θ_i , which is one of $\theta_1, \dots, \theta_k$, where the q th is obtained from the $(q-1)$ st by adding one of the roots $\pm\theta_1, \dots, \pm\theta_k$.*

PROOF. Apply induction on $\sum |a_i|$ to Lemma 7.8. Q.E.D.

THEOREM 7.10. *Let $\pm\theta_1, \dots, \pm\theta_m$ be the set of all the roots of G . Then $\pm\theta_1, \dots, \pm\theta_m$ is the set of roots of a subgroup G' of maximal rank of G if and only if the following two conditions hold:*

(1) *If k is the maximal number of linearly independent vectors in $\theta_1, \dots, \theta_m$, then each can be expressed as a linear combination with integer coefficients of some fixed k of them.*

(2) *A root of G which is a linear combination of $\theta_1, \dots, \theta_m$ with integer coefficients is a root of G' .*

PROOF. Necessity. Designate the fundamental Weyl chamber of G' and take the simple roots $\alpha_1, \dots, \alpha_k$ and then (1) is obvious (Theorem 6.10). Also for (2) it suffices to show that a linear combination of $\alpha_1, \dots, \alpha_k$ with integer coefficients is a root of G' if it is a root of G . To that end, by applying Lemma 7.7 to a sequence in Corollary 7.9 we see that each root of the sequence is a root of G' and so we obtain (2).

Sufficiency. Assume (1) and (2) and let the k roots of (1) be $\alpha_1, \dots, \alpha_k$. Consider

$$\begin{aligned} C &= \{z \in L(T) | \theta_i(z) \equiv 0 \pmod{1}; i = 1, \dots, m'\} \\ &= \{z \in L(T) | \alpha_i(z) \equiv 0 \pmod{1}; i = 1, \dots, k\}, \end{aligned}$$

and let G' be a connected component of the centralizer of $\exp(C)$. Since the invariant subspace V_i corresponding to θ_i for $1 \leq i \leq m$ is contained in $L(G')$, $\pm\theta_i$ are roots of G' .

If we show that C is not contained in $\theta_j^{-1}(Z)$ for $m' < j \leq m$, we will see that θ_j is not a root of G . Since θ_j is not a linear combination of $\alpha_1, \dots, \alpha_k$ with integer coefficients, the argument is divided into the following two cases:

(a) $\theta_j = a_1\alpha_1 + \dots + a_k\alpha_k$ and one of the coefficients, say a_1 , is not an integer.

Then $\theta_j(z)$ is not an integer for a point z given by $\theta_1(z) = 1$, $\theta_2(z) = \dots = \theta_k(z) = 0$, and hence $z \notin \theta_j^{-1}(Z)$, which implies $C \not\subset \theta_j^{-1}(Z)$.

(b) θ_j is independent of $\alpha_1, \dots, \alpha_k$.

Then $l = \text{rank } G > k$, $\bigcap_{i=1}^k \alpha_i^{-1}(0) (\subset C)$ is of dimension $l - k$ and there is a point sufficiently close to 0 which does not belong to $\theta_j^{-1}(Z)$ and hence $C \not\subset \theta_j^{-1}(Z)$. Q.E.D.

THEOREM 7.11. *Let G' be a connected subgroup of G of maximal rank and Z' the center of G' . Then G' is a connected component of the normalizer of Z' in G .*

PROOF. We see in the proof of Theorem 7.10 that $Z' = \exp(C)$ is the center of G' and that G' is the connected component of the normalizer of Z' in G . Consider the connected component $N(Z')_0$ of the centralizer of Z' in G and $A_x: Z' \rightarrow Z'$ for $x \in N(Z')_0$. Joining x and the identity by a path, we have $A_x \simeq 1: Z' \rightarrow Z'$. Thus A_x preserves each component of Z' , which is either a point or a torus, and by using Lemma 3.11 we see that $A_x = 1$. Therefore $N(Z')_0$ coincides with the connected component of the centralizer of Z' . Q.E.D.

COROLLARY 7.12. *The maximal connected, subgroup of maximal rank is a connected component of the normalizer of (= the centralizer) of some element.*

PROOF. The center Z' of a connected subgroup G' of maximal rank contains the center $Z(G)$ of G but does not coincide with it since $G \neq G'$. We have $G' \subset N(z) \subsetneq G$ for $z \in Z' - Z(G)$. If G' is maximal, then $G' = N(z)_0$. Q.E.D.

It will be shown later that the normalizer of any element other than one in the center of G is not necessarily maximal.

Maximal subgroups of maximal rank. We have already stated that studying the (types of) subgroups of maximal rank can be reduced to the case of a simple group G . If one knows maximal subgroups of maximal rank, one can determine the types of subgroups of maximal rank by a finite number of operations (cf. Corollary 7.12). Hereafter, let G be a simple group and we will study the maximal subgroups of maximal rank.

Let $S(G)$ be the closure of the fundamental Weyl cell of a simple group G . Then $S(G)$ is a simplex of dimension l given in terms of the simple roots $\alpha_1, \dots, \alpha_l$ and the dominant root $\tilde{\alpha}$ by

$$S(G) = \{X \in L(T) \mid \alpha_1(X) \geq 0, \dots, \alpha_l(X) \geq 0, \tilde{\alpha}(X) \leq 1\}.$$

An arbitrary point of G is mapped by an inner automorphism to a point $x = \exp X$ of T with $X \in L(T)$, and X is mapped by an element of the extended Weyl group to a point of $S(G)$. Then an arbitrary point of G is mapped by an inner automorphism to $z = \exp Z$ with $Z \in S(G)$. Therefore by Corollary 7.12 we see that in order to study maximal subgroups of maximal rank it suffices to consider only the normalizers of such z .

LEMMA 7.13. *Let $x = \exp X$ with $X \in S(G)$. If a connected component G' of the normalizer of x is a maximal, connected subgroup of maximal rank, X can be one of the following:*

- (1) *a vertex of $S(G)$,*
- (2) *an interior point of the edge whose two vertices belong to the center of $D(G)$,*

where G' is independent of the choice of an interior point in case (2), and is conjugate to a group for which one of two vertices is the origin in (2).

PROOF. If X is an interior point of $S(G)$, then $N(x) = T$, which is not maximal for $l > 1$. If X is an interior point of a face of dimension ≥ 2 of $S(G)$, then the normalizer of the image by $\exp$ of an interior point of an edge of the boundary of the face is larger than $N(x)$ (Theorem 4.15). If X is an interior point of an edge of $S(G)$, and if an endpoint X' of the edge is not a center of $D(G)$, then $N(x') \neq G$ for $x' = \exp X'$ and $N(x')$ is larger than $N(x)$. Thus, we are left with the cases (1) and (2).

Consider case (2) and let Y be an endpoint that belongs to the center of $D(G)$ and $y = \exp Y$. Then $y \in Z(G)$ implies $N(y^{-1}x) = N(x)$, where $y^{-1}x = \exp(X - Y)$ and $X - Y$ is an interior point of an edge issuing from the origin in the closure S' of a Weyl cell, which contains the origin. Considering the action of the Weyl group, which maps S' to $S(G)$, we see that there is an interior point X' of an edge of $S(G)$ issuing from the origin such that $N(y^{-1}x) = N(x')$ with $x' = \exp X'$.

Lastly, independence of the choice of an interior point is obtained as in the proof of Theorem 4.15. Q.E.D.

Incidentally, the following corollary is obtained simultaneously.

COROLLARY 7.14. *If one vertex is the origin in Lemma 7.13(2), then $G' = Z(S)_0$ for a one parameter subgroup (a one-dimensional torus) S , which is obtained by $\exp$ from a straight line containing the edge.*

Express the dominant root $\tilde{\alpha}$ by a linear combination of the roots $\alpha_1, \dots, \alpha_l$ with integral coefficients:

$$\tilde{\alpha} = m_1\alpha_1 + \dots + m_l\alpha_l \quad (m_i: \text{a positive integer}).$$

Let $\mathbf{0}$ be the origin of $S(G)$, define the other vertices X_i of $S(G)$ by

$$\alpha_j(X_i) = 0 \quad \text{for } j \neq i \text{ and } \tilde{\alpha}(X_i) = 1$$

and put $x_i = \exp X_i \in T \subset G$. Then the following theorem holds.

THEOREM 7.15. *Let G be a compact, connected simple Lie group.*

(1) *The normalizer of x_i is maximal if and only if m_i is a prime (> 1).*

(2) *The normalizer of $\exp(X_i/2)$ is maximal if and only if $m_i = 1$. One can find by this all the types of maximal subgroups of maximal rank of G .*

PROOF. (1) *Necessity.* Suppose that $m_i = ab$ with integers $a, b > 1$. $D(N(x_i))$ is a family of the parallel hyperplanes of $D(G)$, which contains X_i . Therefore the set of roots of $N(x_i)$ consists of the roots that are linear combinations of $\alpha_1, \dots, \alpha_l$ such that the coefficient of α_i is either 0 or $\pm m_i$. Similarly, by putting $y_i = \exp(aX_i)$, the set of the roots of $N(y_i)$ consists of the roots such that the coefficient of α_i is a multiple of b . Since there is a root of $N(y_i)$ that has b as a coefficient by Corollary 7.9, we have $N(y_i) \neq N(x_i)$. Since $N(y_i) = N(x_i^a) \supset N(x_i)$, $N(x_i)$ is not maximal. Therefore $N(x_i)$ is maximal and hence m_i is a prime or 1. If $m_i = 1$, then x_i lies in the center of G and hence $N(x_i) = G$. Thus m_i is a prime.

Sufficiency. It suffices to show $N(x_i) = N(y)$ if $N(x_i) \subset N(y) \subsetneq G$ where $y = \exp Y$. $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_l$ and $\tilde{\alpha}$ are roots of $N(x_i)$ as well as $N(y)$. Since y lies in the center of $N(y)$, $\alpha_j(Y)$ for $j \neq i$ and $\tilde{\alpha}(Y)$ are integers. Since y does not lie in the center of G , there is an integer q such that $\alpha_i(Y) = q/m_i$ with $q \not\equiv 0 \pmod{m_i}$. Let Z be a point given by $\alpha_j(Z) = \alpha_j(Y)$ for $j \neq i$ and $\alpha_i(Z) = 0$. If one puts $z = \exp Z$, then z is an element of the center of G such that $y = x_i^q z$.

Since m_i is a prime, there exists an integer r such that $qr \equiv 1 \pmod{m_i}$ and that $y^r = x_i z^r$ with $z^r \in Z(G)$, whence $N(y) \subset N(y^r) = N(x_i z^r) = N(x_i)$ and so $N(x_i) = N(y)$.

(2) *Necessity* follows from Lemma 7.13.

Sufficiency. $Y = X_i/2$ is the midpoint of the edge $\mathbf{0}X_i$. Putting $y = \exp Y$, $D(N(y))$ consists of all the hyperplanes of $D(G)$ that are parallel to $\mathbf{0}X_i$. If N' is a subgroup larger than $N(y)$, then $D(N')$ contains at least one hyperplane that is not parallel to $\mathbf{0}X_i$. So if we denote by $\tilde{\alpha}' = n_1\alpha_1 + \dots + n_l\alpha_l$ the root corresponding to it, we have $n_i \neq 0$. Since $\tilde{\alpha}$ is the dominant root and $m_i = 1$, we have $n_i = \pm m_i = \pm 1$. Thus $\alpha_1 \equiv \dots \equiv \alpha_{i-1} \equiv \alpha_{i+1} \equiv \dots \equiv \alpha_l \equiv \tilde{\alpha}' \equiv 0$ is equivalent to $\alpha_1 \equiv \dots \equiv \alpha_l \equiv 0$, and the center of N' is contained in that of G , whence $N' = G$, that is, $N(y)$ is maximal. Q.E.D.

THEOREM 7.16. *The Dynkin diagram of a maximal subgroup of maximal rank in Theorem 7.13 is obtained from that of G as follows:*

(1) *If m_i is a prime, then the Dynkin diagram of $G_i = N(x_i)$ is obtained by deleting α_i from the extended Dynkin diagram, which is the Dynkin diagram of G with $-\tilde{\alpha}$ added.*

(2) If $m_i = 1$, then $N(\exp(X_i/2))$ is locally a direct product of a one-dimensional torus T and a semisimple group G_{l-1} of rank $l-1$, and the Dynkin diagram of G_{l-1} is obtained by deleting α_i from the Dynkin diagram of G .

PROOF. (1) $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_l, -\tilde{\alpha}$ are the roots of $N(x_i)$. By the fact that the coefficient of α_i of a root of $N(x_i)$ is a multiple of m_i we see that they can be taken as the simple roots of $N(x_i)$ (cf. Theorem 6.10).

(2) The set of the hyperplanes of the Stiefel diagram of $N(\exp(X_i/2))$ consists of those that are parallel to $0X_i$ and they correspond to the roots that can be written as a linear combination of $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_l$, and give a system of simple roots of G_{l-1} . Q.E.D.

If we number the simple roots as in 6.26, the dominant root is given by 6.31 and the following result is obtained.

7.17. The type of a maximal subgroup of maximal rank G' corresponding to each coefficient m_i of the dominant root of a simple group G is given as in Table 5.1.

REMARK 7.18. . If $m_i = 1$ or 2, then G/G' is a symmetric space. The symmetric spaces of exceptional type obtained here are expressed in the notation of E. Cartan as follows:

$$\begin{array}{lll} EII = E_6/(A_1 \times A_5), & EIII = E_6/(D_5 \times T), & EV = E_7/A_7, \\ EVI = E_7/(D_6 \times A_1), & EVII = E_7/(E_6 \times A_1), & EVIII = E_8/D_8, \\ EIX = E_8/(E_7 \times A_1), & FI = F_4/(C_3 \times A_1), & FII = F_4/B_4, \\ G = G_2/(A_1 \times A_1). \end{array}$$

In addition to them there are the following two (irreducible) symmetric spaces of exceptional type:

$$EI = E_6/C_4, \quad EIV = E_6/F_4.$$

REMARK 7.19. If G is an exceptional group, by taking G to be 1-connected, G' is given as follows:

$$\begin{array}{lll} (SU(2) \times SU(6)) \subset E_6, & (\text{Spin}(10) \times T)/Z_4 \subset E_6, & SU(8)/Z_2 \subset E_7 \\ (\text{Spin}(12) \times SU(2))/Z_2 \subset E_7, & (E_6 \times T)/Z_3 \subset E_7, & Ss(16) \subset E_8, \\ (E_7 \times SU(2))/Z_2 \subset E_8, & (Sp(3) \times Sp(1))/Z_2 \subset F_4, & \text{Spin}(9) \subset F_4, \\ SO(4) \subset G_2, \end{array}$$

where a generator of Z_n in the form $(H_1 \times H_2)/Z_n$ is a product form of generators of the centers of H_1 and H_2 . In particular $\text{Spin}(12)$ divided by a generator is $Ss(12)$.

TABLE 5.1

G	i	m_i	G'
A_l	$1 < i < l$	1	$A_{i-l} \times A_{l-i} \times T$
B_l	$i = 1$	1	$B_{i-l} \times T$
	$1 < i < l$	2	$B_{i-l} \times D_{l-i+1}$
	$i = l$	2	D_l
C_l	$1 \leq i < l$	2	$C_{i-l} \times C_{l-i+1}$
	$i = l$	1	$A_{l-1} \times T$
D_l	$i = 1$	1	$D_{i-l} \times T$
	$1 < i < l-1$	2	$B_{i-l} \times D_{l-i+1}$
	$i = l-1, l$	1	$A_1 \times T$
E_6	$i = 1, 6$	1	$D_5 \times T$
	$i = 2, 3, 5$	2	$A-1 \times A_5$
	$i = 4$	3	$A_2 \times A_2 \times A_2$
E_7	$i = 1, 6$	2	$D_6 \times A_1$
	$i = 2$	2	A_7
	$i = 3, 5$	3	$A_2 \times A_5$
	$i = 7$	1	$E_6 \times T$
E_8	$i = 1$	2	D_8
	$i = 2$	3	A_8
	$i = 5$	5	$A_4 \times A_4$
	$i = 7$	3	$E_6 \times A_2$
	$i = 8$	2	$E_7 \times A_1$
F_4	$i = 1$	2	$C_3 \times A_1$
	$i = 2$	3	$A_2 \times A_2$
	$i = 4$	2	B_4
G_2	$i = 1$	3	A_2
	$i = 2$	2	$A_1 \times A_1$

Also for $m_i = 3$ or 5 , by dividing by a central element common to each factor, G' is given as follows:

$$\begin{aligned} (SU(3) \times SU(3) \times SU(3))/Z_3 &\subset E_6, & (SU(3) \times SU(6))/Z_3 &\subset E_7, \\ SU(9)/Z_3 &\subset E_8, & (E_6 \times SU(3))/Z_3 &\subset E_8, \\ (SU(3) \times SU(3))/Z_3 &\subset F_4, & SU(3) &\subset G_2, \\ (SU(5) \times SU(5))/Z_5 &\subset E_8. \end{aligned}$$

Furthermore, we have

$$C_4 = PSp(4) = Sp(4)/Z_2 \quad \text{in } EI.$$

CHAPTER VI

The Bott-Morse Theory

In this chapter we develop Bott-Morse theory and apply it to the topology of the exceptional Lie groups. After introducing classical Morse theory in §1, in particular, the Morse inequalities, we explain how the Poincaré series of a path space in a Riemannian space is expressed in terms of the index of geodesics. In §2 we introduce K -transversal geodesics and their index and variational completeness, which are key notions in the Bott-Morse theory. Variational completeness of symmetric spaces and other spaces are verified in §3. §4 is an application to compact Lie groups G to express the homology of their loop spaces and of homogeneous spaces of G in terms of the indices of geodesic segments. More concretely we determine the rank and the rational type of the exceptional Lie groups in §5. As preparation for the next section we introduce in §6 the notion of a K -cycle, relate it to the Bott-Morse theory, and afterwards assign to each root a subgroup of dimension three in G . In particular we show, for the exceptional groups $G = E_l$ ($l = 6, 7, 8$), that the subgroup S^3 of dimension three represents a generator of $\pi_3(G)$ for any root. In §7 we apply the above theory to calculate the cohomology of the loop space of the exceptional group G and the homotopy groups of G .

§1. Geodesics and Morse theory

Geodesics. Throughout this chapter a manifold M is a connected, paracompact Riemannian manifold.

A curve

$$h: \mathbf{R} \text{ (or a closed interval } [a, b]) \rightarrow M$$

in a manifold M is a C^∞ -map, and consider the differential

$$dh: T\mathbf{R} = \mathbf{R} \times \mathbf{R} \rightarrow TM.$$

NOTATION.

$$\dot{h}(t) = \partial_t h(t) = (\text{the tangent vector to } h \text{ at } h(t) \in T_{h(t)}M) = dh(t, 1),$$

where $(t, 1) \in T\mathbf{R}$ is the standard tangent vector at $t \in \mathbf{R}$.

The length of a curve $h: [a, b] \rightarrow M$ on a Riemannian manifold M is

defined to be $\int_a^b |\dot{h}(t)| dt$. The distance $\rho(x, y)$ between two points $x, y \in M$ is defined to be the infimum of the lengths of curves joining x and y , and M is a metric space with metric ρ .

A *geodesic*

$$h: (a, b) \subset \mathbb{R} \rightarrow M$$

on a Riemannian manifold M can be defined to be a curve that has locally the (shortest) distance as length such that the arc length of h is proportional to the variable t . Using the terms of differential geometry a geodesic is defined as follows.

DEFINITION. $h: (a, b) \rightarrow M$ is called a *geodesic* ($-\infty \leq a < b \leq \infty$) if, when h is expressed as $h(t) = (x^1(t), \dots, x^m(t))$ in local coordinates,

$$\frac{d^2}{dt^2} x^i(t) + \sum_{j,k} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \quad (\forall t \in (a, b), i = 1, \dots, m),$$

where (the Christoffel symbol) $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} \sum_l g^{li} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$ and (g^{li}) is the inverse matrix of the Riemannian metric $g = (g_{ij})$.

1.1. If $c, c+t \in (a, b)$ for a geodesic $g: (a, b) \rightarrow M$, then the length of a geodesic segment $g|_{[c, c+t]}$ is proportional to t .

If the initial conditions

$$g(0) = x \quad \text{and} \quad \dot{g}(0) = X \in T_x M$$

are given for a point x of M , a geodesic $g: (-\varepsilon, \varepsilon) \rightarrow M$ is uniquely obtained in a neighborhood of x as the solution of the ordinary differential equation of the second order in the above. The Riemannian manifold M is said to be *complete* if the geodesic can be extended over $\mathbb{R}$ as the solution curve. As a matter of fact, it turns out that completeness is equivalent to completeness of M as a metric space (the Hopf-Rinow theorem). In particular, a compact M and Euclidean spaces are complete. Hereafter we suppose that M is complete. If we write $g(t) = \exp_x(tX)$ for the geodesic $g: \mathbb{R} \rightarrow M$ thus obtained with $\dot{g}(0) = X$, a map

$$\exp_x: T_x M \rightarrow M$$

is determined, and is called the *exponential map*, with $g(0) = x$ the *initial point* of g . Consider the map $\text{Exp}: TM \rightarrow M \times M$ defined by $\text{Exp}(X) = (p(X), \exp_x X)$, $X \in T_x M$. We see that Exp is a homeomorphism of a neighborhood of the zero vector 0_x of $T_x M$ in TM to a neighborhood of (x, x) , since $d(\exp_x) = 1: T_x M \rightarrow T_x M$. Furthermore, the following is obtained:

1.2. For an arbitrary point of M , there is $\varepsilon > 0$ such that in an ε -neighborhood U of the point there always exists a geodesic (segment) joining any two points of U and it is unique in U .

If one uses the notion of parallel transport along a curve h in a Riemannian manifold, then one can describe a geodesic as a curve on which the parallel transport maps tangent vectors to tangent vectors.

To discuss geodesics on a general Riemannian manifold we need various notions of differential geometry. Our objects are, however, Lie groups and homogeneous spaces; for these cases one can fortunately construct geodesics by using one parameter subgroups as shown in §2 and so based on this in applications, we do not need to enter the deep domains of differential geometry.

The Jacobi vector field

DEFINITION. A vector field $v: \mathbb{R} \rightarrow TM$ along a (C^∞) -curve $h: \mathbb{R} \rightarrow M$ is a C^∞ -map v such that $v(t) \in T_{h(t)}M$ (that is, $\pi \circ v = h$ for the projection $\pi: TM \rightarrow M$).

Let $g: \mathbb{R} \rightarrow M$ be a geodesic in a Riemannian manifold M .

DEFINITION. A map $V: \mathbb{R} \times (-\varepsilon, \varepsilon) \rightarrow M$ is called a (geodesic) variation of the geodesic g if $\varepsilon > 0$ and V is a C^∞ -map such that for each $\alpha \in (-\varepsilon, \varepsilon)$ the map $V_\alpha: \mathbb{R} \rightarrow M$ (given by $V_\alpha(t) = V(\alpha, t)$) is a geodesic of M and $V_0 = g$.

DEFINITION. A vector field v along a geodesic g is called a Jacobi vector field if there exists a variation V of g such that the following is satisfied:

$$v(t) = \partial_\alpha V_\alpha(t). \quad (1.3)$$

REMARK 1.4. The equality (1.3) implies the following about a C^∞ -function $f: M \rightarrow \mathbb{R}$:

$$v(t)f = (\partial/\partial\alpha)f(V_\alpha(t))|_{\alpha=0}.$$

DEFINITION. The restriction of a geodesic $g: \mathbb{R} \rightarrow M$ to a closed interval $[0, a]$ ($0 < a$)

$$s = g|_{[0, a]}: [0, a] \rightarrow M$$

is called a geodesic segment in M and $s(0)$ and $s(a)$ are the initial and terminal points of the geodesic segment respectively.

NOTATION. $J_g = \{\text{Jacobi vector fields along } g\}$.

$J_s = J_g|_s = \{\text{the restrictions to } s \text{ of Jacobi vector fields along } g\}$.

Each geodesic V_α of the variation V of g is determined by (the initial point) $V_\alpha(0)$ and the tangent vector $\dot{V}_\alpha(0)$ there. That is, V is determined as a C^∞ -map of $(-\varepsilon, \varepsilon)$ to a neighborhood of $\dot{g}(0)$ in TM . Moreover, the Jacobi vector field v obtained from V by (1.3) is determined by tangent vectors of the curve. Therefore we see the following:

1.5. J_g is a vector space of dimension $2 \dim M$ and the map given by the restriction

$$s_*: J_g \rightarrow J_s$$

is an isomorphism. Thus we identify J_g with J_s .

NOTATION.

$$\begin{aligned}\Lambda_g(t_0) &= \{v \in J_g | v(t_0) = 0\} \\ &= \{\text{the Jacobi fields along } g \text{ which vanish at } t_0\}.\end{aligned}$$

1.6. Any Jacobi vector field v of $\Lambda_g(t_0)$ is given by $v(t) = \partial_\alpha V_\alpha(t)$ for a variation V satisfying $V(t_0, \alpha) = g(t_0)$.

Further, such V corresponds to a curve $h(\alpha)$ satisfying $h(0) = \dot{g}(t_0)$ in $T_{g(t_0)}M$ by $h(\alpha) = dV(t, \alpha)_{t=t_0}$. Then the Jacobi vector field v given by V is determined by $\theta(v) = \dot{h}(0)$. Thus

$$\theta : \Lambda_g(t_0) \cong T_{g(t_0)}M \quad (\text{an isomorphism}). \quad (1.7)$$

A focal point of a submanifold and the index of a geodesic segment. Let N be a regular submanifold of a Riemannian manifold M .

DEFINITION. $g: \mathbb{R} \rightarrow M$ (or $g: [0, a] \rightarrow M$, $a > 0$) is called a *geodesic* (or a *geodesic segment*) of $M \bmod N$ if g is a geodesic (or a geodesic segment) in M such that $g(0) \in N$ and $\dot{g}(0) \perp N$. ($v \perp N$ means that a vector v of $T_x M$ is perpendicular to $T_x N$, that is, it is a normal vector of N).

DEFINITION. v is called a *Jacobi vector field* of $M \bmod N$ along g if v is given by a variation $V: \mathbb{R} \times (-\varepsilon, \varepsilon) \rightarrow M$ of g satisfying the following properties:

- (1) $v(t) = \partial_\alpha V_\alpha(t)$ ($t \in \mathbb{R}$, $\alpha \in (-\varepsilon, \varepsilon)$).
- (2) Each V_α is a geodesic of $M \bmod N$ ($V_\alpha(0) \in N$, $\dot{V}_\alpha(0) \perp N$).

NOTATION. $J_g^N = \{\text{Jacobi vector fields of } M \bmod N \text{ along } g\}$.

$V_\alpha(0)$ in the above is a path moving in N , and conversely, since V is determined by such a path, we have

$$v \in J_g^N \quad \text{iff} \quad v \in J_g, v(0) \in T_{g(0)}N \subset T_{g(0)}M. \quad (1.8)$$

DEFINITION. Let g be a geodesic and $s = g| [0, a]$ a geodesic segment of $M \bmod N$.

- (1) $\Lambda_s(t) = \Lambda_s^N(t) = J_g^N \cap \Lambda_g(t) = (\text{the focal kernel of } s| [0, t]), 0 < t \leq a$.
- (2) $\lambda_s(t) = \lambda_s^N(t) = \dim \Lambda_s(t)$, $0 < t \leq a$.
- (3) s is called a *focal segment* of $M \bmod N$ if $\lambda_s(a) > 0$. Then the terminal point $s(a)$ is called a *focal point* of $s \bmod N$.
- (4) A point of $M - N$ that is not a focal point of a geodesic segment $\bmod N$ is called a *regular point* of M relative to N .
- (5) s is called *nondegenerate* (in the sense of Morse) if $\lambda_s(a) = 0$ (that is, s is not a focal segment).
- (6) $s(b)$ is called an *exceptional point* of s if the terminal point $s(b)$ of s' is a focal point of $s' \bmod N$ in a subsegment $s' = s| [0, b]$ ($0 < b \leq a$) of s .

REMARK 1.9. A focal point is a generalization of a conjugate point (the case where N is a point) in a geodesic.

DEFINITION. If $s: [0, a] \rightarrow M$ is a geodesic segment of $M \bmod N$, then

$$\lambda_s = \lambda_s^N = \sum_{0 < t \leq a} \lambda_s(t) = \sum_{0 < t \leq a} \dim(J_g^n \cap \Lambda_g(t)) \quad (1.10)$$

is called the *index* of s relative to N (cf. Theorem 1.14(1)).

Path spaces in a Riemannian space. Let M be a compact Riemannian space and let A and B be subspaces of M .

$u: [0, 1] \rightarrow M$ is called a piecewise C^∞ -curve from A to B if there exists $0 = t_0 < t_1 < \dots < t_n = 1$ such that $u|_{[t_{i-1}, t_i]}$ is a C^∞ -curve and $u(0) \in A$ and $u(1) \in B$.

DEFINITION.

$\Omega'(M, A, B) = \{u: [0, 1] \rightarrow M \mid \text{a piecewise } C^\infty\text{-curve from } A \text{ to } B\}$.

$\Omega^*(M, A, B) = \{u \in \Omega'(M, A, B) \mid \text{the arc length is proportional to the parameter } t\}$.

Let $\Omega'(M, A, B)$ be a metric space with distance ρ defined by $\rho(u, u') = \sup_{0 \leq t \leq 1} \{d(u(t), u'(t)) + |L(u) - L(u')|\}$, $u, u' \in \Omega'(M, A, B)$, where d is the distance in M , and L denotes the length of a curve.

We introduce the compact-open topology in

$$\Omega(M, A, B) = \{u: [0, 1] \rightarrow M \mid u \text{ is continuous, } u(0) \in A \text{ and } u(1) \in B\}$$

but it is well known that this is equivalent to the topology given by the metric $\rho'(u, u') = \sup_{0 \leq t \leq 1} d(u(t), u'(t))$. Thus the natural correspondence $i: \Omega'(M, A, B) \rightarrow \Omega(M, A, B)$ is continuous.

For $u \in \Omega'(M, A, B)$ there is determined a monotonously increasing function $\varphi: [0, 1] \rightarrow [0, 1]$ that satisfies $u \circ \varphi \in \Omega^*(M, A, B)$. By $h_\theta(u)(t) = u((1 - \theta)t + \theta\varphi(t))$ we have that

1.11. $\Omega^*(M, A, B)$ is a deformation retract of $\Omega'(M, A, B)$.

THEOREM 1.12. $i: \Omega'(X, A, B) \rightarrow \Omega(X, A, B)$ is a weak homotopy equivalence, that is, i induces isomorphisms of homotopy groups and homology groups.

PROOF. Put

$$\Omega_n = \left\{ u \in \Omega'(X, A, B) \mid u|_{\left[\frac{i-1}{n}, \frac{i}{n} \right]} \text{ for } i = 1, 2, \dots, n \text{ is a geodesic segment} \right\}.$$

Ω_n may be considered a subspace of $M^{n+1} = M \times \dots \times M$ ($(n+1)$ copies) by means of the values at the dividing points of u , and it has the topology of a subspace of $\Omega(M, A, B)$.

Consider an arbitrary $f: I^k \rightarrow \Omega(M, A, B)$. We cover M by sufficiently small neighborhoods U_α such that any two points in U_α can be joined uniquely by a geodesic. Since the image of the adjoint map $F: I^k \times I \rightarrow M$ of f is compact, we can take n sufficiently large so that a cube with sides of length $1/n$ in $I^k \times I$ is mapped by F into some U_α . Then, since $u \in \Omega_n$ is determined such that $f(x)(i/n) = u(i/n)$ ($i = 0, 1, \dots, n$) for every point x of I^k , we put

$$f_\theta(x)(t) = (\text{a point dividing a geodesic segment joining } f(x)(t) \\ \text{and } u(t) \text{ in the ratio of } \theta : (1 - \theta)).$$

$f_\theta: I^k \rightarrow \Omega(M, A, B)$ is a homotopy such that $f_\theta(\partial I^k) = u_0$ if $f(\partial I^k) = u_0 \in \Omega$. Since $f_0 = f$ and $f_1(I^k) \subset \Omega_n$, any element of $\pi_k(\Omega(M, A, B))$ is in the image of i_{n*} with sufficiently large n for

$$i_{n*}: \pi_k(\Omega_n) \rightarrow \pi_k(\Omega(M, A, B)).$$

This shows that $i_*: \pi_k(\Omega'(M, A, B)) \rightarrow \pi_k(\Omega(M, A, B))$ is an epimorphism, since i_{n*} factors through $\pi_k(\Omega'(M, A, B))$.

Since the above argument for f_θ also holds in $\Omega'(M, A, B)$, we may suppose that any element α of $\pi_{k-1}(\Omega'(M, A, B))$ is represented by a map to Ω_m . If $i_*(\alpha) = 0$, we denote by $f: I^{k-1} \times I = I^k \rightarrow \Omega(M, A, B)$ the homotopy to show this triviality and construct f_θ in the above by taking sufficiently large n which is a multiple of m . By the definition of f_θ , we see that if $f(x) \in \Omega_m$ then $f_\theta(x) = f(x)$. Thus $f|I^{k-1} \times 0 = f_1|I^{k-1} \times 0 \simeq f_1|I^{k-1} \times 1 = f|I^{k-1} \times 1 = \text{a constant map}$. Here since the homotopy $f_1|I^{k-1} \times I$ is in Ω_n , α is 0 in $\pi_{k-1}(\Omega_n)$ and hence in $\pi_{k-1}(\Omega'(M, A, B))$. Thus we have also shown that i_* is a monomorphism. Q.E.D.

Since M is arcwise connected, a map

$$*w: \Omega(M, A, p) \rightarrow \Omega(M, A, q)$$

induced by a path w joining two points p and q of M is a homotopy equivalence with the homotopy inverse $*\bar{w}$ induced by the inverse path $\bar{w}$. Therefore, taking the base point x_0 of M arbitrarily, we have that

$$\Omega(M, A) = \Omega(M, A, x_0),$$

by the usual notation, is homotopy equivalent to $\Omega(M, A, p)$ for any p .

COROLLARY 1.13. *For any point r and any subspace N of M , $\Omega^*(M, N, r)$ is weakly homotopy equivalent to $\Omega(M, N)$.*

From Morse theory. We quote the main results from M. Morse's book [42]. Since there are many expository books on the Morse theory, the reader should study the details from them.

Let M be a complete, connected Riemannian manifold and N its regular submanifold.

THEOREM 1.14. (1) *The index is finite for all geodesic segments of $M \bmod N$.*

(2) *The set of regular points of $M \bmod N$ is everywhere dense.*

(3) *If r is a regular point of $M \bmod N$ and l is a positive number, then there are only a finite number of geodesic segments of $M \bmod N$ of length $\leq l$ with r the terminal point.*

NOTATION. $S(M, N, r) = \{s: [0, 1] \rightarrow M | s \text{ is a segment of } M \bmod N \text{ and } s(1) = r\}$.

REMARK. We regard as $S(N, N, r) \subset \Omega^*(M, N, r)$.

If r is a regular point of $M \bmod N$, then $S(M, N, r)$ is a discrete set by Theorem 1.14(3) and coincides with the set of critical points of the function

$$L: \Omega^*(M, N, r) \rightarrow \mathbf{R}$$

taking the length. The following theorem, which is more detailed, is fundamental to Morse theory (cf. Lemma 6.6).

THEOREM 1.15. *Let $s \in S(M, N, r)$ be nondegenerate and of index λ_s . Then there exist a simplex σ_s of dimension λ_s and a map*

$$\phi_s: \sigma_s \rightarrow \Omega^*(M, N, r)$$

such that for the barycenter b_s of σ_s , the following hold:

- (1) $\phi_s(b_s) = s$.
- (2) $L \circ \phi_s: \sigma_s \rightarrow \mathbf{R}$ takes the maximal value c at b_s .
- (3) $L \circ \phi_s$ is a C^∞ -map and the maximal values are nondegenerate near b_s (i.e., the expansion by a suitable coordinate near b_s begins with a negative definite bilinear form).
- (4) Under these σ_s and ϕ_s , $\{u \in \Omega^* | L(u) < c\} \cup s$ has a deformation retract $\{u \in \Omega^* | L(u) \leq c - \varepsilon\} \cup \phi_s(\sigma_s)$ near s .

We simply write

$$\Omega^* = \Omega^*(M, N, r) \quad (1.16)$$

and consider the following subspace for the function L :

$$\{c_k; k = 1, 2, 3, \dots\} = L(S(M, N, r)),$$

where $c_k \in \mathbf{R}$ are arranged in the order of the size $c_1 < c_2 < c_3 < \dots$. We define subspaces Ω_k and Ω_k^- of Ω^* as follows:

$$\Omega_k = \{u \in \Omega^* | L(u) \leq c_k\}, \quad \Omega_k^- = \{u \in \Omega^* | L(u) < c_k\}.$$

By deforming Ω_{k+1}^- to Ω_k along a flow that is transversal to the inverse image by L of each point, Ω_k is a deformation retract of Ω_{k+1}^- . Since $\omega_k = \{s \in S(M, N, r) | L(s) = c_k\}$ is the obstruction to continuing this further, one can deform Ω_k into $\Omega_k^- \cup \omega_k$, hence avoiding it. Using Theorem 1.15 one

sees that Ω_k has a deformation retract $\{u \in \Omega_k | L(u) \leq c_k - \varepsilon\} \cup_{s \in \omega_k} \phi_s(\sigma_s)$. Therefore by the excision and homotopy axioms one has

$$\sum_{s \in \omega_k} H_*(\sigma_s, \partial \sigma_s) \cong H_*(\Omega_k, \Omega_k^-) \cong H_*(\Omega_k, \Omega_{k-1}).$$

Summing up the above the following theorem is deduced:

THEOREM 1.17. $\sum_{s \in \omega_k} H_*(\sigma_s, \partial \sigma_s) \cong H_*(\Omega_k, \Omega_{k-1})$ where $\dim \sigma_s = \lambda_s$.

The Morse series

DEFINITION. $\mathcal{M}(M, N, r; t) = \sum_{s \in S(M, N, r)} t^{\lambda_s}$ is called the *Morse series* of $M \bmod N$, and the series is said to *exist* if every coefficient is finite and definite.

THEOREM 1.18 (The Morse inequality). *Let r be a regular point of $M \bmod N$ and suppose that the Morse series $\mathcal{M}(M, N, r; t)$ exists. Then for any coefficient field $\mathbf{k}$ there exists a formal power series $B(\mathbf{k}; t) = \sum_{i \geq 0} b_i t^i$ with nonnegative integer coefficients b_i such that*

$$\mathcal{M}(M, N, r; t) = P(\Omega(M, N), \mathbf{k}; t) + (1+t)B(\mathbf{k}; t),$$

where $P(\Omega(M, N), \mathbf{k}; t) = \sum_{i \geq 0} \dim H_i(\Omega(M, N); \mathbf{k}) t^i$ is the *Poincaré series*.

REMARK 1.19. When one puts

$$\mathcal{M}(M, N, r; t) = \sum m_i t^i \quad \text{and} \quad P(\Omega(M, N), \mathbf{k}; t) = \sum p_i t^i,$$

one obtains the equalities

$$m_i - p_i = b_i + b_{i-1} \quad \text{for } i = 1, 2, \dots \text{ and with } b_{-1} = 0.$$

In particular,

$$\begin{aligned} m_0 &\geq p_0, \\ m_1 - m_0 &\geq p_1 - p_0, \\ m_2 - m_1 + m_0 &\geq p_2 - p_1 + p_0, \\ &\dots \end{aligned}$$

These are the well-known *Morse inequalities* and the system of the inequalities is equivalent to Theorem 1.18.

Theorem 1.18 is proved algebraically from Theorem 1.17. To simplify the description we introduce the following definition:

DEFINITION. Let $P(t)$ and $Q(t)$ be two formal power series. $P(t)$ is said to *dominate* $Q(t)$, denoted by

$$P(t) \succ Q(t),$$

if there exists a formal power series $B(t)$ with nonnegative coefficients such that

$$P(t) - Q(t) = (1+t)B(t).$$

LEMMA 1.20. If $C = \sum_{i \geq 0} C_i$ is a chain complex of finite type over a field $\mathbf{k}$, the Poincaré series of C dominates the Poincaré series of the homology group $H_*(C) = \sum_{i \geq 0} H_i(C)$:

$$P(C, t) \succ P(H_*(C), t).$$

PROOF. Putting $c_i = \dim C_i$, $h_i = \dim H_i(C)$, $z_i = \dim Z_i(C)$, and $b_i = \dim B_i(C)$, the exact sequence $0 \rightarrow Z_i(C) \rightarrow C_i \xrightarrow{\partial} B_{i-1}(C) \rightarrow 0$ implies $c_i = z_i + b_{i-1}$. $H_i(C) = Z_i(C)/B_i(C)$ implies $h_i = z_i - b_i$. Therefore, for $B(t) = \sum b_i t^i$, we have

$$\begin{aligned} P(C, t) - P(H_*(C), t) &= \sum c_i t^i - \sum h_i t^i = \sum (z_i + b_{i-1} - (z_i - b_i)) t^i \\ &= \sum (b_i + b_{i-1}) t^i = (1+t)B(t). \end{aligned}$$

Q.E.D.

PROOF OF THEOREM 1.18. We consider a subcomplex $A_p = C_*(\Omega_p; \mathbf{k})$ of the singular chain complex $C_*(\Omega^*; \mathbf{k})$ similarly to the construction of the spectral sequence in §2 of Chapter III. Since $A_p \supset A_{p-1}$ and $dA_p \subset A_p$, a spectral sequence $\{E_{p,q}^r\}$ is defined so that

$$\begin{aligned} E_{p,q}^0 &= C_{p+q}(\Omega_p; \mathbf{k}) / C_{p+q}(\Omega_{p-1}; \mathbf{k}) \cong C_{p+q}(\Omega_p, \Omega_{p-1}; \mathbf{k}); \\ E_{p,q}^1 &= H_{p+q}(\Omega_p, \Omega_{p-1}; \mathbf{k}). \end{aligned}$$

Putting $E_n^r = \sum_{p+q=n} E_{p,q}^r$ and $E^r = \sum_n E_n^r$, by Theorem 1.17 we have

$$E_n^1 = \sum_{\lambda_1=n} H_*(\sigma_s, \partial \sigma_s; \mathbf{k}) = \sum_{\lambda_1=n} H_*(\sigma_s, \partial \sigma_s; \mathbf{k})$$

and hence we have that

$$P(E^1, t) = \sum_s t^{\dim \sigma_s} = \sum_s t^{\lambda_s} = \mathcal{M}(M, N, r; t).$$

Since $H(E^r) = E^{r+1}$, by Lemma 1.20 we have

$$P(E^1, t) \succ P(E^2, t) \succ \cdots \succ P(E^r, t) \succ P(E^{r+1}, t) \succ \cdots,$$

where we remark that $\succ$ satisfies the transitivity law. By the assumption that $\mathcal{M}(M, N, r; t)$ exists we see that the series of terms connected by $\succ$ above is constant from some term on. (Recall that for a given n , there exists an integer r_n such that $E^{r_n} = E^{r_n+1} = \cdots$.) On the other hand, since the image of $L \circ f: \Delta^n \rightarrow \mathbf{R}$ is compact for a singular simplex $f: \Delta^n \rightarrow \Omega^*$, the image of f is contained in some Ω_p . Therefore any element of $H_n(\Omega^*; \mathbf{k})$ is in the image from $H_n(\Omega_p; \mathbf{k})$ for some p . Combining these we have

$$P(E^p, t) \equiv P(H_*(\Omega^*; \mathbf{k}), t) \pmod{t^n}$$

by taking p sufficiently large compared to n . Then there is $B_n(t)$ such that $P(E^1, t) = P(E^p, t) + (1+t)B_n(t)$, and we have

$$B_n(t) \equiv B_{n+1}(t) \pmod{t^n}.$$

Therefore there is $B(t)$ such that $B(t) \equiv B_n(t) \pmod{t^n}$, and we have

$$\mathcal{M}(M, N, r; t) = P(E^1, t) = P(H_*(\Omega^*, \mathbf{k}), t) + (1+t)B(t).$$

Since Ω^* can be replaced by $\Omega(M, N)$ by Corollary 1.13, Theorem 1.18 is proved. Q.E.D.

THEOREM 1.21. *If the indices λ_s of geodesics s of $S(M, N, r)$ are all even numbers and if the number of s with the same index is finite, then the following hold:*

- (1) $H_{2i-1}(\Omega(M, N)) = 0$.
- (2) $H_{2i}(\Omega(M, N))$ is a free abelian group.
- (3) $P(\Omega(M, N), t) = \sum_{s \in S(M, N, r)} t^{\lambda_s}$.

PROOF. Since $\mathcal{M}(M, N, r; t) = \sum t^{\lambda_s}$ does not have terms of odd degree, we have $B(\mathbf{k}; t) = 0$ in the equality of Theorem 1.18, and we have $P(\Omega(M, N), \mathbf{k}; t) = \mathcal{M}(M, N, r; t)$, which is true for a field $\mathbf{k}$ of arbitrary characteristic and so the theorem is proved by using the universal coefficient theorem. Q.E.D.

§2. Infinitesimal K -motion and variational completeness

K -transversal geodesics and infinitesimal K -motions. In this chapter we suppose that a compact Lie group K acts on a manifold M by

$$\mu: K \times M \rightarrow M, \quad \mu(k, x) = k \cdot x,$$

and that M is a complete Riemannian manifold with a K -invariant Riemannian metric.

NOTATION. For a point x of M ,

$$K \cdot x = \{k \cdot x | k \in K\} \quad (\text{the } K\text{-orbit of } x),$$

$$K_x = \{k \in K | k \cdot x = x\} \quad (\text{the stabilizer of } x = \text{the isotropy group of } x).$$

The natural correspondence $k \mapsto k \cdot x$ gives rise to the following homeomorphism:

$$K/K_x \cong K \cdot x. \quad (2.1)$$

DEFINITION. A curve (in particular, a geodesic) h on M is called K -transversal if $h(t) \perp K \cdot h(t)$ for any point $h(t)$ on h , that is, h is orthogonal to each K -orbit.

DEFINITION. A Jacobi vector field v along a geodesic g is K -transversal if for a variation V of g such that every V_α is a K -transversal geodesic,

$$v(t) = \partial_\alpha V_\alpha(t).$$

NOTATION. $J_g^K = \{v \in J_g | v \text{ is } K\text{-universal}\}$.

$J_g^K(t_0) = \{v \in J_g^K | v(t_0) \text{ is tangent to } K \cdot g(t_0)\}$.

The assignment $M \ni x \mapsto (\exp tX) \cdot x$ for an element $X \in L(K) = T_e K$ the Lie algebra of K , gives a motion of M . An infinitesimal K -motion is defined as the limit of this.

DEFINITION. A vector field $\dot{\mu}(X)$ on M given by

$$\dot{\mu}(X)(x) = \partial_t(\exp tX \cdot x) \in T_x M$$

is called the *infinitesimal K -motion* determined by X .

Since the projection $K \rightarrow K/K_x \cong K \cdot x$ induces a surjection of the tangent spaces, one has that

2.2. A vector of the tangent space at x of the K -orbit $K \cdot x$ of $x \in M$ is given by the value at x of the infinitesimal K -motion: $T_x(K \cdot x) = \{\dot{\mu}(X)(x) | X \in L(K)\}$.

THEOREM 2.3. If g is a K -transversal geodesic, then the restriction v (given by $v(t) = \dot{\mu}(X)g(t)$, $X \in L(K)$) of the infinitesimal K -motion on g is a K -transversal Jacobi vector field and $v \in J_g^K(t_0)$ for any t_0 .

PROOF. v is given by a variation $V_\alpha(t) = (\exp \alpha X) \cdot g(t)$, and g is K -transversal so $\dot{g}(t) \perp K \cdot g(t)$. Since $x \mapsto (\exp \alpha X) \cdot x$ is an isometric transformation, $\dot{V}_\alpha(t) \perp K \cdot g(t)$ so V_α is K -transversal and v is K -transversal. $\dot{V}_\alpha(t_0)$ is a curve of $K \cdot g(t_0)$ so $v(t_0)$ is tangent to $K \cdot g(t_0)$ and $v \in J_g^K(t_0)$. Q.E.D.

NOTATION. We denote by

$$\pi_g: L(K) = \mathfrak{k} \rightarrow J_g^K(t_0) \subset J_g^K \quad (t_0: \text{arbitrary})$$

the correspondence $X \mapsto v$ given by Theorem 2.3.

LEMMA 2.4. If $\dot{g}(t_0) \perp K \cdot g(t_0)$ for a geodesic $g: \mathbb{R} \rightarrow M$ of M and $t_0 \in \mathbb{R}$, then g is K -transversal. That is, the K -transversality of a geodesic is obtained from K -transversality at a point.

PROOF. It suffices to show $\dot{g}(t_1) \perp K \cdot g(t_1)$ for the case where $g([t_0, t_1])$ lies in the ε -neighborhood U of $g(t_0)$ in which any two points can be joined uniquely by a geodesic. If the length of $g|[t_0, t_1]$ is l , then the following theorem of Gauss holds:

$$\dot{g}(t_0) \perp \{y \in M | \rho(g(t_0), y) = l\}, \quad (2.4')$$

where ρ is the distance measured by a geodesic.

Since $\dot{g}(t_0) \perp K \cdot g(t_0)$, by interchanging t_0 to t_1 in (2.4') we have

$$\rho(x, g(t_1)) = l + o(\rho(x, g(t_0))), \quad x \in K \cdot g(t_0).$$

A similar thing is true if $\dot{g}(t_1) \perp K \cdot g(t_1)$, but supposing that it is not so, there is a curve $k(t)$ starting from e inside K such that the tangent

vector $h(0)$ of $h(t) = k(t) \cdot g(t)$ is not orthogonal to $g(t_1)$. If we choose a parameter t proportional to the length of h , we see the following:

$$\rho(g(t_0), h(t)) = l + at + o(t) \quad \text{with } a \neq 0.$$

Since the action of K is isometric, we have $\rho(g(t_0), h(t)) = \rho(k(t)^{-1} \cdot x, g(t_1))$, hence $l = l + at + o(t)$, which is a contradiction. Thus $\dot{g}(t_1) \perp K \cdot g(t_1)$. Q.E.D.

REMARK 2.5. We need some knowledge of differential geometry to prove the Gauss theorem used in the above proof. For the case of later applications, however, geodesics are given using one parameter subgroup; it is easy to give a direct proof of Lemma 2.4 for that case.

Variational completeness and the index of a K -transversal geodesic

DEFINITION. The action of K on M is called *variationally complete* if $J_g^K(0) \cap J_g^K(1) \subset \pi_g(L(K))$ holds for any geodesic g in M .

LEMMA 2.6. *The action of K on M is variationally complete iff $J_g^K(0) \cap J_g^K(1) \subset \pi_g(L(K))$.*

PROOF. *Necessity* is clear from $\Lambda_g(1) \subset J_g^K(1)$.

Sufficiency. Suppose $v \in J_g^K(0) \cap J_g^K(1)$ and then $v(1) \in T_{g(1)}M$. There exists $X_1 \in L(K)$ by 2.2 such that $v_1(1) = v(1)$ for $v_1 = \pi_g(X_1)$. Since $v - v_1 \in \Lambda_g(1) \cap J_g^K(0)$, $v - v_1 \in \pi_g(L(K))$ and so $v \in \pi_g(L(K))$. Q.E.D.

If we transform a parameter t of a geodesic g by a linear function, it is still a geodesic and so the points $t = 0$ and 1 are not essential in the definition of the variational completeness. That is,

2.7. *The action of K on M is variationally complete iff*

$$J_g^K(t_0) \cap J_g^K(t_1) \subset \pi_g(L(K)) \text{ for any } t_0 \neq t_1$$

which is equivalent to

$$J_g^K(t_0) \cap \Lambda_g(t_1) \subset \pi_g(L(K)) \text{ for any } t_0 \neq t_1.$$

Consider a K -transversal geodesic and a geodesic segment that is its restriction:

$$g: \mathbf{R} \rightarrow M \quad \text{and} \quad s = g|[0, a] \rightarrow M.$$

For the initial point $p = s(0) = g(0)$, the compact submanifold

$$N = K \cdot p \quad (\text{the } K\text{-orbit of the initial point})$$

of M is orthogonal to g at p . That is, s is a geodesic segment of $M \bmod N$. The following is obtained by Lemma 2.4:

2.8. *Let g be a geodesic in M , $s = g|[0, 1]$, $N = K \cdot s(0)$, and $r = s(1)$.*

(1) *g is a geodesic segment of $M \bmod N$ iff s is a K -transversal geodesic segment.*

$$(2) \Lambda_s^N(t) = J_g^N \cap \Lambda_g(t) = J_g^K \cap \Lambda_g(t), \quad 0 < t \leq 1.$$

Therefore the index of $s \in S(M, N, r)$ is

$$\lambda_s = \sum_{0 < t \leq 1} \dim(J_g^K \cap \Lambda_g(t)).$$

If the action of K is variationally complete, then the index λ_s of s can be expressed by the stabilizer of the exceptional point of s . We write

$$K_s = \{k \in K | k \cdot x = x \text{ for every point } x \text{ on } s\}$$

for a geodesic segment s . We use the same notation K_g for a geodesic g .

LEMMA 2.9. *If g is a K -transversal geodesic and s is a geodesic segment that is a part of g , then $K_s = K_g$. Moreover, if the orbit $K \cdot r$ of a point r on s is of maximal dimension (among the K -orbits), then the identity components of K_r and K_s are identical, and $\dim K_r = \dim K_s$.*

PROOF. An element k of K_s maps a geodesic g to a geodesic g' , but they are identical on s and hence $g' = g$, whence $K_s \subset K_g$. Obviously $K_g \subset K_s$. Thus $K_g = K_s$.

We always have $K_x \supset K_g$ for a point x on s . Let q be a regular point on s (Theorem 1.14(2)). If we suppose $\dim K_q > \dim K_s$, then we obtain a nonzero Jacobi vector field that vanishes at q by an infinitesimal K -motion of $X \in L(K_q) - L(K_s)$, which contradicts the regularity of g . Thus $\dim K_q = \dim K_s$, which is minimal for q on s . Since $\dim K_r$ is minimal by the assumption, $\dim K_r = \dim K_s$. Since $K_r \supset K_s$, the identity components of K_r and K_s are identical. Q.E.D.

THEOREM 2.10. *Let $s: [0, 1] \rightarrow M$ be a geodesic segment of $M \bmod K \cdot p$ that passes through a point with the orbit of maximal dimension, where $p = s(0)$. Then*

$$(1) \dim K_{s(t)} - \dim K_s \leq \lambda_s(t) = \dim(J_g^K(0) \cap \Lambda_g(t)) \quad (0 < t \leq 1).$$

(2) *If the action of K is variationally complete, then equality holds in (1).*

PROOF. (1) Consider a map

$$\pi_g: L(K) \rightarrow J_g^K(0)$$

and a map $e_x: J_g^K \rightarrow T_x(K \cdot x)$ that assigns the value at x ,

$$e_x(v) = v(x),$$

where $J_g^K(0) \cap \Lambda_g(t) = \text{Ker}(e_{s(t)})$.

Since $e_x \circ \pi_g: L(K) \rightarrow T_x(K \cdot x)$ is given by the differential of $\cdot x: K \rightarrow K \cdot x \cong K/K_x$, we have $\text{Ker}(e_r \circ \pi_g) = L(K_x)$.

By Lemma 2.9 we have $L(K_r) = L(K_s)$ for $r = s(t_0)$, which has the

orbit of maximal dimension. Obviously $L(K_s) \subset \text{Ker } \pi_g$. Also $L(K_r) = \text{Ker}(e_r \circ \pi_g) \supset \text{Ker } \pi_g$. Thus $L(K_s) = L(K_r) = \text{Ker } \pi_g$. Since $\pi_g(L(K_{s(t)})) \subset \text{Ker } e_{s(t)}$, we have

$$\begin{aligned}\lambda_s(t) &= \dim(J_g^K(0) \cap \Lambda_g(t)) = \dim \text{Ker } e_{s(t)} \geq \dim \pi_g(L(K_{s(t)})) \\ &= \dim L(K_{s(t)}) - \dim \text{Ker } \pi_g = \dim L(K_{s(t)}) - \dim L(K_s) \\ &= \dim K_{s(t)} - \dim K_s.\end{aligned}$$

(2) If the action of K is variationally complete, then $\text{Ker } e_{s(t)} = J_g^K(0) \cap \Lambda_g(t) \subset \pi_g(L(K))$, which implies

$$\text{Ker } e_{s(t)} = \pi_g(\pi_g^{-1}(\text{Ker } e_{s(t)})) = \pi_g(\text{Ker}(e_{s(t)} \circ \pi_g)) = \pi_g(L(K_{s(t)})).$$

Thus equality holds in the above inequality. Q.E.D.

DEFINITION. For a point r with the K -orbit $K \cdot r$ of maximal dimension and an arbitrary point $q \in M$,

$$\delta(q) = \dim K \cdot r - \dim K \cdot q$$

is called the *defect* of q .

Obviously,

$$\delta(q) = \dim K_q - \dim K_r. \quad (2.11)$$

THEOREM 2.12. *If the action of K on M is variationally complete and if r is a point with the K -orbit of maximal dimension, then each geodesic segment s of $M \bmod N \in S(M, K \cdot p, r)$ with the terminal point r is K -transversal and nondegenerate (in the sense of Morse), and its index is given by the following:*

$$\lambda_s = \sum_{0 < t < 1} \delta(s(t)) = \sum_{0 < t < 1} (\dim K_{s(t)} - \dim K_s).$$

PROOF. By Lemma 2.4 s is K -transversal. Theorem 2.10 implies $\lambda_s(t) = \dim K_{s(t)} - \dim K_s$. In particular $\lambda_s(1) = \dim K_r - \dim K_s = 0$, whence s is nondegenerate. By Theorem 1.14(2) and Lemma 2.9 the set of regular points on s is open and dense. Therefore the number of t such that $\lambda_s(t) > 0$ is finite. Thus $\lambda_s = \sum \lambda_s(t) = \sum (\dim K_r - \dim K_s) = \sum \delta(s(t))$. Q.E.D.

Geodesics in Lie groups and homogeneous spaces

THEOREM 2.13. *Suppose that a Lie group G has an invariant Riemannian metric. If we put $x = g(0)^{-1}$ for a geodesic $g: \mathbf{R} \rightarrow G$ of G , $L_x \circ g$ and $R_x \circ g$ are one parameter subgroups. Conversely, a one parameter subgroup $g: \mathbf{R} \rightarrow G$ of G is a geodesic.*

The exponential map $\exp$ defined at the identity e by a geodesic coincides with the exponential map defined by a one parameter subgroup: $\exp_e = \exp: \mathfrak{g} = T_e G \rightarrow G$.

PROOF. We may suppose $g(0) = e$, since L_x and R_x are isometric transformations.

Consider a geodesic $g: \mathbf{R} \rightarrow G$ with $g(0) = e$ and take the ε -neighborhood U of e sufficiently small so that a geodesic joining two points in U is unique. Consider $x = g(a) \in U$ for $0 \leq a < \varepsilon$ and put $y = g(-a)$. We define a path $h: [-a, a] \rightarrow G$ joining y and x by

$$h(t) = \begin{cases} yg(t+a) & (-a \leq t \leq 0), \\ g(t-a)x & (0 \leq t \leq a). \end{cases}$$

Since the length of $g[-a, 0]$ is equal to that of $g[0, a]$ and since L_y and R_x are isometric transformations, the length of h is equal to that of $g[-a, a]$. Since a geodesic segment $g[-a, a]$ is the unique (shortest) curve joining y and x and since the arc lengths of g and h are proportional to a variable t , h is identical with $g[-a, a]$. In particular by $h[0, a] = g[-a, 0]$ we have that if $0 \leq t \leq a$ then $g(t) = g(t-a)$ and $g(a) = g(t-a)^{-1}g(t)$.

Putting $t = 0$ and $a = s$, we have $g(s) = g(-s)^{-1}$. Further putting $a - t = s$, we have

$$s, t, s+t \in [0, \varepsilon] \text{ implies } g(-s) = g(s)^{-1}, g(s)g(t) = g(s+t).$$

It is easily seen that these equalities also hold for $(-\varepsilon, \varepsilon)$, and hence $g|(-\varepsilon, \varepsilon)$ is the restriction of some one parameter subgroup g' . Since $L_{g'(t_0)}$ is an isometric transformation for any t_0 , $g'| (t_0 - \varepsilon, t_0 + \varepsilon)$ and hence the whole g' is a geodesic. If geodesics are partially identical, they are identical in the whole, and hence $g = g'$ is a one parameter subgroup.

Conversely, given a one parameter subgroup g' , there is a geodesic g with $\dot{g}(0)$ the initial vector of g' . g is a one parameter subgroup as above. Since a one parameter subgroup is determined by the initial vector, we have $g = g'$, that is, g' is a geodesic. Also it is obvious from this argument that $\exp_e = \exp$. Q.E.D.

We suppose that a homogeneous space G/H of a Lie group G equipped an invariant Riemannian metric is given by a G -invariant Riemannian metric determined by the following lemma.

LEMMA 2.14. Consider $dp_x: T_x G \rightarrow T_{p(x)}(G/H)$ induced by the projection $p: G \rightarrow G/H$. If we define the inner product of $X, Y \in T_{p(x)}(G/H)$ by $(X, Y) = (\bar{X}, \bar{Y})$ for $\bar{X}, \bar{Y} \in T_x G$ such that $dp_x \bar{X} = X$, $dp_x \bar{Y} = Y$ and $\bar{X} \perp T_x H$, $\bar{Y} \perp T_x H$, then G/H can be a given G -invariant Riemannian metric.

PROOF. If x is fixed in the above definition of the inner product, then $\bar{X}$ and $\bar{Y}$ are uniquely determined. If we replace x by y , which satisfies $p(y) = p(x)$, then R_h is an isometric transformation for $h = x^{-1}y$ and hence $(\bar{X}, \bar{Y}) = (dR_h \bar{X}, dR_h \bar{Y})$. This shows that the definition of the above inner product is determined independently of the choice of x . It is obvious that this Riemannian metric is invariant under the action of G from the left. Q.E.D.

THEOREM 2.15. *Suppose given a G -invariant Riemannian metric on the homogeneous space G/H as in Lemma 2.14. An arbitrary geodesic g of G/H is uniquely expressed in the form $g(t) = p(y \exp(tX))$, where $y \in G$ and $X \in L(H)^\perp = (\text{the orthogonal complement of } L(H) \text{ in } L(G))$.*

PROOF. Taking $y \in G$ such that $g(0) = p(y)$, the left translation L_y is an isometric transformation that preserves the orthogonality of G and G/H and so it suffices to consider the case $y = e$.

For a path h in G/H we can take a path $\tilde{h}$ in G uniquely such that $h = p \circ \tilde{h}$ and that $\tilde{h}$ is orthogonal to every H -orbit. Then the arc lengths of h and $\tilde{h}$ are equal. In particular, if $\tilde{h}$ is a geodesic, so is h . $\tilde{h}(t) = \exp(tX)$ for $X \in L(H)^\perp$ is such a geodesic in G and $g(t) = p(\exp(tX))$ is a geodesic in G/H . By the isomorphism $L(H)^\perp \cong T_{p(e)}(G/H)$, the lemma is proved. Q.E.D.

§3. Variational completeness of a symmetric pair

The Lie algebra of a symmetric pair. Throughout this section let G be a compact, connected Lie group with an invariant Riemannian metric and K a connected, closed subgroup of G .

REVIEW. (G, K) is called a *symmetric pair* if there is an automorphism $\sigma: G \rightarrow G$ satisfying $\sigma^2 = 1_G$ such that K is the identity component of $\{g \in G | \sigma(g) = g\}$.

In a symmetric pair (G, K) , one may consider that the invariant Riemannian metric is also invariant under σ , by taking the mean of the invariant Riemannian metric and the one induced by σ . Then σ is an isomorphic transformation.

Moreover, if G/K is given a Riemannian metric as in Lemma 2.14, then the correspondence

$$\sigma_x: yK \mapsto x\sigma(x)^{-1}\sigma(y)K \quad \text{for } x, y \in G$$

is an isometric transformation that maps a geodesic passing through xK in opposite direction with center xK , that is, G/K is a symmetric space. Denote by

$$\mathfrak{g} = L(G) \quad \text{and} \quad \mathfrak{k} = L(K),$$

the Lie algebra of G and K respectively.

THEOREM 3.1. *(G, K) is a symmetric pair if and only if the following hold:*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k},$$

for the orthogonal complement $\mathfrak{p} = \mathfrak{k}^\perp$ of $\mathfrak{k}$.

PROOF. *Necessity.* We denote by the same σ the automorphism of $\mathfrak{g}$ induced by the automorphism σ , which defines a symmetric pair (G, K) . Obviously $\mathfrak{k} = \{X \in \mathfrak{g} | \sigma(X) = X\}$.

Put $\mathfrak{p} = \{X \in \mathfrak{g} | \sigma(X) = -X\}$. The equality

$$X = (X + \sigma(X))/2 + (X - \sigma(X))/2$$

gives a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since σ preserves the inner product, $(Y, Z) = (\sigma Y, \sigma Z) = (Y, -Z) = -(Y, Z)$ for $Y \in \mathfrak{k}$ and $Z \in \mathfrak{p}$ and hence $(Y, Z) = 0$. Therefore $\mathfrak{k} = \mathfrak{p}^\perp$. Also $\sigma[Y, Z] = [\sigma Y, \sigma Z] = [Y, -Z] = -[Y, Z]$ and hence $[Y, Z] \in \mathfrak{p}$. Therefore $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Similarly proved are $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Sufficiency. By the relation between Lie groups and Lie algebras (Theorem 5.11' of Chapter II) it suffices to show the existence of an automorphism σ of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{k} = \{X \in \mathfrak{g} | \sigma(X) = X\}$ and $\sigma^2 = 1$.

When we decompose $X \in \mathfrak{g}$ as $X = Y + Z$ for $Y \in \mathfrak{k}$ and $Z \in \mathfrak{p}$, we define σ by

$$\sigma(Y + Z) = Y - Z.$$

Obviously $\sigma^2 = 1$ and $\mathfrak{k} = \{X \in \mathfrak{g} | \sigma(X) = X\}$. The fact that σ is a homomorphism follows from

$$\sigma[X, X'] = ([Y, Y'] + [Z, Z']) - ([Y, Z'] + [Z, Y']) = [\sigma X, \sigma X']$$

for $X' = Y' + Z'$ with $Y' \in \mathfrak{k}$ and $Z' \in \mathfrak{p}$. Q.E.D.

For the sake of simplicity we write

NOTATION. $e^X = \exp X \in G$ ($X \in \mathfrak{g} = T_e G = L(G)$).

$$x \cdot A = dL_x(A), \quad A \cdot x = dR_x(A) \quad (x \in G, A \in TG).$$

The adjoint actions are

$$\text{Ad } x : \mathfrak{g} \rightarrow \mathfrak{g} \quad \text{given by} \quad \text{Ad } x(X) = x \cdot X \cdot x^{-1} \quad (x \in G, X \in \mathfrak{g}),$$

$$\text{ad } Y : \mathfrak{g} \rightarrow \mathfrak{g} \quad \text{given by} \quad \text{ad } Y(X) = [Y, X] \quad (X, Y \in \mathfrak{g}),$$

and for the invariant inner product (the invariant Riemannian metric or the Killing form) $(\cdot, \cdot)$ the following hold:

$$\text{Ad } x([X, Y]) = [\text{Ad } x(X), \text{Ad } x(Y)], \quad (3.2.1)$$

$$(X, Y) = (\text{Ad } x(X), \text{Ad } x(Y)), \quad (3.2.2)$$

$$\frac{d}{dt}(\text{Ad } e^{tY}(X))|_{t=0} = \text{ad } Y(X) = [Y, X], \quad (3.2.3)$$

$$([X, Y], Z) + (Y, [X, Z]) = 0. \quad (3.2.4)$$

The first equality shows that $\text{Ad } x$ is a homomorphism of Lie algebras and the second one implies that it preserves the inner product. The third one is 5.15 of Chapter II and the fourth one is (5.34) of Chapter V.

NOTATION. For $a \in G$,

$$\mathfrak{k}_a = \mathfrak{k} \cap \text{Ad } a^{-1}(\mathfrak{k}), \quad \mathfrak{p}_a = \mathfrak{p} \cap \text{Ad } a^{-1}(\mathfrak{p}), \quad \mathfrak{k}^a = \mathfrak{k} + \text{Ad } a^{-1}(\mathfrak{k}).$$

It follows from Theorem 3.1 and (3.2.1) that

$$[\mathfrak{k}_a, \mathfrak{k}_a] \subset \mathfrak{k}_a, \quad [\mathfrak{k}_a, \mathfrak{p}_a] \subset \mathfrak{p}_a, \quad [\mathfrak{p}_a, \mathfrak{p}_a] \subset \mathfrak{k}_a. \quad (3.3)$$

Also $\mathfrak{p} \perp \mathfrak{k}$, $\text{Ad } a^{-1}(\mathfrak{p}) \perp \text{Ad } a^{-1}(\mathfrak{k})$ implies $\mathfrak{p}_a \perp \mathfrak{k}$, $\mathfrak{p}_a \perp \text{Ad } a^{-1}(\mathfrak{k})$ implies $\mathfrak{p}_a \perp \mathfrak{k}^a$ and so $\mathfrak{p}_a \subset (\mathfrak{k}^a)^\perp$. Conversely, $X \in (\mathfrak{k}^a)^\perp$ implies $X \in \mathfrak{k}^\perp = \mathfrak{p}$ and $X \in (\text{Ad } a^{-1}(\mathfrak{k}))^\perp = \text{Ad } a^{-1}(\mathfrak{k})^\perp = \text{Ad } a^{-1}(\mathfrak{p})$ implies $X \in \mathfrak{p}_a$. Thus

$$\mathfrak{p}_a = (\mathfrak{k}^a)^\perp, \text{ namely, } \mathfrak{g} = \mathfrak{k}^a \oplus \mathfrak{p}_a \text{ (an orthogonal decomposition).} \quad (3.4)$$

LEMMA 3.5. $\mathfrak{k}^a = a^{-1} \cdot T_a(KaK)$, $\mathfrak{p}_a = a^{-1} \cdot T_a(KaK)^\perp$.

PROOF. From $T_a(Ka) = dR_a(T_e K) = \mathfrak{k} \cdot a$, $T_a(aK) = dL_a(T_e K) = a \cdot \mathfrak{k}$, it follows that $T_a(KaK) = \mathfrak{k} \cdot a + a \cdot \mathfrak{k}$ and hence $a^{-1} T_a(KaK) = a^{-1} \cdot \mathfrak{k} \cdot a + \mathfrak{k} = \mathfrak{k}^a$. By the isometricity of a^{-1} , $\mathfrak{p}_a = (\mathfrak{k}^a)^\perp = a^{-1} \cdot T_a(KaK)^\perp$. Q.E.D.

It follows immediately from 2.2 and (3.2.3) that

3.6. For the adjoint action $\text{Ad}: K \times \mathfrak{g} \rightarrow \mathfrak{g}$ and $X \in \mathfrak{g}$

$$T_X(K \cdot X) = \text{ad } X(\mathfrak{k}).$$

NOTATION. $\mathfrak{g}_X = \{Y \in \mathfrak{g} | [X, Y] = 0\}$ ($X \in \mathfrak{g}$).

$$\mathfrak{k}_X = \mathfrak{k} \cap \mathfrak{g}_X, \quad \mathfrak{p}_X = \mathfrak{p} \cap \mathfrak{g}_X.$$

It follows easily from the Jacobi identity that $[\mathfrak{g}_X, \mathfrak{g}_X] \subset \mathfrak{g}_X$, and hence

$$[\mathfrak{k}_X, \mathfrak{k}_X] \subset \mathfrak{k}_X, \quad [\mathfrak{k}_X, \mathfrak{p}_X] \subset \mathfrak{p}_X, \quad [\mathfrak{p}_X, \mathfrak{p}_X] \subset \mathfrak{k}_X. \quad (3.7)$$

LEMMA 3.8. (1) $\mathfrak{p} = \text{ad } X(\mathfrak{k}) \oplus \mathfrak{p}_X$ (an orthogonal decomposition) for $X \in \mathfrak{p}$.

(2) $\mathfrak{p}_a = \text{ad } X(\mathfrak{k}_a) \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X)$ (an orthogonal decomposition) for $X \in \mathfrak{p}_a$ and $a \in G$.

(3) $\mathfrak{p}_A = \text{ad } X(\mathfrak{k}_A) \oplus (\mathfrak{p}_A \cap \mathfrak{g}_X)$ (an orthogonal decomposition) for $X \in \mathfrak{p}_A$ and $A \in \mathfrak{g}$.

PROOF. By (3.2.4), $Z \perp [X, Y]$ is equivalent to $[X, Z] \perp Y$. Consider a subspace $\mathfrak{a}$ of $\mathfrak{g}$. By letting Y run over $\mathfrak{a}$ one has

$$Z \in \text{ad } X(\mathfrak{a})^\perp \text{ iff } [X, Z] \in \mathfrak{a}^\perp.$$

Consider a subspace $\mathfrak{b}$ such that $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{b}$. Since $[X, W] \in \mathfrak{a}$ for $X, W \in \mathfrak{b}$, we have

$$Z \in \text{ad } X(\mathfrak{a})^\perp \cap \mathfrak{b} \text{ iff } Z \in \mathfrak{b} \text{ and } [X, Z] \in \mathfrak{a}^\perp \cap \mathfrak{a} = 0 \text{ iff } Z \in \mathfrak{b} \cap \mathfrak{g}_X.$$

Here, observing $\text{ad } X(\mathfrak{a}) \subset [\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{b}$, we have

$$\mathfrak{b} = \text{ad } X(\mathfrak{a}) \oplus (\mathfrak{b} \cap \mathfrak{g}_X) \text{ (an orthogonal decomposition).}$$

(1), (2), (3) are obtained by applying Theorem 3.1, (3.3), (3.7) to the case $(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{k}, \mathfrak{p})$, $(\mathfrak{k}_a, \mathfrak{p}_a)$, $(\mathfrak{k}_A, \mathfrak{p}_A)$ respectively. Q.E.D.

Variational completeness of the left action of $K \times K$. We define a left action $\mu: (K \times K) \times G \rightarrow G$ of $K \times K$ on G by

$$\mu((a, b), x) = axb^{-1} \quad (3.9)$$

in a symmetric pair (G, K) . The invariant Riemannian metric of G is $(K \times K)$ -invariant under the action.

We show that this left action of $K \times K$ on G is variationally complete.

An arbitrary geodesic g with $a \in G$ the initial point ($a = g(0)$) is uniquely expressed in the form $g(t) = ae^{tX}$ (Theorem 2.13). Since the $(K \times K)$ -orbit of a is KaK , the following is obtained by Lemma 3.5:

3.10. A $(K \times K)$ -transversal geodesic $g: \mathbb{R} \rightarrow G$ with the initial point a in G is expressed uniquely in the form

$$g(t) = ae^{tX} \quad \text{for } X \in \mathfrak{p}_a.$$

LEMMA 3.11. $\Lambda_g(0) \cap J_g^{K \times K} = \{(K \times K)\text{-transversal Jacobi vector fields } v \text{ along } g \text{ satisfying } v(0) = 0\}$ consists of the vector fields of the following form:

$$v(t) = (ae^{tX}) \cdot (\text{Ad} e^{-tX}(Y) - Y + tZ),$$

where Y and Z run over $\mathfrak{k}_a$ and $\mathfrak{p}_a \cap \mathfrak{g}_X$ respectively.

PROOF. We define a variation $V = V^{(Y, Z)}$ by

$$V(t, \alpha) = V_\alpha(t) = ae^{\alpha Y} e^{t(X+\alpha Z)} e^{-\alpha Y}$$

for $Y \in \mathfrak{k}_a$ and $Z \in \mathfrak{p}_a \cap \mathfrak{g}_X$. Since $t \mapsto ae^{\alpha Y} e^{t(X+\alpha Z)} e^{-\alpha Y}$ is a one parameter subgroup, V_α is a geodesic with the initial point a and V is a variation of $V_0 = g$. The initial vector of V_α is

$$\dot{V}_\alpha(0) = a \cdot \text{Ad} e^{\alpha Y}(X + \alpha Z).$$

$Y \in \mathfrak{k}_a = \mathfrak{k} \cap \text{Ad} a^{-1}(\mathfrak{k})$ so $e^{\alpha Y} \in K \cap a^{-1}Ka$. Since $\text{Ad} e^{\alpha Y}$ is an orthogonal transformation of $\mathfrak{g}$ that maps $\mathfrak{k} \rightarrow \mathfrak{k}$ and $\text{Ad} a^{-1}(\mathfrak{k}) \rightarrow \text{Ad} a^{-1}(\mathfrak{k})$, $\mathfrak{p} = \mathfrak{k}^\perp \rightarrow \mathfrak{p}$ and $\text{Ad} a^{-1}(\mathfrak{p}) = \text{Ad} a^{-1}(\mathfrak{k})^\perp \rightarrow \text{Ad} a^{-1}(\mathfrak{p})$. By (3.4), $\text{Ad} e^{\alpha Y}(\mathfrak{p}_a) \subset \mathfrak{p}_a = (\mathfrak{k}_a)^\perp$. Since $X + \alpha Z \in \mathfrak{p}_a$ and $V_\alpha(0) = a$, we have that $\dot{V}_\alpha(0) \in a \cdot \mathfrak{p}_a = (a \cdot \mathfrak{k}^\perp)^\perp$ implies $\dot{V}_\alpha(0) \perp KaK$. That is, V gives a $(K \times K)$ -transversal Jacobi vector field v . $Z \in \mathfrak{g}_X$ implies $[Z, X] = 0$ so $e^{t(X+\alpha Z)} = e^{tX} e^{t\alpha Z}$ and hence

$$\begin{aligned} V_\alpha(t) &= ae^{tX} ((e^{-tX} e^{\alpha Y} e^{tX}) e^{t\alpha Z} e^{-\alpha Y}), \\ v(t) &= \partial_\alpha V_\alpha(t) = ae^{tX} \cdot (\text{Ad} e^{-tX}(Y) + tZ - Y). \end{aligned}$$

Thus $v \in \Lambda_g(0) \cap J_g^{K \times K}$.

In the isomorphism $\theta: \Lambda_g(0) \cong T_a M$ of (1.7), since the variation V_α giving $v \in \Lambda_g(0) \cap J_g^{K \times K}$ satisfies $\dot{V}_\alpha(0) \perp KaK$, the corresponding curve $h(\alpha) = \dot{V}_\alpha(0)$ in $T_a M$ is tangent to $T_a(KaK)^\perp = a \cdot \mathfrak{p}_a$ (Lemma 3.5), and $\theta(v) = \dot{h}(0) \in a \cdot \mathfrak{p}_a$, that is,

$$\theta(\Lambda_g(0) \cap J_g^{K \times K}) \subset a \cdot \mathfrak{p}_a.$$

Next, $V_\alpha(t) = ae^{\alpha Y} e^{t(X+\alpha Z)} e^{-\alpha Y}$ implies

$$h(\alpha) = \dot{V}_\alpha(0) = a \cdot \text{Ad} e^{\alpha Y} (X + \alpha Z)$$

so $\dot{h}(0) = a \cdot ([Y, X] + Z)$. By Lemma 3.8(2) the correspondence $(Y, Z) \mapsto v \mapsto \theta(v) = a \cdot ([Y, X] + Z)$ gives a surjection $\mathfrak{k}_a \oplus (\mathfrak{p}_a \cap \mathfrak{g}_X) \rightarrow a \cdot \mathfrak{p}_a$, and hence

$$\theta(\Lambda_g(0) \cap J_g^{K \times K}) \supset a \cdot \mathfrak{p}_a.$$

Thus θ gives rise to an isomorphism $\Lambda_g(0) \cap J_g^{K \times K} \cong a \cdot \mathfrak{p}_a$, and $\Lambda_g(0) \cap J_g^{K \times K}$ is shown to consist of the vector fields of the form in the lemma. Q.E.D.

LEMMA 3.12. (1) An infinitesimal $(G \times G)$ -motion $\dot{\mu}(U, V)$ given at $(U, V) \in \mathfrak{g} \times \mathfrak{g} = L(G \times G)$ with regard to the action $\mu: (G \times G) \times G \rightarrow G$ defined by $\mu((a, b), g) = agb^{-1}$ satisfies

$$\dot{\mu}(U, V)(y) = U \cdot y - y \cdot V \quad \text{for } y \in G.$$

(2) The restrictions of infinitesimal $(K \times K)$ -motions to g (where $g(t) = ae^{tX}$) that vanish at $a = g(t_0)$ consist of the vector fields v of the form:

$$v(t) = (ae^{tX}) \cdot (\text{Ad} e^{-tX}(V) - V), \quad \text{where } V \in \mathfrak{k}_a = \mathfrak{k} \cap \text{Ad} a^{-1} \mathfrak{k}.$$

PROOF. (1) $\dot{\mu}(U, V)$ is obtained by differentiating a one parameter motion $e^{\alpha U} \cdot y \cdot e^{-\alpha V}$ by α . By the equality

$$e^{\alpha U} \cdot y \cdot e^{-\alpha V} = e^{\alpha U} \cdot y \cdot e^{-\alpha V} \cdot y^{-1} y = e^{\alpha U} \cdot e^{-\alpha \text{Ad} y V} \cdot y$$

we have $\dot{\mu}(U, V) = (U - \text{Ad} y(V)) \cdot y = U \cdot y - y \cdot V \cdot y^{-1} \cdot y = U \cdot y - y \cdot V$.

(2) If $U, V \in \mathfrak{k}$ in (1), one obtains an infinitesimal $(K \times K)$ -motion, for which the necessary and sufficient condition to vanish at $a = g(t_0)$ is $U \cdot a - a \cdot V = 0$, that is, $V = \text{Ad} a^{-1}(U) \in \mathfrak{k} \cap \text{Ad} a^{-1} \mathfrak{k} = \mathfrak{k}_a$. Then for $y = g(t) = ae^{tX}$ we have

$$\begin{aligned} U \cdot y - y \cdot V &= (\text{Ad} a)V \cdot y - y \cdot V = y \cdot (\text{Ad} y^{-1})(\text{Ad} a)V - y \cdot V \\ &= y \cdot (\text{Ad}(y^{-1}a)V - V) = y \cdot (\text{Ad} e^{-tX}(V) - V). \quad \text{Q.E.D.} \end{aligned}$$

THEOREM 3.13. If (G, K) is a symmetric pair, then the action of $K \times K$ on G given by (3.9) is variationally complete, that is, an element of $\Lambda_g(0) \cap J_g^{K \times K}(1) = \{(K \times K)\text{-transversal Jacobi vector field along } g \text{ that vanishes at } t = 0 \text{ and is tangent to the } (K \times K)\text{-orbit of } g(1) \text{ at } t = 1\}$ is given by an infinitesimal $(K \times K)$ -motion.

PROOF. We show the second half. Then the variational completeness is given by Lemma 2.6. We put $g(t) = ae^{tX}$ and $g(1) = ae^X = b$. When we express a vector field $v(t)$ of $\Lambda_g(0) \cap J_g^{K \times K}(1) \subset \Lambda_g(0) \cap J_g^{K \times K}$ in the form of Lemma 3.11, the condition for it to be tangent to KbK at $b = g(1)$ is

$$b^{-1} \cdot v(1) = \text{Ad} e^{-X}(Y) - Y + Z \in \mathfrak{k}^b = \mathfrak{k} + \text{Ad} b^{-1}(\mathfrak{k}) \quad (*)$$

by Lemma 3.5.

Here, $Y \in \mathfrak{k}^a$, $Z \in \mathfrak{p}_a \cap \mathfrak{g}_X$ and $\text{Ad } b^{-1} = (\text{Ad } e^{-X})(\text{Ad } a^{-1})$ and so we can write

$$b^{-1} \cdot v(1) = B_1 + \text{Ad } e^{-X}(B_2) \quad \text{with } B_1 \in \mathfrak{k}, B_2 \in \text{Ad } a^{-1}(\mathfrak{k}).$$

$$B_1, B_2 \in \mathfrak{k} + \text{Ad } a^{-1}(\mathfrak{k}) = \mathfrak{k}^a = (\mathfrak{p}_a)^\perp \text{ implies } (Z, B_1) = (Z, B_2) = 0.$$

On the other hand, $[X, Z] = 0$ implies $\text{Ad } e^{-X}(Z) = Z$, and we have

$$(Z, b^{-1} \cdot v(1)) = (Z, \text{Ad } e^{-X}(B_2)) = (\text{Ad } e^{-X}(Z), \text{Ad } e^{-X}(B_2)) = (Z, B_2) = 0.$$

Also, by $\mathfrak{p} = \mathfrak{k}^\perp$ we have $(Z, Y) = 0$ and

$$(Z, \text{Ad } e^{-X}(Y)) = (\text{Ad } e^{-X}(Z), \text{Ad } e^{-X}(Y)) = (Z, Y) = 0.$$

By using (*), we have

$$0 = (Z, b^{-1} \cdot v(1)) = (Z, \text{Ad } e^{-X}(Y) - Y + Z) = (Z, Z),$$

which implies $Z = 0$.

Therefore $v(t) = (ae^{tX}) \cdot (\text{Ad } e^{-X}(Y) - Y)$ for $Y \in \mathfrak{k}_a$. By Lemma 3.12, $v(t)$ is given by an infinitesimal $(K \times K)$ -motion. Q.E.D.

Variational completeness of the adjoint action of K on $\mathfrak{p}$. Since the adjoint action $\text{Ad } x: \mathfrak{g} \rightarrow \mathfrak{g}$ ($x \in K$) of K is an orthogonal transformation and since $\text{Ad } x(\mathfrak{k}) \subset \mathfrak{k}$, we also have $\text{Ad } x(\mathfrak{p}) \subset \mathfrak{p}$ for $\mathfrak{p} = \mathfrak{k}^\perp \cong T_{\pi(e)}(G/K)$.

Next, we prove that the adjoint action

$$\mu = \text{Ad}: K \times \mathfrak{p} \rightarrow \mathfrak{p}$$

is variationally complete.

Since $\mathfrak{p}$ has the same structure as a Euclidean space, a geodesic in it is a straight line, which is uniquely expressed in the form

$$g(t) = A + tX \quad \text{where } A, X \in \mathfrak{p}.$$

The tangent space of the K -orbit $\text{Ad } K(A)$ of $A \in \mathfrak{p}$ at A is spanned by the tangent vectors $(d/dt)(\text{Ad } e^{tX}(A))|_{t=0} = [X, A] = \text{ad } X(A)$ of the curve $\text{Ad } e^{tX}(A)$, where $X \in \mathfrak{k}$. By (3.2.4), $Y \perp [X, A]$ iff $X \perp [Y, A]$. Thus $Y \in \text{ad } \mathfrak{k}(A)^\perp$ iff $[Y, A] \in \mathfrak{k}^\perp = \mathfrak{p}$. On the other hand, $Y, A \in \mathfrak{p}$ implies $[Y, A] \in [\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. It follows from these observations that

$$\begin{aligned} T_A(\text{Ad } K(A)) &= \text{ad } \mathfrak{k}(A), \\ T_A(\text{Ad } K(A))^\perp &= \mathfrak{p}_A \quad (\text{the orthogonal complement in } \mathfrak{p}). \end{aligned} \tag{3.14}$$

LEMMA 3.15. *In the above case, $\Lambda_g(0) \cap J_g^K = \{K\text{-transversal Jacobi vector fields along } g \text{ satisfying } v(0) = 0\}$ consists of the vector fields of the following form ($X \in \mathfrak{p}_A$):*

$$v(t) = t([X, Y] + Z), \quad Y \in \mathfrak{k}_A, Z \in \mathfrak{p}_A \cap \mathfrak{g}_X.$$

PROOF. By (3.14) a K -transversal geodesic with the initial point A is uniquely expressed in the form

$$g(t) = A + tX, \quad \text{where } X \in \mathfrak{p}_A.$$

A K -transversal Jacobi vector field that vanishes at $A = g(0)$ is given by a (linear) variation

$$V_\alpha(t) = A + t(X + \alpha W), \quad \text{where } W \in \mathfrak{p}_A.$$

Thus $v(t)$ is of the form $v(t) = tW$. The lemma is obtained by using the decomposition of Lemma 3.8(3). Q.E.D.

LEMMA 3.16. *The restriction of an infinitesimal K -motion that vanishes at A to $g(t) = A + tX$ where $X \in \mathfrak{p}_A$ is of the form $v(t) = t[V, X]$ where $V \in \mathfrak{k}_A$.*

PROOF. The infinitesimal K -motion given by the action of a one parameter subgroup e^{aV} with $V \in \mathfrak{k}$ assigns $[V, S] \in [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ to $S \in \mathfrak{p}$ by (3.2.3). Restricting to $g(t)$, $v(t) = [V, A + tX]$. To vanish at A , $v(0) = 0$ iff $[V, A] = 0$ which is equivalent to $V \in \mathfrak{g}_A \cap \mathfrak{k} = \mathfrak{k}_A$. Thus $v(t) = t[V, X]$ with $V \in \mathfrak{k}_A$. Q.E.D.

THEOREM 3.17. *The adjoint action of K on $\mathfrak{p}$ is variationally complete, that is, an element $\Lambda_g(0) \cap J_g^K(1) = \{K\text{-transversal vector fields along } g \text{ that vanish at } t=0 \text{ and is tangent to the } K\text{-orbit of } g(1) \text{ at } t=1\}$ is given by the restriction of an infinitesimal K -motion to g .*

PROOF. In order for $v(t)$ of Lemma 3.15 to be tangent to the K -orbit at $t=1$, by (3.14) we have $[X, Y] + Z \in \text{ad } \mathfrak{k}(A+X)$ for $X \in \mathfrak{p}_A$, $Y \in \mathfrak{k}_A$, $Z \in \mathfrak{p}_A \cap \mathfrak{g}_X$. $X \in \mathfrak{p}_A$ and $Y \in \mathfrak{k}_A$ imply that $[X, Y] = [A+X, Y] \in \text{ad } \mathfrak{k}(A+X)$ and hence $Z \in \text{ad } \mathfrak{k}(A+X)$.

On the other hand, $Z \in \mathfrak{p}_A \cap \mathfrak{g}_X$ implies $[A, Z] = [X, Z] = 0$. We have $Z \perp \text{ad } \mathfrak{k}(A+X)$, since

$$([D, A+X], Z) = -([A+X, D], Z) = (D, [A, Z]) + (D, [X, Z]) = 0$$

for any $D \in \mathfrak{k}$. Therefore $Z = 0$, which implies $v(t) = t[X, Y] = t[-Y, X]$. By Lemma 3.16, $v(t)$ is given by an infinitesimal K -motion. Q.E.D.

Variational completeness of the action of K on G/K and of the adjoint action. We consider the action (the left translation) $\mu: K \times (G/K) \rightarrow G/K$ of K on G/K given by

$$\mu(x, yK) = xyK.$$

THEOREM 3.18. *The action of K on G/K in a symmetric pair (G, K) is variationally complete.*

PROOF. Consider the projection $\pi: G \rightarrow G/K$ and $d\pi: TG \rightarrow T(G/K)$.

Let $a \in G$ and $b = \pi(a) \in G/K$. The $(K \times K)$ -orbit KaK of a is mapped by π onto the K -orbit $K \cdot b$ of b and the orthogonal complement $a \cdot \mathfrak{p}_a$ in $T_a(KaK)$ is mapped by $d\pi_a$ isomorphically onto the orthogonal complement of $T_b(K \cdot b)$. But by taking Lemma 2.14 into account, the isomorphism $a \cdot \mathfrak{p}_a \cong T_b(K \cdot b)^\perp$ is isometric. Also by Theorem 2.15, $(K \times K)$ -transversal geodesics g with the initial point a in G are in one-to-one correspondence with K -transversal geodesics with the initial point b in G/K through π . A similar argument works for transversal variations and the Jacobi vector fields; the Jacobi vector fields that are tangent to the $(K \times K)$ -orbit of $a' \in G$ correspond through π to the Jacobi vector fields that are tangent to the K -orbit of $b' = \pi(a')$. It follows that $d\pi$ induces the following isomorphism:

$$\Lambda_g(0) \cap J_g^{K \times K}(1) \cong \Lambda_{\pi \circ g}(0) \cap J_{\pi \circ g}(1).$$

Let $p_1: K \times K \rightarrow K$ be the projection to the first factor and then the diagram is commutative.

$$\begin{array}{ccc} (K \times K) \times G & \xrightarrow{\mu} & G \\ \downarrow p_1 \times \pi & & \downarrow \pi \\ K \times (G/K) & \xrightarrow{\mu} & G/K \end{array}$$

Thus the infinitesimal $(K \times K)$ -motion by $(X_1, X_2) \in \mathfrak{k} \oplus \mathfrak{k}$ is mapped by the projection to the infinitesimal K -motion by $X_1 \in \mathfrak{k}$. We see then that the variational completeness of the action of K on G/K is derived from that of the action of $K \times K$ on G . Q.E.D.

THEOREM 3.19. *The (adjoint) action of G on G by the inner automorphism*

$$A: G \times G \rightarrow G, A(x, y) = xyx^{-1}$$

and the adjoint action of G on its Lie algebra $\mathfrak{g}$

$$\text{Ad}: G \times \mathfrak{g} \rightarrow \mathfrak{g}$$

are both variationally complete.

PROOF. Let $d: G \rightarrow G \times G$ be the diagonal map. Then $(G \times G, d(G))$ is a symmetric pair by $\sigma(x, y) = (y, x)$. By the correspondence $(x, y) \mapsto xy^{-1}$ we obtain an isomorphism $(G \times G)/d(G) \cong G$ and the action of $d(G)$ on $(G \times G)/d(G)$ is equivalent to the inner automorphism. The orthogonal complement of $\mathfrak{k} = L(d(G))$ is given by $\{(X, -X) | X \in \mathfrak{g}\}$; if we identify $\mathfrak{g}$ with this by $X \mapsto (X, -X)$, then the action of G on $\mathfrak{g}$ coincides with the adjoint action. Thus the theorem is deduced from Theorems 3.18 and 3.17. Q.E.D.

§4. Applications to compact Lie groups and symmetric spaces

The orbit and the stabilizer with regard to the adjoint action. By Theorem 3.19 the adjoint actions of G

$$A: G \times G \rightarrow G \quad \text{and} \quad \text{Ad}: G \times \mathfrak{g} \rightarrow \mathfrak{g} = L(G)$$

are variationally complete. We study the G -orbits and the stabilizers of these actions.

NOTATION. $G(y) = A(G \times y) = \{gyg^{-1} | g \in G\}$ denotes the G -orbit of y with regard to A .

LEMMA 4.1. (1) $T_y(G(y)) = \{X \cdot y - y \cdot X | X \in \mathfrak{g}\}$.

(2) $T_y(Z(y)) = \{y \cdot Z | Z \cdot y = y \cdot Z, Z \in \mathfrak{g}\} = \{y \cdot Z | \text{Ad}_y(Z) = Z \in \mathfrak{g}\}$.

(3) $T_y(G) = T_y(Z(y)) \oplus T_y(G(y))$ (an orthogonal decomposition).

(4) $\dim G = \dim Z(y) + \dim G(y)$.

PROOF. (1) By 2.2 a vector of $T_y(G(y))$ is given by an infinitesimal G -motion. (1) is obtained by putting $U = V = X$ in Lemma 3.12(1).

(2) Since $y \cdot Z(y) = Z(y)$, it suffices to show

$$T_e(Z(y)) = L(Z(y)) = \{Z \in \mathfrak{g} | \text{Ad}_y(Z) = Z\}.$$

We have $Z \in L(Z(y))$ iff $ye^{tZ}y^{-1} = e^{tZ}$ iff $e^{\text{Ad}_y(tZ)} = e^{tZ}$ iff $\text{Ad}_y(Z) = Z$.

(3) For $Y \in \mathfrak{g}$, $y \cdot Y \perp G(y)$ iff $(y \cdot Y, X \cdot y - y \cdot X) = 0$ for all $X \in \mathfrak{g}$, which is equivalent to $(Y, \text{Ad}_y^{-1}(X) - X) = 0$, which in turn is equivalent to $(Y, \text{Ad}_y^{-1}(X)) = (Y, X)$, since $y \cdot$ is an orthogonal transformation. By (3.2.2), this is equivalent to $(\text{Ad}_y(Y), X) = (Y, X)$, which is equivalent to $\text{Ad}_y Y = Y$ and to $Y \in T_y(Z(y))$, since $\mathfrak{g} \ni X$ is arbitrary. Thus we obtain $T_y(G(y))^\perp = T_y(Z(y))$ and hence (3). (4) is clear from (3). Q.E.D.

The stabilizer with regard to the action A coincides with the centralizer in G :

$$G_y = Z(y) = \{g \in G | gy = yg\}. \quad (4.2)$$

Similar properties of the action Ad on $\mathfrak{g}$ are obtained for $B \in \mathfrak{g}$.

LEMMA 4.3. (1) $T_B(\text{Ad}(G) \cdot B) = \{B + [X, B] | X \in \mathfrak{g}\}$.

(2) The orthogonal complement of $T_B(\text{Ad}(G) \cdot B)$ is

$$\mathfrak{g}_B = \{Z \in \mathfrak{g} | [Z, B] = 0\}.$$

PROOF. (1) A vector of $T_B(\text{Ad}(G) \cdot B)$ is given as the initial vector of $\text{Ad}(e^{tX})B$. Thus (1) is obtained immediately from (3.2.3).

(2) $T_B(\text{Ad}(G) \cdot B)^\perp = \{B + Y | (Y, [X, B]) = 0 \text{ for } X \in \mathfrak{g}\}$. By (3.2.4) we have $0 = (Y, [X, B]) = -(Y, [B, X]) = ([Y, B], X)$, which is equivalent to $[Y, B] = 0$, namely, to $Y \in \mathfrak{g}_B$ since X is arbitrary. Since $B \in \mathfrak{g}_B$, we have $T_B(\text{Ad}(G) \cdot B)^\perp = B + \mathfrak{g}_B = \mathfrak{g}_B$. Q.E.D.

Next we sum up the results that will be used often below, taken from among those concerning a compact, connected Lie group G and obtained since §3 of Chapter V. () indicates the corresponding result in Chapter V.

4.4.1. Every element in G is in at least one maximal torus (Theorem 3.15).

4.4.2. Any two maximal tori of G are conjugate each other (Corollary 3.16).

4.4.3. $\exp: \mathfrak{g} = L(G) \rightarrow G$ is surjective (Corollary 3.17).

4.4.4. The identity component of the centralizer of $x \in G$ is $Z(x)_0 = \bigcup (\text{maximal tori containing } x)$ (Lemma 4.12).

4.4.5. If $t' = gtg^{-1}$ for $g \in G$ and $t, t' \in T$ (a maximal torus), there exists $n \in N(t)$ such that $t' = ntn^{-1}$ (Theorem 3.18).

4.4.6. For any torus T' of G its centralizer satisfies $Z(T') = \bigcup (\text{maximal tori containing } T')$ and hence $Z(T')$ is connected (Corollary 3.20 and Theorem 3.22).

4.5.1. $G \ni x$ is a regular element (i.e., $\dim Z(x)$ is minimal and equal to $\text{rank } G$) iff there is only one maximal torus which contains x (Theorem 4.13).

The restriction of the adjoint representation Ad to a maximal torus T gives an orthogonal decomposition $L(G) = L(T) \oplus \sum_{r=1}^m V_r$ ($\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$) and the roots

$$\theta_r: L(T) \rightarrow \mathbf{R} \quad \text{or} \quad \bar{\theta}_r: T \rightarrow \mathbf{R}/\mathbf{Z} \quad (r = 1, 2, \dots, m),$$

and closed subgroups $U_r = \bar{\theta}_r^{-1}(0)$ of T and its Lie algebra $L(U_r) = \theta_r^{-1}(0)$ are defined.

4.5.2. If $x \in T$ lies on exactly f pieces of U_r , then $\dim Z(x) = l + 2f$ ($l = \dim T = \text{rank } G$) (Theorem 4.15).

4.5.2'. Therefore, $\{\text{regular elements of } T\} = T - \bigcup_{r=1}^m U_r$.

4.5.3. $Z(G) = \bigcap_{r=1}^m U_r = \text{Ker}(\text{Ad}|_T) = \text{Ker } \text{Ad}$ (Theorem 4.18).

4.6.1. The Weyl group $W(T) = N(T)/T$ acts on $L(T)$ and permutes the roots $\{\theta_r\}$, and hence induces an orthogonal transformation that preserves the infinitesimal diagram $D'(G) = \bigcup_{r=1}^m L(U_r)$ and the Stiefel diagram $D(G) = \bigcup_{r=1}^m \theta_r^{-1}(\mathbf{Z})$ (Theorem 4.21).

4.6.2. The Weyl group $W(G)$ acts on a Weyl chamber (a component of $D'(G)$) simply transitively (Theorem 4.30).

THEOREM 4.7. *The stabilizer G_x of $X \in \mathfrak{g}$ with regard to $\text{Ad}: G \times \mathfrak{g} \rightarrow \mathfrak{g}$ is connected and given by*

$$G_x = \exp(\mathfrak{g}_x) = Z(T'),$$

where T' is the torus obtained as the closure of $\{\exp \alpha X | \alpha \in \mathbf{R}\}$.

PROOF. An element of the identity component $(G_x)_0$ of G_x can be written as $y = e^Y$, $Y \in L(G_x)$. Since $e^{tY} \in (G_x)_0$, we have $\text{Ad } e^{tY} X = X$. By (3.2.3), $[Y, X] = 0$ iff $Y \in \mathfrak{g}_x$. Conversely, $[Y, X] = 0$ implies $e^{tY} e^X e^{-tY} = e^X$ (Lemma 5.22 of Chapter V) so $\text{Ad } e^{tY} X = X$ and hence $e^{tY} \in (G_x)_0$. Thus $(G_x)_0 = \exp(\mathfrak{g}_x)$. Next, $y \in G_x$ iff $\text{Ad } y(tX) = tX$

($\forall t \in \mathbb{R}$) which is equivalent to $ye^{tX}y^{-1} = e^{tX}$ ($\forall t \in \mathbb{R}$) and to $y \in Z(T')$, that is, $G_X = Z(T')$. By 4.4.6, $Z(T') = Z(T')_0$. Q.E.D.

LEMMA 4.8. *Let $X \in \mathfrak{t} = L(T)$. If the number of hyperplanes $L(U_r)$ containing X in the infinitesimal diagram $D'(G)$ is f , then $\dim G_X = l + 2f$.*

PROOF. By the previous theorem $\dim G_X = \dim \mathfrak{g}_X = \dim Z(T')$. We have $G_X = G_{\alpha X}$ for $\alpha > 0$. If we take α sufficiently small, the hyperplanes containing αX in the Stiefel diagram are only those in the infinitesimal diagram. Thus $y = \exp(\alpha X) \in T'$ is contained in exactly f pieces of $U_r = \exp(L(U_r))$ and so by 4.5.2 $\dim Z(y) = l + 2f$. Since the generators of T' are dense, we may suppose that y is an element of T' . Thus $Z(y) = Z(T')$ implies $\dim G_X = \dim Z(y) = l + 2f$. Q.E.D.

REMARK 4.8'. It is obvious from 4.5.2 that the dimension of the stabilizer G_x of $x \in G$ with regard to $A: G \times G \rightarrow G$ is given by $\dim G_x = l + 2f$, where f is the number of hyperplanes of the Stiefel diagram $D(G)$ which pass through $X \in L(T)$ such that $\exp X = x$.

A geodesic transversal with respect to the adjoint action. We suppose that the terminal point r in the set of geodesic segments of $G \bmod N$, where $N = G(p)$,

$$S(G, N, r) \quad (r \notin N)$$

is in the G -orbit $G(r)$ of maximal dimension. Then by Lemma 4.1(4) the dimension of $Z(r)$ is minimal. By 4.5.1 and 4.5.2 we have

4.9. *If r is in an orbit of maximal dimension, $T = Z(r)$ is a maximal torus.*

Let $S(T, N \cap T, r)$ be simply the set of geodesic segments joining a point of $N \cap T$ and r .

LEMMA 4.10. $S(G, N, r) = S(T, N \cap T, r)$.

PROOF. For a geodesic segment $s = g|[0, 1]$ of $S(G, N, r)$, $\dot{g}(1) \perp G(r)$ implies $\dot{g}(1) \in T_r(G(r))^\perp$. By Lemma 4.1(3), $\dot{g}(1) \in T_r(Z(r)) = T_r(T)$, that is, g is tangent to T at r . When g is written as $g(1+t) = re^{tX}$ with $X \in \mathfrak{g}$ and $\dot{g}(1) = r \cdot X \in T_r(T) = r \cdot L(T)$, then $X \in L(T)$ implies that g is a geodesic in T . Thus $S(G, N, r) \subset S(T, N \cap T, r)$.

Conversely, for $s = g|[0, 1] \in S(T, N \cap T, r)$, the G -orbit of $g(0) \in N \cap T$ is N . By 4.4.4, $T \subset Z(x)$ for $x = g(t) \in T$. By Lemma 4.1(3), $\dot{g}(t) \in T_x(T) \subset T_x(Z(x))$ is orthogonal to $G(x)$. Thus g is G -transversal and, in particular, is a geodesic of $G \bmod N$. Since $\dot{g}(0)$ is obviously tangent to $N = G(p)$, we have $s \in S(G, N, r)$. Thus $S(T, N \cap T, r) = S(G, N, r)$. Q.E.D.

In the above proof $N \cap T$ is discrete in T , since N is orthogonal to T

at each intersection point. Since T is compact, we see the following:

4.10'. $N \cap T$ consists of a finite number of points.

To study $S(T, N \cap T, r)$ practically, it is convenient to consider

$$\mathfrak{t} = L(T) \quad (\exp T),$$

which is the universal covering space of T .

NOTATION. $\tilde{N} = (\text{a lift of } N \cap T \text{ by } \exp)$, that is, $\exp|_{\tilde{N}} : \tilde{N} \rightarrow N \cap T$ is a bijection.

$$R = \exp^{-1}(r).$$

$$S(\mathfrak{t}, \tilde{N}, R) = \{s : [0, 1] \rightarrow \mathfrak{t} \text{ a geodesic (segment)} | s(0) \in \tilde{N}, s(1) \in R\}.$$

THEOREM 4.11. $\exp : \mathfrak{t} \rightarrow T \subset G$ gives rise to the following one to one correspondence:

$$S(\mathfrak{t}, \tilde{N}, R) \leftrightarrow S(G, N, r),$$

where $T = Z(r)$, $r \notin N = G(p)$, $R = \exp^{-1}(r)$, and $\tilde{N}$ is a lift of $N \cap T$.

PROOF. In a covering space, a lifting of a path exists uniquely once the initial point is specified, and hence $\exp$ gives a one-to-one correspondence of $S(\mathfrak{t}, \tilde{N}, R)$ to $S(T, N \cap T, r)$. Thus the theorem is obtained from Lemma 4.10. Q.E.D.

Next, consider the adjoint action Ad on $\mathfrak{g} = L(G)$. By Lemma 4.3(2), the dimension of the orbit of $R \in \mathfrak{g}$ is maximal iff the dimension of $\mathfrak{g}_R$ is minimal. Obviously $\mathfrak{g}_R \supset \mathfrak{t} = L(T)$ if $\exp R$ is on a maximal torus T . Using Lemma 4.8 it is easy to see that $\mathfrak{g}_R = \mathfrak{t}$ is equivalent to $R \in \mathfrak{t} - D'(G)$. Therefore

4.12. The orbit of $R (\in \mathfrak{t})$ is of maximal dimension iff $R \in \mathfrak{t} - D'(G)$ which is equivalent to $\mathfrak{g}_R = \mathfrak{t}$ and to $T_R(\text{Ad}(G) \cdot R) = \mathfrak{t}^\perp = R + \mathfrak{m}$. For a general $R \in \mathfrak{t}$, $\mathfrak{t} \subset \mathfrak{g}_R$.

In the set of (geodesic) segments of $\mathfrak{g} \bmod N$, where $N = \text{Ad}(G) \cdot P = G(P)$,

$$S(\mathfrak{g}, N, R) \quad (R \notin N),$$

let $r \in \mathfrak{t} = l(\mathfrak{t})$ (and hence r be on an orbit of maximal dimension).

Let $S(\mathfrak{t}, N \cap \mathfrak{t}, R)$ simply be the set of (geodesic) segments joining a point of $N \cap \mathfrak{t}$ to R .

THEOREM 4.13. For $N, R, \mathfrak{t}$ as in the above,

$$S(\mathfrak{g}, N, R) = S(\mathfrak{t}, N \cap \mathfrak{t}, R).$$

PROOF. Since a segment s of $S(\mathfrak{g}, N, R)$ is orthogonal to $T_R(\text{Ad}(G) \cdot R) = \mathfrak{t}^\perp$ at $s(1) = R$, s is contained in $\mathfrak{t}$, and hence $S(\mathfrak{g}, N, R) \subset S(\mathfrak{t}, N \cap \mathfrak{t}, R)$.

Conversely, since $\dot{s}(t) \in \mathfrak{t} \subset \mathfrak{g}_X = (\text{Ad}(G) \cdot X)^\perp$ for a point $X = s(t)$ on a segment s of $S(\mathfrak{t}, N \cap \mathfrak{t}, R)$, s is G -transversal and, in particular, is a geodesic segment of $\mathfrak{g} \bmod N$. Thus $S(\mathfrak{t}, N \cap \mathfrak{t}, R) \subset S(\mathfrak{g}, N, R)$. Q.E.D.

Homology of path spaces on compact Lie groups. The homology of the path space $\Omega(G, N)$ is characterized by the Morse series $\mathcal{M}(G, N, r; t)$, which is determined by the indices of the geodesic segments of $S(G, N, r)$.

LEMMA 4.14. *The index of λ_s of $s \in S(T, N \cap T, r) = S(G, N, r)$ is even and*

$$\frac{1}{2}\lambda_s = \sum_{i=1}^m \# \{U_i \cap s(\text{Int } I)\}, \quad \text{Int } I = (0, 1).$$

PROOF. Consider the formula $\lambda_s = \sum_{0 < i < 1} \delta(s(t))$ in Theorem 2.12. The stabilizer G_x of $x \in G$ by the adjoint action A of G is $G_x = Z(x)$ by (4.2). Also by observing $Z(r) = T$, we have

$$\delta(s(t)) = \dim Z(x) - \dim T = l + 2f - l = 2f$$

for δ in (2.11) and the integer f in 4.5.2. The lemma follows immediately from this formula and Theorem 2.12. Q.E.D.

THEOREM 4.15. *Let G be a 1-connected Lie group and $N = G(p) = \{xpx^{-1} | x \in G\}$ for a point p of G . Then the homology group of the path space $\Omega(G, N)$ is determined by the following:*

- (1) $H_{2j-1}(\Omega(G, N)) = 0$;
- (2) $H_{2j}(\Omega(G, N))$ is a finitely generated free abelian group;
- (3) $P(\Omega(G, N), t) = \sum t^{2\nu(s)} (s \in S(\mathfrak{t}, \tilde{N}, R))$, where $R = \exp^{-1}(r)$ for the exponential map $\exp: \mathfrak{t} \rightarrow T$ (a maximal torus of G) and a regular point r of T , which is not contained in N ; $\tilde{N}$ is a lifting of $N \cap T$ over $\mathfrak{t}$, $S(\mathfrak{t}, \tilde{N}, R) = \{\text{segments } s \text{ in } \mathfrak{t} \text{ joining } \tilde{N} \text{ and } R\}$, and $\nu(s) = \#\{\text{hyperplanes } L_{i,n} \text{ of } D(G) \text{ with which the segment } s \text{ intersects}\}$ for the Stiefel diagram $D(G) = \bigcup_{i,n} L_{i,n}$ with $L_{i,n} = \theta_i^{-1}(n)$, $n \in \mathbb{Z}$.

PROOF. Apply Lemma 4.10 and Theorem 1.21. Use Lemma 4.14 and $\exp^{-1}(U_i) = L_i = \bigcap_n L_{i,n}$ to obtain $\nu(s)$. Q.E.D.

THEOREM 4.16. *For the loop space ΩG of a 1-connected Lie group G ,*

- $H_{2j-1}(\Omega G) = 0$;
- $H_{2j}(\Omega G)$ is a free abelian group;
- $P(\Omega G, t) = \sum_{\Delta} t^{2\nu(\Delta)}$, where Δ runs over the Weyl cells contained in the fundamental Weyl chamber $\mathcal{F}$ and $\nu(\Delta) = \#\{\text{hyperplanes } L_{i,n} \text{ of the Stiefel diagram } D(G) \text{ that intersect segments } s \text{ joining the original and (interior) points of } \Delta\}$.

PROOF. We have $N = \{e\}$ by putting $p = e$ in Theorem 4.15. Thus the results on H_{2j-1} and H_{2j} are obvious. Take $\tilde{N}$ to be the origin 0 . Since the Weyl group $W(G)$ permutes the Weyl chambers simply transitively by 4.6.2, there is $\phi \in W(G)$ for each $s \in S(\mathfrak{t}, 0, R)$ such that $\phi(s) \subset \mathcal{F}$. Since ϕ is a

congruent transformation that permutes the hyperplanes of $D(G)$ (Theorem 5.1 of Chapter V), we may replace s with $\varphi \circ s$. The terminal point $\in R$ of $\varphi \circ s$ is an interior point (regular point) of a Weyl cell Δ contained in $\mathcal{F}$. We see from Theorems 5.1 and 5.8(4) and from $\Gamma/\Gamma_0 = \pi_1(G) = 0$ ($\Gamma = \exp^{-1}(e)$) in Theorem 5.19 of Chapter V that there is exactly one point of R in every $\Delta \subset \mathcal{F}$. Moreover we easily see that $\nu(\Delta)$ is independent of the choice of an interior point of Δ and we obtain $\nu(s) = \nu(\Delta)$. Q.E.D.

THEOREM 4.17. *Let G be a compact, connected Lie group. Then the following hold:*

(1) $\pi_2(G) = 0$.

(2) $\pi_3(G)$ is a free abelian group and the number of the generators is equal to the number of simple factors of G . In particular, $\pi_3(G) \cong \mathbf{Z}$ if G is simple.

PROOF. We have $\pi_i(\tilde{G}) \cong \pi_i(G)$ for $i > 1$ in a covering group $(\tilde{G}, p, G)$. By Corollary 5.31 of Chapter V we can take the direct product of a torus T and several 1-connected simple Lie groups as $\tilde{G}$. Since $\pi_i(T) = 0$ for $i > 1$, it suffices to show the theorem for the case G is a 1-connected, simple Lie group. By Theorem 4.16 and the Hurewicz theorem, $\pi_2(G) \cong \pi_1(\Omega G) \cong H_1(\Omega G) = 0$ and $\pi_3(G) \cong \pi_2(\Omega G) \cong H_2(\Omega G)$ is a free abelian group and the number of the generators is the number of Weyl cells Δ of the fundamental chamber $\mathcal{F}$ such that the segment joining the origin and a point of Δ intersects exactly one hyperplane of $D(G)$. A halfline issuing from the origin in $\mathcal{F}$ crosses through the fundamental Weyl cell Δ_0 in the first place and the first hyperplane it intersects is $\tilde{\alpha} = 1$ (where $\tilde{\alpha}$ is the dominant root); there are only two Weyl cells in $\mathcal{F}$ that have $\tilde{\alpha} = 1$ as a face, that is, Δ_0 and its reflection Δ in $\tilde{\alpha} = 1$. It follows that the number of the generators is 1 and $\pi_3(G) \cong \mathbf{Z}$. Q.E.D.

COROLLARY 4.18. *If G is a 1-connected, simple Lie group, then*

$$\pi_2(\Omega G) \cong H_2(\Omega G) \cong H^2(\Omega G) \cong \mathbf{Z}.$$

Homology of $G/Z(T')$. Let T' be a torus of G , T a maximal torus containing T' , and $\mathfrak{t} = L(T) \subset \mathfrak{g} = L(G)$.

LEMMA 4.19. *Let $X \in \mathfrak{t}$ be a point such that $\exp X$ is a generator of T' and $N = \text{Ad}(G) \cdot X$ be the orbit of X . Then*

(1) N is homeomorphic to $G/Z(T')$ and homotopy equivalent to $\Omega(\mathfrak{g}, N)$.

(2) *If we choose X to be a point sufficiently close to the origin, then $\mathfrak{t} \cap N$ coincides with the orbit $W(G) \cdot X$ of X by the Weyl group $W(G)$.*

PROOF. (1) By Theorem 4.7 the stabilizer of X is $G_X = Z(T')$. Thus

$$N = \text{Ad}(G) \cdot X \cong G/G_X = G/Z(T').$$

The second half holds in general if $\mathfrak{g}$ is contractible.

(2) Since the action of $\varphi = [n] \in W(G) = N(T)/T$ on t is given by $\text{Ad } n$, we have $W(G) \cdot X \subset t \cap N$. To show the converse inclusion; let $Y \in t \cap N$, then there is $g \in G$ such that $\text{Ad } g(X) = Y$. If we put $x = \exp X$ and $y = \exp Y$, then we have $x, y \in T$ and $gxg^{-1} = y$. By 4.4.5 there exists $n \in N(T)$ such that $ntn^{-1} = y$. Putting $\text{Ad } n(X) = Y'$, we have $\exp Y' = \exp(\text{Ad } n(X)) = nxn^{-1} = gxg^{-1} = \exp(\text{Ad}(g(X)) = \exp Y$, whence $Y - Y' \in \Gamma = \exp^{-1}(e)$. $\text{Ad } g$ and $\text{Ad } n$ are orthogonal transformations of $\mathfrak{g}$, and if we take X sufficiently close to the origin, $Y - Y'$ is close to the origin and hence $Y - Y' = 0$ and so $Y = Y' = \text{Ad } n(X) \in W(G) \cdot X$. Thus $t \cap N \subset W(G) \cdot X$ is shown. Q.E.D.

THEOREM 4.20. *If T' is a torus of a compact, connected Lie group G , then*

(1) $H_{2j-1}(G/Z(T')) = 0$.

(2) $H_*(G/Z(T'))$ is a finitely generated free abelian group.

(3) $P(G/Z(T'), t) = \sum t^{2\nu(Q)} \quad (Q \in W(G) \cdot x)$, where X is a point of $t = L(T)$ such that $\exp X$ generates T' , and $\nu(Q)$ is given as follows when a point R in a Weyl chamber $\mathcal{F} (\subset L(T) - D'(G))$ is fixed:

$\nu(Q) = \# \{ \text{the hyperplanes in the infinitesimal diagram } D'(G)$

that intersect at an interior point of a segment s

joining Q and R }.

PROOF. By Lemma 4.19, $G/Z(T')$ can be replaced by $\Omega(\mathfrak{g}, N)$ with $N = \text{Ad}(G) \cdot X$ and we have $N \cap t = W(G) \cdot X$ by taking X sufficiently close to the origin. By Theorems 1.21 and 4.13 it suffices to show that the index $\lambda_s = \sum_{0 < t < 1} \delta(s(t))$ is even for a segment s of $S(t, N \cap t, R)$ and that the following equality is satisfied for the initial point $Q \in W(G) \cdot X$ of s :

$$\lambda_s = 2\nu(Q). \quad (4.20')$$

By Lemma 4.8 and (2.11) we have

$$\delta(Y) = \dim G_Y - \dim G_R = l + 2f - l = 2f$$

for $Y \in t$, where f is the number of the hyperplanes of $D'(G)$ that pass through Y . If we sum up these numbers for $Y \in s((0, 1))$, we obtain the equality of (4.20'). Q.E.D.

THEOREM 4.21. *Let G be a compact, connected Lie group and T a maximal torus of G . Then*

(1) $H_{2j-1}(G/T) = H^{2j-1}(G/T) = 0$.

(2) $H_*(G/T) \cong H^*(G/T)$ is a free abelian group, the rank of which coincides with the order of the Weyl group $W(G)$.

PROOF. We have $Z(T) = T$, since T is a maximal torus. In Theorem 4.20 we can take X to be a point in a Weyl cell. By 4.6.2 the number of the orbits $W(G) \cdot X$ of X is equal to $|W(G)|$. Thus, the theorem follows easily from Theorem 4.20. Q.E.D.

Transversal geodesics with respect to a symmetric pair and the defect. Consider the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ of the Lie algebra by an automorphism σ of G , which determines a symmetric pair (G, K) . As the Lie algebra $\mathfrak{t} = L(T)$ of a maximal torus T of G is called a Cartan subalgebra, we give the following

DEFINITION. A maximal abelian Lie subalgebra $\mathfrak{s}$ in $\mathfrak{p}$ is called a *Cartan subalgebra* of (G, K) .

If we put $S = \exp(\mathfrak{s})$, we have the following for the Lie algebra $\mathfrak{t} = L(T)$ of a maximal torus T containing S .

4.22. We have $\mathfrak{s} = \{X \in \mathfrak{t} | \sigma(X) = -X\} = \mathfrak{t} \cap \mathfrak{p}$, and S is a torus of G , which coincides with the identity component of $\{x \in T | \sigma(x) = x^{-1}\}$.

DEFINITION. $S = \exp(\mathfrak{s})$ is called the *maximal torus* of a symmetric pair (G, K) .

$A \in \mathfrak{s}$ is called a *regular point* of $\mathfrak{s}$ if $\mathfrak{s} = \mathfrak{p}_A (= \{X \in \mathfrak{p} | [A, X] = 0\})$.

$a \in S$ is called a *regular point* of S if $\mathfrak{s} = \mathfrak{p}_a (= \mathfrak{p} \cap \text{Ad } a^{-1}(\mathfrak{p}))$.

LEMMA 4.23. If $N = \text{Ad}(K) \cdot B$ is the orbit of $B \in \mathfrak{p}$ with respect to the adjoint action of K on $\mathfrak{p}$ in Theorem 3.17 and if $A \notin N$ is a regular point of $\mathfrak{s}$, then we have $S(\mathfrak{p}, N, A) = S(\mathfrak{s}, N \cap \mathfrak{s}, A)$.

The proof is similar to Theorem 4.13.

COROLLARY 4.24. $S(\mathfrak{p}, N, A) \neq \emptyset$ and hence $N \cap \mathfrak{s} \neq \emptyset$.

PROOF. Since N is compact, there is the closest point to A such that any straight line joining the point and A is perpendicular to N . Thus $S(\mathfrak{p}, N, A) \neq \emptyset$. Q.E.D.

COROLLARY 4.25. The Cartan subalgebra of (G, K) are conjugate to one another with respect to the adjoint action of K .

PROOF. Consider a Cartan subalgebra $\mathfrak{s}'$ in addition to $\mathfrak{s}$ and let B be a regular point of $\mathfrak{s}'$ and N the orbit of B . Then, since $N \cap \mathfrak{s} \neq \emptyset$, there is $k \in K$ such that $\text{Ad } k(B) \in \mathfrak{s}$ hence $\text{Ad } k(\mathfrak{s}') = \mathfrak{s}$. Q.E.D.

LEMMA 4.26. The orbit N of an arbitrary point B of $\mathfrak{p}$ is orthogonal to $\mathfrak{s}$ at each intersection point and $\dim N = \dim \mathfrak{p} - \dim \mathfrak{p}_B$. If A is a regular point, then the dimension of its orbit is maximal.

PROOF. We may suppose $B \in \mathfrak{s}$ by Corollary 4.24. By Lemma 3.8(1) and (3.2.3), the orthogonal complement of $T_B N$ is $\mathfrak{p}_B$. Also, $\mathfrak{p}_B \supset \mathfrak{s} = \mathfrak{p}_A$. The lemma is obtained from these. Q.E.D.

THEOREM 4.27. The defect $\delta_K(B) = \dim \text{Ad}(K) \cdot A - \dim \text{Ad}(K) \cdot B$ of $B \in \mathfrak{p}$ satisfies the following equality:

$$\delta_K(B) = \dim \mathfrak{p}_B - \dim \mathfrak{p}_A = \frac{1}{2}(\dim \mathfrak{g}_B - \dim \mathfrak{g}_A).$$

PROOF. It follows immediately from Lemma 4.26 that $\delta_K(B) = \dim \mathfrak{p}_B - \dim \mathfrak{p}_A$.

Obviously, $\mathfrak{g}_B = \mathfrak{k}_B \oplus \mathfrak{p}_B$. By Lemma 3.8(1), $\mathfrak{p} = \text{ad } B(\mathfrak{k}) \oplus \mathfrak{p}_B$. If $\mathfrak{g}_B^\perp$ is the orthogonal complement of $\mathfrak{g}_B$ in $\mathfrak{g}$, we have $\text{ad } B(\mathfrak{k}) = \mathfrak{g}_B^\perp \cap \mathfrak{p}$ and $\text{ad } B: \mathfrak{g}_B^\perp \cap \mathfrak{k} \cong \mathfrak{g}_B^\perp \cap \mathfrak{p}$. $\mathfrak{k} = \mathfrak{k}_B \oplus (\mathfrak{g}_B^\perp \cap \mathfrak{k})$ and $\mathfrak{p} = \mathfrak{p}_B \oplus (\mathfrak{g}_B^\perp \cap \mathfrak{p})$ imply $\dim \mathfrak{k} - \dim \mathfrak{k}_B = \dim \mathfrak{p} - \dim \mathfrak{p}_B$. Similarly for A , and we have

$$\dim \mathfrak{k}_B - \dim \mathfrak{p}_B = \dim \mathfrak{k} - \dim \mathfrak{p} = \dim \mathfrak{k}_A - \dim \mathfrak{p}_A.$$

Thus $2\delta_K(B) = (\dim \mathfrak{p}_B - \dim \mathfrak{p}_A) + (\dim \mathfrak{k}_B - \dim \mathfrak{k}_A) = \dim \mathfrak{g}_B - \dim \mathfrak{g}_A$. Q.E.D.

A similar result on the action of $K \times K$ on G is obtained. Let $N = KbK$ be the orbit of a point b on a maximal torus S of (G, K) . By Lemma 3.5 and (3.4), one has a decomposition $T_b G = T_b N \oplus b \cdot \mathfrak{p}_b$. The following lemma is obtained similarly to Theorem 4.13.

LEMMA 4.28. *If $a \notin N$ is a regular point of S , then the dimension of the orbit of a is maximal (among the points of S), $N \cap S \neq \emptyset$, N is orthogonal to S , and $S(G, N, a) = S(\mathfrak{s}, N \cap \mathfrak{s}, a)$.*

NOTATION. $\mathfrak{g}|a = \{X \in \mathfrak{g} | \text{Ad } a \cdot X = X\}$.

LEMMA 4.29. *If $a \in S$ (in general $\sigma(a) = a^{-1}$), we have the following orthogonal decomposition:*

$$\mathfrak{g}|a^2 = \mathfrak{k}_a \oplus \mathfrak{p}_a, \quad \mathfrak{p} = (\text{Ad } a - \text{Ad } a^{-1})\mathfrak{k} \oplus \mathfrak{p}_a.$$

PROOF. Obviously $\sigma(\mathfrak{g}|a^2) = \mathfrak{g}|a^2$ and hence $\mathfrak{g}|a^2 = (\mathfrak{g}|a^2) \cap \mathfrak{k} \oplus (\mathfrak{g}|a^2) \cap \mathfrak{p}$. $X \in \mathfrak{p}_a = \mathfrak{p} \cap \text{Ad } a^{-1} \cdot \mathfrak{p}$ implies $\text{Ad } a \cdot X \in \mathfrak{p}$ which implies $\text{Ad } a \cdot X = -\sigma(\text{Ad } a \cdot X) = \text{Ad } a^{-1} \cdot X$. Thus $\text{Ad } a^2 \cdot X = X$ and we have $X \in \mathfrak{g}|a^2$, whence $\mathfrak{p}_a \subset (\mathfrak{g}|a^2) \cap \mathfrak{p}$. Conversely, if $X \in (\mathfrak{g}|a^2) \cap \mathfrak{p}$ then

$$\sigma(\text{Ad } a^{-1} \cdot X) = -\text{Ad } a \cdot X = -\text{Ad } a^{-1} \cdot X$$

which implies $\text{Ad } a^{-1} \cdot X \in \mathfrak{p}$, whence $\mathfrak{p}_a = (\mathfrak{g}|a^2) \cap \mathfrak{p}$. Similarly $\mathfrak{k}_a = (\mathfrak{g}|a^2) \cap \mathfrak{k}$ is shown and so the first equality is obtained.

Next, since $\sigma((\text{Ad } a - \text{Ad } a^{-1})X) = -(\text{Ad } a - \text{Ad } a^{-1})\sigma(X)$, we have $(\text{Ad } a - \text{Ad } a^{-1})\mathfrak{k} \subset \mathfrak{p}$ and $(\text{Ad } a - \text{Ad } a^{-1})\mathfrak{p} \subset \mathfrak{k}$. If $Z \in \mathfrak{p}$ is orthogonal to $(\text{Ad } a - \text{Ad } a^{-1})\mathfrak{k}$, $0 = ((\text{Ad } a - \text{Ad } a^{-1})\mathfrak{k}, Z) = (\mathfrak{k}, (\text{Ad } a - \text{Ad } a^{-1})Z)$ so $(\text{Ad } a - \text{Ad } a^{-1})Z$ belongs to $\mathfrak{p}$ as well as $\mathfrak{k}$. Therefore $\text{Ad } a^{-1} \cdot Z = \text{Ad } a \cdot Z$, which implies $Z \in (\mathfrak{g}|a^2) \cap \mathfrak{p} = \mathfrak{p}_a$. Conversely, it is similarly shown that an element of $\mathfrak{p}_a$ is orthogonal to $(\text{Ad } a - \text{Ad } a^{-1})\mathfrak{k}$ and so the second equality is obtained. Q.E.D.

COROLLARY 4.30. *If $a \in S$, then*

$$\dim((\mathfrak{g}|a^2) \cap \mathfrak{p}) = \dim((\mathfrak{g}|a^2) \cap \mathfrak{k}), \quad \dim \mathfrak{k}_a - \dim \mathfrak{p}_a = \dim \mathfrak{k} - \dim \mathfrak{p}.$$

The corollary is shown similarly to the proof of Theorem 4.27 by using $\mathfrak{k}_a = \text{Ker}(\text{Ad } a - \text{Ad } a^{-1}: \mathfrak{k} \rightarrow \mathfrak{k})$ and the following equality is obtained as well.

4.3.1. *The defect of $b \in S$ with respect to the action of $K \times K$ on G is given by*

$$\delta_K(b) = \dim \mathfrak{k}^a - \dim \mathfrak{k}^b = \frac{1}{2}(\dim(\mathfrak{g}|b^2) - \dim(\mathfrak{g}|a^2)),$$

where a is a regular point of S , $\mathfrak{k}^b = \mathfrak{k} \oplus \text{Ad } b^{-1}(\mathfrak{k})$ and $\mathfrak{k}^a = \mathfrak{k} \oplus \text{Ad } a^{-1}(\mathfrak{k})$.

Homology of K/K_s and $\Omega(G/K)$. When S is a maximal torus of a symmetric pair (G, K) and T is a maximal torus of G containing S , we define the infinitesimal diagram $D'(G, K) \subset L(T)$ of (G, K) as follows:

DEFINITION. Those hyperplanes of the infinitesimal diagram $D'(G)$ of G that do not contain $s = L(S)$ are called *hyperplanes* of $D'(G, K)$ and the *infinitesimal diagram* $D'(G, K)$ is defined to be the union of these hyperplanes. Each connected component of $L(T) - D'(G, K)$ is called a *Weyl chamber* of $D'(G, K)$.

NOTATION. $K_s = \{k \in K | \text{Ad } k \cdot X = X, \forall X \in \mathfrak{s}\}$ (the stabilizer of $\mathfrak{s}$).

$N_K(\mathfrak{s}) = \{k \in K | \text{Ad } k \cdot \mathfrak{s} = \mathfrak{s}\}$ (the normalizer of $\mathfrak{s}$).

DEFINITION. $W(G, K) = N_K(\mathfrak{s})/K_s$ is called the *Weyl group* of (G, K) .

These infinitesimal diagrams and Weyl groups have properties similar to infinitesimal diagrams and Weyl groups of (compact) Lie groups. For example, it follows easily from Lemma 4.28 that

4.32. *$W(G, K)$ is a finite group and $N \cap \mathfrak{s}$ coincides with the orbit of B under $W(G, K)$ where $N = \text{Ad } K \cdot B$ is the K -orbit of a regular point B of $\mathfrak{s}$.*

The following theorem holds analogously to Theorem 4.20.

THEOREM 4.33. *Under the assumption and the notation of Lemma 4.23 the Morse series of $\Omega^*(\mathfrak{p}, N, A)$ is given by*

$$\mathcal{M}(\mathfrak{p}, N, A; t) = \sum_s t^{\lambda(s)} \quad (s \in S(\mathfrak{s}, N \cap \mathfrak{s}, A)),$$

where $\lambda(s) = \#\{\text{the hyperplanes of } D'(G, K) \text{ that intersect the geodesic } s\}$. Therefore there exists a spectral sequence that begins with a chain complex $\sum_i C_i$ over an (arbitrary) field $\mathbf{k}$ such that $\sum \dim_{\mathbf{k}} C_i t^i = \sum t^{\lambda(s)}$ and converges to $H_*(N; \mathbf{k}) \cong H_*(K/K_s; \mathbf{k})$.

REMARK. According to Theorem 6.7 the spectral sequence in Theorem 4.33 is trivial if $\mathbf{k} = \mathbb{Z}_2$. In particular, it follows from $H_0(N; \mathbb{Z}_2) = \mathbb{Z}_2$ that there is only one s such that $\lambda(s) = 0$.

Therefore by using 4.32 one sees the following:

4.34. $W(G, K)$ permutes the Weyl chambers of (G, K) simply transitively.

DEFINITION. The Stiefel diagram $D(G, K)$ of $D(G)$ is defined to be the union of those hyperplanes of $D(G)$ that are parallel to the hyperplanes of $D'(G, K)$ (which are called *hyperplanes* of $D(G, K)$). Each connected component of $L(S) - D(G, K)$ is called a *Weyl cell* of (G, K) .

A result similar to Theorem 4.15 is obtained by using

$$p: \mathfrak{s} \rightarrow S \quad \text{where} \quad p(X) = \exp(\tfrac{1}{2}X), \quad X \in \mathfrak{s}$$

(in place of $\exp: \mathfrak{t} \rightarrow T$). Then the defect δ_K with regard to the action of $K \times K$ on G is given by

$$\delta_K(p(X)) = \#\{\text{the hyperplanes of } D(G, K) \text{ containing } X\} \quad (4.35)$$

and the following theorem is obtained.

THEOREM 4.36. Under the assumption and with the notation of Lemma 4.28, the Morse series of $\Omega^*(G, N, a)$ is given by

$$\mathcal{M}(G, N, a; t) = \sum_s t^{\nu(s)} \quad (s \in S(\mathfrak{s}, p^{-1}(N \cap \mathfrak{s}), A), \quad a = p(A)),$$

where $\nu(s) = \#\{\text{the hyperplanes of } D(G, K) \text{ that } s \text{ intersects}\}$.

If N is the orbit of the identity e , then we have $N = KeK = K$, and the projection $G \rightarrow G/K$ of the fibration induces a homotopy equivalence $\Omega(G, K) \rightarrow \Omega(G/K)$ and hence the following theorem, which corresponds to Theorem 4.16, is obtained.

THEOREM 4.37. Let G be 1-connected. Fix one Weyl chamber $\mathcal{F}$ of (G, K) . Suppose that a Weyl cell Δ of (G, K) runs over those that are contained in the closure of $\overline{\mathcal{F}}$ of $\mathcal{F}$. Let

$$\nu(\Delta) = \#\{\text{the hyperplanes of } D(G, K) \text{ that a segment joining the origin and a point of } \Delta \text{ transverse}\}.$$

Then there is a spectral sequence, which begins with a module $\sum_i C_i$ over an (arbitrary) field $\mathbf{k}$, such that $\sum_i \dim_{\mathbf{k}} C_i t^i = \sum_{\Delta} t^{\nu(\Delta)}$ and converges to $H_*(\Omega(G/K); \mathbf{k})$.

REMARK 4.38. According to Theorem 6.7 the spectral sequence in Theorem 4.27 is trivial if $\mathbf{k} = \mathbf{Z}_2$ and the Poincaré series of $H_*(\Omega(G/K); \mathbf{Z}_2)$ coincides with $\sum_{\Delta} t^{\nu(\Delta)}$. Similarly in Theorem 4.36, the Poincaré series of $H_*(\Omega(G, N); \mathbf{Z}_2)$ coincides with $\sum_s t^{\nu(s)}$.

§5. The rank and the type of a compact connected Lie group

The rank of a compact connected Lie group. In this section we take the rational field $\mathbb{Q}$ as coefficients for cohomology. Observe that the results of this section hold for any field k of characteristic 0, similarly to the case of $\mathbb{Q}$, by using the isomorphism

$$H^*(X; k) \cong H^*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} k. \quad (5.1)$$

LEMMA 5.2. *If $\tilde{G}$ and G are compact, connected Lie groups and if $(\tilde{G}, p, G)$ is a covering group, then*

$$p^*: H^*(G; \mathbb{Q}) \cong H^*(\tilde{G}; \mathbb{Q}) \quad (\text{an isomorphism}).$$

PROOF. $\text{Ker } p$ is a finite subgroup contained in the center of $\tilde{G}$. Since $(\tilde{G}, p, G)$ is a $\text{Ker } p$ -bundle, there is the classifying space $B(\text{Ker } p)$ with a map $f: G \rightarrow B \text{Ker } p$ such that

$$\tilde{G} \xrightarrow{p} G \xrightarrow{f} B \text{Ker } p$$

is a fibration. Here we see that the action of $\pi_1(B \text{Ker } p) \cong \text{Ker } p$ on $\tilde{G}$ is given by inner automorphisms by elements of $\text{Ker } p$ and that it is trivial. Therefore there is a spectral sequence: $E_2 = H^*(B \text{Ker } p; \mathbb{Q}) \otimes H^*(\tilde{G}; \mathbb{Q}) \Rightarrow H^*(G; \mathbb{Q})$. Since $\text{Ker } p$ is a finite abelian group, we see $H^*(B \text{Ker } p; \mathbb{Q}) \cong \mathbb{Q}$. Thus the spectral sequence in the above is trivial and p^* is an isomorphism. Q.E.D.

THEOREM 5.3. *For a compact, connected Lie group G there exist a finite number of elements $x_{2m_i+1} \in H^{2m_i+1}(G; \mathbb{Q})$ of odd degrees such that*

$$H^*(G; \mathbb{Q}) = \Lambda(x_{2m_1+1}, \dots, x_{2m_l+1}).$$

PROOF. By Corollary 5.3 of Chapter V and Lemma 5.2 one may suppose that G is a direct product of a torus $T_0 = S^1 \times \dots \times S^1$ and a 1-connected Lie group G' . The theorem is valid for G' by Theorem 6.26 of Chapter IV. Since it is also obvious for T_0 , we see that the theorem holds by the Künneth isomorphism. Q.E.D.

DEFINITION. The sequence $(2m_1+1, \dots, 2m_l+1)$ of positive odd integers in the above theorem is called the *(rational) type* of G (cf. Chapter IV, §6).

Next we will show that l coincides with $\text{rank } G = \dim T$, where T is a maximal torus.

LEMMA 5.4. *Let X be a 1-connected space of finite type (i.e., $H^i(X)$ is finitely generated for all i). If $H^*(\Omega X; \mathbb{Q}) = \Lambda(x_{2m_1+1}, \dots, x_{2m_l+1})$ for the loop space ΩX of X , then*

$$H^*(X; \mathbb{Q}) = \mathbb{Q}[x_{2m_1+2}, \dots, x_{2m_l+2}],$$

where x_{2m_i+1} is chosen to be transgressive so that $\tau(x_{2m_i+1}) = x_{2m_i+2}$.

PROOF. The case $l = 1$ is easily proved by using the Gysin exact sequence. Since $\pi_i(S^{2m+1})$ is a finite group for $i \neq 2m+1$, a map $f: S^{2m+1} \rightarrow K(\mathbb{Z}, 2m+1)$ representing a generator of π_{2m+1} of the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2m+1)$ is a 0-equivalence and hence we have

$$f^{*-1}: H^*(S^{2m+1}; \mathbb{Q}) \cong H^*(K(\mathbb{Z}, 2m+1); \mathbb{Q}) = \Lambda(u_{2m+1}).$$

From the case $l = 1$ we have

$$H^*(K(\mathbb{Z}, 2m+2); \mathbb{Q}) = \mathbb{Q}[u_{2m+2}] \quad (5.4')$$

since $\Omega K(\mathbb{Z}, 2m+2) = K(\mathbb{Z}, 2m+1)$.

Let $l > 1$ and suppose that the theorem is true for $l-1$. We have

$$H^{2m_1+1}(\Omega X; \mathbb{Q}) \cong H^{2m_1+2}(X; \mathbb{Q})$$

in the minimum degree. Choose the generator x_{2m_1+1} in the integral cohomology class and let $f: X \rightarrow K(\mathbb{Z}, 2m_1+2)$ be a map corresponding to x_{2m_1+2} through the bijection $[X, K(\mathbb{Z}, 2m_1+2)] \cong H^{2m_1+2}(X)$, and then $f^*(u_{2m_1+2}) = x_{2m_1+2}$. We replace f by the projection of a fibration and denote by Y the fibre:

$$Y \xrightarrow{i} X \xrightarrow{f} K(\mathbb{Z}, 2m_1+2). \quad (*)$$

Taking loop spaces, we also obtain a fibration

$$\Omega Y \xrightarrow{\Omega i} \Omega X \xrightarrow{\Omega f} K(\mathbb{Z}, 2m_1+1) = \Omega K(\mathbb{Z}, 2m+2),$$

where we have a 0-equivalence $g: \prod_{i>1} S^{2m_i+1} \rightarrow \Omega X$ by Theorem 6.7 of Chapter IV. It is easy to see that g can be chosen to satisfy $\Omega f \circ (g|S^{2m_i+1}) \simeq 0$ for $i > 1$ and hence there exists $h: \prod_{i>1} S^{2m_i+1} \rightarrow \Omega Y$ so that $\Omega i \circ h \simeq g| \prod_{i>1} S^{2m_i+1}$. Comparing the homotopy groups mod $\mathcal{E}_0$ (a class of finite groups), we see that h is a 0-equivalence and so $H^*(\Omega Y; \mathbb{Q}) \cong H^*(\prod_{i>1} S^{2m_i+1}; \mathbb{Q}) = \Lambda(x_{2m_2+1}, \dots, x_{2m_{l+1}})$. By the inductive hypothesis we have $H^*(Y; \mathbb{Q}) = \mathbb{Q}[x_{2m_2+2}, \dots, x_{2m_{l+2}}]$. Since the rational cohomology groups of the base space and the fibre of the fibration $(*)$ consist of the elements of even degree, the associated spectral sequence is trivial and so

$$\begin{aligned} H^*(X; \mathbb{Q}) &\cong H^*(K(\mathbb{Z}, 2m_1+2); \mathbb{Q}) \otimes H^*(Y; \mathbb{Q}) \\ &\cong \mathbb{Q}[x_{2m_1+2}, \dots, x_{2m_{l+2}}]. \end{aligned}$$

If one takes x_{2m_1+1} to be the suspension (which is the converse of the transgression and unique) image of x_{2m_1+2} , the theorem is obtained. Q.E.D.

REMARK. The lemma is valid even with $\mathbb{Z}_p$ -coefficients (cf. Theorem 2.14 of Chapter VII).

THEOREM 5.5. *Let G be a compact, connected Lie group and T a maximal torus of G . If G is of the type $(2m_1 + 1, \dots, 2m_l + 1)$, then the following hold:*

- (1) l coincides with $\text{rank } G = \dim T$.
- (2) The Poincaré polynomial of G/T is given as follows:

$$P(G/T, t) = \prod_{i=1}^l \frac{1 - t^{2m_i+2}}{1 - t^2} = \prod_{i=1}^l (1 + t^2 + t^4 + \dots + t^{2m_i}).$$

- (3) The order of the Weyl group $W(G)$ of G is $|W(G)| = \prod_{i=1}^l (m_i + 1)$.

- (4) $P(\Omega G, t) = \prod_{i=1}^l (1 - t^{2m_i})^{-1}$, if G is 1-connected.

- (5) $H^*(BG; \mathbf{Q}) = \mathbf{Q}[x_{2m_1+2}, \dots, x_{2m_l+2}]$, $x_{2m_i+2} = \tau(x_{2m_i+1})$.

PROOF. (1) Put $\text{rank } G = \dim T = l'$. In the $\mathbf{Q}$ -coefficient spectral sequence associated with the fibre bundle

$$G/T \rightarrow BT \rightarrow BG,$$

$H^*(G/T; \mathbf{Q})$ does not have elements of odd degrees by Theorem 4.21. The same thing is also valid for BG by Lemma 5.4 and the fact that G is (weakly) homotopy equivalent to ΩBG . Thus the spectral sequence associated with the above fibration is trivial and hence we have the following equation:

$$P(BT, t) = P(BG, t) \cdot P(G/T, t) = \prod_{i=1}^l (1 - t^{2m_i+2})^{-1} \cdot P(G/T, t).$$

$$H^*(BT; \mathbf{Q}) = \mathbf{Q}[t_1, \dots, t_{l'}], \quad t_i \in H^2 \text{ implies } P(BT, t) = (1 - t^2)^{-l'} \text{ so}$$

$$P(G/T, t)(1 - t^2)^{l'-l} = \prod_{i=1}^l \frac{(1 - t^{2m_i+2})}{(1 - t^2)} = \prod_{i=1}^l (1 + t^2 + t^4 + \dots + t^{2m_i}).$$

Putting $t = 1$, we have $l = l' = \text{rank } G$, since $|W(G)| = P(G/T, 1)$ is a positive integer by Theorem 4.21. Thus (2) and (3) are obtained.

(4) and (5) are obtained from the second equation of Corollary 6.29 of Chapter IV and Lemma 5.4 respectively. Q.E.D.

COROLLARY 5.5'. *If $Z(T')$ is of type $(2n_1 + 1, \dots, 2n_l + 1)$, then*

$$P(G/Z(T'), t) = \prod_{i=1}^l \frac{1 - t^{2m_i+1}}{1 - t^{2n_i+1}}.$$

PROOF. By a similar argument to the above using Theorem 4.20 we have $P(G/T, t) = P(G/Z(T'), t) \cdot P(Z(T')/T, t)$. Q.E.D.

The relation between the type of a simple Lie group and the roots. Let $\theta_r: L(T) \rightarrow \mathbf{R}$ for $r = 1, 2, \dots, m$ be the positive roots of a compact, connected, simple Lie group and define by

$$\mathcal{F} = \{X \in L(T) | 0 < \alpha_i(X) < 1 \text{ for } i = 1, 2, \dots, l\}$$

the parallelootope in the fundamental Weyl chamber

$$\mathcal{F} = \{X \in L(T) | \theta_r(X) > 0 \text{ for } r = 1, 2, \dots, m\},$$

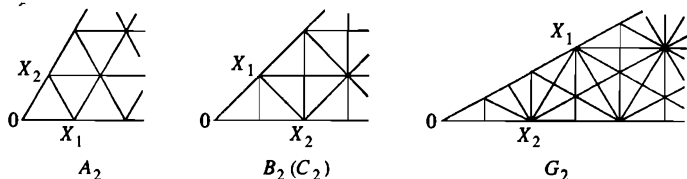


FIGURE 6.1

which determines the simple roots α_i for $i = 1, 2, \dots, l$. We also define vectors $X_1, \dots, X_l$ of $\mathcal{F}$ by

$$\alpha_i(X_j) = \delta_{ij} \quad \text{for } i, j = 1, 2, \dots, l$$

using the simple roots α_i .

Let $\mathbf{Z}_+$ be the set of 0 and the positive integers and $\mathbf{Z}_+^l$ the direct product of l copies of $\mathbf{Z}_+$. We define $X_n \in \mathcal{F}$ for $n = (n_1, \dots, n_l) \in \mathbf{Z}_+^l$ by the following equality:

$$X_n = n_1 X_1 + \dots + n_l X_l.$$

LEMMA 5.6. (1) Any Weyl cell Δ contained in $\mathcal{F}$ is uniquely expressed as $\Delta = X_n + \Delta_0 = \{X_n + X | X \in \Delta_0\}$ with a Weyl cell Δ_0 contained in $\mathcal{F}$ and an element n of $\mathbf{Z}_+^l$.

(2) If $\tilde{\alpha} = a_1 \alpha_1 + \dots + a_l \alpha_l$ is a dominant root, then the number of the Weyl cells contained in $\mathcal{F}$ is $l! \prod_{i=1}^l a_i$.

(3) Let $\sum_{r=1}^m \theta_r = d_1 \alpha_1 + \dots + d_l \alpha_l$ be the sum of the positive roots θ_r . Then the following formula holds for the integer $\nu(\Delta)$ of Theorem 4.16:

$$\nu(X_n + \Delta) = \nu(\Delta) + \sum_{i=1}^l d_i n_i \quad \text{where } n = (n_1, \dots, n_l).$$

PROOF. (1) Parallel translation by X_n obviously preserves $D(G)$ and so maps a Weyl cell to a Weyl cell. If we put n_i to be the integer part of $\alpha_i(X)$ for $X \in \Delta$, we have $X - X_n \in \Delta_0$. Also it is clear that $X_n + \Delta_0 \subset \mathcal{F}$ for $\Delta_0 \subset \mathcal{F}$ and $n \in \mathbf{Z}_+^l$. (1) is obtained from these facts.

(The inside of the thick lines of Figure 6.1 shows $\mathcal{F}$.)

(2) The fundamental Weyl cell is given by $\Delta_0 = \{X \in \mathcal{F} | \tilde{\alpha}(X) < 1\}$. The length of an edge of Δ_0 in the direction $\overrightarrow{0X_i}$ is $1/a_i \overrightarrow{0X_i}$. Furthermore, the ratio of the volumes of the parallelotope $\mathcal{F}$ and the simplex with vertices $0, X_1, \dots, X_l$ is $l! : 1$, hence the ratio of the volumes of $\mathcal{F}$ and Δ_0 is $l! \prod_{i=1}^l a_i : 1$. Since Weyl cells are congruent each other, (2) is obtained.

(3) For $X \in \Delta$ the number of the points at which the line joining 0 and X intersects the hyperplanes of $L_r = \theta_r^{-1}(\mathbf{Z})$ is the integer part of $\theta_r(X)$. Since $\nu(\Delta)$ is the sum of them, we see that the difference between $\nu(X_n + \Delta)$

and $\nu(\Delta)$ is $\sum_r \theta_r(X_{\mathbf{n}})$. Hence

$$\begin{aligned}\nu(X_{\mathbf{n}} + \Delta) - \nu(\Delta) &= \sum_r \theta_r(X_{\mathbf{n}}) = \sum_{i=1}^l d_i \alpha_i \left(\sum_{j=1}^l n_j X_j \right) \\ &= \sum_{i,j} d_i n_j \alpha_i(X_j) = \sum_{i,j} d_i n_j \delta_{ij} = \sum_{i=1}^l d_i n_i. \quad \text{Q.E.D.}\end{aligned}$$

THEOREM 5.7. *Let G be a compact, connected Lie group of type $(2m_1 + 1, \dots, 2m_l + 1)$, $\tilde{\alpha} = a_1 \alpha_1 + \dots + a_l \alpha_l$ the dominant root, and $\sum_r \theta_r = d_1 \alpha_1 + \dots + d_l \alpha_l$ the sum of the positive roots. Then the following relation holds:*

$$\prod_{i=1}^l d_i = l! \prod_{i=1}^l a_i m_i.$$

PROOF. By Theorems 5.5(4) and 4.16 we have the expressions of $P(\Omega G, t)$ in the two ways that are equal, and hence

$$\prod_{i=1}^l (1 - t^{2m_i})^{-1} = \sum_{\Delta \in \mathcal{F}} t^{2\nu(\Delta)}.$$

By making use of (1) and (3) of Lemma 5.6 we divide the sum of the right-hand side into Δ belonging to each $X_{\mathbf{n}} + \mathcal{F}$ and take the partial sums and so

$$\prod_{i=1}^l (1 - t^{2m_i})^{-1} = \sum_{\Delta \in \mathcal{F}} \left(t^{2\nu(\Delta)} \sum_{\mathbf{n}} \prod_{i=1}^l t^{2d_i n_i} \right) = \sum_{\Delta \in \mathcal{F}} t^{2\nu(\Delta)} \prod_{i=1}^l (1 - t^{2d_i})^{-1},$$

and

$$\prod_{i=1}^l (1 + t^2 + t^4 + \dots + t^{2d_i - 2}) = \sum_{\Delta \in \mathcal{F}} t^{2\nu(\Delta)} \prod_{i=1}^l (1 + t^2 + t^4 + \dots + t^{2m_i - 2}).$$

Here we put $t = 1$. Since $\sum 1^{2\nu(\Delta)}$ is the number of Δ contained in $\mathcal{F}$, we have by using Lemma 5.6(2),

$$\prod_{i=1}^l d_i = l! \prod_{i=1}^l a_i \prod_{i=1}^l m_i = l! \prod_{i=1}^l a_i m_i.$$

Q.E.D.

The types of the exceptional Lie groups. When we consider the isomorphism of Corollary 5.31 of Chapter V,

$$(T \times \tilde{G}_1 \times \dots \times \tilde{G}_k) / K \cong G$$

for a compact, connected Lie group G , we have by Lemma 5.2 and the Künneth formula 5.8.

5.8. The type of G is expressed as the union of the types of $T, \tilde{G}_1, \dots, \tilde{G}_k$.

Thus, the problem is to find the type of a simple Lie group. This has already been examined for the case of the classical groups in Chapter III. So we determine the types of the exceptional Lie groups.

THEOREM 5.9. *The dimension and the sum $\sum_{r=1}^m \theta_r$ of the positive roots of a (compact) exceptional Lie group G are given as follows:*

$$\dim E_6 = 78, \quad \sum \theta_r = 16\alpha_1 + 22\alpha_2 + 30\alpha_3 + 42\alpha_4 + 30\alpha_5 + 16\alpha_6; \quad (E_6)$$

$$\dim E_7 = 133, \quad \sum \theta_r = 34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7; \quad (E_7)$$

$$\dim E_8 = 248, \quad \sum \theta_r = 92\alpha_1 + 136\alpha_2 + 182\alpha_3 + 270\alpha_4 + 220\alpha_5 + 168\alpha_6 + 114\alpha_7 + 58\alpha_8; \quad (E_8)$$

$$\dim F_4 = 52, \quad \sum \theta_r = 16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4; \quad (F_4)$$

$$\dim G_2 = 14, \quad \sum \theta_r = 10\alpha_1 + 6\alpha_2. \quad (G_2)$$

PROOF. The decomposition $L(G) = L(T) \oplus \sum_{r=1}^m V_r$ with $\dim V_r = 2$ of $L(G)$ implies $\dim G = l + 2m$. One can determine $\dim G$ by counting the number of the positive roots in the table of the positive roots, the terms in $\sum (-1)^{\nu(i)} e_i$ for the case E_l ($l = 6, 7, 8$) cancel each other and $\sum_{i < j} (e_i + e_j - e_i + e_j) = 2 \sum_{i < j} e_j$, and so we obtain the following:

$$\begin{aligned} \sum \theta_r &= 8(e_8 - e_7 - e_6) + 2 \sum_{1 \leq i < j \leq 5} e_j \\ &= 8(e_8 - e_7 - e_6) + 2 \sum_{j=1}^5 (j-1)e_j; \end{aligned} \quad (E_6)$$

$$\sum \theta_r = 17(e_8 - e_7) + 2 \sum_{1 \leq i < j \leq 6} e_j = 17(e_8 - e_7) + 2 \sum_{j=1}^6 (j-1)e_j; \quad (E_7)$$

$$\sum \theta_r = 32e_8 + 2 \sum_{1 \leq i < j \leq 8} e_j = 46e_8 + 2 \sum_{j=1}^7 (j-1)e_j; \quad (E_8)$$

$$\begin{aligned} \sum \theta_r &= \sum_{i=1}^4 e_i + 2 \sum_{1 \leq i < j \leq 4} e_i + \frac{1}{2} \sum (e_1 \pm e_2 \pm e_3 \pm e_4) \\ &= 11e_1 + 5e_2 + 3e_3 + e_4; \end{aligned} \quad (F_4)$$

$$\sum \theta_r = -2e_1 + 4e_2 + 6e_3. \quad (G_2)$$

By expressing these in terms of the simple roots $\alpha_1, \dots, \alpha_l$, we obtain the equality for $\sum \theta_r$ in the theorem. Q.E.D.

THEOREM 5.10. *The type $(2m_1 + 1, \dots, 2m_l + 1)$ of the exceptional Lie groups G are those given below and we have:*

$$H^*(G; \mathbf{Q}) = \Lambda(x_{2m_1+1}, \dots, x_{2m_l+1}).$$

The type of $E_6 = (3, 9, 11, 15, 17, 23)$;

The type of $E_7 = (3, 11, 15, 19, 23, 27, 35)$;

The type of $E_8 = (3, 15, 23, 27, 35, 39, 47, 59)$;

The type of $F_4 = (3, 11, 15, 23)$;

The type of $G_2 = (3, 11)$.

PROOF. Since $\dim G = \sum_{i=1}^l (2m_i + 1)$, the following equality holds:

$$\sum_{i=1}^l m_i = \frac{1}{2}(\dim G - l).$$

The following values are obtained from Theorem 5.9 and 6.31 of Chapter V:

$$\begin{aligned} \sum m_i = 36, \quad \prod d_i = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11, \\ \prod a_i = 2^3 \cdot 3, \quad l! = 2^4 \cdot 3^2 \cdot 5; \end{aligned} \quad (E_6)$$

$$\begin{aligned} \sum m_i = 63, \quad \prod d_i = 2^9 \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17, \\ \prod a_i = 2^5 \cdot 3^2, \quad l! = 2^4 \cdot 3^2 \cdot 5 \cdot 7; \end{aligned} \quad (E_7)$$

$$\begin{aligned} \sum m_i = 240, \quad \prod d_i = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29, \\ \prod a_i = 2^7 \cdot 3^3 \cdot 5, \quad l! = 2^7 \cdot 3^2 \cdot 5 \cdot 7; \end{aligned} \quad (E_8)$$

$$\begin{aligned} \sum m_i = 24, \quad \prod d_i = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, \\ \prod a_i = 2^4 \cdot 3, \quad l! = 2^3 \cdot 3; \end{aligned} \quad (F_4)$$

$$\begin{aligned} \sum m_i = 6, \quad \prod d_i = 2^2 \cdot 3 \cdot 5, \\ \prod a_i = 2 \cdot 3, \quad l! = 2. \end{aligned} \quad (G_2)$$

Since $\prod m_i = \prod d_i / l! \prod a_i$ by Theorem 5.7, $\prod m_i$ is as follows:

$$\prod m_i = 2^5 \cdot 5 \cdot 7 \cdot 11; \quad (E_6)$$

$$\prod m_i = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17; \quad (E_7)$$

$$\prod m_i = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29; \quad (E_8)$$

$$\prod m_i = 5 \cdot 7 \cdot 11; \quad (F_4)$$

$$\prod m_i = 5. \quad (G_2)$$

Since $H_3(G) \cong \pi_3(G) \cong \mathbf{Z}$ by Theorem 4.17, we have $m_1 = 1$ and $m_i > 1$ for $i > 1$. Therefore the decomposition of $\prod m_i$ is uniquely determined for

E_8 , F_4 , and G_2 . Also for the cases of E_6 and E_7 we see by combining the condition on $\sum m_i$ that $(m_1, \dots, m_l)$ is

$$(1, 4, 5, 7, 8, 11), \quad (E_6)$$

$$(1, 5, 7, 9, 11, 13, 17). \quad (E_7)$$

Thus the theorem is proved. Q.E.D.

§6. The Bott-Samelson K -cycle

K -cycles and their homology classes. In this section we suppose that a compact, connected Lie group K acts on a Riemannian manifold M by isometric transformations and that the action is variationally complete.

Let N be a K -orbit of maximal dimension and $r \in M$ and consider a (K -transversal) geodesic segment of $M \bmod N$

$$s \in S(M, N, r), \quad s: (I, 0, 1) \rightarrow (M, N, r), \quad \dot{s}(0) \perp N.$$

Let $x_i = s(t_i)$ be an exceptional point on s and denote the identity component of its stabilizer by

$$K_i = (K_{x_i})_0 = (K_{s(t_i)})_0,$$

where t_i is arranged so that $0 < t_1 < t_2 < \dots < t_n < 1$. On the points other than exceptional points, namely, regular points $x = s(t)$, we have

$$K_s = (K_x)_0 = (K_{s(t)})_0 \quad (t \neq t_i, \quad i = 1, 2, \dots, n).$$

We have $K_s \subset K_i$ and the defect $\delta(x_i)$ and index $\lambda(s)$ of s are given by the following:

$$\begin{aligned} \delta(x_i) &= \delta(s(t_i)) = \dim K_i - \dim K_s > 0, \\ \lambda(s) &= \sum_{i=1}^n \delta(x_i) = \sum_{i=1}^n (\dim K_i - \dim K_s). \end{aligned} \quad (6.1)$$

We put

$$W_s = K_1 \times K_2 \times \dots \times K_n, \quad K_s^n = K_s \times K_s \times \dots \times K_s \quad (n \text{ copies})$$

and define the action of K_s^n on W_s from the right by the following:

$$\begin{aligned} \bar{a} \cdot b &= (b_2^{-1} a_1 b_1, \dots, b_n^{-1} a_{n-1} b_{n-1}, a_n b_n) \\ \text{for } \bar{a} &= (a_1, \dots, a_n) \in W_s, \quad b = (b_1, \dots, b_n) \in K_s^n. \end{aligned} \quad (6.2)$$

DEFINITION. We denote by $\Gamma_s = W_s / K_s^n$ the quotient space of W_s by the action of (6.2) and call it the K -cycle of s .

We define a path in $\Omega^* = \Omega^*(M, N, r)$

$$f_s(a): (I, 0, 1) \rightarrow (M, N, r) \quad \text{such that } \dot{f}_s(0) \perp N$$

for every point $a = \bar{a} \cdot K_s^n$ of Γ_s and $\bar{a} \in W_s$ as follows. First, for a representative $\bar{a} = (a_1, \dots, a_n) \in W_s$ of a we define

$$\bar{f}_s(\bar{a}): I \rightarrow M$$

on every interval $[t_{i-1}, t_i]$ ($i = 1, 2, \dots, n+1$) by the equality

$$\bar{f}_s(\bar{a})(t) = (a_n \cdots a_i) \cdot s(t) = \prod_{j=n}^i a_j \cdot s(t), \quad t_{i-1} \leq t \leq t_i, \quad (6.3)$$

where $t_0 = 1$ and $t_{n+1} = 1$ and, in particular, we let $\bar{f}_s(\bar{a})|_{[t_n, 1]} = s|_{[t_n, 1]}$.

LEMMA 6.4. (1) Equality (6.3) defines a path $\bar{f}_s(\bar{a}): I \rightarrow \Omega^*$ in $\Omega^* = \Omega^*(M, N, r)$ and hence a map $\bar{f}_s: W_s \rightarrow \Omega^*$. The length of $\bar{f}_s(\bar{a})$ is equal to that of s .

(2) $\bar{f}_s(\bar{a})$ is determined independently of a representative $\bar{a}$ of $a = \bar{a} \cdot K_s^n$ and the map $f_s: \Gamma_s \rightarrow \Omega^*$ defined by $f_s(a) = \bar{f}_s(\bar{a})$ is a homeomorphism into.

PROOF. (1) In order to show that $\bar{f}_s(\bar{a})$ is a path, it suffices to show that the two definitions $\bar{f}_s(\bar{a})(t_i) = (a_n \cdots a_i) \cdot s(t_i)$ and $\bar{f}_s(\bar{a})(t_i) = (a_n \cdots a_{i-1}) \cdot s(t_i)$ given by (6.3) coincide at $t = t_i$. Since $a_i \in K_i = K_{s(t_i)}$ implies $a_i \cdot s(t) = s(t_i)$, the two definitions are identical. Clearly $\bar{f}_s(\bar{a})(1) = r$. We have $\bar{f}_s(\bar{a})(0) \in N$, and that $\bar{f}_s(\bar{a})$ is orthogonal to N at this point is obvious from the facts that $\bar{f}_s(\bar{a})(t) = (a_n \cdots a_1) \cdot s(t)$ at t near 0 and that the action of $a_n \cdots a_1 \in K$ preserves the Riemannian metric.

Since the action of K preserves the lengths of curves, the length of $\bar{f}_s(a)$ is equal to that of s .

(2) If we put $\bar{a} \cdot b = \bar{c} = (c_1, \dots, c_n)$ for $b = (b_1, \dots, b_n)$, we have $c_n \cdots c_i = a_n \cdots a_i b_i$ by (6.2). Since $b_i \cdot s(t) = s(t)$ for any t , we have $\bar{f}_s(\bar{a}) = \bar{f}_s(\bar{c}) = \bar{f}_s(\bar{a} \cdot b)$ by (6.3). Thus $f_s: \Gamma_s \rightarrow \Omega^*$ is defined by $f_s(a) = \bar{f}_s(\bar{a})$.

Suppose $\bar{f}_s(\bar{a}) = \bar{f}_s(\bar{c})$ for $\bar{a} = (a_1, \dots, a_n)$ and $\bar{c} = (c_1, \dots, c_n) \in W_s$. (6.3) implies $(a_n \cdots a_i) \cdot s(t) = (c_n \cdots c_i) \cdot s(t)$ for $t \in [t_{i-1}, t_i]$, and hence $b_i = (a_n \cdots a_i)^{-1}(c_n \cdots c_i)$ belongs to K_s . Since $a_n \cdots a_i b_i = c_n \cdots c_i$, solving successively we obtain $c_n = a_n b_n$ and $c_i = b_{i+1}^{-1} a_i b_i$ for $i = n-1, \dots, 2, 1$ and hence $\bar{a} \cdot K_s^n = \bar{c} \cdot K_s^n$. This shows that f_s is injective. Since Γ_s is a compact, Hausdorff space, f_s is a homeomorphism into. Q.E.D.

Since $\dim W_s - \dim K_s^n = \sum_{i=1}^n (\dim K_i - \dim K_s) = \lambda(s)$, we have

6.5. Γ_s / K_s^n is a manifold of dimension $\lambda(s)$.

DEFINITION. The image of the fundamental homology class $[\Gamma_s] \in H_{\lambda(s)}(\Gamma_s; \mathbb{Z}_2)$ of Γ_s by $f_{s*}: H_{\lambda(s)}(\Gamma_s; \mathbb{Z}_2) \rightarrow H_{\lambda(s)}(\Omega^*; \mathbb{Z}_2)$ is denoted by $\gamma_s = f_{s*}[\Gamma_s]$ and called the K -homology class of s .

If Γ_s is an orientable manifold, we similarly write $\gamma_s = f_{s*}[\Gamma_s] \in H_{\lambda(s)}(\Omega^*)$ and call it the integral K -homology class of s .

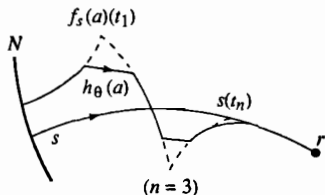


FIGURE 6.2

K-cycles and the Morse theory. Let $L(s) = c$ be the length of $s \in S(M, N, r)$ and consider

$$\Omega_c = \{u \in \Omega^* | L(u) \leq c\} \quad \text{and} \quad \Omega_c^- = \{u \in \Omega^* | L(u) < c\}$$

as in §1. By the second half of Lemma 6.4(1) we have $f_s(\Gamma_s) \subset \Omega_c^-$.

Let $e_* \in \Gamma_s$ be the class of $(e, e, \dots, e)$ so that $f_s(e_*) = s$.

LEMMA 6.6. *There exists a homotopy $h_\theta: (\Gamma_s, e_*) \rightarrow (\Omega_c, s)$, $0 \leq \theta \leq 1$, satisfying $h_0 = f_s$ such that $h_\theta(\Gamma_s) \subset \Omega_c^- \cup \{s\}$ in the range of $0 < \theta \leq 1$ and that the length function $L \circ h_\theta$ is a nondegenerate C^∞ -function at e_* .*

PROOF. Let ε be a sufficiently small positive number and define a homotopy h_θ by putting $h_\theta(a)[t_i - \theta\varepsilon, t_i + \theta\varepsilon]$ to be a geodesic segment from $f_s(a)(t_i - \theta\varepsilon)$ to $f_s(a)(t_i + \theta\varepsilon)$ and to be identical with $f_s(a)$ in the remaining parts. (See Figure 6.2.) Obviously $L(h_\theta(a)) \leq c$. The length of $h_\theta(a)[t_i - \theta\varepsilon, t_i + \theta\varepsilon]$ is shorter than that of $f_s(a)[t_i - \theta\varepsilon, t_i + \theta\varepsilon]$ for i such that $a_i \neq e$. Therefore $a \neq e_*$ implies $L(h_\theta(a)) < c$.

Moreover, if a is sufficiently close to e_* , then $f_s(a)$ and $h_\theta(a)$ on $[t_i - \theta\varepsilon, t_i + \theta\varepsilon]$ are approximated by an isosceles triangle with side $b\theta$; if the angle at a vertex is $\pi - 2\alpha$, the difference of the lengths $2b\theta - 2b\theta \cos \alpha$ is approximated by $b\theta\alpha^2$. It follows from this that $L \circ h_\theta$ is nondegenerate. Q.E.D.

THEOREM 6.7. *Let the action of K on M be variationally complete, $r \in M$ a point on the orbit of maximal dimension and $N = K \cdot p$ the orbit of p . Consider $\Omega^* = \Omega^*(M, N, r) \supset S(M, N, r)$.*

(1) $H^*(\Omega^*; \mathbf{Z}_2)$ has $\{\gamma_s = f_{s*}[\Gamma_s] | s \in S(M, N, r)\}$ as a basis over $\mathbf{Z}_2$.

(2) If Γ_s is an orientable manifold for all $s \in S(M, N, r)$, then $H^*(\Omega^*)$ is a $\mathbf{Z}$ -free module and has $\{\gamma_s \in f_{s*}[\Gamma_s] | s \in S(M, N, r)\}$ as a basis.

PROOF. First for case (2): $\{\gamma_s\}$ is seen by Lemma 6.6 to form a basis over $\mathbf{Z}$ of $\sum_k H_*(\Omega_k, \Omega_{k-1})$ in Theorem 1.17, that is, $E^1 = \sum E_{p,q}^1$ in the proof of Theorem 1.18 has a basis $\{\gamma_s\}$, where γ_s is in the image of $H_*(\Omega_k)$ and so the differential d' in the spectral sequence is trivial: $E^1 = E^\infty$ implies that $\{\gamma_s\}$ is a $\mathbf{Z}$ -basis of $H^*(\Omega^*)$.

For case (1), the argument can be carried out with $\mathbf{Z}_2$ -coefficients. Q.E.D.

The bundle of a K -cycle. A K -cycle Γ was given by

$$\Gamma = W/K_s^n, \quad W = K_1 \times K_2 \times \cdots \times K_n$$

by the action of (6.2) for subgroups $K_1, \dots, K_n$ of K and K_s contained in them. For $n \geq 2$ we put

$$\Gamma' = W'/K_s^{n-1}, \quad W' = K_2 \times \cdots \times K_n$$

by omitting the first factor.

THEOREM 6.8. (1) *The natural map $p: W \rightarrow W'$ induces a projection $p: \Gamma \rightarrow \Gamma'$ of a bundle with fibre K_1/K_s . The map $\sigma: \Gamma' \rightarrow \Gamma$ induced by the natural inclusion $W' \hookrightarrow W$ is a cross section with respect to $p: p \circ \sigma = 1_{\Gamma'}$.*

(2) *The action $(a_1, a_2, \dots, a_n) \mapsto (a_1 h^{-1}, a_2, \dots, a_n)$ of $h \in N_{K_1}(K_s)$ on W induces a map $\varphi: \Gamma \rightarrow \Gamma$ on Γ . Moreover, $p \circ \varphi = \varphi$ and $\varphi|(K_1/K_s)$ is the usual action.*

PROOF. By the same equality as (6.2) the action of K_s^n on W is extendable over that of $K_1 \times K_s^{n-1}$ on W , and a projection

$$p: \Gamma = W/K_s^n \rightarrow W/(K_1 \times K_s^{n-1}) = W'/K_s^{n-1} = \Gamma'$$

of a fibre bundle is obtained, with fibre $(K_1 \times K_s^{n-1})/K_s^n = K_1/K_s$. The rest is easily seen. Q.E.D.

It follows immediately from $p \circ \sigma = 1$ and $p \circ \varphi = \varphi$ that:

6.9. $p^*: H^*(\Gamma) \rightarrow H^*(\Gamma')$ is a monomorphism to a direct summand and the image is invariant under φ^* ($\varphi^* \circ p^* = p^*$). We have $i^* \circ p^* = 0$ for the inclusion $i: K_1/K_s \hookrightarrow \Gamma$.

LEMMA 6.10. Suppose that Γ' is 1-connected and that the fibre is a sphere $S^m = K_1/K_s$ for $m \geq 1$.

(1) For a generator u of $H^m(S^m)$, $z \in H^m(\Gamma)$ is determined such that $i^*(z) = u$ and $\sigma^*(z) = 0$, and the correspondence $(y, y') \mapsto z \cdot p^*(y) + p^*(y')$ gives rise to the isomorphism:

$$H^{*-m}(\Gamma') \oplus H^*(\Gamma') \cong H^*(\Gamma).$$

(2) If the action φ of Theorem 6.8 satisfies $(\varphi|S^m)^* = -1$ and $H^*(\Gamma')$ is 2-torsion free, then

$$\text{Im } p^* = H^*(\Gamma)^\varphi = \{x \in H^*(\Gamma) | \varphi^*(x) = x\}.$$

Moreover, if $\varphi^2 = 1$ and m is even, then $z \cdot \varphi^*(z) = 0$, and $H^*(\Gamma)$ is determined by this equation and $H^*(\Gamma')$.

PROOF. (1) Consider the spectral sequence $E_2 = H^*(\Gamma') \otimes H^*(S^m) \Rightarrow H^*(\Gamma)$.

By the exactness of the Gysin sequence

$$H^m(\Gamma) \xrightarrow{i^*} H^m(S^m) \xrightarrow{\tau} H^{m+1}(\Gamma') \xrightarrow{p^*} H^{m+1}(\Gamma)$$

and 6.9, $i^*: H^*(\Gamma) \rightarrow H^*(S^m)$ is epimorphic. Thus the spectral sequence is trivial and $H^*(\Gamma') \xrightarrow{\otimes u} E_2^{*,m} = E_2^{*,m} = H^{**m}(\Gamma)/p^* H^{**m}(\Gamma')$ is an isomorphism of $H^*(\Gamma')$ -modules. It follows that the second half of (1) holds for any z satisfying $i^*(z) = u$. By putting $z = z_0 - p^* \sigma^*(z_0)$ for such z_0 satisfying $i^*(z_0) = u$, we have $i^*(z) = u$ and $\sigma^*(z) = 0$, since $\sigma^* \circ p^* = 1$ and $i^* \circ p^* = 0$.

(2) By the second half of 6.9 we have $\text{Im } p^* \subset H^*(\Gamma)^\varphi$. If we put $\varphi^*(z) = a \cdot z + p^* y$ with $a \in \mathbb{Z}$, we have $a \cdot u = i^* \varphi^*(z) = (\varphi|S^m)^* u = -u$ by applying i^* , so $a = -1$. Therefore

$$\varphi^* = -z + p^*(y), \quad \text{where } y \in H^*(\Gamma'). \quad (6.10')$$

If we express $x \in H^*(\Gamma)^\varphi$ as $x = z \cdot p^*(y_1) + p^*(y_2)$, we have

$$z \cdot p^*(y_1) + p^*(y_2) = \varphi^*(z) p^*(y_1) + p^*(y_2) = z \cdot p^*(-y_1) + p^*(y y_1 + y_2)$$

by $x = \varphi^*(x)$ and (6.10'). Thus $y_1 = -y_1$ and so $2y_1 = 0$. By the assumption, $y_1 = 0$ so $x \in \text{Im } p^*$. Thus one obtains $H^*(\Gamma)^\varphi = \text{Im } p^*$.

If $\varphi^2 = 1$ and m is even, $\varphi^*(x) = \varphi^*(z) \cdot z = x$ for $x = z \cdot \varphi(z)$ implies that there is a y such that $x = p^*(y)$. Using $\sigma^* \circ p^* = 1$ and $\sigma^*(z) = 0$, one obtains

$$y = \sigma^* p^*(y) = \sigma^*(x) = \sigma^*(z \cdot \varphi^* z) = \sigma^*(z) \cdot \sigma^* \varphi^*(z) = 0.$$

Therefore $z \cdot \varphi^*(z) = x = p^*(y) = 0$ holds and by (6.10') we obtain a relation of the form

$$z^2 = z \cdot p^*(y), \quad y = p^{*-1}(z + \varphi^*(z)). \quad (6.10'')$$

The ring structure of $H^*(\Gamma)$ is determined by this and (1). Q.E.D.

Roots and a three-dimensional subgroup. Let G be a compact, 1-connected Lie group, T its maximal torus, and $\theta_r \in \text{Hom}(L(T), \mathbb{R})$ a root. Then $\Gamma = \Gamma_0$ by Theorem 5.19 of Chapter V, where the integer lattice Γ is the kernel of $\exp: L(T) \rightarrow T$ and Γ_0 is the subgroup generated by the inverse roots θ_r^* ($\theta_r^* \perp L(U_r)$ and $\theta_r(\theta_r^*) = 2$).

LEMMA 6.11. $\theta_r^*/2 \notin \Gamma = \Gamma_0$.

PROOF. Suppose $\theta_r^*/2 \in \Gamma$. By Theorem 6.16 of Chapter V there is $\varphi \in W(G)$ such that $\varphi(\theta_r) = \alpha_i$ is a simple root. Since φ preserves Γ , we have $\alpha_i^*/2 \in \Gamma$, which contradicts Theorem 6.35 of Chapter V. Thus $\theta_r^*/2 \notin \Gamma$. Q.E.D.

$$S^3 = Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\} \supset S^1 = U(1) = \{c \in \mathbb{C} \mid |c| = 1\}$$

is a three-dimensional compact, 1-connected Lie group with its maximal torus, and if we decompose as given in

$$L(S^3) = L(S^1) \oplus V, \quad L(S^1) = (2\pi i) \cdot \mathbb{R}, \quad V = \mathbf{j} \cdot \mathbb{R} + \mathbf{k} \cdot \mathbb{R},$$

we have $e^{it} j e^{-it} = j \cos 2t + k \sin 2t$ and hence its root is given by

$$\theta: L(S^1) \rightarrow \mathbf{R} \quad \text{where } \theta(2\pi it) = 2t. \quad (6.12)$$

Therefore

6.12'. $\Gamma(S^3) = \Gamma_0(S^3)$ is generated by $\theta^* = 2\pi i$. $W(S^3)$ is generated by the class $\varphi = [j]$ of j , and its action on $L(S^1)$ is given by $\varphi(it) = -it$.

By Theorem 4.7, Lemma 4.8, and 4.4.6 one has

6.13. If $X \in L(U_r)$ and $X \notin L(U_s)$ for $s \neq r$, then G_X is a connected, closed subgroup of dimension $l+2$ containing T and it coincides with $Z(U_r)_0$.

THEOREM 6.14. For every root θ_r of (1-connected) G , there exists a monomorphism

$$\lambda_r: S^3 \rightarrow G$$

of Lie groups satisfying $\lambda_r(S^1) = \lambda_r(S^3) \cap T$ such that $G_X = Z(U_r)_0$ in 6.13 is generated by $\lambda_r(S^3)$ and T such that $\theta = \theta_r \circ d\lambda_r: L(S^1) \rightarrow L(T) \rightarrow \mathbf{R}$ holds. Moreover the class $\varphi_r \in W(G)$ of $\lambda_r(j)$ gives rise to a reflection in $\theta_r = 0$.

PROOF. We have $L(G_X) = L(X) \oplus V_r$ in $L(G) = L(T) \oplus \sum_{i=1}^m V_i$. For the straight line L_0 spanned by θ_r^* , $L(T) = L_0 \oplus L(U_r)$. Since $\theta_r = 0$ on $L(U_r)$, $L(G_X) = (L_0 + V_r) \oplus L(U_r)$ yields a direct decomposition of the Lie algebra, where $L_0 + V_r$ is isomorphic to $L(S^3)$, since it is of A_1 -type. Thus there exists $\lambda_r: S^3 \rightarrow G_X$ such that $\lambda_r \times 1: S^3 \times T_r \rightarrow G$ with $T_r = \exp L(U_r) = (U_r)_0$ a torus of dimension $l-1$ is a local isomorphism and that $\theta = \theta_r \circ d\lambda_r$. The center of S^3 consists of 1 and $-1 = \exp(\theta^*/2)$, which is mapped to $\exp(\theta_r^*/2)$ by λ_r . By Lemma 6.11 we have $\lambda_r(-1) \neq e$, which shows that λ_r is a monomorphism. The rest is easy. (The kernel of $\lambda_r \times 1$ is either 0 or $\mathbf{Z}_2$; if it is $\mathbf{Z}_2$, it is not a subgroup of S^3 or of T_r .) Q.E.D.

COROLLARY 6.15. λ_r induces a homeomorphism $\bar{\lambda}_r: S^3/S^1 \cong Z(U_r)_0/T$.

If one identifies the inverse root θ_r^* with the class $\in \pi_1(T) = H_1(T)$ of the image by $\exp$ of a path joining the origin and θ_r^* , one has $\Gamma_0 = H_1(T)$. We also regard a root $\theta_s: L(T) \rightarrow \mathbf{R}$ as an element of $H^1(T) = \text{Hom}(H_1(T), \mathbf{Z})$ through the restriction $\theta_s|_{\Gamma}: \Gamma = \Gamma_0 \rightarrow \mathbf{Z}$.

LEMMA 6.16. (1) $\theta_s \in H^1(T)$ satisfies $\theta_s(\theta_r^*) = 2(\theta_s, \theta_r)/(\theta_r, \theta_r)$.

(2) $\lambda_r^*: H^1(T) \rightarrow H^1(S^1)$ satisfies $\lambda_r^*(\theta_s) = 2(\theta_s, \theta_r)t/(\theta_r, \theta_r)$, where t is a generator of $H^1(S^1)$ represented by $\theta/2$.

PROOF. If we decompose as $\theta_s = a\theta_r + Y$ where $Y \perp \theta_r$, we have $a = (\theta_s, \theta_r)/(\theta_r, \theta_r)$. Then (1) is obtained from $Y(\theta_r^*) = 0$ and $\theta_r(\theta_r^*) = 2$. By

Theorem 6.14 we have $\lambda_r^*(\theta_s) = \theta_s \circ d\lambda_r = a(\theta_r \circ d\lambda_r) + Y \circ d\lambda_r = a\theta$. (6.12) implies that $\theta/2$ is a generator of $H^1(S^1)$. Q.E.D.

Since $S^3/S^1 = S^2$, by Theorem 6.14 we obtain the following commutative diagram concerning the transgressions τ and τ_r :

$$\begin{array}{ccc} H^1(T) & \xrightarrow{\lambda_r^*} & H^1(S^1) \\ \downarrow \tau & \cong \downarrow \tau & (S_r^2 = Z(U_r)_0/T \cong S^2). \\ H^2(S_r^2) & \xrightarrow[\cong]{\bar{\lambda}_r^*} & H^2(S^2) \end{array} \quad (6.17)$$

COROLLARY 6.18. If a generator u_r of $H^2(S_r^2) \cong \mathbb{Z}$ is given by $\bar{\lambda}_r^*(u_r) = \tau(t)$ with $t = [\theta/2]$, then

$$\tau_r(\theta_s) = \frac{2(\theta_s, \theta_r)}{(\theta_r, \theta_r)} u_r.$$

The cohomology ring of K -cycles with respect to the adjoint action. Hereafter let $K = G$ be compact, 1-connected and suppose that the action of K is the adjoint action on G or $\mathfrak{g} = L(G)$, and that a geodesic segment s is a line segment in the Lie algebra $\mathfrak{t} = L(T)$ of a maximal torus such as in Theorems 4.1.6, 4.15, or 4.20. Since the exceptional points of s are the intersection points with $D(G)$ or $D'(G)$ and since the terminal point of s can be suitably rechosen, s can be chosen so that:

6.19. *The exceptional points of s are the intersection points by hyperplanes of $D(G)$ or $D'(G)$ and the hyperplane passing through the exceptional points is unique. The terminal point of s is a regular point.*

Then the K -cycle corresponding to s is

$$\Gamma_s = (K_1 \times \cdots \times K_n)/T^n, \text{ where } K_i = Z(U_i)_0 \text{ if the hyperplane passing through the exceptional point } x_i = s(t_i) \text{ is given by } \theta_i = n_i \in \mathbb{Z}. \quad (6.20)$$

Consider the bundle with a cross section σ in Theorem 6.8:

$$S_1^2 = Z(U_1)_0 \xrightarrow{i} \Gamma_s \xrightarrow{p} \Gamma' = (K_2 \times \cdots \times K_n)/T^{n-1}. \quad (6.20')$$

THEOREM 6.21. (1) *The K -cycle Γ_s corresponding to s , which satisfies the condition of 6.19, is a 1-connected, orientable manifold of dimension $2n$ and*

$$H^*(\Gamma_s) = \Delta(x_1, \dots, x_n), \quad x_i \in H^2(\Gamma_s).$$

(2) *For every Weyl cell Δ contained in the fundamental Weyl chamber $\mathcal{F}$ in Theorem 4.16, a line segment s joining the origin and a point of Δ can be chosen so that it satisfies 6.19. The set $\gamma_s = f_{s*}(\Gamma_s) \in H_{2\nu(\Delta)}(\Omega G)$ for all such s forms a basis of the free abelian group $H_*(\Omega G)$.*

(3) A segment s joining $Q \in W(G) \cdot X$ and R in Theorem 4.20 satisfies 6.19, and the set $\gamma_s = f_{s*}(\Gamma_s) \in H_{2\nu(Q)}(\Omega(\mathfrak{g}, G/Z(T'))) \cong H_{2\nu(Q)}(G/Z(T'))$ forms a basis of the free abelian group $H_*(G/Z(T'))$.

PROOF. (1) is easily obtained by using the induction on the bundle of (6.20') (cf. Lemma 6.10).

(2) and (3) are easily obtained from Theorem 6.7(2). Q.E.D.

THEOREM 6.22. If the roots θ_i are numbered (admitting multiplicity) so that $K_i = Z(U_i)_0$ in $\Gamma_s = (K_1 \times \cdots \times K_n)/T^n$, the cohomology of Γ_s is given by

$$H^*(\Gamma_s) = \mathbb{Z}[x_1, \dots, x_n]/(\rho_1, \dots, \rho_n),$$

where a generator $x_i \in H^2(\Gamma_s)$ is inductively determined as follows: $x_i = p^*(x_i)$ for $x_i \in H^2(\Gamma')$ given in Γ' for $1 < i \leq n$ and x_1 is such that $\sigma^*(x_1) = 0$ and $i^*(x_1) = u_1$ is a generator in Corollary 6.18 by Lemma 6.10(1); moreover, ρ_i is given by the following equality:

$$\rho_i = x_i \left\{ x_i + \sum_{j=i+1}^n \frac{2(\theta_i, \theta_j)}{(\theta_j, \theta_j)} x_j \right\}.$$

REMARK 6.22'. The theorem will not be used later and so the proof is omitted (cf. Lemma 6.16). Using the equation, Bott and Samelson have determined the structure of $H^*(G_2/T)$, the range where $\pi_i(E_l)$ vanishes for $i > 3$ ($l = 6, 7, 8$), $\pi_{10}(G_2) \cong \mathbb{Z}_3$, and so on. The calculation of this ring structure, however, is so complicated that it is not appropriate to consider it in this book.

THEOREM 6.23. Let G be a compact 1-connected, simple Lie group, $\tilde{\alpha}$ the dominant root, and θ_r a root. Denote by $S_r^3 = \lambda_r(S^3)$ the image of the monomorphism $\lambda_r: S^3 \rightarrow G$ in Theorem 6.14. If there is an element φ of the Weyl group $W(G)$ such that $\varphi(\theta_r) = \tilde{\alpha}$, then the inclusion $i_r: S_r^3 \rightarrow G$ induces an isomorphism $i_{r*}: \pi_3(S_r^3) \cong \pi_3(G)$.

PROOF. Since the action $\varphi: G \rightarrow G$ of φ is given by an inner automorphism and since G is connected, we have $\varphi \simeq 1_G$. If we denote by $\tilde{\lambda}: S^3 \rightarrow G$ the map corresponding to $\tilde{\alpha}$ in Theorem 6.14, $\varphi \circ \lambda_r = \tilde{\lambda}$ implies $\lambda_r \simeq \tilde{\lambda}$ and hence it suffices to show the theorem when $\theta_r = \tilde{\alpha}$.

If Δ_0 is the fundamental Weyl cell, the faces of Δ_0 inside $\mathcal{F}$ lie in the hyperplane $\tilde{\alpha} = 1$. If Δ is the reflection of Δ_0 in the hyperplane, then $\gamma_s = f_{s*}[\Gamma_\Delta]$ determined by a line segment s joining the origin and an interior point of Δ is a generator of $H_2(\Omega G) \cong \pi_2(\Omega G) \cong \mathbb{Z}$. Obviously $\Gamma_\Delta = S^2$, and $f_s: S^2 \rightarrow \Omega G$ represents a generator of $\pi_2(\Omega G)$. On the other hand, $f_s(S^2) \subset S_r^3 \cdot T$ since $\tilde{\alpha}$ is a root of S_r^3 ($\theta_r = \tilde{\alpha}$). Thus $\pi_2(\Omega(S_r^3 \cdot T)) \rightarrow \pi_2(\Omega G)$ is epimorphic, and so is $\pi_3(S_r^3 \cdot T) \rightarrow \pi_3(G)$. We

have $\pi_3(S_r^3) \cong \pi_3(S_r^3 \cdot T)$, since $(S_r^3 \cdot T)/S_r^3$ is a torus with homotopy group $\pi_i = 0$ for $i \neq 1$. The composite of these homomorphisms is i_{r*} , which is an epimorphism between the groups isomorphic to $\mathbb{Z}$, and hence it is an isomorphism. Q.E.D.

LEMMA 6.24. *The Weyl group of E_l acts on the set of the roots transitively for $l = 6, 7, 8$.*

PROOF. Looking up with the root system of 6.26 of Chapter V, the reflection φ_i corresponding to $\alpha_i = e_{i-1} - e_{i-2}$ acts as a transposition of e_{i-1} and e_{i-2} ; φ_2 corresponding to $\alpha_2 = e_1 + e_2$ maps $(e_1, e_2) \mapsto (-e_2, -e_1)$ and φ_1 corresponding to α_1 satisfies $\varphi_1(e_2 - e_1) = \frac{1}{2}(-e_1 + e_2 - e_3 - \cdots - e_7 + e_8)$ and $\varphi_1(e_8 - e_7) = \frac{1}{2}(-e_1 + e_2 + \cdots + e_6 - e_7 + e_8)$. It follows from these that $W(E_l)$ acts on the root system transitively. Q.E.D.

COROLLARY 6.25. *The inclusion $i_r: S_r^3 \rightarrow G$ of the subgroup $S_r^3 = \lambda_r(S^3)$ corresponding to an arbitrary root of E_l represents a generator of $\pi_3(G)$ for $l = 6, 7, 8$.*

§7. Loop spaces and homotopy groups of the exceptional groups

The exceptional groups F_4 and G_2

NOTATION. $\mathcal{R}(G) = \{\text{the roots } \pm\theta_1, \dots, \pm\theta_m \text{ of } G\}$,

$\mathcal{R}_+(G) = \{\text{the positive roots } \theta_1, \dots, \theta_m \text{ of } G\}$.

If G' is a closed subgroup of G of maximal rank, one can regard

$$\mathcal{R}_+(G') = \{\text{the roots of } G' \text{ belonging to } \mathcal{R}_+(G)\} \subset \mathcal{R}_+(G) \quad (7.1)$$

as in §7 of Chapter V.

THEOREM 7.2. *Let G be a 1-connected Lie group and G' a 1-connected, closed subgroup of G of maximal rank. If we put*

$$\lambda = \#\{\theta' \in \mathcal{R}(G') \mid \theta'(x) > \theta(x), \forall x \in \mathcal{F}, \forall \theta \in \mathcal{R}_+(G) - \mathcal{R}_+(G')\}$$

with respect to the fundamental Weyl chamber $\mathcal{F}$ of G , the homomorphism

$$\Omega i_*: H_*(\Omega G') \rightarrow H_*(\Omega G)$$

induced by the inclusion $\Omega i: \Omega G' \rightarrow \Omega G$ is an epimorphism for degrees $< 2(\lambda + 1)$.

PROOF. For each cell Δ of $\mathcal{F}$ the K -cycle Γ_Δ of a line segment s joining the origin and an interior point of Δ is an orientable manifold (Theorem 6.21). By Theorem 6.7, $H_*(\Omega G)$ has the K -homology class $f_{s*}[\Gamma_0] \in H_{2\lambda(s)}(\Omega G)$ as a $\mathbb{Z}$ -basis, where $\lambda(s)$ is the number of hyperplanes of $D(G)$ intersecting s . As is easily seen from the definition of λ , a hyperplane of $D(G)$ intersecting s is a hyperplane of $D(G')$ if $\lambda(s) \leq 2\lambda$.

Obviously, the fundamental Weyl chamber $\mathcal{F}'$ of G' contains $\mathcal{F}$. Therefore if $\lambda(s) \leq 2\lambda$, the K -cycle Γ'_Δ for s constructed for G' satisfies $\Gamma'_\Delta = \Gamma_\Delta$, and $f_s(\Gamma'_\Delta) \subset \Omega G'$, which shows that Ωi_* is an epimorphism for degrees $< 2(\lambda + 1)$. Q.E.D.

COROLLARY 7.3. *In Theorem 7.2 $\Omega i^*: H^*(\Omega G) \rightarrow H^*(\Omega G')$ is a monomorphism to a direct summand for degrees $< 2(\lambda + 1)$.*

PROOF. Since $H_*(\Omega G)$ and $H_*(\Omega G')$ are free abelian groups (Theorem 4.16), the kernel of Ωi_* is a direct summand for degrees $< 2(\lambda + 1)$. Regarding as $H^*(\Omega G) = \text{Hom}(H_*(\Omega G), \mathbb{Z})$ and so on, the corollary is obtained as Ωi^* is the dual map of Ωi_* . Q.E.D.

When G is an exceptional group, the following are the two cases for which Theorem 7.2 can be applied (cf. §7 of Chapter V). Here we use the expressions of the roots in 6.26 of Chapter V.

THEOREM 7.4. (1) *There is a subgroup of F_4 isomorphic to $\text{Spin}(9)$ such that*

$$\mathcal{R}_+(\text{Spin}(9)) = \{e_i \ (1 \leq i \leq 4); \ e_i + e_j, e_i - e_j \ (1 \leq i < j \leq 4)\},$$

$$\mathcal{R}_+(F_4) - \mathcal{R}_+(\text{Spin}(9)) = \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

(2) *There is a subgroup of G_2 isomorphic to $SU(3)$ such that*

$$\mathcal{R}_+(SU(3)) = \{-2e_1 + e_2 + e_3, \ e_1 - 2e_2 + e_3, \ -e_1 - e_2 + 2e_3\},$$

$$\mathcal{R}_+(G_2) - \mathcal{R}_+(SU(3)) = \{e_1 - e_2, \ -e_1 + e_3, \ -e_2 + e_3\}.$$

PROOF. Since the root system $\mathcal{R}(\text{Spin}(9))$ given by (1) satisfies Theorem 7.10 of Chapter V, there exists a subgroup G' of B_4 -type of F_4 such that $\mathcal{R}(\text{Spin}(9))$ is its root system. G' is obtained as the normalizer of a certain element (Theorem 7.16 of Chapter V) and is a closed subgroup $\text{Spin}(9)$ or $SO(9) = \text{Spin}(9)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is the center of $\text{Spin}(9)$). On the other hand, since the point $e_1 = (1, 0, 0, 0)$ of $L(T) = \{(x_1, x_2, x_3, x_4) | x_i \in \mathbb{R}\}$ takes an integer value on every root of $\text{Spin}(9)$, $\exp(e_1)$ lies in the center of $\text{Spin}(9)$. Also, the value of e_1 by $\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ is $\frac{1}{2}$ which is not integral and hence $\exp(e_1) \neq 0$, which shows $G' = \text{Spin}(9)$.

(2) is similarly shown; $\exp(\frac{1}{3}, -\frac{1}{3}, 0) \neq 0$ implies $G' = SU(3)$. Q.E.D.

LEMMA 7.5. (1) $\Omega i^*: H^*(\Omega F_4) \rightarrow H^*(\Omega \text{Spin}(9))$ is a monomorphism to a direct summand for degrees < 10 .

(2) $\Omega i^*: H^*(\Omega G_2) \rightarrow H^*(\Omega SU(3))$ is a monomorphism to a direct summand for degrees < 6 .

PROOF. (1) The fundamental Weyl chamber $\mathcal{F}$ of F_4 is given by

$$x_2 > x_3 > x_4 > 0 \quad \text{and} \quad x_1 > x_2 + x_3 + x_4.$$

Since the values of $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathcal{F}$ by the roots of $\mathcal{R}_+(F_4)$ satisfy

$$\begin{aligned} x_1 + x_2 &> x_1 + x_3 > x_1 + x_4 > x_1 > \frac{1}{2}(x_1 + x_2 + x_3 + x_4) \\ &\geq \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4), \end{aligned}$$

we have $\lambda \geq 4$ in Theorem 7.2, and (1) is obtained from Corollary 7.3.

(2) The fundamental Weyl chamber $\mathcal{F}$ of G_2 is given by

$$\frac{1}{2}(x_2 + x_3) > x_1 > x_2 \quad (x_1 + x_2 + x_3 = 0),$$

and furthermore

$-x_1 - x_2 + 2x_3 > x_1 - 2x_2 + x_3 > (\text{the values of the other roots of } \mathcal{R}_+(G_2))$
in $\mathcal{F}$ and so $\lambda \geq 2$ in Theorem 7.2. Q.E.D.

Applications of Hopf algebras. We abstract the situation in Lemma 7.5 and discuss the structure of the Hopf algebras.

THEOREM 7.6. *Let A^* and B^* be connected ($A^0 = B^0 = \mathbf{k}$ and $A^i = B^i = 0$ for $i < 0$) and commutative quasi Hopf algebras of finite type over a field $\mathbf{k}$ and $f: B^* \rightarrow A^*$ a homomorphism of quasi Hopf algebras.*

(1) *If f is a monomorphism for degrees $< N$, then there exists a $\mathbf{k}$ -submodule M^* of A^* such that $\mu \circ (f \otimes 1): B^* \otimes M^* \rightarrow A^*$ is an isomorphism (of B^* -modules) for degrees $< N$ where μ is the product on A^* . Thus we have*

$$A^*/(f(B^+)) \cong M^* \quad (\text{isomorphism of } \mathbf{k}\text{-modules})$$

for degrees $< N$, where $B^+ = \sum_{i>0} B^i$.

(2) *If B is 1-connected in a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ and if the assumption of (1) is satisfied with $B^* = H^*(B; \mathbf{k})$, $A^* = H^*(E; \mathbf{k})$ and $f = p^*$, then i^* induces an isomorphism $H^*(E; \mathbf{k})/(p^*(H^+(B; \mathbf{k}))) \cong H^*(F; \mathbf{k})$ and $H^*(E; \mathbf{k})$ is isomorphic with $H^*(B; \mathbf{k}) \otimes H^*(F; \mathbf{k})$ as $H^*(B; \mathbf{k})$ -modules for degrees $< N$.*

PROOF. (1) Since the projection $\pi: A^* \rightarrow A^*/(f(B^+))$ is an epimorphism of $\mathbf{k}$ -modules, there is a submodule M^* of A^* such that $\pi|_{M^*}$ is an isomorphism. The composite

$$B^* \otimes M^* \xrightarrow{f \otimes 1} A^* \otimes M^* \xrightarrow{\mu} A^* \xrightarrow{\phi} A^* \otimes A^* \xrightarrow{1 \otimes \pi} A^* \otimes A^*/(f(B^+))$$

maps $b \otimes m$ to

$$\begin{aligned} & (1 \otimes \pi)(\phi(f(b))\phi(m)) \\ &= (1 \otimes \pi)(f(b \otimes 1 + \cdots + 1 \otimes b)(m \otimes 1 + \cdots + 1 \otimes m)) \\ &= f(b) \otimes \pi(m) + \sum_{\alpha} x_{\alpha} \otimes y_{\alpha}, \end{aligned}$$

where $\deg y_{\alpha} < \deg m_{\alpha}$. This shows that the composite and hence $\mu \circ (f \otimes 1)$ is a monomorphism for degrees $< N$. On the other hand, since A^* is multiplicatively generated by $f(B^+)$ and M^* , $\mu \circ (f \otimes 1)$ is an epimorphism. Thus (1) is obtained.

(2) It suffices to show that the spectral sequence

$$E_2 = H^*(B; \mathbf{k}) \otimes H^*(F; \mathbf{k}) \Rightarrow H^*(E; \mathbf{k})$$

is trivial for degrees $< N$, which can be shown by induction on the degree of $H^*(F; \mathbf{k})$. The details are left to the reader. Q.E.D.

COROLLARY 7.7. *Theorem 7.6 holds similarly if $\mathbf{k}$ is replaced by $\mathbf{Z}$ and if A^* and B^* are free abelian groups with f replaced by a monomorphism into a direct summand.*

PROOF. If A^* and B^* are replaced by $A^* \otimes \mathbf{k}$ and $B^* \otimes \mathbf{k}$, then $f \otimes 1: B^* \otimes \mathbf{k} \rightarrow A^* \otimes \mathbf{k}$ is a monomorphism. After applying the theorem one can consider A^* and B^* . Q.E.D.

THEOREM 7.8. (1) $H^*(F_4/\text{Spin}(9)) = \mathbf{Z}[x_8]/(x_8^3)$, $x_8 \in H^8$.

(2) $H^*(G_2/SU(3)) = \Lambda(x_6)$, $x_6 \in H^6$.

PROOF. (1) Since F_4 is of type (3, 11, 15, 23) and $\text{Spin}(9)$, that is, $SO(9)$ is of type (3, 7, 11, 15), the Poincaré series of ΩF_4 and $\Omega \text{Spin}(9)$ are $1 + t^2 + t^4 + t^6 + t^8 + \dots$ and $1 + t^2 + t^4 + 2t^6 + 2t^8 + \dots$ respectively by Theorem 5.5(3). Applying Corollary 7.7 and Lemma 7.5(1) to the fibration

$$\Omega^2(F_4/\text{Spin}(9)) \rightarrow \Omega \text{Spin}(9) \rightarrow \Omega F_4,$$

one obtains $H^*(\Omega^2(F_4/\text{Spin}(9))) = \Lambda(x_6)$ for degrees < 10 . Using the suspension isomorphism $H^*(\Omega^2(F_4/\text{Spin}(9))) \cong H^{*+2}(F_4/\text{Spin}(9))$ for degrees < 11 one has $H^*(F_4/\text{Spin}(9)) = \mathbf{Z}[x_8]$ for degrees < 12 . Since $\dim(F_4/\text{Spin}(9)) = \dim F_4 - \dim \text{Spin}(9) = 52 - 36 = 16$, one obtains (1) by using the Poincaré duality theorem (Theorem 3.10 of Chapter VII).

(2) follows similarly from the fact that G_2 is of type (3, 11), $SU(3)$ of type (3, 5), and $\dim(G_2/SU(3)) = 6$. Q.E.D.

REMARK 7.8'. $F_4/\text{Spin}(9) = \varphi = S^8 \cup e^{16}$ (the Cayley plane) and $G_2/SU(3) = S^6$.

LEMMA 7.9. (1) *If y_2 is a generator of $H^2(\Omega SU(n); \mathbf{k})$ for an arbitrary field $\mathbf{k}$, then $y_2^{n-1} \neq 0$.*

(2) *If y_2 is a generator of $H^2(\Omega \text{Spin}(n); \mathbf{k})$ for a field $\mathbf{k}$ of characteristic $\neq 2$, then $y_2^{n-1} \neq 0$ for n odd, and $y_2^{n-2} \neq 0$ for n even.*

(3) *If $n \geq 7$, then $H^*(\Omega \text{Spin}(n)) = \mathbf{Z}[y_2]$ for degrees < 6 .*

PROOF. (1) By Lemma 5.5 of Chapter IV there is a map $f: P_{n-1}(\mathbf{C}) \rightarrow \Omega SU(n)$ such that $f^*: H^*(\Omega SU(n); \mathbf{k}) \rightarrow H^*(P_{n-1}(\mathbf{C}); \mathbf{k}) = \mathbf{k}[t]/(t^n)$ is an isomorphism for degree 2. Since one may suppose $t = f^*(y_2)$, $f^*(y_2^{n-1}) = t^{n-1} \neq 0$, which implies $y_2^{n-1} \neq 0$.

(2) Let n be odd. By Theorem 6.23 of Chapter IV there is a p -equivalence $(SU(n)/SO(n)) \times SO(n) \rightarrow SU(n)$ for $p \neq 2$, and so is $\Omega(SU(n)/SO(n)) \times \Omega \text{Spin}(n) \rightarrow \Omega SU(n)$, which implies

$$H^*(\Omega SU(n); \mathbf{k}) \cong H^*(\Omega(SU(n)/SO(n)); \mathbf{k}) \otimes H^*(\Omega \text{Spin}(n); \mathbf{k}).$$

Hence $1 \otimes y_2$ is a generator of $H^*(\Omega SU(n); \mathbf{k})$ and so $(1 \otimes y_2)^{n-1} \neq 0$ by (1) hence $y_2^{n-1} \neq 0$.

If n is even, one can argue similarly by restricting to $\Omega SU(n-1)$.

(3) By Theorem 6.11 of Chapter III and Theorem 5.10 of Chapter IV we have $H^*(\Omega \text{Spin}) \cong H^*(\Omega_0 \text{SO}) \cong H^*(\text{SO}/U) \cong \mathbb{Z}[y_2]$ for degrees < 6 , from which (3) follows easily. Q.E.D.

LEMMA 7.10. Let A^* be a quasi Hopf algebra over a field of characteristic $p > 0$ such that $A^1 = 0$. If $y_2 \in A^2$ satisfies $y_2^n \neq 0$, then $y_2^{p^f-1} \neq 0$ for a minimal f such that $n < p^f$.

PROOF. y_2 is primitive: $\phi(y_2) = y_2 \otimes 1 + 1 \otimes y_2$. Putting $a = p^{f-1}$ one has $y_2^a \neq 0$ by the assumption. We have $\phi(y_2^a) = (y_2 \otimes 1 + 1 \otimes y_2)^a = y_2^a \otimes 1 + 1 \otimes y_2^a$, since a is a power of p . Furthermore, $\phi(y_2^{ai}) = (y_2^a \otimes 1 + 1 \otimes y_2^a)^i = y_2^{ai} \otimes 1 + i y_2^{ai-a} \otimes y_2^a + \dots$ for $1 < i < p$. The second term is not 0 by the induction on i and so $y_2^{ai} \neq 0$ for $1 \neq i < p$. In particular, $\phi(y_2^{a(p-1)}) = y_2^{a(p-1)} \otimes 1 - y_2^{a(p-2)} \otimes y_2^a - \dots \neq 0$. Also, $\phi(y_2^{ap-1}) = \phi(y_2^{a(p-1)} y_2^{a-1}) = y_2^{ap-1} \otimes 1 + y_2^{a(p-1)} \otimes y_2^{a-1} + \dots \neq 0$, which implies $y_2^{ap-1} \neq 0$. Since $ap-1 = p^f-1$, the lemma is proved. Q.E.D.

REMARK. This lemma and Theorem 7.6 are the models for the general theory of Hopf algebras in §1 of the next chapter.

THEOREM 7.11. (1) $H^*(\Omega G_2) = \mathbb{Z}[y_2]$ for degrees < 6 .

(2) $H^*(\Omega F_4) = \mathbb{Z}[y_2]$ for degrees < 8 .

(3) If $\mathbf{k}$ is a field of characteristic $p \neq 2$, then

$$H^*(\Omega F_4; \mathbf{k}) = \mathbf{k}[y_2, y_{10}, y_{14}] \text{ for degrees } < 18.$$

PROOF. (1) The Poincaré series of ΩG_2 is $1 + t^2 + t^4 + t^6 + \dots$. By Lemmas 7.5(2) and 7.9(1) one sees that y_2^2 is a generator of $H^4(\Omega G_2) \cong \mathbb{Z}$ and so one obtains (1).

(2) The Poincaré series of ΩF_4 is $1 + t^2 + t^4 + t^6 + t^8 + \dots$. Hence it suffices to show $y_2^3 \neq 0$ ($\in H^6(\Omega F_4; \mathbf{k})$) for a field $\mathbf{k}$ of arbitrary characteristic. When $\mathbf{k}$ is of characteristic 2, it suffices by Lemma 7.10 to show $y_2^2 \neq 0$, which follows from Lemmas 7.5(1) and 7.9(3). The case where the characteristic of $\mathbf{k}$ is not 2 is contained in the following.

(3) A generator y_2 of $H^2(\Omega F_4; \mathbf{k})$ is mapped to a generator y'_2 of $H^2(\Omega \text{Spin}(9); \mathbf{k})$ by Lemma 7.5(1). By Lemma 7.9(2), $(y'_2)^8 \neq 0$ so $y_2^8 \neq 0$. Since y_2 is primitive, the subring B^* generated by y_2 is a Hopf subalgebra of $A^* = H^*(\Omega F_4; \mathbf{k})$, and $B^* = \mathbf{k}[y_2]$ holds for degrees < 18 by Lemma 7.10. Since the Poincaré series of ΩF_4 is

$$(1 + t^2 + t^4 + \dots + t^{16} + \dots)(1 + t^{10} + t^{14} + t^{20} + \dots),$$

(3) is obtained by using Theorem 7.6(1). Q.E.D.

THEOREM 7.12. (1) $\pi_i(G_2) = 0$ if $i \neq 3$ and $i < 6$; ($\pi_3(G_2) \cong \mathbb{Z}$).

(2) $\pi_i(F_4) = 0$ if $i \neq 3$ and $i < 8$; ($\pi_3(F_4) \cong \mathbb{Z}$).

(3) *There is a CW-complex $S^{11} \cup e^{15}$ and a map $f: S^{11} \cup e^{15} \rightarrow F_4$ such that $f_*: \pi_i(S^{11} \cup e^{15}) \rightarrow \pi_i(F_4)$ is an isomorphism modulo 2-primary components for $3 < i < 18$.*

PROOF. (1) Since $\pi_2(\Omega G_2) \cong H_2(\Omega G_2) \cong \mathbb{Z}$, there is a map $f: \Omega G_2 \rightarrow K(\mathbb{Z}, 2)$, which induces an isomorphism of π_2 . If we replace f by the projection of a fibration and denote the fibre by $\Omega \tilde{G}_2$, we obtain a fibration

$$\Omega(K(\mathbb{Z}, 2)) \rightarrow \Omega \tilde{G}_2 \rightarrow \Omega G_2,$$

where $\Omega(K(\mathbb{Z}, 2)) = K(\mathbb{Z}, 1)$ is homotopy equivalent to S^1 and $\Omega \tilde{G}_2$ is 2-connected and hence by the Gysin exact sequence one obtains $H^i(\Omega \tilde{G}_2) = 0$ for $0 < i < 5$ and hence $H_i(\Omega \tilde{G}_2) = 0$ for $0 < i < 5$, so $\pi_i(\Omega \tilde{G}_2) = 0$ for $0 < i < 5$ by the Hurewicz theorem. By the construction of $\Omega \tilde{G}_2$ we have $\pi_i(\Omega \tilde{G}_2) \cong \pi_i(\Omega G_2)$ for $i \neq 2$, where $\pi_i(G_2) \cong \pi_{i-1}(\Omega G_2)$, and hence (1) is obtained.

(2) is shown similarly to (1) by using a fibration $K(\mathbb{Z}, 1) \rightarrow \Omega \tilde{F}_4 \rightarrow \Omega F_4$.

(3) By making use of the Gysin exact sequence with coefficients in a field $\mathbf{k}$ of characteristic $\neq 2$, one has $H^*(\Omega \tilde{F}_4; \mathbf{k}) = \mathbf{k}[x_{10}, x_{14}]$ for degrees < 17 from Theorem 7.11(3). The fibration in (2) can be constructed as the one obtained by applying the loop functor to a fibration

$$K(\mathbb{Z}, 2) \rightarrow \tilde{F}_4 \xrightarrow{p} F_4 (\rightarrow K(\mathbb{Z}, 3)).$$

By the suspension isomorphism $H_*(\tilde{F}_4; \mathbf{k}) \cong H^*(\tilde{F}_4; \mathbf{k}) = \mathbf{k}[x_{11}, x_{15}]$ for degrees < 18 . Using $\mathcal{G}$ -theory one sees the existence of $S^{11} \cup e^{15}$ and a map $\tilde{f}: S^{11} \cup e^{15} \rightarrow \tilde{F}_4$ which induces an isomorphism modulo the 2-component of π_i for $i < 17$. One may put $f = p \circ \tilde{f}$. Q.E.D.

REMARK 7.12'. $\pi_6(G_2) \cong \mathbb{Z}_3$, $\pi_8(F_4) \cong \mathbb{Z}_2$.

The exceptional group E_8

LEMMA 7.13. *There exist homomorphisms*

$$f: SU(9) \rightarrow E_8 \quad \text{and} \quad g: \text{Spin}(16) \rightarrow E_8$$

of Lie groups which induce an isomorphism of π_3 . Hence

$$\Omega f^*: H^2(\Omega E_8) \cong H^2(\Omega SU(9)), \quad \Omega g^*: H^2(\Omega E_8) \cong H^2(\Omega \text{Spin}(16)).$$

PROOF. By Theorem 7.16 and 7.17 of Chapter V, $A_8 = N(\exp(X_3/2)) \subset E_8$. If we consider $S_r^3 \subset A_8 \subset E_8$ corresponding to a root θ_r of A_8 , we have $\pi_3(S_r^3) \cong \pi_3(E_8)$ by Corollary 6.25, hence $i_*: \pi_3(A_8) \rightarrow \pi_3(E_8)$ is an epimorphism. Since the universal covering map $p: SU(9) \rightarrow A_8$ induces an isomorphism of π_3 , $f = i \circ p$ induces an epimorphism of π_3 and hence an isomorphism of π_3 .

For $D_8 = N(\exp(X_1/2))$ the result for g is obtained similarly. Q.E.D.

THEOREM 7.14. *In degrees < 32*

$$H^*(\Omega E_8) = \mathbb{Z}[y_2, y_{14}, y_{22}, y_{26}]$$

or

$$H^*(\Omega E_8) = \mathbb{Z}[y_2, y_{14}, y_{22}, y_{26}, y_{28}]/(r),$$

where $r \equiv 2y_{28} - y_{14}^2$ and $\bar{\phi}(y_{28}) \equiv y_{14} \otimes y_{14} \pmod{(y_2)}$.

PROOF. A generator $y_2 \in H^2(\Omega \text{Spin}(16); \mathbf{k})$ satisfies $y_2^{14} \neq 0$ for a field $\mathbf{k}$ of characteristic $\neq 2$ (Lemma 7.9(2)). By Ωg^* in Lemma 7.13 we see that the generator y_2 of $H^2(\Omega E_8; \mathbf{k})$ satisfies $y_2^{14} \neq 0$. Furthermore using Lemma 7.10,

$$y_2^{16} \neq 0 \in H^{32}(\Omega E_8; \mathbf{k}) \text{ with } \mathbf{k} \text{ of characteristic } \neq 2. \quad (7.14')$$

Also, for a field of characteristic 2, by Ωf^* in Lemmas 7.9(1) and 7.13 we have $y_2^8 \neq 0$, whence $y_2^{15} \neq 0$ by Lemma 7.10.

It follows from the above arguments that the natural map $\mathbb{Z}[y_2] \rightarrow H^*(\Omega E_8)$ is an isomorphism into a direct summand for degrees < 32 . Since E_8 is of type (3, 15, 23, 27, 35, 39, 47, 59), there is

$$M^* = \{1, y_{14}, y_{22}, y_{26}, y_{28}\}$$

by Theorem 7.6 such that $H^*(\Omega E_8) = \mathbb{Z}[y_2] \otimes M^*$ holds for degrees < 32 . We have $\phi(y_{14}) \equiv y_{14} \otimes 1 + 1 \otimes y_{14}$ so $\phi(y_{14}^2) \equiv y_{14}^2 \otimes 1 + 2y_{14} \otimes y_{14} + 1 \otimes y_{14}^2 \pmod{(y_2)}$, which shows that $y_{14}^2 \not\equiv 0 \pmod{(y_2)}$ and that $a = \pm 1$ or ± 2 in $y_{14}^2 \equiv ay_{28} \pmod{(y_2)}$. Thus the theorem is shown. Q.E.D.

THEOREM 7.15. *There is a CW-complex $K = S^{15} \cup e^{23} \cup e^{27}$ and a map $f: K \rightarrow E_8$ such that $f_*: \pi_i(K) \cong \pi_i(E_8)$ holds for $i \neq 3$ and $i < 28$. In particular,*

$$\pi_i(E_8) = 0 \text{ for } 3 < i < 15 \text{ and } \pi_{15}(E_8) \cong \mathbb{Z}.$$

PROOF. The theorem is obtained from Theorem 7.14 in a similar fashion to the proof of Theorem 7.12(3). Q.E.D.

COROLLARY 7.16.

$$\begin{aligned} \pi_{16}(E_8) &\cong \pi_{17}(E_8) \cong \mathbb{Z}_2, \pi_{18}(E_8) \cong \mathbb{Z}_{24}, \\ \pi_{19}(E_8) &= \pi_{20}(E_8) = 0, \pi_{21}(E_8) \cong \mathbb{Z}_2. \end{aligned}$$

REMARK 7.16'. $\pi_{22}(E_8) = 0, \pi_{23}(E_8) \cong \mathbb{Z}$.

The exceptional groups E_6 and E_7 . Denote by E_6 the compact, 1-connected Lie group of type E_6 . In addition, $\text{Ad}(E_6) \cong E_6/\mathbb{Z}_3$ is also of type E_6 . Similarly E_7 denotes the compact, 1-connected Lie group of type E_7 . $\text{Ad}(E_7) \cong E_7/\mathbb{Z}_2$ is also of type E_7 . (When $G = E_8, F_4$, and G_2 , there is no need for such a remark, since $\text{Ad}(G) \cong G$.)

LEMMA 7.17. (1) *There exist homomorphisms $f: SU(8) \rightarrow E_7$ and $g: \text{Spin}(12) \rightarrow E_7$ such that*

$$\Omega f^*: H^2(\Omega E_7) \cong H^2(\Omega SU(8)), \quad \Omega g^*: H^2(\Omega E_7) \cong H^2(\Omega \text{Spin}(12)).$$

(2) *There exist homomorphisms $f: SU(6) \rightarrow E_6$ and $g: \text{Spin}(10) \rightarrow E_6$ such that*

$$\Omega f^*: H^2(\Omega E_6) \cong H^2(\Omega SU(6)), \quad \Omega g^*: H^2(\Omega E_6) \cong H^2(\Omega \text{Spin}(10)).$$

PROOF. (1) f is obtained similarly to Lemma 7.13 using $A_7 \subset E_7$.

Since E_7 contains $N(\exp(\overrightarrow{0X_1})) = D_6 \cdot T^1$, one sees using a root belonging to $D_6 \cdot T^1$ that there is S_r^3 contained in $D_6 \cdot T^1$. Since $\pi_i(T^1) = 0$ for $i \neq 1$, the inclusion $D_6 \subset E_7$ induces an isomorphism of π_3 . Similarly for $g: \text{Spin}(12) \rightarrow E_7$.

(2) E_6 contains $N(\exp(\overrightarrow{0X_2})) = A_5 \cdot T^1$. Using a root belonging to A_5 one obtains $S_r^3 \subset A_5 \cdot T^1$ and sees the existence of f . It is shown similarly for g by using $N(\overrightarrow{0X_1}) = D_5 \cdot T^1$. Q.E.D.

THEOREM 7.18. (1) $H^*(\Omega E_7) = \mathbb{Z}[y_2, y_{10}, y_{14}]$ for degrees < 16 . The generator y_2 of $H^2(\Omega E_7; \mathbb{Z}_p)$ satisfies $y_2^{10} \neq 0$ for odd prime p .

(2) $H^*(\Omega E_6) = \mathbb{Z}[y_2, y_8, y_{10}, y_{14}]$ for degrees < 16 .

The generator y_2 of $H^2(E_6; \mathbb{Z}_p)$ satisfies $y_2^8 \neq 0$ for odd prime p .

PROOF. (1) Through Ωf^* and Ωg^* in Lemmas 7.9 and 7.17(1) we see $y_2^7 \neq 0 \pmod{2}$ and $y_2^{10} \neq 0 \pmod{p}$ respectively. Thus $\mathbb{Z}[y_2]$ is a direct summand of $H^*(\Omega E_7)$ for degrees < 16 , and the first half is also obtained by Theorem 7.6.

(2) Similarly one has $y_2^5 \neq 0 \pmod{2}$ using Ωf^* in Lemma 7.17(2). By Lemma 7.10 one has $y_2^7 \neq 0 \pmod{2}$. Also, using Ωg^* one has $y_2^8 \neq 0 \pmod{p}$. The rest is similar. Q.E.D.

THEOREM 7.19. (1) *There is a CW-complex $K = S^{11} \cup e^{15}$ and a map $f: K \rightarrow E_7$ such that $f_*: \pi_i(K) \cong \pi_i(E_7)$ holds for $i \neq 3$ and $i < 16$. In particular,*

$$\pi_i(E_7) = 0 \text{ for } 3 < i < 11 \text{ and } \pi_{11}(E_7) \cong \mathbb{Z}.$$

(2) *There is a CW-complex $K = S^9 \cup e^{11} \cup e^{15}$ and a map $f: K \rightarrow E_6$ such that $f_*: \pi_i(K) \cong \pi_i(E_6)$ holds for $i \neq 3$ and $3 < i < 16$. In particular,*

$$\pi_i(E_6) = 0 \text{ for } 3 < i < 9 \text{ and } \pi_9(E_6) \cong \mathbb{Z}.$$

PROOF. The proof is similar to Theorem 7.15. Q.E.D.

COROLLARY 7.20. $\pi_{12}(E_7) \cong \pi_{13}(E_7) \cong \mathbb{Z}_2$.

REMARK 7.20'. (1) $\pi_{14}(E_7) = 0$, $\pi_{15}(E_7) \cong \mathbb{Z}$.

$$(2) \pi_{10}(E_6) = \pi_{13}(E_6) = \pi_{14}(E_6) = 0, \pi_{11}(E_6) \cong \pi_{15}(E_6) \cong \mathbb{Z},$$

$$\pi_{12}(E_6) \cong \mathbb{Z}_{12}.$$

CHAPTER VII

Cohomology of Exceptional Groups

The first half of this chapter is devoted to the general theory of compact Lie groups and the second half focuses on the determination of the mod p cohomology rings of exceptional groups. In §1 we describe some of the general properties of Hopf algebras with applications to the cohomology rings of Lie groups, and in §2 we prove various transgression theorems, using the comparison theorem as the main tool, and prepare for applications to classifying spaces and for later use. §3 contains a general theory of classifying spaces, characteristic classes, and actions of Weyl groups. In §4 we describe the theory of cohomology operations, in particular the properties of the Steenrod operations, as preparation for later use and also introduce the cohomology rings of Eilenberg–Mac Lane spaces. In §5 we first describe a new method for calculating the cohomology ring of a 1-connected simple Lie group. The method is useful particularly for an exceptional group, for example, to decide whether or not it is p torsion free. Next we determine the mod p cohomology rings of exceptional groups for p odd. In §6 we determine the mod 2 cohomology rings of exceptional groups by a completely new method.

§1. Hopf algebras and cohomology of H -spaces

The structure of a connected quasi Hopf algebra. In this section we consider the structure of a quasi Hopf algebra and its application to H -spaces (in particular, to Lie groups). To begin with we review Hopf algebras from Chapter IV, §2.

Unless otherwise stated, the ground ring R is either $\mathbb{Z}$ or a field K_p (may be regarded as $\mathbb{Z}_p$) of characteristic p .

Recall that $A = (A, \mu, \phi)$ is called a *quasi Hopf algebra* if

$A = \sum_{i \geq 0} A^i$ has an associative product μ and a diagonal map (a coproduct) ϕ ,

A is *connected* (i.e., $A^0 = R$, the unit of which is 1 of A) and of *finite type* (i.e., each A^i is finitely generated),

A is *commutative* ($xy = (-1)^{|x||y|}yx$, where $xy = \mu(x \otimes y)$ and $|x|, |y|$ denote the degrees).

In particular, the diagonal map $\phi: A \rightarrow A \otimes A$ is a ring homomorphism

satisfying $\phi(1) = 1 \otimes 1$ and if $|x| > 0$ then

$$\phi(x) = x \otimes 1 + 1 \otimes x + \sum_{\alpha} x'_{\alpha} \otimes x''_{\alpha}, \quad \text{where } |x'_{\alpha}|, |x''_{\alpha}| > 0.$$

The following definition is also given in §2 of Chapter IV:

A is called a *Hopf algebra* if ϕ in the quasi Hopf algebra A is associative:

$$(\phi \otimes 1)\phi = (1 \otimes \phi)\phi.$$

ϕ is called *commutative* if $T \circ \phi = \phi$ ($T(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$).

NOTATION. $\bar{A} = \sum_{i \geq 0} A^i$,

$\bar{\mu}: \bar{A} \otimes \bar{A} \rightarrow \bar{A}$, $\bar{\phi}: \bar{A} \rightarrow \bar{A} \otimes \bar{A}$ (induced by μ, ϕ).

$P(A) = \text{Ker } \bar{\phi} = \{x \in \bar{A} \mid \phi(x) = x \otimes 1 + 1 \otimes x\}$ is the module of *primitive elements*.

$Q(A) = \text{Coker } \bar{\mu} = \bar{A}/\bar{A} \cdot \bar{A}$. An element (or its representative) of $Q(A)$ is called an *indecomposable element* or a *generator*.

An element of $\bar{A}^2 = \bar{A} \cdot \bar{A}$ is said to be *decomposable*.

THEOREM 1.1. *If A is a quasi Hopf algebra generated by $P(A)$, then the diagonal map ϕ of A is associative and commutative, and A is a Hopf algebra.*

PROOF. Since ϕ is a ring homomorphism, it suffices to show $(\phi \otimes 1) \circ \phi = (1 \otimes \phi) \circ \phi$ and $T \circ \phi = \phi$ for primitive elements, which is obvious. Q.E.D.

DEFINITION. A is called *monogenic* if the R -algebra A is generated by one generator x with $|x| > 0$.

The *height* $h(x)$ of $x \in \bar{A}$ is the maximal positive integer such that $x^{h(x)-1} \neq 0$ ($h(x) = \infty$ if $x^k \neq 0$ for all positive integers k).

THEOREM 1.2. *The height $h(x)$ of an element x with positive degree in a quasi Hopf algebra A over a field K_p of characteristic p satisfies the following:*

(1) $h(x) = 2$, if $|x|$ is odd and $p \neq 2$,

(2) $h(x) = p^f$ ($p \neq 0$) or $h(x) = \infty$, if $|x|$ is even or $p = 2$.

In particular, if A is monogenic, the generator x is primitive and

$$A = K_p[x]/(x^{h(x)}).$$

PROOF. If $|x|$ is odd then $x^2 = -x^2$ and we have (1).

(2) is similar to the proof of Lemma 7.10 of Chapter VI. Q.E.D.

THEOREM 1.3. (The Hopf-Borel Theorem). *Suppose that the ground ring of a quasi Hopf algebra A is a perfect field. Then there are monogenic Hopf algebras B_n and a ring isomorphism $\bigotimes_n B_n \cong A$.*

PROOF. Take a basis $\{\bar{x}_i\}$ of $Q(A)$ and a representative $x_i \in A$ of $\bar{x}_i$ so that $|x_1| \leq |x_2| \leq \dots$. Denote by A_n the subring of A generated by $\{x_1, \dots, x_n\}$. Obviously ϕ is closed in A_n , and so A_n is a quasi Hopf

algebra, which is a direct summand of A and that $B_n = A_n/(A_{n-1}) = A_n/A_{n-1} \cdot \bar{A}_n$ is a monogenic Hopf algebra generated by $\bar{x}_n$. Making use of the assumption that the ground ring is a perfect field (i.e., it has all p th power roots), one sees the existence of a representative x_n of $\bar{x}_n$ such that $h(\bar{x}_n) = h(x_n)$. The assignment $\bar{x}_n \mapsto x_n$ induces a monomorphism $B_n \rightarrow A_n$. Consider the composite

$$A_{n-1} \otimes B_n \xrightarrow{\text{inj}} A_n \otimes A_n \xrightarrow{\mu} A_n \xrightarrow{\phi} A_n \otimes A_n \xrightarrow{\text{proj}} A_n \otimes B_n;$$

then it is a monomorphism as in the proof of Theorem 7.6 of Chapter VI. On the other hand, as $A_{n-1} \otimes B_n \rightarrow A_n$ is an epimorphism, this is an isomorphism. By induction on n , we obtain the theorem (cf. Theorem 2.26 of Chapter IV). Q.E.D.

COROLLARY 1.4. *If in Theorem 1.3 the ground field is a field of characteristic p , we have the following ring isomorphisms:*

(1) For $p = 0$; $A \cong (\bigotimes_{\alpha} \Lambda(x_{\alpha})) \otimes (\bigotimes_{\beta} K_0[x_{\beta}])$, where $|x_{\alpha}|$ is odd, $|x_{\beta}|$ is even.

(2) For $p = 2$; $A \cong (\bigotimes_{\alpha} K_2[x_{\alpha}]/(x_{\alpha}^{h(x_{\alpha})})) \otimes (\bigotimes_{\beta} K_2[x_{\beta}])$, where $h(x_{\alpha})$ is a power of 2.

(3) For $p \neq 0, 2$; $A \cong (\bigotimes_{\alpha} \Lambda(x_{\alpha})) \otimes (\bigotimes_{\beta} K_p[x_{\beta}]) \otimes (\bigotimes_{\gamma} K_p[x_{\gamma}]/(x_{\gamma}^{h(x_{\gamma})}))$, where $|x_{\alpha}|$ is odd, $|x_{\beta}|$ and $|x_{\gamma}|$ are even, and $h(x_{\gamma})$ is a power of p . Here if $\dim A < \infty$, then there is no term of $K_p[x_{\beta}]$.

REMARK. The case $p = 0$ and $\dim A < \infty$ is the classical Hopf theorem.

Notice that the isomorphism in Theorem 1.3 is not necessarily one of Hopf algebras. This is seen from the fact that in general A need not be primitively generated, although $\bigotimes_n B_n$ is generated by primitive elements.

Hereafter let K_p be a perfect field.

DEFINITION. A is said to have a *simple system of generators* $\{x_1, x_2, \dots\}$. (Notation. $A = \Delta(x_1, x_2, \dots)$) if A as a K_p -module has a basis $\{x_{i_1} x_{i_2} \cdots x_{i_r} \text{ for } 1 \leq i_1 < i_2 < \cdots < i_r; r \geq 0\}$ of elementary monomials.

THEOREM 1.5. *A quasi Hopf algebra over K_2 has a simple system of generators.*

PROOF. Consider $\{x_{\alpha}^{2^i} \text{ for } 2^i < h(x_{\alpha}); x_{\beta}^{2^j}\}$ in Corollary 1.4 (2). Q.E.D.

THEOREM 1.6. *Let $p \neq 2$. The following conditions are equivalent for a quasi Hopf algebra A of finite dimension over K_p :*

(1) A is isomorphic (as a ring) to an exterior algebra $\Lambda(x_1, \dots, x_m)$, where $|x_i|$ is odd.

(2) A has a simple system of generators: $A = \Delta(x_1, \dots, x_m)$.

(3) $\dim A = 2^m$.

PROOF. Clearly (1) implies (2) and (2) implies (3).

(3) implies (1). In fact, the case $p = 0$ is clear from Corollary 1.4. The case $p > 2$ is also clear since $\dim K[x_{\gamma}]/(x_{\gamma}^{h(x_{\gamma})}) = h(x_{\gamma}) = p^f$. Q.E.D.

THEOREM 1.7. *If the Poincaré polynomial of a quasi Hopf algebra A of finite dimension over K_p is $P(A, t) = \prod_{i=1}^m (1 + t^{k_i})$ with k_i odd, then $A = \Lambda(x_1, \dots, x_m)$, where $|x_i| = k_i$.*

PROOF. The case $p \neq 2$ is clear from Theorem 1.6. Let $p = 2$. Writing $P(A, t) = \prod (1 + t^{l_j})$ by Theorem 1.5, an even l_j appears only when there is an α such that $h(x_\alpha) > 2$. By the assumption there is no such α , and so $h(x_\alpha) = 2$ implies $x_\alpha^2 = 0$. Q.E.D.

THEOREM 1.8. *If a quasi Hopf algebra over $\mathbf{Z}$ is a finitely generated free abelian group, then*

$$A = \Lambda(x_1, \dots, x_m), \quad \text{where } |x_i| \text{ is odd.}$$

PROOF. Set (we regard $\mathbf{Z}_0 = \mathbf{Q}$)

$D^i =$ (the group of decomposable elements in A^i),

$D_p^i =$ (the group of decomposable elements in $A^i \otimes \mathbf{Z}_p$).

We choose a basis $\{y_{i,1}, \dots, y_{i,s_i}, x_{i,1}, \dots, x_{i,t_i}\}$ of A^i so that $\{m_{ij}y_{i,j}\}$ forms a basis of D^i with suitable integers $m_{ij} \neq 0$.

Since the rank of A is finite, so is that of $A \otimes \mathbf{Q}$, and hence by Corollary 1.4(1), we have

$$A \otimes \mathbf{Q} = \Lambda(x'_1, \dots, x'_m), \quad \text{where } |x'_i| \text{ is odd and } x'_i = x_i \otimes 1.$$

Thus the Poincaré polynomial is $P(A \otimes \mathbf{Q}, t) = \prod (1 + t^{|x'_i|})$.

For $p \neq 0$, $A \otimes \mathbf{Z}_p = A/pA$ and D^i is mapped onto D_p^i . $A \otimes \mathbf{Z}_p$ is a quasi Hopf algebra with the diagonal map induced from the diagonal map ϕ of A . Since A is torsion free, we have $\dim(A^i \otimes \mathbf{Z}_p) = \dim(A^i \otimes \mathbf{Q}) = \text{rank } A^i$ and hence

$$P(A \otimes \mathbf{Z}_p, t) = P(A \otimes \mathbf{Q}, t) = \prod (1 + t^{|x'_i|}), \quad \text{where } |x'_i| \text{ is odd.}$$

By Theorem 1.7, $A \otimes \mathbf{Z}_p = \Lambda(x_{p,1}, \dots, x_{p,m})$ with $|x_{p,i}| = |x'_i|$.

It follows that $\dim D_p^i = \dim D_0^i$. By the way of choosing a basis of A^i , we have $s_i = \dim D_p^i$ and $t_i = \dim(A^i \otimes \mathbf{Z}_p) - \dim D_p^i$ for all p . Therefore $m_{ij} = \pm 1$, and one obtains $t_i = 0$ for even i and $\sum_{i>0} t_i = m$. Thus A is generated by m pieces of generators $x_1, \dots, x_m$ of odd degrees. If $|x_i|$ is odd then $x_i^2 = -x_i^2$ and so $2x_i^2 = 0$. Since A is torsion free, we have $x_i^2 = 0$ and obtain an epimorphism $\Lambda(x_1, \dots, x_m) \rightarrow A$. The epimorphism is a monomorphism since $A \otimes \mathbf{Q} = \Lambda(x'_1, \dots, x'_m)$ and hence it is an isomorphism. Q.E.D.

Primitive elements and generators. As was seen in Theorem 1.1, a quasi Hopf algebra A , which is generated by primitive elements, is a Hopf algebra with ϕ commutative. We call A a *primitively generated Hopf algebra*.

Consider the natural morphism

$$\iota: P(A) \subset \bar{A} \rightarrow \bar{A}/\bar{A}^2 = Q(A).$$

Since representatives of elements in $Q(A)$ generate A , we have

1.9. A is primitively generated iff $\iota: P(A) \rightarrow Q(A)$ is surjective.

LEMMA 1.10. (1) $(\phi \times 1)\phi = (1 \times \phi)\phi$ iff $(\bar{\phi} \times 1)\bar{\phi} = (1 \times \bar{\phi})\bar{\phi}$.

(2) If $A = \Delta(x_1, x_2, \dots, x_m)$ and each x_i is primitive then $P(A)$ has a basis $\{x_1, x_2, \dots, x_m\}$.

(3) If $A = \Lambda(x_1, x_2, \dots, x_m)$ then $\iota: P(A) \rightarrow Q(A)$ is a monomorphism.

PROOF. (1) Write $\bar{\phi}(x) = \sum x'_\alpha \otimes x''_\alpha$ and calculate $(\phi \otimes 1)\phi - (\bar{\phi} \otimes 1)\bar{\phi}$ and $(1 \otimes \phi)\phi - (1 \otimes \bar{\phi})\bar{\phi}$; their values coincide with the expression below, from which (1) is obtained:

$$\sum (x'_\alpha \otimes x''_\alpha \otimes 1 + x'_\alpha \otimes 1 \otimes x''_\alpha + 1 \otimes x'_\alpha \otimes x''_\alpha) - x \otimes 1 \otimes 1 - 1 \otimes x \otimes 1 - 1 \otimes 1 \otimes x.$$

(2) When $1 \leq i_1 < i_2 < \dots < i_r \leq m$ and $r \geq 2$, we have

$$\bar{\phi}(x_{i_1} x_{i_2} \dots x_{i_r}) = x_{i_1} \otimes x_{i_2} \dots x_{i_r} + \sum x'_\alpha \otimes x''_\alpha$$

for elementary monomials $x'_\alpha \neq x_{i_1}$. Thus $\bar{\phi}$ is a monomorphism of the submodule spanned by $\{x_{i_1} x_{i_2} \dots x_{i_r}; r \geq 2\}$, and $\bar{\phi}(x_i) = 0$ is trivial. Therefore (2) is obtained.

(3) If we arrange the generators so that $|x_1| \geq |x_2| \geq \dots \geq |x_m|$, the term $x_{i_1} \otimes x_{i_2} \dots x_{i_r}$ does not appear in the image of the other monomials of some degree by $\bar{\phi}$ in (2). Therefore, decomposable elements are not primitive and primitive elements are linear combinations of x_i modulo $\bar{A}^2$. Thus (3) is obtained. Q.E.D.

Let us consider a condition for a Hopf algebra to be primitively generated.

LEMMA 1.11. Let A be a Hopf algebra and $x_1, \dots, x_m$ primitive elements of A . Suppose that $A = \Delta(x_1, \dots, x_m)$ for degrees $< N$. An arbitrary element x of $\bar{A}$ of degree N can be chosen modulo decomposable elements so that it satisfies the following equality:

$$\bar{\phi}(x) = \sum a_{ij} x_i \otimes x_j, \quad \text{for } |x_i| + |x_j| = |x|,$$

where a_{ij} are constants, and $a_{ji} = 0$ if $i \neq j$ and $a_{ii} \neq 0$.

PROOF. We put

$$\Psi = \bar{\phi} \otimes 1 - 1 \otimes \bar{\phi}, \quad \Delta_i = \Delta(x_i, \dots, x_m).$$

We have $\bar{\phi}(x) \in \bar{\Delta}_1 \otimes \bar{\Delta}_1$. By Lemma 1.10(1) and by the primitiveness of x_i we have

$$\Psi(\bar{\phi}(x)) = \Psi(x_i \otimes x_j) = 0, \quad \bar{\phi}(x_i x_j) = x_i \otimes x_j \pm x_j \otimes x_i.$$

Thus, to prove the lemma, it suffices to show that $\bar{\phi}(A^N) \cap (\bar{\Delta}_1 \otimes \bar{\Delta}_1)$ is spanned by $\{x_i \otimes x_j\}$ and $\bar{\phi}(\bar{\Delta}_1)$. Since $\bar{\Delta}_1 \otimes \bar{\Delta}_1$ is spanned by $\{x_i \otimes x_j\}$ and $\bar{\Delta}_1 \otimes \bar{\Delta}_1^2 + \bar{\Delta}_1^2 \otimes \bar{\Delta}_1$, it suffices to show

$$\bar{\phi}(A^N) \cap (\bar{\Delta}_i \otimes \bar{\Delta}_i^2 + \bar{\Delta}_i^2 \otimes \bar{\Delta}_i) \subset \bar{\phi}(\bar{\Delta}_i) \quad (i = m, \dots, 2, 1) \quad (*)_i$$

by induction on decreasing i , where $\bar{\Delta}_i^2$ is the subspace spanned by

$$\{x_{i_1} \cdots x_{i_r} \mid i \leq i_1 < \cdots < i_r \leq m; r \geq 2\}.$$

$(*)_m$ is trivial. We will show that $(*)_{i+1}$ implies $(*)_i$.

Let $\bar{\phi}(x)$ be an element of the left-hand side of $(*)_i$. If $\bar{\phi}(x)$ contains the term $x_i \otimes f_0$ with $f_0 \in \bar{\Delta}_{i+1}^2$, we replace x by $x' = x - x_i f_0$ to kill $x_i \otimes f_0$, and $\bar{\phi}(x')$ becomes

$$\sum x_i g_\gamma \otimes x_i h_\gamma + x_i g \otimes x_i + x_i \otimes x_i h + f \otimes x_i + \sum x_i g_\alpha \otimes h_\alpha + \sum g_\beta \otimes x_i h_\beta + R,$$

where $f \in \bar{\Delta}_{i+1}^2$; $g_\gamma, h_\gamma, g, h, g_\alpha, h_\alpha, g_\beta, h_\beta \in \bar{\Delta}_{i+1}$ and $R \in \bar{\Delta}_{i+1} \otimes \bar{\Delta}_{i+1}^2 + \bar{\Delta}_{i+1}^2 \otimes \bar{\Delta}_{i+1}$. Observe that $\Psi(\bar{\phi}(x')) = 0$ and $\bar{\phi}(\bar{\Delta}_{i+1}) \subset \bar{\Delta}_{i+1} \otimes \bar{\Delta}_{i+1}$. We apply $\Psi = \bar{\phi} \otimes 1 - 1 \otimes \bar{\phi}$ to the above equality and express it in terms of elementary monomials. Considering the terms where x_i appears singly, we have

$$\begin{aligned} 0 &= x_i \otimes \left(\sum g_\gamma \otimes x_i h_\gamma + g \otimes x_i - x_i \otimes h \right) \\ &\quad + (\text{the other terms containing two } x_i) + \bar{\phi}(f) \otimes x_i \\ &\quad + \sum (x_i \otimes g_\alpha \pm g_\alpha \otimes x_i) \otimes h_\alpha - \sum g_\beta \otimes (x_i \otimes h_\beta \pm h_\beta \otimes x_i). \end{aligned}$$

Thus $\sum g_\gamma \otimes x_i h_\gamma = g = h = \sum g_\alpha \otimes h_\alpha = 0$, and hence $\sum g_\beta \otimes x_i h_\beta = 0$, which implies $\sum g_\beta \otimes h_\beta = 0$. Therefore we have $\bar{\phi}(x') = f \otimes x_i + R$ and $\bar{\phi}(f) = 0$. Since Lemma 1.10 (2) holds even when restricting the degrees, $\bar{\phi}(f) = 0$ and $f \in \bar{\Delta}_{i+1}^2$ and hence $f = 0$. Then it follows that

$$\bar{\phi}(x') = R \in \bar{\phi}(A^N) \cap (\bar{\Delta}_{i+1} \otimes \bar{\Delta}_{i+1}^2 + \bar{\Delta}_{i+1}^2 \otimes \bar{\Delta}_{i+1})$$

and we see that $(*)_i$ is reduced to $(*)_{i+1}$. The proof is completed. Q.E.D.

THEOREM 1.12. *If a Hopf algebra A has a simple system of generators of odd degrees, i.e., $A = \Delta(x_1, \dots, x_n)$ with $|x_i|$ odd, then A is primitively generated and an exterior algebra.*

PROOF. We order the generators x_i by degree and show by induction on m that $x_1, \dots, x_m$ can be replaced by primitive elements modulo decomposable elements. By Lemma 1.11 we have $\bar{\phi}(x_m) = \sum a_{ij} x_i \otimes x_j$ where $|x_m|$ is odd and $|x_i \otimes x_j|$ is even, and hence $\bar{\phi}(x_m) = 0$. That A is an exterior algebra follows from Theorems 1.7 and 1.8. Q.E.D.

COROLLARY 1.13 (The Hopf-Samelson theorem). *If a Hopf algebra A is an exterior algebra generated by the elements of odd degree, then A is primitively generated.*

THEOREM 1.14. *Let A be a finitely generated Hopf algebra over $\mathbb{Z}$. For the diagonal map in $A/\text{Tor } A$ induced from ϕ in A , $A/\text{Tor } A$ is a primitively generated Hopf algebra and is an exterior algebra generated by the elements of odd degree.*

PROOF. Since ϕ satisfies $\phi(\text{Tor } A) \subset \text{Tor}(A \otimes A) = A \otimes \text{Tor } A + \text{Tor } A \otimes A \subset A \otimes A$, ϕ induces a diagonal map $\phi': A/\text{Tor } A \rightarrow (A/\text{Tor } A) \otimes (A/\text{Tor } A)$, and hence $A/\text{Tor } A$ is a torsion free Hopf algebra over $\mathbb{Z}$. $A/\text{Tor } A$ is an exterior algebra by Theorem 1.8 and is primitively generated by Theorem 1.12. Q.E.D.

DEFINITION. The map $\xi: A \rightarrow A$ is defined by $\xi(x) = x^p$, where A is a Hopf algebra over K_p , $p \neq 0$.

Since ξ is a linear map commuting with μ and ϕ , we have

1.15. *The image ξA of ξ is a Hopf algebra and ξ maps $P(A)$ to $P(\xi A)$.*

THEOREM 1.16. *Let A be a Hopf algebra over K_2 .*

(1) $0 \rightarrow P(\xi A) \rightarrow P(A) \xrightarrow{i} Q(A)$ is exact.

(2) A has a simple system of generators $\{x_1, x_2, \dots\}$, a subset of which is a basis of $P(A)$.

PROOF. (1) $P(\xi A) \subset \text{Ker } i$ is clear. So we show $\text{Ker } i \subset P(\xi A)$.

First we consider the case $\xi \bar{A} = 0$, i.e., $x^2 = 0$ for $x \in \bar{A}$. Then we have $A = \bigotimes_{\alpha} \Lambda(x_{\alpha})$ by Corollary 1.4(2) and the lemma claims $\text{Ker } i = 0$. Hence it suffices to show that a decomposable element $x (\in \text{Ker } i)$ is not primitive.

When we express x as a sum of elementary monomials $x_{\alpha_1} \cdots x_{\alpha_n}$ (for $\alpha_1 < \cdots < \alpha_n$, $|x_{\alpha_1}| \leq \cdots \leq |x_{\alpha_n}|$; $n \geq 2$), we consider one of them with maximal $|x_{\alpha_n}|$. $\phi(x_{\alpha_1} \cdots x_{\alpha_n})$ contains the term $x_{\alpha_1} \cdots x_{\alpha_{n-1}} \otimes x_{\alpha_n}$, which does not appear from any other monomials. Thus x is not primitive.

Second we consider A in general and suppose that $x \in P(A)$ is decomposable.

$C = A/\xi \bar{A} \cdot A$ is a Hopf algebra satisfying $\xi C = 0$, and the class of x in C is 0, since it is primitive and decomposable. If we choose $B \subset A$ so that the image in C is a basis of C , we can write $x = \sum a_i^2 b_i$, where $a_i \in A$ and $b_i \in B$. $\phi(a_i^2 b_i)$ contains the term $a_i^2 \otimes b_i$ but we see that it does not come from another $\phi(a_j^2 b_j)$ with $i \neq j$ if $|b_j|$ is maximal, since we have $\phi(a_i^2) = \phi(a_i)^2$. If $|b_j| > 0$, then x is not primitive, and so $b_i = 1$, which implies $x = \sum a_i^2 = (\sum a_i)^2 \in \xi A \cap P(A) = P(\xi A)$. Thus it is shown that $\text{Ker } i \subset P(\xi A)$.

(2) As is seen from Corollary 1.4(2), if we take $B_n \subset \xi^n A$ for $n = 0, 1, 2, \dots$ so that its image forms a basis of $Q(\xi^n A)$, then B_n are

mutually disjoint and $\bigcup B_n$ is a basis of the generators of A . By (1) we can choose $B'_n = B_n \cap P(\xi^n A)$ so that it is a basis of $P(\xi^n A)/P(\xi^{n+1} A)$. Then it is clear that $\bigcup B'_n = \{x_1, x_2, \dots\}$ satisfies the property of (2). Q.E.D.

REMARK 1.16'. Let A be a Hopf algebra over K_p . Then $0 \rightarrow P(\xi A) \rightarrow P(A) \xrightarrow{1} Q(A)$ is also exact for an odd prime p .

COROLLARY 1.17. In the simple system of generators of Theorem 1.16 (2) we have $x_i^2 = 0$, if $x_i \in P(A)$ and there is no j such that $|x_j| = 2|x_i|$.

PROOF. We have $x_i^2 \in P(\xi A) \subset P(A)$ by (1.15). The corollary follows from Theorem 1.16 (2). Q.E.D.

Cohomology of H -spaces. Let X be an arcwise connected H -space of finite type (i.e., $H_i(X)$ is finitely generated for all i) with multiplication $\mu: X \times X \rightarrow X$. Then the cohomology ring $H^*(X; R)$ is connected, commutative, and associative. If R is a field or $H^*(X; R)$ is R -free, using the Künneth formula (Theorem 2.15 of Chapter IV) we have

1.18. $H^*(X; R)$ is a quasi Hopf algebra with the diagonal map $\phi = \mu^*: H^*(X; R) \rightarrow H^*(X; R) \otimes H^*(X; R)$. Thus the results obtained so far in this section hold for

$$A = H^*(X; R), \quad (R = \mathbb{Z} \text{ and } H^*(X) \text{ is } R\text{-free, or } R = K_p).$$

1.18'. If X is homotopy associative, in particular if $X = G$ is a topological group or a Lie group, then $H^*(X; R)$ is a Hopf algebra ($\phi = \mu^*$ is associative).

If X is of finite type, using the universal coefficient theorem ((1.5) and (1.7) of Chapter III) we have

1.19. (1) (The rank of $H_n(X)$ (= the number of generators of the free part)) = (the rank of $H^n(X)$).

(2) (The torsion part of $H_n(X)$) $\cong$ (the torsion part of $H^{n+1}(X)$).

In particular, the following holds for the p -torsion.

1.20. The following (1) ~ (8) are equivalent for a prime p , a field k of characteristic p , and X of finite type:

- (1) $H_*(X)$ is p -torsion free.
 - (2) $H^*(X)$ is p -torsion free.
 - (3) $H_*(X) \otimes k \cong H_*(X; k)$.
 - (4) $H^*(X) \otimes k \cong H^*(X; k)$.
 - (5) $H_*(X; k) \cong H^*(X; k)$.
 - (6) $P(H_*(X; k), t) = P(H^*(X; k), t)$.
 - (7) $\dim_k H_n(X; k) = \dim_Q H_n(X; Q)$ for any n .
 - (8) $\dim_k H^n(X; k) = \dim_Q H^n(X; Q)$ for any n .
- (We say that such X is p -torsion free.)

Let X be an H -space of finite type. The last term in the Künneth formula

$$H^*(X \times X) \cong \text{Hom}(H_*(X), H^*(X)) \oplus \text{Ext}(H_*(X), H^*(X))$$

is a torsion group, and the term Hom corresponds to the cross product $H^*(X) \otimes H^*(X) \rightarrow H^*(X \times X)$. Furthermore we obtain from them an isomorphism

$$(H^*(X)/\text{Tor } H^*(X)) \otimes (H^*(X)/\text{Tor } H^*(X)) \cong H^*(X \times X)/\text{Tor } H^*(X \times X).$$

$\mu^*: H^*(X) \rightarrow H^*(X \times X)$ induces a diagonal map in $H^*(X)/\text{Tor } H^*(X)$. Thus

1.21. *If X is an H -space of finite type, then $H^*(X)/\text{Tor } H^*(X)$ is a quasi Hopf algebra and a free abelian group.*

By applying Theorems 1.8 and 1.14 to this we obtain the following:

THEOREM 1.22. *Let X be a 0-connected H -space. If $H_*(X)$ is finitely generated, then as a quasi Hopf algebra $H^*(X)/\text{Tor } H^*(X)$ is an exterior algebra $\Lambda(x_1, \dots, x_l)$ generated by primitive elements x_i of odd degree. Here $H^*(X; \mathbb{Q}) = \Lambda(x_1, \dots, x_l) \otimes \mathbb{Q}$.*

COROLLARY 1.23. *If $H_*(X)$ is p -torsion free in Theorem 1.22, then $H^*(X; \mathbb{Z}_p)$ is an exterior algebra $\Lambda(x_1, \dots, x_l)$ generated by primitive elements of odd degree.*

PROOF. p -torsion free implies $H^*(X; \mathbb{Z}_p) \cong (H^*(X)/\text{Tor } H^*(X)) \otimes \mathbb{Z}_p$. Q.E.D.

The following holds even for the case with p -torsion.

THEOREM 1.24. *Let p be an odd prime. For X and l in Theorem 1.22 there are at least l generators of odd degree among the generators $\{x_\alpha\}$ of a quasi Hopf algebra $A = H^*(X; \mathbb{Z}_p)$ in Corollary 1.4(3).*

PROOF. By Theorem 1.22 $H^*(X)/\text{Tor } H^*(X) = \Lambda(x_1, \dots, x_l)$ with $|x_i|$ odd. Let $\bar{x}_i \in H^*(X)$ be a representative of x_i and then $\bar{x}_1 \cdots \bar{x}_l$ generates a free direct summand of $H^*(X)$. Thus we have $y_1 \cdots y_l \neq 0$ for $y_i = (\bar{x}_i \bmod p) \in H^*(X; \mathbb{Z}_p)$. Expressing y_i as a polynomial in the generators x_α (the power of x_α not exceeding its height) by Corollary 1.4, each term (a monomial) of the polynomial contains x_α of odd degree, since $|y_i|$ is odd. Therefore $y_1 \cdots y_l$ is a sum of monomials containing at least l pieces of x_α of odd degree. p is an odd prime so $x_\alpha^2 = 0$, and hence there must be at least l generators x_α of odd degree in order that $y_1 \cdots y_l \neq 0$. Q.E.D.

§2. The comparison theorem and the transgression theorem

The comparison theorem. When one knows the cohomologies of any two of the spaces F , E , B in a fibration $F \rightarrow E \rightarrow B$, one can use the Serre spectral sequence $\{E_r, d_r\}$ (cf. Chapter III)

$$E_2 = H^*(B; H^*(F; R)) \Rightarrow H^*(E; R)$$

in order to obtain the cohomology of the third one. In the case where the cohomology of B or F is simple, one can use the Wang or Gysin exact sequence, but in the case where the structure is complicated, one uses the method of first constructing a formal spectral sequence $\{^i E_r, d_r\}$, conjecturing the result, and then studying to what extent it coincides with the original spectral sequence $\{E_r, d_r\}$.

DEFINITION. $\{E_r = \sum E_r^{p,q}, d_r\}$ for $r = 2, 3, \dots$ is called a (cohomology) spectral sequence if R -modules $E_r^{p,q}$ and homomorphisms $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ are given for integers p, q and $r \geq 2$, and satisfy the following:

$$(i) \quad d_r \circ d_r = 0: E_r^{p-r, q+r-1} \xrightarrow{d_r} E_r^{p, q} \xrightarrow{d_r} E_r^{p+r, q-r+1},$$

$$(ii) \quad E_{r+1}^{p,q} \cong H(E_r^{p,q}, d_r) = Z_r^{p,q} / B_r^{p,q},$$

where $Z_r^{p,q} = \text{Ker } d_r$ and $B_r^{p,q} = \text{Im } d_r$ in $E_r^{p,q}$.

DEFINITION. A spectral sequence $\{E_r^{p,q}, d_r\}$ is called *positive* if

$$E_r^{p,q} = 0 \quad \text{for } p < 0 \text{ or } q < 0.$$

The (Serre) spectral sequence of a fibration is a positive spectral sequence.

Hereafter we always consider positive spectral sequences. Then, since we have

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \dots, \quad \text{for } r > \text{Max}(p, q+1), \quad (2.1)$$

we denote the limit by $E_\infty^{p,q}$, that is, $E_\infty^{p,q} = E_r^{p,q}$ for $r > \text{Max}(p, q+1)$.

DEFINITION. $f = \{f_r^{p,q}: ^i E_r^{p,q} \rightarrow E_r^{p,q}\}$ is called a *homomorphism* between the spectral sequences $\{^i E_r^{p,q}\}$ and $\{E_r^{p,q}\}$ if f commutes with d_r ($d_r \circ f_r^{p,q} = f_r^{p+r, q-r+1} \circ d_r$) and $f_r^{p,q}$ induces $f_{r+1}^{p,q}$.

We put

$$f_\infty^{p,q} = f_r^{p,q}: ^i E_\infty^{p,q} \rightarrow E_\infty^{p,q} \quad \text{for } r > \text{Max}(p, q+1). \quad (2.1')$$

Here we have the following:

ASSUMPTION 2.2. If $f_2^{p,0}: ^i E_2^{p,0} \rightarrow E_2^{p,0}$ and $f_2^{0,q}: ^i E_2^{0,q} \rightarrow E_2^{0,q}$ are both epimorphisms (resp. monomorphisms), then $f_2^{p,q}: ^i E_2^{p,q} \rightarrow E_2^{p,q}$ is an epimorphism (resp. a monomorphism).

The following case is of practical use.

2.2'. If $^i E_2^{p,q} \cong ^i E_2^{p,0} \otimes ^i E_2^{0,q}$ and $E_2^{p,q} \cong E_2^{p,0} \otimes E_2^{0,q}$ and $f_2^{p,q} = f_2^{p,0} \otimes f_2^{0,q}$ through these isomorphisms and in addition, $f_2^{p,0}$ and $f_2^{0,q}$ are split homomorphisms (i.e., the kernel and image are direct summands, e.g., R is a field), then Assumption 2.2 is satisfied.

If the local system of coefficient rings is trivial and $H^*(B; R)$ or $H^*(F; R)$ is R -free and of finite type in the spectral sequence of a fibration $F \rightarrow E \rightarrow B$, then

$$E_2^{p,q} = H^p(B; R) \otimes H^q(F; R). \quad (2.3)$$

Therefore, $f_2^{p,0}$ and $f_2^{0,q}$ give information on $H^p(B; R)$ and $H^q(F; R)$ respectively, and $\{f_\infty^{p,q}, p+q=n\}$ gives information on $H^n(E; R)$.

THEOREM 2.4 (The Zeeman comparison theorem). *Suppose that a homomorphism $\{f_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p,q}\}$ of positive cohomology spectral sequences satisfies Assumption 2.2. Consider the following statements for positive integers P, Q , and N :*

- (B)_P $f_2^{p,0}$ is a bijection for $p < P$ and an injection for $p = P$;
- (F)_Q $f_2^{0,q}$ is a bijection for $q < Q$ and an injection for $q = Q$;
- (E)_N $f_\infty^{p,q}$ is a bijection for $p+q < N$ and an injection for $p+q = N$.

There are the following relations among these statements:

- (1) (B)_P and (F)_Q imply (E)_N if $N \leq \min(P, Q)$;
- (2) (E)_N and (F)_Q imply (B)_P if $P \leq \min(N, Q+1)$;
- (3) (E)_N and (B)_P imply (F)_Q if $N \leq \min(N, P-1)$,

where one may suppose that $f_r^{p,q}$ is defined only for

$$p+q \leq \min(P, Q+1, N).$$

PROOF. (1) First the following statement (*) holds:

$$\begin{aligned} &\text{If } f_r^{p,q} \text{ is surjective and } f_{r+1}^{p+r, q-r+1} \text{ is injective then} \\ &f_{r+1}^{p,q} \text{ is surjective and } f_{r+1}^{p+r, q-r+1} \text{ is injective.} \end{aligned} \quad (*)$$

For arbitrary $\bar{x} \in E_{r+1}^{p,q}$ and its representative $x \in E_r^{p,q}$ ($d_r(x) = 0$), $f_r^{p,q}$ being an epimorphism implies $x = f_r(y)$ for $y \in E_r^{p,q}$. $f_r d_r(y) = d_r f_r(y) = d_r(x) = 0$ and $f_r^{p+r, q-r+1}$ being a monomorphism imply $d_r(y) = 0$. We have $f_{r+1}(\bar{y}) = \bar{x}$ for the class $\bar{y} \in E_r^{p,q}$ of y , that is, $f_{r+1}^{p,q}$ is an epimorphism.

The proof that $f_{r+1}^{p+r, q-r+1}$ is a monomorphism is dual to this.

The following is proved by induction on $r \geq 2$ using (*).

2.4'. When (B)_P and (F)_Q hold, then for $r \geq 2$

- (i)_r (a) $p \leq P, q \leq Q - r + 1$ or (b) $p \leq P, p+q \leq Q$ implies that $f_r^{p,q}$ is a monomorphism
- (ii)_r (c) $p \leq P - r + 1, q < Q$ or (d) $q < Q, p+q < P$ implies that $f_r^{p,q}$ is an epimorphism.

Here if we let $r = \infty$ (r is sufficiently large), we have (E)_N for $N \leq \min(P, Q)$.

(2) The case $P \leq 1$ is trivial. Suppose (E)_{P+1}, (F)_P, and (B)_P by induction on $P \geq 1$. Then it suffices to show that $f_2^{p,0}$ is an epimorphism

and $f_2^{p+1,0}$ is a monomorphism. First we show

$$r \geq q+2 \text{ implies that } f_r^{p-q,q} \text{ is an epimorphism } (r \geq 2). \quad (2.4'')$$

Since $(2.4'')$ is true by $(E)_{p+1}$, it suffices to show $(2.4'')$ by assuming $(2.4'')$. $d_r(x) \in E_r^{p-q+r, q-r+1} = 0$ for any $x \in E_r^{p-q,q}$ implies the class $[x] \in E_{r+1}^{p-q,q}$ of x . By $(2.4'')$ $[x] = f_{r+1}[y]$ for $y \in E_r^{p-q,q}$ implies $x = f_r(y) + d_r(z)$ for $z \in E_r^{p-q-r, q+r-1}$. If $z \neq 0$, then $P \geq q+r$, hence $f_r^{p-q-r, q+r-1}$ is an epimorphism by $(2.4'')(ii)_r$, (c) so $z = f_r(w)$. Thus, $x = f_r(y) + d_r f_r(w) = f_r(y + d_r w)$. That is $(2.4'')$ holds.

By considering $(2.4'')$ with $q=0$, we see that $f_2^{p,0}$ is an epimorphism.

To show that $f_2^{p+1,0}$ is a monomorphism, it suffices to show that $f_r^{p+1,0}$ is a monomorphism (i.e., $f_r(x) = 0$ for $x \in E_r^{p+1,0}$ implies $x = 0$) by assuming that $f_{r+1}^{p+1,0}$ is a monomorphism for $P+1 \geq r \geq 2$, since $f_\infty^{p+1,0}$ is a monomorphism.

$f_{r+1}[x] = [f_r(x)] = 0$ implies $[x] = 0$, and hence $x = d_r(y)$ for $y \in E_r^{p+1-r, r-1}$. $d_r f_r(y) = f_r d_r(y) = f_r(x) = 0$ implies $[f_r(y)] \in E_{r+1}^{p+1-r, r-1}$. By $(2.4'')$ with $q=r-1$, $[f_r(y)] = f_{r+1}[z]$ for $z \in E_r^{p+1-r, r-1}$ implies $f_r(y) = f_r(z) + d_r(w)$ for $w \in E_r^{p+1-2r, 2r-2}$. If $w \neq 0$, then $P \geq 2r-1$. Then by $(2.4'')(ii)_r$, (d), $f_r^{p+1-2r, 2r-2}$ is an epimorphism so $w = f_r(u)$ and we have $f_r(y) = f_r(z + d_r(u))$. By $(2.4')(i)_r$, (b), $f_r^{p+1-r, r-1}$ is a monomorphism, and hence $y = z + d_r(u)$, which implies $x = d_r(y) = d_r(z) + d_r d_r(u) = 0$.

Therefore $f_2^{p+1,0}$ is a monomorphism.

(3) It suffices to show that $(E)_{Q+1}$, $(B)_{Q+2}$, and $(F)_Q$ imply that $f_2^{0,Q}$ is an epimorphism and $f_2^{Q,0+1}$ is a monomorphism. This is shown dually to the proof of (2), and the condition corresponding to $(2.4'')$ is

$$r \geq q+1 \text{ implies that } f_r^{p, Q+1-p} \text{ is a monomorphism for } r \geq 2. \quad (2.4''')$$

The details are left to the reader. Q.E.D.

The Borel transgression theorem. As a typical example of a spectral sequence with E_∞ trivial, we have

$$'E_2 = \mathbf{k}[y] \otimes \Delta(x), \quad d_r(1 \otimes x) = y \otimes 1, \quad d_r \text{ is distributive,}$$

where the degrees are $|y| = r$ and $|x| = r-1$, and we suppose that r is even if $\mathbf{k}$ is of characteristic $\neq 2$. Since we have

$$d_r(y^n \otimes 1) = 0 \quad \text{and} \quad d_r(y^n \otimes x) = y^{n+1} \otimes 1,$$

we obtain

$$'E_2 = 'E_r, \quad 'E_{r+1} = H('E_r) \cong \mathbf{k}, \quad 'E_\infty \cong \mathbf{k}.$$

By the Künneth formula $H(C_1^* \otimes_k C_2^*) = H(C_1^*) \otimes_k H(C_2^*)$, the tensor product of a finite number of such spectral sequences is also a spectral sequence. That is,

2.5. By the distributivity of d_r in $'E_2 = (\otimes_\alpha k[y_\alpha]) \otimes_k (\otimes_\alpha \Delta(x_\alpha))$ and $d_r(1 \otimes x_\alpha) = y_\alpha \otimes 1$, where $|y_\alpha| = |x_\alpha| + 1$ (if k is of characteristic $\neq 2$ we assume $|y_\alpha|$ is even) the spectral sequence $\{ 'E_r, d_r \}$ is defined, and

$$'E_r = \left(\bigotimes_{\beta} k[y_{\beta}] \right) \bigotimes_k \left(\bigotimes_{\beta} \Delta(x_{\beta}) \right) \quad (\text{where } \beta \text{ satisfies } |y_{\beta}| \geq r).$$

In particular, $'E_{\infty} = k$.

Now we recall the relations between the transgression τ and the suspension σ in the cohomology of a fibration and the cohomology spectral sequence (§2 of Chapter III).

In the spectral sequence $\{E_r, d_r\}$ of a fibration $F \rightarrow E \xrightarrow{p} B$ where B is 1-connected and F is 0-connected, we may take

$$H^p(B; R) = E_2^{p,0} \quad \text{and} \quad H^q(F; R) = E_2^{0,q}.$$

When we consider

$$\delta^*: H^{r-1}(F; R) \rightarrow H^*(E, F; R) \quad \text{and} \quad p^*: H^r(B; R) \rightarrow H^r(E, F; R)$$

for $r > 1$, an element x of $\delta^{*-1}(\text{Im } p^*)$ is said to be *transgressive* and an element y of $p^{*-1}(\text{Im } \delta^*)$ is *suspensive*. If we have

$$\delta^*(x) = p^*(y),$$

then y is called the *transgression (image)* of x and x is called the *suspension (image)* of y , denoted by

$$y = \tau(x) \quad \text{and} \quad x = \sigma(y).$$

Note that τ and σ are not univalent morphisms. The relation with the spectral sequence $\{E_r, d_r\}$ is given by the following:

2.6. (1) $x \in H^{r-1}(F; R) = E_2^{0,r-1}$ is transgressive iff $x \in E_r^{0,r-1} (\subset E_2^{0,r-1})$.

(2) $y \in H^r(B; R) = E_2^{r,0}$ is suspensive iff the class of y in $E_r^{r,0}$ (denoted by the same y) is in the image of $d_r: E_r^{0,r-1} \rightarrow E_r^{r,0}$, which holds iff the class of y in $E_{r+1}^{r,0} = E_{\infty}^{r,0}$ is 0.

(3) $d_r(1 \otimes x) = y \otimes 1$ iff $y = \tau(x)$, which holds iff $x = \sigma(y)$.

(4) $\tau(x) = 0$ iff $x \in \text{Im}(i^*: H^*(E; R) \rightarrow H^*(F; R))$.

NOTATION. We denote $\otimes_\alpha k[y_\alpha]$ simply by $k[y_\alpha]$. Similarly for $\Delta(x_\alpha)$, $\Lambda(x_\alpha)$, and so on.

THEOREM 2.7. *Suppose that B is 1-connected and of finite type in a fibration $F \rightarrow E \rightarrow B$ and that $H^*(E; \mathbf{k}) \cong \mathbf{k}$ in the range of the degrees $\leq N$. Then the following are equivalent:*

(1) *There are transgressive homogeneous elements x_α such that the natural map $\Delta(x_\alpha) \rightarrow H^*(F; \mathbf{k})$ is bijective for degrees $< N$ and injective for degree $= N$,*

(2) *There are homogeneous elements y_α such that the natural map $k[y_\alpha] \rightarrow H^*(B; \mathbf{k})$ is bijective for degrees $< N + 1$ and injective for degree $= N + 1$, where x_α and y_α satisfy $|y_\alpha| = |x_\alpha| + 1 > 1$ and correspond to one another through $y_\alpha = \tau(x_\alpha)$ and $x_\alpha = \sigma(y_\alpha)$; $|y_\alpha|$ is even if $\mathbf{k}$ is of characteristic $\neq 2$.*

PROOF. By the assumption of (1) and 2.6 we obtain the existence of y_α such that $d_r(1 \otimes x_\alpha) = y_\alpha \otimes 1$ where $r = |y_\alpha|$. Also, for $1 < r \leq N$, $H^r(E; \mathbf{k}) = 0$ hence $E_\infty^{r,0} = 0$. For y_α with $|y_\alpha| \leq N$, there exists x_α such that $d_r(1 \otimes x_\alpha) = y_\alpha \otimes 1$ by 2.6(2). Then since we can naturally define a homomorphism of the spectral sequences $\{E_r\}$ in 2.5 to $\{E_r\}$, the theorem is easily deduced, using the comparison theorem (Theorem 2.4). Q.E.D.

COROLLARY 2.8. *Theorem 2.7 can be applied to $F = \Omega B$, or to $B = BG$ and $F = G$, and $H^*(E; \mathbf{k}) = \mathbf{k}$ holds for all degrees. Therefore for these B and F we have*

(1) *If $H^*(F; \mathbf{k}) = \Delta(x_\alpha)$, where x_α is transgressive, then $H^*(B; \mathbf{k}) = \mathbf{k}[\tau(x_\alpha)]$.*

(2) *If $H^*(B; \mathbf{k}) = \mathbf{k}[y_\alpha]$, then $H^*(F; \mathbf{k}) = \Delta(\sigma(y_\alpha))$.*

REMARK 2.8'. Corollary 2.8 is sometimes called the *Borel little (transgression) theorem*, and in order to apply it, one has to show that x_α is transgressive by using cohomology operations and so on; however, this is not needed in the following theorem.

THEOREM 2.9 (The Borel transgression theorem). *In a fibration $F \rightarrow E \rightarrow B$ of Theorem 2.7, suppose that $H^*(E; \mathbf{k}) = \mathbf{k}$ for degrees $\leq N + 2$ and that there exist elements $x_i \in H^*(F; \mathbf{k})$ of odd degrees with $|x_i| \leq N$ such that the natural morphism*

$$\Delta(x_1, \dots, x_l) \rightarrow H^*(F; \mathbf{k})$$

is bijective for degrees $\leq N$ and injective for degree $= N + 1$.

Then we can choose the generators x_i so as to be transgressive and

$$\mathbf{k}[y_1, \dots, y_l] \rightarrow H^*(B; \mathbf{k}), \quad \text{where } y_i = \tau(x_i),$$

is bijective for degrees $\leq N + 1$ and injective for degree $= N + 1$.

PROOF. We prove the theorem by induction on N . The case $N = 0$ is trivial. Supposing that the theorem is true for $N - 1$, we prove the theorem for N , where we may suppose that $0 < |x_1| \leq |x_2| \leq \dots \leq |x_l| \leq N$.

The case N even. If $|x_i|$ is odd then $|x_i| \leq N-1$, and hence by the inductive hypothesis one can choose $x_1, \dots, x_l$ to be transgressive. Then the theorem is clear from Theorem 2.7.

The case N odd. Suppose that $|x_m| \leq N-2$ and $|x_{m+1}| = \dots = |x_l| = N$. Then, since

$$\Delta(x_1, \dots, x_m) \rightarrow H^*(F; \mathbf{k})$$

is bijective for degrees $\leq N-1$ and injective for degree $= N$, by the inductive hypothesis we can choose $x_1, \dots, x_m$ transgressive such that

$$\mathbf{k}[y_1, \dots, y_m] \rightarrow H^*(B; \mathbf{k})$$

is bijective for degrees $\leq N$ and injective for degree $= N+1$, where $y_i = \tau(x_i)$ for $i = 1, \dots, m$. Therefore there are elements $y_{m+1}, \dots, y_n$ of degree $N+1$ (even) such that the natural morphism

$$\mathbf{k}[y_1, \dots, y_m, y_{m+1}, \dots, y_n] \rightarrow H^*(B; \mathbf{k})$$

is bijective for degrees $\leq N+1$, where the morphism is injective for degree $= N+2$, since $N+2$ is odd. From (2) implying (1) of Theorem 2.7 we have that

$$\Delta(x_1, \dots, x_m, \sigma(y_{m+1}), \dots, \sigma(y_n)) \rightarrow H^*(F; \mathbf{k})$$

is bijective for degrees $\leq N$. This shows that $l = n$ and that one can choose a generator x_{m+j} so that $x_{m+j} = \sigma(y_{m+j})$. Then we have $y_{m+j} = \tau(x_{m+j})$ and have shown that the theorem is true. Q.E.D.

REMARK. The original Borel transgression theorem is stated in the algebraic form of a spectral sequence. The assumption is

$$H^*(F; \mathbf{k}) = \Lambda(x_1, \dots, x_l),$$

with $|x_i| \leq M$ and $H^*(E; \mathbf{k}) = \mathbf{k}$ for degrees $\leq N$, where $N \geq 2M+1$; the conclusion is $H^*(B; \mathbf{k}) = \mathbf{k}[y_1, \dots, y_l]$ for degrees $\leq N$. It does not, however, make any difference from the view point of application.

Applications to classifying spaces. Let G be a Lie group and $\gamma = (EG, p, BG, G)$ the universal G -bundle. Then the equivalence classes of G -bundles $\xi = (E_\xi, p, X, G)$ with base space X , which is (a space having the same homotopy type as) a CW -complex, are in one to one correspondence with the homotopy classes of maps $f: X \rightarrow BG$ satisfying $\xi = f^* \gamma$ (cf. Theorem 6.6 of Chapter II). Now, for a homogeneous element

$$c \in H^n(BG; R) \quad \text{where } n > 0$$

the element

$$c(\xi) = f^*(c) \in H^n(X; R)$$

is determined independently of the choice in the homotopy class of f . Further, when G acts on F effectively, the element

$$c(\xi_F) = c(\xi) \in H^n(X; R)$$

is determined by c also for the associated bundle $\xi_F = (E, p, X, F, G)$ of ξ .

DEFINITION. $c(\xi) = c(\xi_F)$ is called the *characteristic class* of ξ (or ξ_F) corresponding to c .

For Chern classes, symplectic Pontrjagin classes, Stiefel Whitney classes, Pontrjagin classes, and the Euler class in Chapter III, the characteristic classes of bundles whose structure groups are the corresponding classical groups are called by the same names. A characteristic class $c(\tau_M)$ of the tangent bundle τ_M of a manifold M is called a *characteristic class* of M .

2.10. A characteristic class $\{c(\xi)\}$ is characterized by the following property: Let ξ and η be G -bundles (or its associated bundles) over CW -complexes X and Y respectively. For a map $f: X \rightarrow Y$,

$$\text{if } f^*\xi \equiv \eta \text{ (equivalent), then } f^*(c(\xi)) = c(\eta) = c(f^*\xi).$$

Of course, this characteristic class corresponds to $c = c(\gamma) \in H^n(BG; R)$.

Below, the base space of a bundle is assumed to be (a space having the same homotopy type as) a CW -complex.

DEFINITION. $x \in H^n(G; R)$ for $n > 0$ is called *universally transgressive* if x is transgressive with respect to any G -bundle $\xi = (E_\xi, p, X, G)$.

By the naturality of the transgression (i.e., $f^*(\tau(x)) = \tau(x)$ for f in 2.10), one sees the following:

2.11. $x \in H^n(G; R)$ is *universally transgressive* iff x is transgressive in the universal G -bundle $\gamma = (EG, p, BG, G)$.

THEOREM 2.12. Let G be a compact, connected Lie group and R be $\mathbb{Z}$ or a field $\mathbf{k}$ of characteristic p . Suppose that $H_*(G)$ is torsion free if $R = \mathbb{Z}$ or is p -torsion free if $R = \mathbf{k}$ is a field of characteristic $p \neq 0$. Then there are universally transgressive elements $x_i \in H^{n_i}(G; R)$ such that

$$H^*(G; R) = \Lambda(x_1, \dots, x_l), \quad \text{where } |x_i| = n_i \text{ is odd,}$$

$$H^*(BG; R) = R[y_1, \dots, y_l], \quad \text{where } y_i = \tau(x_i).$$

Here a sequence of the degrees $(n_1, \dots, n_l)$ is the type of G (for $R = \mathbb{Q}$).

PROOF. By Theorem 1.22 and Corollary 1.23 we have $H^*(G; \mathbf{k}) = \Lambda(x_1, \dots, x_l)$, where $|x_i|$ is odd. By applying Theorem 2.9 with $N = \infty$ to this, the theorem is obtained. Q.E.D.

COROLLARY 2.12'. If $H^*(\Omega X)$ is finitely generated for a 1-connected topological space X , then Theorem 2.12 also holds for $G = \Omega X$ and $BG = X$.

COROLLARY 2.13. If $H_*(G)$ is p -torsion free, then $H_*(BG)$ is p -torsion free.

We calculated the cohomology of G and BG for each case of the classical groups G in Chapter III, but we can discuss them systematically using the above theorem for the p -torsion free case. The theorem is also useful for cases such as the exceptional groups where one can not make concrete computations.

THEOREM 2.14. *Let $R = \mathbb{Z}$ or $R = \mathbf{k}$ a field of characteristic $p \neq 0$. For a compact, connected Lie group G , the following conditions are equivalent:*

(1) $H_*(G)$ is torsion free if $R = \mathbb{Z}$, or p -torsion free if $R = \mathbf{k}$ a field of characteristic p .

(2) $H^*(G; R)$ is an exterior algebra generated by elements of odd degrees.

(3) $H^*(BG; R)$ is a polynomial ring generated by elements of even degrees.

PROOF. That (1) implies (3) follows from Theorem 2.12 and that (3) implies (2) follows from Corollary 2.8(2).

To show that (2) implies (1) first note that (3) is obtained by Theorem 2.9. $H_*(BG)$ is torsion (or p -torsion) free, since $H^*(BG; R)$ consists only of elements of even degrees. Thus $H^*(BG; \mathbb{Q})$ is a polynomial ring having similar generators (of the same degrees) to $H^*(BG; R)$. By Corollary 2.8 (2), $H^*(G; \mathbb{Q})$ is an exterior algebra having the generators of the same degrees as those of $H^*(G; R)$. We obtain (1) from the proof that (7) implies (1) in 1.20. Q.E.D.

Applications to loop spaces. Conversely to the spectral sequence studied in 2.5, we consider the case where $E_2^{0,*}$, which corresponds to the fibre, is a polynomial ring and suppose that

$$'E_2 = \Delta(x) \otimes \mathbf{k}[y], \quad d_r(1 \otimes y) = x \otimes 1, \quad d_r \text{ is distributive,}$$

where $|y|$ is even. Since we have

$$d_r(x \otimes y^n) = 0, \quad d_r(1 \otimes y^n) = n(x \otimes y^{n-1}), \quad 'E_r = 'E_2,$$

by calculating $'E_{r+1} = H('E_r)$ we obtain the following:

2.15. (1) $\mathbf{k}$ is of characteristic 0, then $'E_\infty = \mathbf{k}$.

(2) If $\mathbf{k}$ is of characteristic $p \neq 0$, then $'E_\infty = \mathbf{k}$ in the range of total degrees $< p|y|$.

By using the spectral sequence obtained as the tensor product of such spectral sequences and the spectral sequence of the form 2.5, the following theorem is deduced from the comparison theorem.

THEOREM 2.16. *Let $\mathbf{k}$ be a field of characteristic p and X a 1-connected topological space of finite type. Suppose that there are elements x_α of odd degrees and elements x'_β of even degrees in $H^*(X; \mathbf{k})$ such that the natural morphism*

$$\Delta(x_\alpha) \otimes \mathbf{k}[x'_\beta] \rightarrow H^*(X; \mathbf{k})$$

is bijective for degrees $\leq N$ and injective for degree $= N + 1$. If we put

$$M = \begin{cases} \text{Min}(N-1, p(|x_\alpha| - 1) - 2) & \text{for } p \neq 0, \\ N-1 & \text{for } p = 0, \end{cases}$$

then the natural morphism

$$\mathbf{k}[\sigma(x_\alpha)] \otimes \Delta(\sigma(x'_\beta)) \rightarrow H^*(\Omega X; \mathbf{k}),$$

where $\sigma(x_\alpha)$ and $\sigma(x'_\beta) \in H^*(\Omega X; \mathbf{k})$ are the suspension images, is bijective for degrees $\leq M$ and injective for degree $= M + 1$.

No converse to this theorem is known in general but the following holds for an H -space X .

THEOREM 2.17. Let $\mathbf{k}$ be a field of characteristic $p \neq 0$ and X a 1-connected H -space of finite type. Suppose that there are elements y_α of even degrees and elements y'_β of odd degrees in $H^*(\Omega X; \mathbf{k})$ such that

$$H^*(\Omega X; \mathbf{k}) = \mathbf{k}[y_\alpha] \otimes \Lambda(y'_\beta) \text{ for degrees } < N.$$

Then we can replace y_α and y'_β by transgressive elements so that

$$H^*(X; \mathbf{k}) = \Delta(x_\alpha) \otimes \mathbf{k}[x'_\beta] \text{ where } x_\alpha = \tau(y_\alpha) \text{ and } x'_\beta = \tau(y'_\beta)$$

in the range of degrees $\leq \min(N, p|y_\alpha| - 2, p(|y'_\beta| + 1) - 2)$.

PROOF. By Theorem 1.3 (Corollary 1.4) we can express $H^*(X; \mathbf{k}) = \Delta(x_\gamma) \otimes \mathbf{k}[x'_\delta]$ in the range of degrees $\leq \min(p|x'_\delta| - 1)$. By Theorem 2.16, $H^*(\Omega X; \mathbf{k}) = \mathbf{k}[\sigma(x_\gamma)] \otimes \Delta(\sigma(x'_\delta))$ holds in the range of degrees $\leq \min(p|x'_\delta| - 2, p(|x_\gamma| - 1) - 2)$. By replacing with $y_\alpha = \sigma(x_\gamma)$ and $y'_\beta = \sigma(x'_\delta)$ and by comparing with the assumption, the theorem is obtained. Q.E.D.

REMARK. In the case $p = 2$ the theorem only gives the suspension isomorphism. In order to obtain results in higher degrees, one uses Theorem 2.28.

THEOREM 2.17'. If $\mathbf{k}$ is a field of characteristic 0 in Theorem 2.17, then

$$H^*(X; \mathbf{k}) = \Lambda(x_\alpha) \otimes \mathbf{k}[x'_\beta] \text{ in the range of degrees } \leq N.$$

The spectral sequence associated with a principal fibration. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a principal fibration. That is, F is an H -space acting on E (from right) and the following diagram is commutative:

$$\begin{array}{ccccc} F \times F & \xrightarrow{i'} & E \times F & \xrightarrow{p'} & B \\ \downarrow \mu & & \downarrow \bar{\mu} & & \downarrow \text{id} \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array} \quad \left(\begin{array}{l} i' = i \times 1_F \\ p' : E \times F \xrightarrow{\text{proj}} E \xrightarrow{p} B \end{array} \right) \quad (2.18)$$

where μ is the multiplication of F and $\bar{\mu}$ is the action.

The examples of a principal fibration are $G \rightarrow EG \rightarrow BG$, $H \rightarrow G \rightarrow G/H$, $\Omega X \rightarrow PX \rightarrow X$, and $\Omega X \rightarrow \Omega(X, A) \rightarrow A$.

Let $\mathbf{k}$ be a field, B 1-connected, and F of finite type. Denote by $\{E_r, d_r\}$ the cohomology spectral sequence associated with the lower sequence of (2.18):

$$E_2 = H^*(B; \mathbf{k}) \otimes H^*(F; \mathbf{k}) \Rightarrow H^*(E; \mathbf{k}).$$

Since the upper sequence in (2.18) can be constructed from the lower sequence and the trivial fibration $F \xrightarrow{1_F} F \rightarrow *$ (a point), its spectral sequence $\{{}^tE_r, {}^t d_r\}$ such that

$${}^tE_2 = H^*(B; \mathbf{k}) \otimes H^*(F \times F; \mathbf{k}) \Rightarrow H^*(E \times F; \mathbf{k})$$

is seen to be given by

$${}^tE_r = E_r \otimes H^*(F; \mathbf{k}), \quad {}^t d_r = d_r \otimes 1 \quad (2.19)$$

by the Künneth isomorphism.

$\bar{\mu}$ induces also the homomorphism between the spectral sequences

$$\bar{\mu}_r^*: E_r \rightarrow {}^tE_r = E_r \otimes H^*(F; \mathbf{k}) \left(E_r^{p,q} \rightarrow \sum_{a+b=q} E_r^{p,a} \otimes H^b(F; \mathbf{k}) \right),$$

which satisfies

- 2.20. (1) $\bar{\mu}_r^* \circ d_r = (d_r \otimes 1) \circ \bar{\mu}_r^*, \quad \bar{\mu}_{r+1}^* = H(\bar{\mu}_r^*)$.
 (2) $\bar{\mu}_2^* = 1 \otimes \bar{\mu}^*: H^*(B; \mathbf{k}) \otimes H^*(F; \mathbf{k}) \rightarrow H^*(B; \mathbf{k}) \otimes H^*(F; \mathbf{k}) \otimes H^*(F; \mathbf{k})$.
 (3) $\bar{\mu}^*: H^*(E; \mathbf{k}) \rightarrow H^*(E; \mathbf{k}) \otimes H^*(F; \mathbf{k})$ induces $\bar{\mu}_\infty^*$.

LEMMA 2.21. *The product*

$$E_r^{p,0} \otimes E_r^{0,q} \rightarrow E_r^{p,q} \quad \text{for } r \geq 2$$

is a monomorphism in the spectral sequence of a principal fibration.

PROOF. Consider the image by $\bar{\mu}_r^*$ of the product of $f \in E_r^{0,q} \subset E_2^{0,q}$ and $b \in E_2^{p,0} \rightarrow E_r^{p,0}$ (an epimorphism) and then $\bar{\mu}_r^*(b \cdot f)$ is expressed by $\bar{\mu}_2^*(b \cdot f) = (b \otimes 1) \cdot \mu^*(f) = b \otimes f + \cdots + b \cdot f \otimes 1$ by 2.20. Take the first term and consider the composition

$$E_r^{p,0} \otimes E_r^{0,q} \xrightarrow{\text{product}} E_r^{p,q} \xrightarrow{\bar{\mu}_r^*} \sum_{a+b=q} E_r^{p,a} \otimes H^b(F; \mathbf{k}) \rightarrow E_r^{p,0} \otimes H^b(F; \mathbf{k})$$

The composition is seen to be a monomorphism, and hence the product is a monomorphism. Q.E.D.

The transgression theorem in a principal fibration

DEFINITION. The set of homogeneous elements $\{b_1, \dots, b_n\}$ of $H^*(B; \mathbf{k})$ is called a *regular sequence* (in the range of degrees $< N$) if the following conditions (α) and (β) are satisfied:

- (α) $0 < |b_1| \leq |b_2| \leq \cdots \leq |b_m| (\leq N)$
 (β) the correspondence given by the product with b_i

$$b_i \cdot: H^*(B; \mathbf{k})/(b_1, \dots, b_{i-1}) \rightarrow H^*(B; \mathbf{k})/(b_1, \dots, b_{i-1})$$

is a monomorphism for $1 \leq i \leq m$ (in the range of degrees $\leq N$).

When b_i is given by the transgression image $b_i = \tau(x_i)$ for $x_i \in H^*(F; \mathbf{k})$, we say that the set of transgression images $\{\tau(x_1), \dots, \tau(x_m)\}$ is a *regular sequence*, if the set of representatives b_i of $\tau(x_i)$ forms a regular sequence.

THEOREM 2.22. *Let B be 1-connected and F 0-connected and of finite type in a fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$. Suppose that there are transgressive elements $x_\alpha, x_\beta \in H^*(F; \mathbf{k})$ such that the natural morphism $\Delta(x_\alpha) \otimes \Delta(x_\beta) \rightarrow H^*(F; \mathbf{k})$ is bijective for degrees $< N$ and injective for degree $= N$, that $\tau(x_\alpha) = 0$ and that $\{\tau(x_\beta)\}$ is a regular sequence in $H^*(B; \mathbf{k})$.*

Then there are $\bar{x}_\alpha \in H^*(E; \mathbf{k})$ satisfying $i^*(\bar{x}_\alpha) = x_\alpha$ such that the following holds:

$$H^*(E; \mathbf{k}) = \text{Im } \pi^* \otimes \Delta(\bar{x}_\alpha), \quad \text{Im } i^* = \Delta(x_\alpha) \text{ for degrees } < N, \\ \text{Im } \pi^* \cong H^*(B; \mathbf{k})/(\tau(x_\beta)) \text{ for degrees } \leq N.$$

PROOF. First we construct a spectral sequence $\{^r E_r\}$ from $^1 E_2 = H^*(B; \mathbf{k}) \otimes (\Delta(x_\beta) \otimes \Delta(x_\alpha))$ by $d_r(H^*(B; \mathbf{k}) \otimes 1) = 0$, $d_r(1 \otimes x_\alpha) = 0$, $d_r(1 \otimes x_\beta) = 0$ for $r \leq |x_\beta|$, $d_r(1 \otimes x_\beta) = \tau(x_\beta) \otimes 1$ for $r = |x_\beta| + 1$, and by the distributivity of d_r . Then we have

$$^r E_r = H^*(B; \mathbf{k})/(\tau(x_\beta); |x_\beta| < r - 1) \otimes \Delta(x_\beta; |x_\beta| \geq r - 1) \otimes \Delta(x_\alpha) \quad (2.23)$$

and a homomorphism of the spectral sequences $\{f_r^{p,q}: ^r E_r^{p,q} \rightarrow E_r^{p,q}\}$ is induced by the natural morphism, where $\{E_r\}$ is the cohomology spectral sequence with coefficient $\mathbf{k}$ associated with a fibration in the theorem.

In fact, in order to calculate $^r E_{r+1} = H(^r E_r)$, it suffices to calculate d_r on $A = H^*(B; \mathbf{k})/(\tau(x'_\beta); \beta' < \alpha) \otimes \Delta(x_\beta)$ for each β with $|x_\beta| = r - 1$, arranging x_β in the order of degree. Since we have $H(A, d_r) = H^*(B; \mathbf{k})/(\tau(x'_\beta); \beta' \leq \beta) \otimes 1$ by the assumption of a regular sequence, the first half of (2.23) is obtained. The second half is trivial. By applying the comparison Theorem 2.4 (1) with $P = \infty$, $Q = N$ to $\{f_r^{p,q}\}$ we see $(E)_N$, that is,

$$f_\infty = \sum f_\infty^{p,q}: ^1 E_\infty = (H^*(B; \mathbf{k})/(\tau(x_\beta))) \otimes \Delta(\bar{x}_\alpha) \rightarrow E_\infty$$

is bijective for $p + q < N$ and injective for $p + q = N$.

Since $\text{Im } \pi^*$ is identified with $E_\infty^{*,0}$ and π^* induces an epimorphism $H^*(B; \mathbf{k})/(\tau(x_\beta)) \rightarrow \text{Im } \pi^*$, we have $\text{Im } \pi^* \cong H^*(B; \mathbf{k})/(\tau(x_\beta))$ for degrees $\leq N$. Since $\text{Im } i^*$ is identified with $E_\infty^{0,*}$ and $^1 E_\infty^{0,*} = 1 \otimes \Delta(x_\alpha)$, we have $\text{Im } i^* = \Delta(x_\alpha)$ for degrees $< N$. In addition, $D^{*,q} \supset \text{Im } \pi^*$ (the part of degree q in $\Delta(\bar{x}_\alpha)$) and $D^{*,q}/D^{*,q-1} \cong E_\infty^{*,q}$ and hence we see that $H^*(E; \mathbf{k}) = \text{Im } \pi^* \otimes \Delta(\bar{x}_\alpha)$ for degrees $< N$. Q.E.D.

It is not known in general whether or not the transgressiveness for x_α and x_β holds in the above theorem, but it can sometimes be checked inductively by the following theorem for the case of a principal fibration.

THEOREM 2.24. *Let the fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$ in Theorem 2.22 be a principal fibration and make the following assumption:*

- $H^*(F; \mathbf{k}) = \Delta(x_\gamma)$ with $|x_\gamma| \leq N$ for degrees $\leq N$;
 if $|x_\gamma| < N$, then x_γ is transgressive and $\{\tau(x_\gamma) \mid |x_\gamma| < N; \tau(x_\gamma) \neq 0\}$ is a regular sequence of $H^*(B; \mathbf{k})$.

Then, by replacing generators x_γ of degree N suitably, the following hold:

- (0) x_γ is transgressive;
- (1) $\text{Im } i^* = \Delta(x_\delta)$ for degrees $\leq N$, where $\{x_\delta\} = \{x_\gamma \mid \tau(x_\gamma) = 0\}$;
- (2) $H^*(E; \mathbf{k}) = \text{Im } \pi^* \otimes \Delta(\overline{x}_\delta)$ as $\text{Im } \pi^*$ -modules for degrees $\leq N$, where $i^*(\overline{x}_\delta) = x_\delta$;
- (3) $\text{Im } \pi^* \cong H^*(B; \mathbf{k})/(\tau(x_\gamma))$ for degrees $\leq N + 1$.

PROOF. If one puts $\{x_\alpha\} = \{x_\gamma \mid \tau(x_\gamma) = 0 \text{ for } |x_\gamma| < N\}$ and $\{x_\beta\} = \{x_\gamma \mid \tau(x_\gamma) \neq 0 \text{ for } |x_\gamma| < N\}$, then the assumption of Theorem 2.22 is satisfied and $f_r^{p,q}: {}'E_r^{p,q} \rightarrow E_r^{p,q}$ in (2.23) can be considered. Putting $P = \infty$ and $Q = N$ in 2.4', one sees that $f_r^{p,q}$ is injective for $p + q \leq N$ and surjective for $q < N$. $f_r^{p,q}$ is obtained by composing $f_r^{p,0} \circ f_r^{0,q}$ with the product in Lemma 2.21. Since $f_r^{p,0}$ and $f_r^{0,q}$ are injective for $p \leq N$ and $q \leq N$ respectively, so is $f_r^{p,q}$ by Lemma 2.21. Thus

$$f_r^{p,q}: {}'E_r^{p,q} \rightarrow E_r^{p,q} \text{ is a bijection for } p \leq N \text{ and } q < N. \quad (2.24')$$

When $|x_\gamma| = N$, we consider a minimal $r \geq 2$ such that $d_r(1 \otimes x_\gamma) \neq 0$ and let $r \leq N$. If $d_r(1 \otimes x_\gamma) \in E_r^{r, N-r+1}$ is not in the image of $d_r({}'E_r^{0,N})$ by $f_r^{r, N-r+1}$, then $f_{r+1}^{r, N-r+1}$ is not a monomorphism, which contradicts (2.24'). So we can choose x_γ modulo $'E_r^{0,N} \subset {}'E_2^{0,N} \subset \Delta(x_\gamma \mid |x_\gamma| < N)$ so that $d_r(1 \otimes x_\gamma) = 0$. Repeating this, $d_r(1 \otimes x_\gamma) = 0$ for $2 \leq r \leq N$, that is, x_γ is chosen to be transgressive.

In this way we may suppose that all the x_γ are transgressive. Therefore, if we take a maximal number of linearly independent elements out of $\{\tau(x_\gamma) \mid |x_\gamma| = N\}$, the remaining x_γ with $|x_\gamma| = N$ can be chosen modulo these elements so that $\tau(x_\gamma) = 0$.

If we choose x_γ with $|x_\gamma| = N$ as above and put $\{x_\delta\} = \{x_\gamma \mid \tau(x_\gamma) = 0\}$ and $\{x_\epsilon\} = \{x_\gamma \mid \tau(x_\gamma) \neq 0\}$, then x_γ is transgressive and $\{\tau(x_\epsilon)\}$ is a regular sequence for degrees $\leq N + 1$. In addition, the natural injection

$$\Delta(x_\delta) \otimes \Delta(x_\epsilon) \rightarrow H^*(F; \mathbf{k})$$

is bijective for degrees $\leq N$. It is seen from the form in Corollary 1.4 that the inclusion is injective for all degrees. Therefore by calculating the spectral sequence $\{E_r\}$ (cf. the proof of Theorem 2.22) we get the theorem. Q.E.D.

The theorem is generalized as follows, using the spectral sequence in 2.5(2).

THEOREM 2.25. *We replace the assumption of Theorem 2.24 by the following:*

$$H^*(F; \mathbf{k}) = \Delta(x_\gamma) \otimes \mathbf{k}[y_\lambda] \otimes \mathbf{k}[y_\nu], \quad (N+2)/p < |y_\nu| \leq N,$$

for degrees $\leq N$, where x_y with $|x_y| < N$ and y_ν are transgressive, and $y_\lambda = i^* \bar{y}_\lambda$; there is a subring B^* containing $\tau(x_y)$ such that $H^*(B; \mathbf{k}) = B^* \otimes \Delta(\tau(y_\nu))$, and $\{\tau(x_y) \mid \tau(x_y) \neq 0 \text{ for } |x_y| < N\}$ is a regular sequence.

Then the conclusion of Theorem 2.24 holds, with the following modifications:

- (1) $\text{Im } i^* = \Delta(x_\delta) \otimes \mathbf{k}[y_\lambda]$,
- (2) $H^*(E; \mathbf{k}) = \text{Im } \pi^* \otimes \Delta(\bar{x}_\delta) \otimes \mathbf{k}[\bar{y}_\lambda]$ for degrees $\leq N$,
- (3) $\text{Im } \pi^* \cong B^*/(\tau(x_y))$ for degrees $\leq N+1$.

The diagonal map and the differential in the spectral sequence. Let us study the relation between the diagonal map

$$\phi = \mu^*: H^*(F; \mathbf{k}) \rightarrow H^*(F; \mathbf{k}) \otimes H^*(F; \mathbf{k})$$

in the Hopf algebra $H^*(F; \mathbf{k})$ and the differential

$$d_r: E_r^{0,q} (\subset E_2^{0,q} \cong H^q(F; \mathbf{k})) \rightarrow E_r^{r, q-r+1}$$

in the spectral sequence $\{E_r\}$, with coefficients in a field $\mathbf{k}$, of a principal fibration $F \rightarrow E \rightarrow B$, where B is 1-connected, F is 0-connected and of finite type:

$$E_2 = H^*(B; \mathbf{k}) \otimes H^*(F; \mathbf{k}) \Rightarrow H^*(E; \mathbf{k}).$$

When we express

$$\bar{\phi}(x) = \sum_i x_i \otimes x'_i,$$

where $\bar{\phi}(x) = \phi(x) - x \otimes 1 - 1 \otimes x$ for $|x| > 0$, we take $\{x'_i\}$ so that they form (a part of) the basis of $H^*(F; \mathbf{k})$. Then the expression in $\bar{\phi}(x)$ is unique.

THEOREM 2.26. *If we suppose $1 \otimes x \in E_{r+1}^{0,q}$, for $x \in H^q(F; \mathbf{k})$, then the following holds for $\bar{\phi}(x) = \sum_i x_i \otimes x'_i$:*

(1) $1 \otimes x_i \in E_{r+1}^{0,*}$. Hence $x_i \in \text{Im}(i^*: H^*(E; \mathbf{k}) \rightarrow H^*(F; \mathbf{k}))$, if $|x_i| < r$; x_i is transgressive if $|x_i| = r$.

(2) If we put $d_{r+1}(1 \otimes x) = \sum_i y_i \otimes x'_i$ where $y_i \in H^{r+1}(B; \mathbf{k})$ and $|x'_i| = q - r$, then y_i is the transgressive image $\tau(x_i) = y_i$ of x_i .

PROOF. (1) By 2.20 $\bar{\mu}_{r+1}^*(1 \otimes x) = 1 \otimes \mu^*(x) = 1 \otimes \phi(x) \in E_{r+1}^{0,*} \otimes H^*(F; \mathbf{k})$. We have $1 \otimes x_i \in E_{r+1}^{0,*}$, since $\{x'_i\}$ are linearly independent.

(2) By 2.20 $\bar{\mu}_{r+1}^* d_{r+1}(1 \otimes x) = (d_{r+1} \otimes 1)(1 \otimes \phi(x)) = d_{r+1}(1 \otimes x) \otimes 1 + \sum d_{r+1}(1 \otimes x_i) \otimes x'_i$ and $\bar{\mu}_{r+1}^* d_{r+1}(1 \otimes x) = \bar{\mu}_{r+1}^* \sum y_i \otimes x'_i = \sum y_i \otimes \phi(x'_i)$. Comparing the terms of the form $1 \otimes 1 \otimes g$, one obtains (2). Q.E.D.

REMARK 2.26'. y_i in Theorem 2.26 is determined not by x_i but modulo the kernel of the transgression. For example, to show $y_i = 0$ from $x_i = 0$, it suffices to observe that $p^*: H^{r+1}(B; \mathbf{k}) \rightarrow H^{r+1}(E, F; \mathbf{k})$ is a monomorphism (i.e., $E_2^{r+1,0} \cong E_{r+1}^{r+1,0}$).

The following is obtained similarly to Lemma 2.16 of Chapter IV.

2.27. If $x \in H^q(F; \mathbf{k})$ is transgressive and $H^q(E; \mathbf{k}) = H^{q+1}(E; \mathbf{k}) = 0$, then x is primitive (i.e., $\bar{\phi}(x) = 0$). In particular, if x is universally transgressive, it is primitive.

THEOREM 2.28. Let X be a 1-connected topological space of finite type. Suppose that

$$H^*(\Omega X; \mathbf{k}) = \Delta(x_\alpha) \text{ with } |x_\alpha| \leq N \text{ for degrees } \leq N,$$

where $\mathbf{k}$ is a field and put $M = \text{Min}(|x_\alpha|)$.

- (1) If $|x_\alpha| \leq 3M$ and x_α is primitive, then x_α is (universally) transgressive.
- (2) We make the following assumptions:
 - (a) Except for one γ , x_β is transgressive for $\beta \neq \gamma$ and $y_\beta = \tau(x_\beta)$;
 - (b) In $\bar{\phi}(x_\gamma) = \sum x'_i \otimes x''_i \neq 0$ each x'_i and x''_i is transgressive with $|x'_i| \geq |x''_i|$;
 - (c) For nonzero $f \in \mathbf{k}[y_\beta]$ and $g \in \Delta(x_\alpha)$ satisfying $|f| + |g| = |x_\gamma| + 1$, f is not decomposable if $0 < |f| < \text{Min}|x'_i|$.

Then there is an element r_γ of $\mathbf{k}[y_\beta]$ with $|r_\gamma| = |x_\gamma| + 2$ such that the following holds:

$$H^*(X; \mathbf{k}) = \mathbf{k}[y_\beta]/(r_\gamma) \text{ for degrees } \leq N + 1,$$

where $r_\gamma \equiv \sum \tau(x'_i) \cdot \tau(x''_i)$ modulo the ideal $(y_\alpha; |x_\alpha| < \text{Max}|x'_i|)$.

PROOF. (1) $H^*(\Omega X; \mathbf{k}) = 0$ for $0 < i < M$ implies $H^*(X; \mathbf{k}) = 0$ for $0 < i \leq M$. Thus $E_r^{p,q} = 0$ for $0 < p \leq M$ or $0 < q < M$. It follows easily that $E_2^{r+1,0} \cong E_{r+1}^{r+1,0}$ for $r \leq 2M$. By Theorem 2.26 (Remark 2.26'), $\bar{\phi}(x_\alpha) = 0$ implies $d_{r+1}(1 \otimes x_\alpha) = 0$ for $r \leq 2M$. Also, $2M < r < |x_\alpha|$ implies $0 < q = |x_\alpha| < M$, which implies $d_{r+1}(1 \otimes x_\alpha) \in E_{r+1}^{r+1,q} = 0$ and hence (1) is proved.

(2) Put $q = |x_\gamma|$ and $s = \text{Min}|x'_i|$ and then $s + 1 \geq q - s + 1$. By Theorem 2.7 we have $H^*(X; \mathbf{k}) = \mathbf{k}[y_\beta]$ for degrees $\leq q$, and $\tau(x_\beta)$ is determined modulo decomposables in $\mathbf{k}[y_\beta]$. Applying Theorem 2.26 (2) to $x = x_\gamma$ for $r < s$ and putting $d_{r+1}(1 \otimes x_\gamma) = \sum f \otimes g$, we have $f = \tau(0)$ modulo 0 by the assumption of (2) and so $f = 0$, that is, $d_{r+1}(1 \otimes x_\gamma) = 0$ for $r < s$. When $r = s$, $d_{s+1}(1 \otimes x_\gamma) = \sum_i \tau(x'_i) \otimes x''_i$ and hence $d_{q-s+1}(\sum_i \tau(x'_i) \otimes x''_i) = 0$. This shows that the relation $r_\gamma = 0$ satisfying $r_\gamma \equiv \sum \tau(x'_i) \cdot \tau(x''_i)$ also holds in $H^*(X; \mathbf{k})$. Using the transgressiveness of x_β and the differential formula for this x_γ , we construct a formal spectral sequence and then we can prove (2) by virtue of the comparison theorem. Q.E.D.

REMARK 2.29. By constructing the projective plane associated with ΩX , one can show that the concrete form of r_γ is given by

$$r_\gamma = \sum (-1)^{|x'_i|} \tau(x'_i) \cdot \tau(x''_i) \text{ if } \bar{\phi}(x_\gamma) = \sum x'_i \otimes x''_i \text{ for } |x_\gamma| \leq 3M.$$

§3. The action of the Weyl group and the classifying space

The Euler characteristic. The Euler characteristic is defined by taking the value of the variable to be -1 in the Poincaré polynomial. That is, the Poincaré series (polynomial) of a positive, graded module of finite type $C = \sum_{n \geq 0} C_n$ over $\mathbf{k}$ is given by

$$P(C, t) = P_{\mathbf{k}}(C, t) = \sum_{n \geq 0} \dim_{\mathbf{k}} C_n t^n$$

and the Euler characteristic of it is given by

$$\chi(C) = \sum_{n \geq 0} (-1)^n \dim_{\mathbf{k}} C_n = P_{\mathbf{k}}(C, -1)$$

if C is of finite dimension. Obviously,

$$P(C \oplus C', t) = P(C, t) + P(C', t), \quad \chi(C \oplus C') = \chi(C) + \chi(C'), \quad (3.1)$$

$$P(C \otimes_{\mathbf{k}} C') = P(C, t) \cdot P(C', t), \quad \chi(C \otimes_{\mathbf{k}} C') = \chi(C) \cdot \chi(C').$$

If C is a (co)chain complex, we have

$$\chi(H(C)) = \chi(C), \quad (3.2)$$

where $H(C)$ is the (co)homology of C . (3.2) is obtained by putting $t = -1$ in the following

3.2'. There is $B(t) = \sum_{n \geq 0} b_n t^n$ with $b_n \geq 0$ such that $P(C, t) = P(H(C), t) + (1+t)B(t)$, that is,

$$P(C, t) \succ P(H(C), t).$$

3.2' is obtained by putting $b_n = \dim_{\mathbf{k}} \partial C_{n+1}$ if C is a chain complex or $b_n = \dim_{\mathbf{k}} \delta C_{n-1}$ if C is a cochain complex (cf. Lemma 1.20 of Chapter VI).

The Euler characteristic $\chi(X)$ of a topological space X is defined to be that of the (co)homology group of X , if $H_*(X)$ is finitely generated. Here, using the universal coefficient theorem, one has the following remark.

3.3. $\chi(X) = \chi(H_*(X; \mathbf{k})) = \chi(H^*(X; \mathbf{k}))$ is independent of the choice of a field $\mathbf{k}$.

For a finite CW -complex K , it follows from (3.2) and (3.3) that

$$\chi(K) = \sum_n (-1)^n a_n, \quad (3.3')$$

where $a_n =$ (the number of n -cells in K).

THEOREM 3.4. Let $H_*(X)$, $H_*(B)$, and $H_*(E)$ be finitely generated in a fibration $F \rightarrow E \rightarrow B$. If the local coefficient ring $H^*(F; \mathbf{k})$ is trivial for some field $\mathbf{k}$ or B is (weakly) homotopy equivalent to a CW -complex, then the following equality holds:

$$\chi(E) = \chi(B) \cdot \chi(F).$$

PROOF. In the case where the local coefficient ring is trivial, we have the spectral sequence $\{E_r\}$:

$$E_2 = H^*(B; \mathbf{k}) \otimes_{\mathbf{k}} H^*(F; \mathbf{k}) \Rightarrow H^*(E; \mathbf{k}).$$

Putting $\chi(E_r) = \sum (-1)^{p+1} \dim_{\mathbf{k}} E_r^{p,q}$, we have $\chi(E_2) = \chi(B) \cdot \chi(F)$ by (3.1). Also we have $\chi(E_r) = \chi(E_{r+1})$ by (3.2), and so $\chi(E_\infty) = \chi(B) \cdot \chi(F)$. Since E_∞ and $H^*(E; \mathbf{k})$ are isomorphic as graded modules over $\mathbf{k}$, we have $\chi(E_\infty) = \chi(E)$. Thus we obtain the equality of the theorem in this case. In the latter case we may suppose that B is a CW -complex. Then we have the spectral sequence

$$E_1 = C^*(B; H^*(F; \mathbf{k})) \Rightarrow H^*(E; \mathbf{k})$$

and hence the theorem is proved similarly. Q.E.D.

LEMMA 3.5. Let $H_*(X)$ be finitely generated and

$$f: H^*(X; \mathbf{k}) \otimes_{\mathbf{k}} H^*(X; \mathbf{k}) \rightarrow \mathbf{k}$$

be a homomorphism having the following property: $f(y \otimes x) = (-1)^{|x| \cdot |y|} \cdot f(x \otimes y)$ for homogeneous elements x, y and $f(x \otimes y) = 0$ if $|x| + |y|$ is odd. We take a basis $\{x_i\}$ of $H^*(X; \mathbf{k})$ consisting of homogeneous elements of $H^*(X; \mathbf{k})$. If the square matrix $B = (f(x_i \otimes y_j))$ is regular, then $B^{-1} \cdot {}^t B = ((-1)^{|x_i|} \delta_{ij})$, and we have

$$\chi(X) = \text{tr}(B^{-1} \cdot {}^t B)$$

regarding it as a matrix of integer elements.

PROOF. Since $\text{tr}(B^{-1} \cdot {}^t B)$ is independent of the basis, we prove the lemma arranging the basis $\{x_1, \dots, x_{r+s}\}$ so that $x_1, \dots, x_r$ are of even degrees and $x_{r+1}, \dots, x_{r+s}$ are of odd degrees. By the property of f , $b_{ij} = f(x_i \otimes y_j)$ satisfies $b_{ij} = b_{ji}$ for $i \leq r$ and $j \leq r$; $b_{ij} = -b_{ji}$ for $i > r$ and $j > r$; $b_{ij} = 0$ for $i \leq r < j$ or $j \leq r < i$. Thus B is decomposed into a direct sum as follows:

$$B = B_1 \oplus B_2, \quad {}^t B_1 = B_1, \quad {}^t B_2 = -B_2.$$

Therefore $B^{-1} \cdot {}^t B = (B_1^{-1} \cdot {}^t B_1) \oplus (B_2^{-1} \cdot {}^t B_2) = I_r \oplus (-I_s)$. Also we have

$$\chi(X) = \sum \dim_{\mathbf{k}} H^{2k}(X; \mathbf{k}) - \sum \dim_{\mathbf{k}} H^{2k-1}(X; \mathbf{k}) = r - s = \text{tr}(B^{-1} \cdot {}^t B).$$

Q.E.D.

REMARK 3.5'. Lemma 3.5 holds even when replacing $\mathbf{k}$ with $\mathbf{Z}$. In the case where $\mathbf{k} = \mathbf{Q}$, $\text{tr}(B^{-1} \cdot {}^t B) \in \mathbf{Z} \subset \mathbf{Q}$, which is equal to $\chi(X)$.

The Euler class of a manifold and the Poincaré duality theorem. Let M^n be an n -dimensional, compact, connected, differentiable manifold and denote the tangent bundle by

$$\tau_M = (E, p, M^n, \mathbf{R}^n, G), \quad G \subset GL(n, \mathbf{R}).$$

Here we suppose that M^n is orientable and M has a Riemannian metric, and hence we may assume $G \subset SO(n)$.

Let $E_0 = E - s_0(M^n)$ be the complement of the image of the zero-section $s_0: M^n \rightarrow E$ (i.e., $s_0(x) = 0_x \in \mathbf{R}_x^n$). Then by Theorem 3.1 of Chapter III there exists a Thom class $U \in H^n(E, E_0; R)$, where $R = \mathbf{Z}$ or $\mathbf{k}$ and U corresponds through the natural morphism $\mathbf{Z} \rightarrow R$, so that we have the Thom isomorphism

$$\Phi: H^i(M^n; R) \cong H^{i+n}(E, E_0; R) \quad \text{given by} \quad \Phi(x) = p^*(x)U.$$

Here the Thom class U is characterized by the fact that $i_x^*(U)$ is a generator of $H^n(\mathbf{R}_x^n, \mathbf{R}_x^n - 0_x; R) \cong R$, where $i_x: (\mathbf{R}_x^n, \mathbf{R}_x^n - 0_x) \rightarrow (E, E_0)$ is the inclusion.

The Euler class $\chi(\tau_M)$ of τ_M is given by

$$\tau_M = s^* j^*(U) \quad \text{for } H^n(E, E_0; R) \xrightarrow{j^*} H^n(E; R) \xrightarrow{s^*} H^n(M^n; R), \quad (3.6)$$

where $j: (E, \emptyset) \rightarrow (E, E_0)$ is the inclusion and $s: M^n \rightarrow E$ is an arbitrary cross section. Observe that s is a homotopy equivalence and the projection p is its inverse ($s \circ p = 1_E$). Therefore we have $\Phi\chi(\tau_M) = p^* s^* j^*(U)U = j^*(U)U = U^2$; namely,

$$\chi(\tau_M) = \Phi^{-1}(U^2). \quad (3.6')$$

NOTATION. $E(\varepsilon) = \{v \in E \mid |v| \leq \varepsilon\}$ and $E_0(\varepsilon) = E(\varepsilon) \cap E_0$ for $\varepsilon > 0$.

Let $v \in E(\varepsilon)$. Consider a geodesic segment $\exp_x(tv)$ for $0 \leq t \leq 1$ with $x = p(v)$ the initial point and v the initial vector. We define a map

$$g: E(\varepsilon) \rightarrow M^n \times M^n \quad \text{by} \quad g(v) = (x, \exp_x(v)), \quad \text{where } x = p(v).$$

g gives a homeomorphism: $E(\varepsilon) \cong D \subset M^n \times M^n$ for sufficiently small ε , and D is a neighborhood of the diagonal set $\Delta = \{(x, x) \mid x \in M^n\} \subset M^n \times M^n$. Consider the inclusions

$$j_\varepsilon: (E(\varepsilon), E_0(\varepsilon)) \rightarrow (E, E_0)$$

and

$$j_D: (D, D - \Delta) \rightarrow (M^n \times M^n, M^n \times M^n - \Delta).$$

Define an isomorphism ψ by the commutativity of the diagram:

$$\begin{array}{ccc} H^*(E, E_0; R) & \xrightarrow{\psi} & H^*(M^n \times M^n, M^n \times M^n - \Delta; R) \\ \cong \downarrow j_\varepsilon^* & & \cong \downarrow j_D^* \\ H^*(E(\varepsilon), E_0(\varepsilon); R) & \xleftarrow[\cong]{s^*} & H^*(D, D - \Delta; R). \end{array} \quad (3.7)$$

Consider the homomorphisms k^* and k_x^* induced by the inclusions $k: (M^n \times M^n, \emptyset) \rightarrow (M^n \times M^n, M^n \times M^n - \Delta)$ and $k_x: M^n \rightarrow M^n \times M^n$ (where $k_x(y) = (x, y)$).

NOTATION. $\bar{U} = k^* \psi(U) \in H^*(M^n \times M^n; R)$, $\mu = k_x^* \bar{U} \in H^n(M^n; R)$.

LEMMA 3.8. We have $H^n(M^n; R) \cong R$ and μ is its generator (the fundamental class of M^n).

PROOF. The first half is well known and the isomorphism is given by

$$H^n(M^n; R) \xleftarrow{\cong} H^n(M^n, M^n - x; R) \xrightarrow{\cong} H^n(V_x, V_x - x; R) \cong R,$$

where V_x is a neighborhood of x , which is given by $k_x(V_x) = D \cap k_x(M^n)$ and is homeomorphic to $\mathbf{R}^n$. The second isomorphism is the excision isomorphism.

By restricting the left-hand and right-hand sides of (3.7) to $\mathbf{R}_x^n$ and $k_x(M^n) = x \times M^n$ respectively, we see that μ is the image of U through the following correspondence (the coefficient ring R is omitted):

$$\begin{aligned} H^n(E, E_0) &\xrightarrow{i_x^*} H^n(\mathbf{R}_x^n, \mathbf{R}_x^n - 0_x) \xrightarrow{\cong} H^n(\mathbf{R}_x^n \cap E(e), \mathbf{R}_x^n \cap E_0(e)) \\ &\xleftarrow[(\cong)]{(k_x \circ g)^*} H^n(V_x, V_x - x) \xrightarrow{\cong} H^n(M^n, M^n - x) \xrightarrow{\cong} H^n(M^n). \end{aligned}$$

We obtain the latter half of the lemma, since $i_x^*(U)$ is a generator of $H^n(\mathbf{R}_x^n, \mathbf{R}_x^n - 0_x)$. Q.E.D.

LEMMA 3.9. Let $\{x_1 = 1, x_2, \dots, x_m = \mu\}$ be a basis of $H^*(M^n; \mathbf{k})$ over $\mathbf{k}$ consisting of the homogeneous elements. We define $a_{ij}, b_{ij} \in \mathbf{k}$ by the equations

$$\bar{U} = \sum a_{ij}(x_i \times x_j) \quad \text{and} \quad x_i x_j = b_{ij} \mu$$

when $|x_i| + |x_j| = n$, and by $a_{ij} = b_{ij} = 0$ when $|x_i| + |x_j| \neq n$. Then the following holds:

$$\sum_{j=1}^m a_{ij} b_{jk} = (-1)^{n|x_k|} \delta_{ik}.$$

In particular $A = (a_{ij})$ is the inverse matrix of $B = (b_{ij})$ if n is even.

PROOF. First from $x_m = \mu = k_x^* \bar{U}$ we see that

$$a_{1m} = 1, \quad a_{im} = 0 \quad \text{for } i > 1, \quad (*)$$

since $k_x^*(1 \times x_j) = x_j$ and $k_x^*(x_i \times x_j) = 0$ for $|x_i| > 0$. Next we show

$$\bar{U}(x_k \times 1) = \bar{U}(1 \times x_k). \quad (**)$$

We have $p_1^*(x_k) = x_k \times 1$ and $p_2^*(x_k) = 1 \times x_k$ for the projections $p_1, p_2: M^n \times M^n \rightarrow M^n$ (where $p_1(x, y) = x$, $p_2(x, y) = y$). Δ is a deformation retract of D and $p_1|_{\Delta} = p_2|_{\Delta}$ and hence $p_1|_D \simeq p_2|_D$. Thus

$$j_D^* \circ p_1^* = j_D^* \circ p_2^*: H^*(M^n; \mathbf{k}) \rightarrow H^*(M^n \times M^n; \mathbf{k}) \rightarrow H^*(D; \mathbf{k}).$$

So we have $j_D^*(\psi(U) \cdot (x_k \times 1)) = j_D^*(\psi(U)) \cdot j_D^* p_1^*(x_k) = j_D^*(\psi(U)) \cdot j_D^* p_2^*(x_k) = j_D^*(\psi(U) \cdot (1 \times x_k))$, where we have

$$\psi(U) \cdot (x_k \times 1) = \psi(U) \cdot (1 \times x_k) \in H^*(M^n \times M^n, M^n \times M^n - \Delta; \mathbf{k})$$

since j_D^* in (3.7) is the excision isomorphism.

By applying k^* to this equality one obtains (**). Rewriting (**) one has

$$\sum_{i,j} (-1)^{|x_j| \cdot |x_k|} a_{ij}(x_i x_k \times x_j) = \sum_{i,j} a_{ij}(x_i \times x_j x_k).$$

Comparing the parts belonging to the direct summand $H^*(M^n; \mathbf{k}) \otimes H^*(M^n; \mathbf{k})$ in the Künneth isomorphism $\times: H^*(M^n; \mathbf{k}) \otimes H^*(M^n; \mathbf{k}) \cong H^*(M^n \times M^n; \mathbf{k})$, we have

$$(-1)^{n|x_k|}(x_k \times \mu) = \sum_{i,j} a_{ij}(x_i \times x_j x_k) = \sum_i \left(\sum_j a_{ij} b_{jk} \right) (x_i \times \mu)$$

since there remain by (*) only the cases when $x_j = x_m = \mu$ and $x_i = x_1 = 1$ in the left-hand side.

We have obtained the equality of the lemma, since $\{x_j \times \mu\}$ are linearly independent. Q.E.D.

THEOREM 3.10 (The Poincaré duality theorem). *Let M^n be a compact, connected, orientable differential manifold. Then the isomorphism*

$$\varphi: H^i(M^n; \mathbf{k}) \cong \text{Hom}_{\mathbf{k}}(H^{n-i}(M^n; \mathbf{k}), \mathbf{k})$$

can be defined by $(\varphi(x)(y)) \cdot \mu = xy$ for any field $\mathbf{k}$. Thus as vector spaces over $\mathbf{k}$, $H^i(M^n; \mathbf{k})$, $H_i(M^n; \mathbf{k})$, $H^{n-i}(M^n; \mathbf{k})$, and $H_{n-i}(M^n; \mathbf{k})$ are isomorphic to one another.

PROOF. Let the dual basis of $\{x_i\}$ be

$$\{x_i^* \in \text{Hom}(H^*(M^n; \mathbf{k}), \mathbf{k}) \mid x_i^*(x_j) = \delta_{ij}\},$$

and then $\varphi(x_i) = \sum_j b_{ij} x_j^*$. Since Lemma 3.9 shows that the matrix (b_{ij}) is regular, φ defines an isomorphism $H^*(M^n; \mathbf{k}) \cong \text{Hom}(H^*(M^n; \mathbf{k}), \mathbf{k})$. Considering it in each degree, we obtain the theorem, since $b_{ij} \neq 0$ implies $|x_i| + |x_j| = n$. Q.E.D.

REMARK 3.10'. Theorem 3.10 holds not only for a differentiable manifold but also in a topological manifold. It holds also in the nonorientable case, if $\mathbf{k}$ is of characteristic 2.

COROLLARY 3.11. $\chi(M^n) = 0$ if n is odd.

The reason why we use the name "Euler" both for the Euler characteristic and the Euler class is due to the following theorem.

THEOREM 3.12. *Let M^n be a compact, orientable, connected manifold and τ_M be its tangent bundle. Then*

$$\chi(M^n) \cdot \mu = \chi(\tau_M) \in H^n(M^n) \cong \mathbf{Z},$$

where the fundamental class μ and the Euler class $\chi(\tau_M)$ are defined from the same Thom class U .

PROOF. Since $U^2 \in H^{2n}(E, E_0) \cong H^n(M^n) \cong \mathbf{Z}$, we have $U^2 = -U^2$ so $U^2 = 0$ if n is odd, and hence $\chi(\tau_M) = \phi^{-1}(U^2) = 0$ by (3.6'). Thus the theorem is true for n odd by Corollary 3.11.

Let n be even. Extending the coefficient ring $\mathbf{Z}$ to $\mathbf{Q}$, the morphism $H^n(M^n) = H^n(M^n; \mathbf{Z}) \rightarrow H^n(M^n; \mathbf{Q}) \cong \mathbf{Q}$ is injective, and hence it suffices to show the theorem with $\mathbf{Q}$ -coefficients. Consider the diagram:

$$\begin{array}{ccccc} H^n(E; \mathbf{Q}) & \xleftarrow[p^*]{\cong} & H^n(M^n; \mathbf{Q}) & \xleftarrow{d^*} & H^n(M^n \times M^n; \mathbf{Q}) \\ \cong \downarrow j_* & & & & \downarrow j_D^* \\ H^n(E(e); \mathbf{Q}) & \xleftarrow[g^*]{\cong} & & & H^n(D; \mathbf{Q}) \end{array}$$

where $d: M^n \rightarrow M^n \times M^n$ is the diagonal map. All the morphisms in the diagram except those in the upper sequence are obtained from the corresponding ones in (3.7) by restrictions. By the definitions of ψ and $\bar{U}$ see that

$$j_* j^*(U) = g^* j_D^*(\bar{U})$$

for $j^*: H^n(E, E_0; \mathbf{Q}) \rightarrow H^n(E; \mathbf{Q})$. The above diagram is commutative, since $s_0(M^n)$ is a deformation retract of $E(e)$ and g satisfies $g|_{s_0(M^n)} = d \circ p$. $s \circ p = 1_E$ implies $p^* = s^{*-1}$ and so by using (3.6), the above facts, and the relations in Lemma 3.9 we have

$$\begin{aligned} \chi(\tau_M) &= s^* j^*(U) = p^{*-1} j^*(U) = p^{*-1} j_e^{*-1} g^* j_D^*(\bar{U}) = d^*(\bar{U}) \\ &= d^* \sum a_{ij}(x_i \times x_j) = \sum a_{ij} x_i x_j = \sum a_{ij} \cdot b_{ij} \mu = \text{tr}(A \cdot {}^t B) \cdot \mu. \end{aligned}$$

Here we have $A = B^{-1}$ by Lemma 3.9. The map f determined by $f(x_i \otimes x_j) = b_{ij}$ satisfies the property in Lemma 3.5 for $X = M^n$ and $\mathbf{k} = \mathbf{Q}$, and hence by Lemma 3.5

$$\chi(M^n) \cdot \mu = \text{tr}(B^{-1} \cdot {}^t B) \cdot \mu = \text{tr}(A \cdot {}^t B) \cdot \mu = \chi(\tau_M). \quad \text{Q.E.D.}$$

The order of the Weyl group and the p -torsion

THEOREM 3.13. Let G be a compact, connected Lie group, T a maximal torus of G and U a connected, closed subgroup of maximal rank of G . Then

- (1) the Euler characteristic $\chi(G/T) = |W(G)|$ (the order of the Weyl group),
- (2) the Euler characteristic $\chi(G/U) = [W(G): W(U)]$ (the index).

PROOF. (1) This is Theorem 3.14 of Chapter V, as $W(G) = N(T)/T$.

(2) Without loss of generality one may suppose $G \supset U \supset T$. Using

Theorem 3.4 for the fibre bundle $U/T \rightarrow G/T \rightarrow G/U$, we have $\chi(G/T) = \chi(G/U)\chi(U/T)$. By (1) we have $\chi(G/U) = \chi(G/T)/\chi(U/T) = |W(G)|/|W(U)| = [W(G):W(U)]$. Q.E.D.

Consider a fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$.

LEMMA 3.14. *Let F and B be 0-connected and k a field. Suppose that an integer q satisfies the following conditions:*

$H^j(F; k) = 0$ for any $j > q$, $i^*H^q(E; k) \neq 0$ and the local coefficient $H^q(F; k)$ is trivial.

Then $\pi^*: H^*(B; k) \rightarrow H^*(E; k)$ is a monomorphism.

PROOF. By the assumption we have

$$E_2^{i,q} = H^i(B; k) \otimes H^q(F; k) \quad \text{and} \quad E_2^{i,j} = 0 \quad \text{for } j > q$$

and that there is $1 \otimes x \neq 0$ belonging to $i^*H^q(E; k) = D^{0,q} = E_\infty^{0,q} \subset E_2^{0,q} = 1 \otimes H^q(F; k)$ such that $d_r(1 \otimes x) = 0$ for $r \geq 2$ in the spectral sequence $\{E_r\}$ of the fibration $F \rightarrow E \rightarrow B$. Now, suppose there is a nonzero $b \in H^i(B; k)$ for which $\pi^*(b) = 0$. Since π^* is equivalent to $E_2^{i,0} \rightarrow E_\infty^{i,0} = D^{i,0} \subset H^i(E; k)$, there exist $r \geq 2$ and $y \in E_2^{i-r, r-1}$ such that $d_r(y) = b \otimes 1 \neq 0$ and hence $d_r(y \cdot (1 \otimes x)) = b \otimes x$.

This shows that $b \otimes x \neq 0$ ($\in E_2^{i,q}$) represents a class in $E_r^{i,q}$, which is 0 in $E_{i+1}^{i,q}$. Therefore $b \otimes x \neq 0$ belongs to $d_s E_s^{i-s, q+s-1}$ for some $s \geq 2$. On the other hand, $q+s-1 > q$ implies $E_s^{i-s, q+s-1} = 0$, which is a contradiction. Thus there is no b as above, and π^* is a monomorphism. Q.E.D.

THEOREM 3.15. *Let k be a field of characteristic p . Suppose that F is a compact, connected, orientable, differentiable manifold, that the action of G on F is a C^∞ -map preserving the orientation and that E has the homotopy type of a connected CW-complex in a fibre bundle (E, π, B, F, G) . Then, if $\chi(F)$ is prime to p ($\chi(F) \neq 0$ when $p = 0$), $\pi^*: H^*(B; k) \rightarrow H^*(E; k)$ is a monomorphism.*

PROOF. Let $q = \dim F$. Using coordinate functions of the given orientable bundle, one can construct a q -dimensional real vector bundle over E whose total space is the union $E^* = \bigcup_x T(F_x)$ of tangent bundles $T(F_x)$ of the fibres F_x over $x \in B$. By the classifying theorem (Theorem 6.10 of Chapter II) this vector bundle is induced by some map $f: E \rightarrow BSO(q)$. Therefore the tangent bundle τ_F of one fibre $F (= F_{x_0})$ is induced by $f \circ i: F \hookrightarrow E \rightarrow BSO(q)$, where $i: F \rightarrow E$ is the inclusion. Then the Euler class $\chi(\tau_F)$ of τ_F is the image of the Euler class $\chi \in H^q(BSO(q); k)$ of the universal bundle

$$\chi(\tau_F) = (f \circ i)^* \chi = i^* f^* \chi \in H^q(F; k).$$

By the assumption and Theorem 3.12, $\chi(\tau_F) = \chi(F) \cdot \mu \neq 0$ and hence

$i^*H^q(E; \mathbf{k}) \neq 0$ and $i^*: H^q(E; \mathbf{k}) \rightarrow H^q(F; \mathbf{k})$ is an epimorphism. Therefore the local coefficient system $H^q(F; \mathbf{k})$ is trivial (Theorem 4.4 of Chapter III). Obviously $H^j(F; \mathbf{k}) = 0$ for $j > q$. Thus by using Lemma 3.14 we see that $\pi^*: H^*(B; \mathbf{k}) \rightarrow H^*(E; \mathbf{k})$ is a monomorphism. Q.E.D.

Let G be a compact, connected Lie group and T be its maximal torus. Since the centralizer $Z(T)$ of T coincides with T , we obtain by Theorem 4.20 (or Theorem 5.5(2)) of Chapter VI:

3.16. $H^*(G/T)$ is torsion free and $H^{\text{odd}}(G/T) = 0$.

THEOREM 3.17. *If the order of the Weyl group $W(G)$ is prime to p a prime, then $H_*(BG)$ and hence $H^*(BG)$ is p -torsion free.*

PROOF. Consider the fibre bundle $(BT, \rho(T, G), BG, G/T)$, where $\rho(T, G): BT \rightarrow BG$ is the projection of the fibre bundle induced by the inclusion $T \subset G$. Since G/T is orientable by 3.16 and since $(\chi(G/T), p) = 1$ by the assumption and Theorem 3.13(1), we can apply Theorem 3.15 so that $\rho(T, G)^*: H^*(BG; \mathbf{Z}_p) \rightarrow H^*(BT; \mathbf{Z}_p)$ is a monomorphism. Observe that $H^{\text{odd}}(BT; \mathbf{Z}_p) = 0$ implies $H^{\text{odd}}(BG; \mathbf{Z}_p) = 0$. By the universal coefficient theorem, $H_*(BG)$ is p -torsion free. Q.E.D.

The characteristic ring. Let $\xi = (E, p, B, G)$ be an arbitrary G -bundle. By the classifying theorem (Theorem 6.10 of Chapter II) there exists a map $f_\xi: B \rightarrow BG$ of B to the classifying space BG , unique up to homotopy, such that ξ is induced by f_ξ . Thus the homomorphism

$$f_\xi^*: H^*(BG; R) \rightarrow H^*(B; R), \quad \text{where } R \text{ is the coefficient ring,}$$

is uniquely determined by the bundle structure of ξ .

DEFINITION. f_ξ^* is called the *characteristic map* and $\text{Im } f_\xi^*$ is the *characteristic ring* of ξ . We write

$$\Gamma(\xi) = \text{Im } f_\xi^* \quad (\subset H^*(B; R)).$$

Suppose that a Lie group G acts on E and F in a fibre bundle $\xi = (E, \pi, B, F, H)$ and that by the action we have the following G -bundles:

$$\zeta = (E, p, E/G, G) \quad \text{and} \quad \eta = (F, p, F/G, G).$$

Further, suppose that the action of G on E preserves each fibre F_b for $b \in B$ and the action of G commutes with admissible maps $F \rightarrow F_b$. For example, this property is satisfied, if ξ is a principal bundle ($F = H$) and G is a closed subgroup of H .

Then a new fibre bundle $\bar{\xi} = (E/G, \bar{\pi}, B, F/G)$ can be constructed. The G -bundle induced by the map

$$f_\eta = f_\zeta \circ i: F/G \rightarrow E/G \rightarrow BG$$

is equivalent to η , where $f_\zeta: E/G \rightarrow BG$ is a map inducing ζ and $i: F/G \rightarrow E/G$ is the inclusion determined by an admissible map. Therefore we have

$$\Gamma(\eta) \subset \text{Im}\{i^*: H^*(E/G; R) \rightarrow H^*(F/G; R)\}. \quad (3.18)$$

By Theorem 4.4 of Chapter III we have

3.19. If R is a field and $H^*(F/G; R) = \Gamma(\eta)$, then F/G is totally nonhomologous to zero in the bundle $\bar{\xi} = (E/G, \pi, B, F/G)$.

THEOREM 3.20. Let G be a compact, 1-connected Lie group, T its maximal torus, $\zeta = (G, p, G/T, T)$, and $\mathbf{k}$ a field of characteristic p . Then the following conditions are equivalent.

- (1) $H^*(BG)$ is p -torsion free (no condition when $p = 0$).
- (2) $H^*(G/T; \mathbf{k}) = \Gamma(\zeta)$.
- (3) $H^*(G/T; \mathbf{k})$ is (multiplicatively) generated by $H^2(G/T; \mathbf{k})$.
- (4) $H^*(BG; \mathbf{k})$ consists only of the elements of even degree ($H^{\text{odd}}(BG; \mathbf{k}) = 0$).

PROOF. To show that (1) implies (2), note that since the Poincaré series of $H^*(BG; \mathbf{k})$ and $H^*(BG; \mathbf{Q})$ are equal by 1.20(7), $H^{\text{odd}}(BG; \mathbf{Q}) = 0$ implies $H^{\text{odd}}(BG; \mathbf{k}) = 0$. Also we have $H^{\text{odd}}(G/T; \mathbf{k}) = 0$ by 3.16. Therefore the E_2 -term of the spectral sequence associated with the fibre bundle $(BT, \rho(T, G), BG, G/T)$ is concentrated in even degrees and hence $E_2 = E_\infty$, which implies that $\rho(T, G)^*$ is an epimorphism and so we have (2).

To show that (2) implies (3), note that since $H^*(BT; \mathbf{k})$ and hence $\Gamma(\zeta) = H^*(G/T; \mathbf{k})$ is generated by elements of degree 2, we obtain (3).

To show that (3) implies (4), note that since G is 1-connected, G is 2-connected (Theorem 4.12 of Chapter VI) and hence BG is 3-connected. Since BT and G/T are 1-connected, the exact sequence

$$H^2(BT; \mathbf{k}) \xrightarrow{i^*} H^2(G/T; \mathbf{k}) \rightarrow H^3(BG; \mathbf{k}) = 0$$

associated with the bundle ζ implies that i^* is an epimorphism. By the assumption of (3) $i^*: H^*(BT; \mathbf{k}) \rightarrow H^*(G/T; \mathbf{k})$ is an epimorphism. Hence G/T is totally nonhomologous to zero and so $\rho(T, G)^*: H^*(BG; \mathbf{k}) \rightarrow H^*(BT; \mathbf{k})$ is a monomorphism. Since $H^{\text{odd}}(BT; \mathbf{k}) = 0$, we have $H^{\text{odd}}(BG; \mathbf{k}) = 0$.

That (4) implies (1) is clear from the universal coefficient theorem. Q.E.D.

Let U be a connected, closed subgroup of maximal rank of G , and let the type of G be $(2r_1 - 1, \dots, 2r_l - 1)$, that of U be $(2s_1 - 1, \dots, 2s_l - 1)$ and $l = \text{rank } G$ in the following theorem.

THEOREM 3.21. Suppose that $H^*(U)$ is p -torsion free. Then the following hold:

- (1) $H^*(G/U)$ is p -torsion free, and for a field $\mathbf{k}$ of characteristic p ,

$$P_p(G/U, t) = P(H^*(G/U; \mathbf{k}), t) = \prod (1 - t^{2r_i}) / (1 - t^{2s_i}).$$

- (2) Further if $(\chi(G/U), p) = 1$, $H^*(BG)$ is p -torsion free and $H^*(G/U; \mathbf{k}) = \Gamma(\nu)$, where $\nu = (G, G/U, U)$.

PROOF. (1) Since $H^*(U)$ is a p -torsion free by the assumption, so is $H^*(BU)$ by Corollary 2.13. Consider $\zeta = (U, p, U/T, T)$ for a maximal torus $T \subset U$ and then $H^*(U/T; \mathbf{k}) = \Gamma(\zeta)$ by Theorem 3.20(2). Therefore by 3.19 U/T is mod p totally nonhomologous to zero in the fibre bundle $(G/T, \pi, G/U, U/T)$, and hence the cohomology spectral sequence $\{E_r\}$ of this bundle is trivial:

$$E_2 = H^*(G/U; \mathbf{k}) \otimes H^*(U/T; \mathbf{k}) = E_\infty = \mathcal{Z}H^*(G/T; \mathbf{k}).$$

It follows that $P_p(G/U, t)P_p(U/T, t) = P_p(G/T, t)$. Now we can apply Theorem 5.5(2) of Chapter VI.

(2) Since $(\chi(G/U), p) = 1$, we can apply Theorem 3.15 to the fibre bundle $(BU, \rho(U, G), BG, G/U)$, and we have that $\rho(U, G)^*: H^*(BG; \mathbf{k}) \rightarrow H^*(BU; \mathbf{k})$ is a monomorphism. By Theorem 3.20 $H^{\text{odd}}(BU; \mathbf{k}) = 0$, which implies $H^{\text{odd}}(BG; \mathbf{k}) = 0$ and hence $H^*(BG)$ is p -torsion free. Using (1) $E_2 = H^*(BG; \mathbf{k}) \otimes H^*(G/U; \mathbf{k})$ in the spectral sequence of the above fibre bundle consists only of the elements of even degrees and hence $E_2 = E_\infty$ and hence $H^*(G/U; \mathbf{k}) = \Gamma(\nu)$. Q.E.D.

The action of the Weyl group. Let G be a compact Lie group and U a connected, closed subgroup of G . Consider principal bundles $\xi = (E, \pi, B, G)$ and $\eta = (E, \pi', E/U, U)$, where E/U is the left quotient space. If N is a subgroup of G such that $N \subset N(U)$, then N acts on U by inner automorphisms $u \mapsto nun^{-1}$. Since $nU = Un$ for $n \in N$, the left action of N gives a permutation of the cosets Ux of E and hence N acts on η . Therefore N acts on $H^*(E; R)$, $H^*(E/U; R)$, and the spectral sequence $\{E_r\}$ of η as well. The action of $N \cap U$ on E/U is trivial and the action on $H^*(E; R)$ is also trivial, since U is connected. Then the following properties are satisfied:

3.22.1. The action of N induces that of $N/(N \cap U)$ on $H^*(E; R)$ and on $H^*(E/U; R)$, which preserves the fibration associated with η and commutes with the one induced by a map of fibre bundles.

3.22.2. The action of N on $E_2 = H^*(E/U; H^*(U; R))$ commutes with the isomorphisms

$$E_2^{p,0} \cong H^p(E/U; R) \quad \text{and} \quad E_2^{0,q} \cong H^q(U; R).$$

In particular, $N \cap U$ acts trivially and hence $N/(N \cap U)$ acts on E_r for $r \geq 2$.

3.22.3. The action of $N/(N \cap U)$ on E_∞ coincides with the one induced by the action on $H^*(E; R)$. In particular, if R is a field, there is an isomorphism $H^*(E; R) \cong E_\infty$ as vector spaces, which commutes with the actions on $H^*(E; R)$ and E_∞ .

Next, consider the fibre bundle $\xi/U = (E/U, p, B, G/U)$, where G/U is the left quotient space. Then N acts on G/U and E/U by the right translation and the action is compatible with the structure of ξ/U and the

action on B is trivial. Therefore N acts on $H^*(G/U; R)$ and hence $N/(N \cap U)$ acts on $H^*(G/U; R)$ since U is connected. Further, it acts on the spectral sequence of ξ/U and possesses the following properties:

3.22.1'. $N/(N \cap U)$ acts on $H^*(E/U; R)$ and the action on $H^*(BU; R)$ is trivial. Further, the action preserves the filtration associated with ξ/U and commutes with the one induced by a map of fibre bundles.

3.22.2'. If G is connected, N acts on the spectral sequence $\{E_r\}$ of ξ/U and the action on $E_2 \cong H^*(B; H^*(G/U; R))$ commutes with the isomorphisms

$$E_2^{p,0} \cong H^p(B; R) \quad \text{and} \quad E_2^{0,q} \cong H^q(G/U; R).$$

Since $N \cap U$ acts trivially, $N/(N \cap U)$ acts on E_r for $r \geq 2$.

3.22.3'. The action of $N/(N \cap U)$ on E_∞ coincides with the one induced by the action on $H^*(E/U; R)$. In particular, if R is a field, there is an isomorphism $H^*(E/U; R) \cong E_\infty$ as vector spaces which commutes with the actions on $H^*(E/U; R)$ and E_∞ .

In the above argument we put $U = T =$ (a maximal torus of G) and $N = N(T) =$ (the normalizer of T in G) and consider the principal bundle $\zeta = (ET, p, BT, T)$, where $ET = EG$. Then the Weyl group $W(G) = N(T)/T$ of G satisfies the properties 3.22.1, 3.22.2, and 3.22.3. Since $H^*(T)$ is torsion free, we have $E_2 = H^*(BT) \otimes H^*(T)$ in the spectral sequence $\{E_r\}$ of ζ . $EG = ET$ is contractible so $E_3^{0,1} = E_\infty^{0,1} = 0 = E_\infty^{2,0} = E_3^{2,0}$. Thus $d_2: E_2^{0,1} \rightarrow E_2^{2,0}$ is an isomorphism, and the transgression τ gives the following isomorphism:

$$\tau: H^1(T) \cong H^2(BT). \quad (3.23)$$

Here, if we denote by t_i the image of each generator of $H^1(T)$ by the transgression, $H^*(BT) = \mathbb{Z}[t_1, \dots, t_l]$ with $l = \dim T = \text{rank } G$. The operation to identify $H^*(BT)$ with the polynomial algebra over $H^*(T)$ through the transgression τ commutes with the action of $W(G)$.

Regular representations. Let X be a polyhedron and a basis of $H^q(X; \mathbb{Q})$ be $\{e(q)_i, 1 \leq i \leq b_q\}$. Then we define the *Lefschetz number* $L(f)$ for a map $f: X \rightarrow X$ by

$$L(f) = \sum (-1)^q \text{tr}(A_q(f)),$$

where $\text{tr}(A_g(f))$ is the trace of the coefficient matrix $A_g(f) = (a_{ij}(q))$ when we express $f^*(e(q)_i) = \sum a_{ij}(q)e(q)_j$. Here we quote again the following fixed point theorem without proof (cf. (3.13) of Chapter V).

THEOREM 3.24 (The Lefschetz fixed point theorem). *If a map $f: X \rightarrow X$ for a polyhedron X does not have a fixed point, then $L(f) = 0$.*

Let the order of a finite group W be r and let V be an r -dimensional

vector space with a basis $\{e_w \mid w \in W\}$. Defining a linear map $\rho_u: V \rightarrow V$ by $\rho_u(e_w) = e_{uw}$ for each $u \in W$, we obtain a linear representation $\rho: W \rightarrow GL(V)$, which is called the *regular representation* of W . If $u \neq e$ (the identity), we have $\rho_u e_w = e_{uw}$ for $uw \neq w$, and so the diagonal elements of the matrix ρ_u are 0, i.e., $\text{tr}(\rho_u) = 0$. On the other hand, we have $\text{tr}(\rho_e) = \text{tr}(I_r) = r = \dim V$. Since the character of the representation ρ is given by $\chi_W(u) = \text{tr}(\rho_u)$, by the general theory of characters in representation theory we have:

3.25. *The character χ_W of the regular representation is characterized by*

$$\chi_W(u) = \begin{cases} r = |W| & \text{for } u = e, \\ 0 & \text{for } u \neq e. \end{cases}$$

Let G be a compact, connected Lie group, T a maximal torus of G , and $N(T)$ the normalizer of T . For each $n \in N(T)$ we define a homeomorphism

$$f_n: G/T \rightarrow G/T \quad \text{by} \quad f_n(gT) = ngT.$$

Further, we define a linear representation $\rho: W(G) \rightarrow \text{Aut}(H^*(G/T; \mathbb{Q}))$ of the Weyl group $W(G)$ by

$$\rho_w = f_n^*: H^*(G/T; \mathbb{Q}) \rightarrow H^*(G/T; \mathbb{Q}), \quad \text{where } \{n\} = w.$$

Then by definition, the Lefschetz number $L(f_n)$ of f_n is

$$\begin{aligned} L(f_n) &= \sum (-1)^q (\text{tr}(\rho_w) \text{ in } H^q(G/T; \mathbb{Q})), \quad \text{where } \{n\} = w \\ &= \sum (\text{tr}(\rho_w) \text{ in } H^q(G/T; \mathbb{Q})) = (\text{tr}(\rho_w) \text{ in } H^*(G/T; \mathbb{Q})). \end{aligned}$$

If $n \notin T$ (i.e., $w \neq e$), f_n does not have a fixed point, and so by Theorem 3.24 $L(f_n) = 0$, whence the character is $\chi_{W(G)}(w) = \text{tr}(\rho_w) = 0$. On the other hand, if $n \in T$ (i.e., $w = e$), f_n is the identity map, and so the character is $\chi_{W(G)}(w) = \text{tr}(\rho_w) = \dim H^*(G/T; \mathbb{Q}) = |W(G)|$. Since this is the character of a regular representation by 3.25, we have:

THEOREM 3.26 (Leray). *The natural representation ρ of $W(G)$ in $H^*(G/T; \mathbb{Q})$ is a regular representation.*

COROLLARY 3.27. (1) *The set of elements of $H^*(G/T; \mathbb{Q})$ invariant under the action of $W(G)$ coincides with $H^0(G/T; \mathbb{Q})$.*

(2) (1) and Theorem 3.26 also hold by replacing $\mathbb{Q}$ with any field $\mathbf{k}$ of characteristic 0.

Invariants of the Weyl group. For the moment, let G be a compact, connected Lie group and T a maximal torus of G .

NOTATION. $I(G) = H^*(BT)^{W(G)}$ = (the ring of invariants of $W(G)$ in $H^*(BT)$),

$$I^+(G) = (\text{the invariants of positive degree}) \subset I(G).$$

If $kx \in I(G)$ for $x \in H^*(BT)$ and $0 \neq k \in \mathbb{Z}$, then we have $kx = w(kx) = k(wx)$ for any $w \in W(G)$. Since $H^*(BT)$ is free, $x = wx$ implies $x \in I(G)$. Thus 3.28.

3.28. $I(G)$ is a direct summand. In particular

$$I(G) \otimes R \subset H^*(BT) \otimes R = H^*(BT; R).$$

THEOREM 3.29. If $H^*(G)$ is p -torsion free for a field $\mathbf{k}$ of characteristic p , then

- (1) $\rho_p(T, G)^*: H^*(BG; \mathbf{k}) \rightarrow H^*(BT; \mathbf{k})$ is an isomorphism onto $I(G) \otimes \mathbf{k}$;
- (2) $H^*(G/T; \mathbf{k}) = \Gamma(\xi) = H^*(BT; \mathbf{k})/(I^+(G) \otimes \mathbf{k})$, $\xi = (G, G/T, T)$.

PROOF. Consider the cohomology spectral sequence $\{E_r\}$:

$$E_2 = H^*(BG; \mathbf{k}) \otimes H^*(G/T; \mathbf{k}) \Rightarrow H^*(BT; \mathbf{k})$$

with $\mathbf{k}$ -coefficients of the fibre bundle $(BT, BG, G/T)$. Since $H^*(G)$ is p -torsion free, $H^*(BG; \mathbf{k})$ consists only of elements of even degrees by Theorem 2.12. Since $H^*(G/T; \mathbf{k})$ also consists only of elements of even degrees by 3.16, so does E_2 and hence the spectral sequence is trivial: $E_2 = E_\infty$. Thus we see that

$$\rho_p(T, G)^*: H^*(BG; \mathbf{k}) \rightarrow H^*(BT; \mathbf{k}) \text{ is a monomorphism.}$$

It remains to determine the image of $\rho_p(T, G)^*$.

The case $p = 0$. By 3.22.1' $W(G)$ acts trivially on $H^*(BG; \mathbf{k})$. On the other hand, by Theorem 3.26 and Corollary 3.27 $W(G)$ acts on $H^*(G/T; \mathbf{k})$ as a regular representation and the set of the invariants is $H^0(G/T; \mathbf{k})$. It follows that $H^*(BG; \mathbf{k}) \otimes H^0(G/T; \mathbf{k})$ is the set of the invariants of $W(G)$ in E_2 , the projection of which to E_∞ is $\text{Im } \rho_0(T, G)^*$. Since $E_2 = E_\infty$, by identifying by the isomorphism of vector spaces in 3.22.3', $E_\infty = H^*(BT; \mathbf{k})$. Hence $H^*(BG; \mathbf{k}) \otimes H^0(G/T; \mathbf{k}) = I(G) \otimes \mathbf{k}$ and so $\text{Im } \rho_0(T, G)^*$ is the set of the invariants $I(G) \otimes \mathbf{k}$.

The case $p \neq 0$. Since $I(G)$ is a direct summand, $\dim(I(G) \otimes \mathbf{k})^i = \text{rank}(I(G))^i = \dim H^i(BG; \mathbf{Q})$. Since $H^i(G)$ is p -torsion free, so is $H^*(BG)$ by Corollary 2.13 and $\dim H^i(BG; \mathbf{k}) = \dim H^i(BG; \mathbf{Q})$. Moreover $H^*(BG; \mathbf{k}) = H^*(BG) \otimes \mathbf{k}$, which implies $\rho_p(T, G)^* = \rho_z(T, G)^* \otimes \mathbf{k}$ hence $\text{Im } \rho_p(T, G)^* \subset I(G) \otimes \mathbf{k}$. By 3.22.1' and the above facts,

$$\dim(\text{Im } \rho_p(T, G)^* \cap H^i(BT; \mathbf{k})) = \dim(I(G) \otimes \mathbf{k})^i$$

so $\text{Im } \rho_p(T, G)^* \cap H^*(BT; \mathbf{k}) = (I(G) \otimes \mathbf{k})^i$. Q.E.D.

REMARK 3.30. If $H^*(G)$ has no torsion, the above theorem also holds with $\mathbf{Z}$ -coefficients.

THEOREM 3.31. Let G be a (not necessarily connected) compact Lie group, T its maximal torus, and $\mathbf{k}$ a field of characteristic 0. Then

$$\rho_0(T, G)^*: H^*(BG; \mathbf{k}) \cong I(G) \otimes \mathbf{k}.$$

PROOF. Since $G_0 = (\text{the connected component of the identity } e) \supset T$, we have the natural projection $EG/T \rightarrow EG/G_0 \rightarrow EG/G$, i.e.,

$$BT \xrightarrow{\rho(T, G_0)} BG_0 \xrightarrow{\lambda} BG,$$

which satisfies $\rho(T, G) = \lambda \circ \rho(T, G_0)$. The map λ is a regular covering with covering transformation group G/G_0 . $W(G_0)$ is a normal subgroup of $W(G)$ such that $W(G)/W(G_0) = G/G_0$. Moreover we have $\rho(T, G)^* = \rho(T, G_0)^* \circ \lambda^*$, and it suffices to apply the following and Theorem 3.29:

λ^* is a monomorphism and $\text{Im } \lambda^* = (\text{the invariants of the covering transformation group in } H^*(BG_0; \mathbf{k}))$.

We will prove this. $\Pi = G/G_0$ acts on $C_0 = C_*(BG_0; \mathbf{k})$ and λ_* can be regarded as the projection $C_0 \rightarrow C = C_0/\Pi = C_*(BG; \mathbf{k})$. Define a chain map $t: C \rightarrow C_0$ by $t(s) = \frac{1}{n} \sum_{g \in \Pi} g \cdot \bar{s}$ where $n = |\Pi|$ and $\lambda_*(\bar{s}) = s$, and put $P = t \circ \lambda_*$. Since $\lambda_* \circ t = \text{id}$, $t^* \circ \lambda^* = \text{id}$, which implies that λ^* is a monomorphism and t^* is an epimorphism. Since $P^*(x) = \frac{1}{n} \sum g \cdot x$, x is Π -invariant iff $P^*(x) = x$ which is equivalent to $x \in \text{Im } P^*$. $P^* = \lambda^* \circ t^*$ and t^* an epimorphism imply $\text{Im } \lambda^* = \text{Im } P^* = (\text{the } \Pi\text{-invariants in } H^*(BG_0; \mathbf{k}))$. Q.E.D.

REMARK 3.32. Let $\mathbf{k}$ be a field of characteristic p . If $(|G/G_0|, p) = 1$, then $I(G) \otimes \mathbf{k}$ is the set of the invariants of G/G_0 in $I(G_0) \otimes \mathbf{k}$. Using this fact one can show the following:

If $H^*(G_0)$ is p -torsion free and $(|G/G_0|, p) = 1$, then

$$\rho_p(T, G)^*: H^*(BG; \mathbf{k}) \cong I(G) \otimes \mathbf{k}.$$

THEOREM 3.33. Let U be a connected, closed subgroup of maximal rank of G . If $H^*(G)$ and $H^*(U)$ are p -torsion free, we have

$$H^*(G/U; \mathbf{k}) = \Gamma(\eta) = (I(U) \otimes \mathbf{k}) / (I^+(G) \otimes \mathbf{k})$$

for the U -bundle $\eta = (G, G/U, U)$ and $\mathbf{k}$ a field of characteristic p .

PROOF. Consider the cohomology spectral sequence $\{E_r\}$:

$$E_2 = H^*(BG; \mathbf{k}) \otimes H^*(G/U; \mathbf{k}) \Rightarrow H^*(BU; \mathbf{k})$$

with $\mathbf{k}$ -coefficients of the fibre bundle $(BU, \rho(U, G), BG, G/U)$. By Theorems 2.12 and 3.21(1) each term of E_2 has only elements of even degree and so $E_2 = E_\infty$. Thus G/U is totally nonhomologous to zero, and $i^*: H^*(BU; \mathbf{k}) \rightarrow H^*(G/U; \mathbf{k})$ is an epimorphism for the inclusion $i: G/U \rightarrow BU$. Here we have $H^*(G/U; \mathbf{k}) = \Gamma(\eta)$, since i^* is the characteristic map. It follows that

$$H^*(G/U; \mathbf{k}) = H^*(BU; \mathbf{k}) / (\rho_p(U, G)^* H^+(BG; \mathbf{k})).$$

Applying $\rho_p(T, U)^*$, which is a monomorphism by Theorem 3.29(1), we have

$$\begin{aligned} H^*(G/U; \mathbf{k}) &= (I(U) \otimes \mathbf{k})/(\rho_p(T, U)^* \rho_p(U, G)^* H^+(BG; \mathbf{k})) \\ &= (I(U) \otimes \mathbf{k})/(\rho_p(T, G)^* H^+(BG; \mathbf{k})) \\ &= (I(U) \otimes \mathbf{k})/(I^+(G) \otimes \mathbf{k}). \quad \text{Q.E.D.} \end{aligned}$$

REMARK 3.34. A similar result also holds with $\mathbf{Z}$ -coefficients. If $H^*(G)$, $H^*(U)$, and $H^*(G/U)$ are torsion free, then $H^*(G/U) = I(U)/(I^+(G))$.

§4. Cohomology operations and the cohomology of Eilenberg-Mac Lane spaces

Eilenberg-Mac Lane spaces and cohomology groups. Let G be an abelian group and n a positive integer. We denote (cf. §1 of Chapter IV) the *Eilenberg-Mac Lane complex* of type (G, n) by

$$K(G, n) \text{ where } \pi_i(K(G, n)) = 0 \text{ for } i \neq n, \quad \pi_n(K(G, n)) \cong G.$$

By the Hurewicz theorem we have

$$H_n(K(G, n)) \cong \pi_n(K(G, n)) \cong G.$$

Regarding this isomorphism as an element of $\text{Hom}(H_n(K(G, n)), G) \cong H^n(K(G, n); G)$, the element

$$u_n \in H^n(K(G, n); G) \cong \text{Hom}(G, G)$$

is called the *fundamental cohomology class* of $K(G, n)$.

THEOREM 4.1. *Let X be an (arcwise) connected (or a topological space having the same homotopy type as a) CW-complex. If we assign $f^*(u_n) \in H^n(X; G)$ to the homotopy class $[f] \in [X, K(G, n)]_0$ of $f: X \rightarrow K(G, n)$, then we have a natural isomorphism*

$$[X, K(G, n)]_0 \cong H^n(X; G),$$

where the group structure on the left-hand side is obtained by identifying $K(G, n) = \Omega K(G, n+1)$.

OUTLINE OF PROOF. The result is trivial if $\dim X < n$ or if X^n is a point. $X^n/X^{n-1} = \vee S_\alpha^n$ (a wedge) so

$$[X^n/X^{n-1}, K(G, n)]_0 \cong H^n(X^n/X^{n-1}; G) = C^n(X; G)$$

(a cochain complex of a cell complex). Then one can see that δ^* in the exact sequence of Theorem 1.28(2) of Chapter IV associated with $X^n/X^{n-1} \hookrightarrow X^{n+1}/X^{n-1} \rightarrow X^{n+1}/X^n$ corresponds to $\delta: C^n(X; G) \rightarrow C^{n+1}(X; G)$. Thus one obtains

$$[X, K(G, n)]_0 \cong [X^{n+1}/X^{n-2}, K(G, n)] = \text{Ker } \delta / \text{Im } \delta \cong H^n(X; G)$$

and the correspondence can be given as in the theorem. Q.E.D.

It follows from this theorem that the homotopy type of $K(G, n)$ is determined by G and n alone, and so one can define the following:

NOTATION. $H^q(\Pi, n; G) = H^q(K(\Pi, n); G)$.

A cohomology operation $\Psi = \{\Psi_X: H^n(X; \Pi) \rightarrow H^q(X; G)\}$ of (n, Π, q, G) -type is defined (§1 of Chapter III) so that it satisfies the naturality condition $\Psi_X \circ f^* = f^* \circ \Psi_Y$ for $f: X \rightarrow Y$.

THEOREM 4.2. Cohomology operations Ψ of (n, Π, q, G) -type are in one to one correspondence with the elements of $H^q(\Pi, n; G)$ by $\Psi \mapsto \Psi(u_n)$.

PROOF. The inverse map $\alpha \mapsto \Psi_\alpha$ (for $\alpha \in H^q(\Pi, n, G)$) is defined as follows. For $x \in H^n(X; \Pi)$ there is a map $f: X \rightarrow K(\Pi, n)$ such that $f^*(u_n) = x$. (Theorem 4.1). Then we put $\Psi_\alpha(x) = f^*(\alpha) \in H^q(X; G)$. It is easily verified that Ψ_α is a cohomology operation. Clearly $\alpha \mapsto \Psi_\alpha \mapsto \Psi_\alpha(u_n) = \alpha$. It is also shown by the naturality of Ψ that $\Psi = \Psi_\alpha$ if $\Psi \mapsto \Psi(u_n) = \alpha$. Q.E.D.

A stable cohomology operation of (Π, G) -type $\Psi: H^n(X; \Pi) \rightarrow H^{n+t}(X; G)$ of degree t is defined so that it is given for any n and X and commutes with δ^* or with the suspension $\sigma^*: H^m(SX;) \rightarrow H^{m-1}(X;)$. The following theorem is easily obtained from the definition:

THEOREM 4.3. Stable cohomology operations of (Π, G) -type Ψ of degree t are in one to one correspondence with sequences $\{\alpha_n; n = 1, 2, 3, \dots\}$ of elements $\alpha_n \in H^{n+t}(\Pi, n; G)$ satisfying $\sigma^* \alpha_n = \alpha_{n-1}$.

Since $K(\Pi, n+1)$ is n -connected, the following property of the cohomology suspension σ associated with the fibration

$$\Omega K(\Pi, n+1) = K(\Pi, n) \xrightarrow{i} PK(\Pi, n+1) \xrightarrow{p} K(\Pi, n+1) \quad (4.4)$$

is obtained:

4.4'. $\sigma: H^{i+1}(\Pi, n+1; G) \rightarrow H^i(\Pi, n; G)$ is an isomorphism for $i \leq 2n-1$ and a monomorphism for $i = 2n$.

NOTATION. $A^t(\Pi, G) = H^{n+t}(\Pi, n; G)$ for $n > t$.

4.5. Stable cohomology operations of (Π, G) -type of degree t are in one to one correspondence with the elements of $A^t(\Pi, G)$.

As stated above, cohomology operations have a close relation with the cohomology of Eilenberg-Mac Lane spaces, and the properties of the former are clarified by studying the latter.

The following theorem is often used for stable cohomology operations.

THEOREM 4.6. A stable cohomology operation Ψ commutes with the transgression τ and the suspension σ ($\Psi \circ \tau = \tau \circ \Psi$ and $\Psi \circ \sigma = \sigma \circ \Psi$) in a

fibration. That is, the following diagram is commutative.

$$\begin{array}{ccccc}
 H^n(F; \Pi) & \xrightarrow{\delta} & H^{n+1}(E, F; \Pi) & \xleftarrow{p^*} & H^{n+1}(B; \Pi) \\
 \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi \\
 H^{n+i}(F; G) & \xrightarrow{\delta} & H^{n+i+1}(E, F; G) & \xleftarrow{p^*} & H^{n+i+1}(B; G)
 \end{array}
 \quad \left(\begin{array}{l} \tau = p^{*-1} \circ \delta \\ \sigma = \delta^{-1} \circ p^* \end{array} \right)$$

Squaring operations and $H^*(\Pi, n; \mathbb{Z}_2)$

LEMMA 4.7. $u_i^2 \in \text{Im } \sigma$ and $\sigma(u_i^2) = 0$ for $u_i \in H^t(\mathbb{Z}_2, t; \mathbb{Z}_2)$.

PROOF. For the first half; in the spectral sequence

$$E_2 = H^*(\mathbb{Z}_2, t+1; \mathbb{Z}_2) \otimes H^*(\mathbb{Z}_2, t; \mathbb{Z}_2) \Rightarrow E_\infty = \mathbb{Z}_2$$

associated with (4.4) when $\Pi = G = \mathbb{Z}_2$ and $n = t$, for degree reasons, $r \neq t+1$ and $r < 2t+1$ implies $d_r(1 \otimes u_i^2) = 0$. Also by $d_{t+1}(1 \otimes u_i) = u_{t+1} \otimes 1$ and the distributivity of d_{t+1} , $d_{t+1}(1 \otimes u_i^2) = 2u_{t+1} \otimes u_i = 0$, which shows that u_i^2 is transgressive, that is, it is in the image of the suspension σ .

For the second half; consider the case $n = t-1$, then $d_t(u_i \otimes u_{t-1}) = u_i^2 \otimes 1$ implies $\sigma(u_i^2) = 0$. Q.E.D.

THEOREM 4.8. For any positive integer t , there exists a stable cohomology operation of degree t

$$Sq^t: H^n(X; \mathbb{Z}_2) \rightarrow H^{n+t}(X; \mathbb{Z}_2)$$

that possesses the properties:

- (1) $Sq^t \circ f^* = f^* \circ Sq^t$ for $f: Y \rightarrow X$ (i.e., it is a cohomology operation);
- (2) $Sq^t \circ \delta^* = \delta^* \circ Sq^t$, $Sq^t \circ \tau = \tau \circ Sq^t$, $Sq^t \circ \sigma = \sigma \circ Sq^t$ (i.e., it is stable);
- (3) $Sq^t(x) = x^2$, if $n = |x| = t$.

In addition, by putting $Sq^0 = \text{id}$ and $Sq^t = 0$ for $t < 0$, the following properties are satisfied:

- (4) $Sq^t(x) = 0$ if $n = |x| < t$;
- (5) (The Cartan formula) $Sq^t(xy) = \sum_{i+j=t} Sq^i(x)Sq^j(y)$;
- (6) Sq^1 coincides with the Bockstein operator associated with the short exact sequence $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$.

PROOF. Consider $\sigma: H^{n+t+1}(\mathbb{Z}_2, n+1; \mathbb{Z}_2) \rightarrow H^{n+t}(\mathbb{Z}_2, n; \mathbb{Z}_2)$. If one puts $\alpha_t = u_t^2$, then there exists α_{t+1} such that $\sigma(\alpha_{t+1}) = \alpha_t$ by Lemma 4.7 and α_{t+1} is unique by the second half of 4.4'. By the first half of 4.4', $\alpha_n \in H^{n+t}(\mathbb{Z}_2, n; \mathbb{Z}_2)$ for $n \geq t$ is uniquely determined, satisfying $\sigma(\alpha_{n+1}) = \alpha_n$. If $n < t$, then $\alpha_n = 0$ by the second half of Lemma 4.7. If we denote by Sq^t the stable cohomology operation corresponding to $\alpha = \{\alpha_n\}$, we obtain the unique existence of Sq^t and properties (1)–(4).

(5) is equivalent to

(5') (The Cartan formula) $Sq^l(x \times y) = \sum_{i+j=l} Sq^i(x) \times Sq^j(y)$ with respect to the cross product $\times$. It suffices to show (5') for the case $x = u_n$ and $y = u_m$. In particular we have $Sq^l(u_n \times u_m) = 0 = \sum_{i+j=l} Sq^i(u_n) \times Sq^j(u_m)$ for $l > n + m$ and $Sq^{n+m}(u_n \times u_m) = (u_n \times u_m)^2 = u_n^2 \times u_m^2 = Sq^n u_n \times Sq^m u_m = \sum_{i+j=n+m} Sq^i(u_n) \times Sq^j(u_m)$, and hence (5') is true for $l \geq n + m$. The rest can be shown by induction on $n + m - l$ related with σ , but the details are left to the reader. Q.E.D.

DEFINITION. For a sequence $I = (i_1, i_2, \dots, i_l)$ of integers $i_j > 0$ we put $|I| = \sum i_j$, $|\emptyset| = 0$ and

$$Sq^\emptyset = \text{id},$$

$$Sq^I = Sq^{i_1} Sq^{i_2} \dots Sq^{i_l} \quad (\text{a stable cohomology operation of degree } |I|).$$

We say that

I is *admissible* if $I = \emptyset$ ($l = 0$), $I = (i)$ ($l = 1$) or $i_j \geq i_{j+1}$ ($1 \leq j < l$), and $e(I) = i_l + \sum_{1 \leq j < l} (i_j - 2i_{j+1}) = 2i_1 - |I|$ is called the *excess* of I .

THEOREM 4.9. $H^*(Z_2, n; Z_2) = Z_2[Sq^I u_n; I \text{ is admissible and } e(I) < n]$.

PROOF. By induction on n . For the case $n = 1$; regarding as $K(Z_2, 1) = P^\infty(\mathbf{R})$, we have $H^*(Z_2, 1; Z_2) = Z_2[u_1]$ and hence the theorem is true. Supposing that the theorem is true for n , we can write

$$H^*(Z_2, n; Z_2) = Z_2[Sq^I u_n] = \Delta((Sq^I u_n)^{2^k}) = \Delta(Sq^{2^{k-1}L} \dots Sq^{2L} Sq^L Sq^I u_n)$$

using the simple system of generators, where $L = n + |I|$. Checking the conditions for admissibility and excess, we have

$$H^*(Z_2, n; Z_2) = \Delta(Sq^J u_n; J \text{ is admissible and } e(J) < n + 1).$$

Since $\tau(u_n) = u_{n+1}$, we have $\tau(Sq^J u_n) = Sq^J u_{n+1}$ by Theorem 4.8. Applying Corollary 2.8(1) to (4.4) we obtain $H^*(Z_2, n + 1; Z_2) = Z_2[Sq^J u_{n+1}]$. Q.E.D.

THEOREM 4.10. $H^*(Z, n; Z_2) = Z_2[Sq^I u_n; I = (i_1, \dots, i_l) \text{ is admissible, } i_j > 1 \text{ and } e(I) < n] \text{ for } n > 1$.

PROOF. For $n = 2$; regarding as $K(Z, 2) = P^\infty(\mathbf{C}) = BU(1)$, we have $H^*(Z, 2; Z_2) = Z_2[u_2]$ and hence the theorem is true. The rest is proved similarly to Theorem 4.9. Q.E.D.

For convenience of later use, we write down the cases $n = 3$ and 4.

$$H^*(Z_2, 3; Z_2) = Z_2[u_3, Sq^2 u_3, Sq^4 Sq^2 u_3, Sq^8 Sq^4 Sq^2 u_3, \dots], \quad (4.10')$$

$$H^*(Z_2, 4; Z_2) = Z_2[u_4, Sq^2 u_4, Sq^3 u_4, Sq^4 Sq^2 u_3, Sq^5 Sq^2 u_4, Sq^6 Sq^3 u_4, Sq^8 Sq^4 Sq^2 u_4, \dots]. \quad (4.10'')$$

DEFINITION. $\mathcal{A}_2 = A^*(\mathbb{Z}_2, \mathbb{Z}_2) = \{\text{stable cohomology operations of } (\mathbb{Z}_2, \mathbb{Z}_2)\text{-type}\}$ equipped with product by composition of operations is called the mod 2 Steenrod algebra.

COROLLARY 4.11. $\mathcal{A}_2$ has a basis $\{Sq^I; I \text{ is admissible}\}$ over $\mathbb{Z}_2$.

THEOREM 4.12. The mod 2 Steenrod algebra $\mathcal{A}_2$ is a Hopf algebra with the diagonal map ϕ satisfying $\phi(Sq^I) = \sum_{i+j=I} Sq^i \otimes Sq^j$ (and with non-commutative product μ). The structure of the dual Hopf algebra $\mathcal{A}_2^*$ of $\mathcal{A}_2$ is given by

$$\mathcal{A}_2^* = \mathbb{Z}_2[\xi_1, \xi_2, \xi_3, \dots]; \quad \mu^*(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{2^i} \otimes \xi_i \quad (\xi_0 = 1),$$

where ξ_n is the element of degree $2^n - 1$ given by $\xi_n(Sq^I) = 1$ for $I = (2^{n-1}, \dots, 2, 1)$ and $\xi_n(Sq^I) = 0$ for the other I .

(The proof is omitted; see [55].)

Sq^J for nonadmissible J is uniquely expressed as a linear combination of Sq^I for admissible I by Corollary 4.11. The following formula is used. Write elements Sq^I (I not admissible) in terms of the basis.

4.13 (The Adem relations). If $a < 2b$, then

$$Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i.$$

Among the proofs of the formula are Adem's method going back to the algebraic definition of Sq^I , Serre's method studying the values of $u_1 \times \dots \times u_1$ at $K(\mathbb{Z}_2, 1) \times \dots \times K(\mathbb{Z}_2, 1)$, Milnor's method making use of Theorem 4.12, and so on, but we omit the proof and only state some examples for future convenience.

$$4.14. (1) Sq^1 Sq^{2n} = Sq^{2n+1}, \quad Sq^1 Sq^{2n+1} = 0.$$

$$(2) Sq^2 Sq^n = Sq^{n+2} + Sq^{n+1} Sq^1 \quad (n \equiv 0, 3 \pmod{4}, n \geq 3);$$

$$Sq^2 Sq^n = Sq^{n+1} Sq^1 \quad (n \equiv 1, 2 \pmod{4}, n \geq 1).$$

$$(3) Sq^4 Sq^n = Sq^{n+4} + Sq^{n+3} Sq^1 + Sq^{n+2} Sq^2 \quad (n \equiv 0, 5 \pmod{8}, n \geq 5);$$

$$Sq^4 Sq^n = Sq^{n+4} + Sq^{n+2} Sq^2 \quad (n \equiv 6, 7 \pmod{8}, n \geq 6);$$

$$Sq^4 Sq^n = Sq^{n+3} Sq^1 + Sq^{n+2} Sq^2 \quad (n \equiv 1, 4 \pmod{8}, n \geq 4);$$

$$Sq^4 Sq^n = Sq^{n+2} Sq^2 \quad (n \equiv 2, 3 \pmod{8}, n \geq 2).$$

$$(4) Sq^{2^k} Sq^{2^{k+1}a} = Sq^{2^k(2a+1)} + \sum_{i=0}^{k-1} Sq^{2^k(2a+1)-2^i} Sq^{2^i}.$$

Reduced power operations and $H^*(\Pi, n; \mathbb{Z}_p)$. Let p be an odd prime. We have results similar to the case $p = 2$ but we omit the proofs because they are fairly complicated. We add some explanations to the results in place of proofs.

Observe that $K(\mathbb{Z}_p, n) = \Omega K(\mathbb{Z}_p, n+1)$ is an H -space. The following lemma corresponds to Lemma 4.7.

LEMMA 4.15. We have $u_{2t}^p \in \text{Im } \sigma$ and $\sigma(u_{2t}^p)$ for $u_{2t} \in H^{2t}(\mathbb{Z}_p, 2t; \mathbb{Z}_p)$.

The second half is clear but the first half needs the following proposition, which is a variation of (4.4').

4.16. $\sigma: H^{i+1}(\mathbb{Z}_p, n+1, \mathbb{Z}_p) \rightarrow H^i(\mathbb{Z}_p, n; \mathbb{Z}_p)$ is an isomorphism for $i \leq pn-1$ and a monomorphism for $i \leq pn$ between the primitive elements. In addition, there exists a primitive element $\alpha_{2t+1} \in H^{2pt+1}(\mathbb{Z}_p, 2t+1; \mathbb{Z}_p)$ such that $u_{2t}^p = \sigma(\alpha_{2t+1})$.

Using this, the following theorem is proved similar to Theorem 4.8.

THEOREM 4.17. For any positive integer t , there exists a unique stable cohomology operation of degree $2t(p-1)$

$$\wp^t: H^n(X; \mathbb{Z}_p) \rightarrow H^{n+2t(p-1)}(X; \mathbb{Z}_p)$$

with the properties

- (1) $\wp^t \circ f^* = f^* \circ \wp^t$.
- (2) $\wp^t \circ \delta^* = \delta^* \circ \wp^t$, $\wp^t \circ \tau = \tau \circ \wp^t$, $\wp^t \circ \sigma = \sigma \circ \wp^t$.
- (3) $\wp^t(x) = x^p$, if $|x| = 2t$.

In addition, by putting $\wp^0 = \text{id}$ and $\wp^t = 0$ for $t < 0$, the following properties are satisfied:

- (4) $\wp^t(x) = 0$ if $n = |x| < 2t$.
- (5) (The Cartan formula) $\wp^t(xy) = \sum_{i+j=t} \wp^i(x) \wp^j(y)$.

Corresponding to Theorem 4.8(6), we have the Bockstein operation

$$\beta: H^n(X; \mathbb{Z}_p) \rightarrow H^{n+1}(X; \mathbb{Z}_p)$$

as a basic cohomology operation. Obviously

$$\beta\beta = 0, \quad \beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y). \quad (4.18)$$

Omitting the general form of the theorem corresponding to Theorems 4.9 and 4.10, we list some examples:

$$4.19. (1) H^*(\mathbb{Z}, 2; \mathbb{Z}_p) = \mathbb{Z}_p[u_2].$$

$$(2) H^*(\mathbb{Z}, 3; \mathbb{Z}_p) = \Lambda(u_3, \wp^1 u_3, \wp^p \wp^1 u_3, \dots) \otimes \mathbb{Z}_p[\beta \wp^1 u_3, \beta \wp^p \wp^1 u_3, \dots].$$

$$(3) \quad H^*(\mathbb{Z}, 4; \mathbb{Z}_p) = \mathbb{Z}_p[u_4, \wp^1 u_4, \wp^p \wp^1 u_4, \beta \wp^{p+1} \wp^1 u_4, \dots] \\ \otimes \Lambda(\beta \wp^1 u_4, \beta \wp^p \wp^1 u_4, \wp^{p+1} \beta \wp^1 u_4, \dots).$$

The result in (1) is obvious. In the spectral sequence deducing (2) from (1) the differentials

$$d_3(1 \otimes u_2) = u_3 \otimes 1, \quad d_{2p+1}(1 \otimes u_2^p) = \wp^1 u_3 \otimes 1, \\ d_{2p^2+1}(1 \otimes u_2^{p^2}) = \wp^p \wp^1 u_3 \otimes 1, \dots$$

are obtained from $u_2^p = \varphi^1 u_2$, $u_2^{p^2} = \varphi^p \varphi^1 u_2, \dots$. If we construct a formal spectral sequence $\{''E_r\}$ by using these, we have

$$''E_\infty = \Lambda(u_3 \otimes u_2^{p-1}, \varphi^1 u_3 \otimes u_2^{p(p-1)}, \dots).$$

Since we should have $E_\infty = Z_p$ practically, we see the existence of d_r as follows:

$$\begin{aligned} d_{2(p-1)+1}(u_3 \otimes u_2^{p-1}) &= \beta \varphi^1 u_3 \otimes 1, \\ d_{2p(p-1)+1}(\varphi^1 u_3 \otimes u_2^{p(p-1)}) &= \beta \varphi^p \varphi^1 u_3 \otimes 1, \dots \end{aligned}$$

However, to show that these images are expressed in terms of β , we need some special devices (such as considerations with integral coefficients). One can obtain 4.19(2) in this manner, but one needs the following result to deduce 4.19(3) or more general results.

4.20 (The Kudo transgression theorem). *If an element x of degree $2t$ is transgressive such that $d_{2t+1}(1 \otimes x) = y \otimes 1$, then $d_{2pt+1}(1 \otimes x^p) = \varphi^t y \otimes 1$ and $d_{2t(p-1)+1}(y \otimes x^{p-1}) = \beta \varphi^t y \otimes 1$.*

DEFINITION. $\mathcal{A}_p = A^*(Z_p, Z_p)$ is called the mod p Steenrod algebra where the product μ is composition.

4.21. $\mathcal{A}_p$ is a noncommutative Hopf algebra generated by β and φ^t (to state more in detail one needs the definition of the admissible φ^t), and the structure of the dual algebra $\mathcal{A}_p^*$ is given by

$$\begin{aligned} \mathcal{A}_p^* &= Z_p[\xi_1, \xi_2, \xi_3, \dots] \otimes \Lambda(\tau_0, \tau_1, \tau_2, \dots), \\ \mu^*(\xi_n) &= \sum_{i=0}^n \xi_{n-i}^{p^i} \otimes \xi_i, \quad \mu^*(\tau_n) = \sum_{i=0}^n \xi_{n-i}^{p^i} \otimes \tau_i + \tau_n \otimes 1 \quad (\xi_0 = 1), \end{aligned}$$

where $|\xi_n| = 2p^n - 2$ and $|\tau_n| = 2p^n - 1$.

4.22 (The Adem Relations)

(1) If $a < pb$, then

$$\varphi^a \varphi^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(b-i)(p-1)-1}{a-pi} \varphi^{a+b-i} \varphi^i,$$

(2) If $a \leq pb$, then

$$\begin{aligned} \varphi^a \beta \varphi^b &= \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(b-i)(p-1)}{a-pi} \beta \varphi^{a+b-i} \varphi^i \\ &\quad + \sum_{i=0}^{[(a-1)/p]} (-1)^{a+i-1} \binom{(b-i)(p-1)-1}{a-pi-1} \varphi^{a+b-i} \beta \varphi^i. \end{aligned}$$

In particular,

$$\varphi^1 \varphi^b = (b+1) \varphi^{b+1}. \quad (4.23)$$

Some applications of cohomology operations. If the homology of G is p -torsion free, then $H^*(BG; \mathbb{Z}_p)$ is a polynomial ring by Theorem 2.14, on which $\mathscr{A}_p$ acts in the sense of (3) ~ (5) of Theorem 4.17.

The following theorem is used to study the behavior of $\wp^1$ in this case. Notice that the results for $H^*(BG; \mathbb{Z}_p)$ give those for $H^*(G; \mathbb{Z}_p)$ through σ .

THEOREM 4.24. Suppose that a graded algebra A^* over $\mathbb{Z}_p$ is a polynomial ring $A^* = \mathbb{Z}_p[y_1, \dots, y_m]$ generated by the elements $y_1, \dots, y_m$ of even degrees > 0 for degrees $\leq 2pn$ and that it satisfies the conditions (3)-(5) of Theorem 4.17. Let the degree of y_i be $|y_i| = 2a_i$.

(1) If $a_i \leq n$ and $a_i \not\equiv 0 \pmod{p}$, then there exist k and j such that $ka_i = a_j + p - 1$ and $1 \leq k \leq p$. In particular, for an i , if there is only one such pair (k, j) and $k = 1$, we can replace a generator y_i with $\wp^1 y_j$.

(2) If there is only one with $k = 1$ among the pairs (k, j) satisfying the condition in (1) and if there exists always l such that $\wp^1 y_j = y_l$ with $k > 1$, we can replace y_i with $\wp^1 y_j$ for j corresponding to $k = 1$, similarly to (1).

PROOF. It follows from Theorem 4.17(3) and (4.23) that $y_i^p = \wp^{a_i} y_i = a_i \wp^1(\wp^{a_i-1} y_i)$. By the assumption of the theorem we have $a_i \not\equiv 0$ and $y_i^p \neq 0$. Here recall that $\wp^1$ is distributive by the Cartan formula:

$$\wp^1(xy) = \wp^1(x)y + x\wp^1(y).$$

Therefore, by expressing $\wp^{a_i-1} y_i$ as a polynomial in y_j for $j < i$ and operating with $\wp^1$, we see that $\wp^1 y_j$ for some j must contain a term y_i^k ($1 \leq k \leq p$) with a nonzero coefficient. Comparing degrees, we have $ka_i = a_j + p - 1$. If there is only one such pair (k, j) and $k = 1$, we must have $\wp^1 y_j = by_j + (\text{decomposables})$, where $b \not\equiv 0 \pmod{p}$. Thus (1) is shown.

We can exclude the terms $\wp^1 y_j = y_l$ for $l < i$ in (2). Q.E.D.

Applications for the case $p = 2$ will be found in §6.

Consider the Hopf algebra $H^*(G; \mathbb{Z}_p)$ for a Lie group G (an H -space in general). By the Cartan formula

$$\bar{\phi}(x) = \sum_{\alpha} x'_{\alpha} \otimes x''_{\alpha} \quad \text{implies} \quad \bar{\phi}(\wp^1 x) = \sum_{i+j=t} \sum_{\alpha} \wp^i x'_{\alpha} \otimes \wp^j x''_{\alpha}. \quad (4.25)$$

Similarly for Sq^t ; in particular,

THEOREM 4.26. If $x \in H^*(G; \mathbb{Z}_p)$ is primitive, so are $\wp^t(x)$ for $p > 2$ and $Sq^t(x)$ for $p = 2$.

§5. Mod p cohomology of the exceptional groups for $p \neq 2$

A general method. Hereafter in this chapter our aim is to determine the mod p cohomology rings of exceptional groups, but first let us consider a compact, 1-connected Lie group G . By Corollary 5.3 of Chapter V, G is decomposed into a direct product of simple groups. Thus it suffices for us to consider the case when G is simple. By Theorem 4.17 of Chapter VI we have

$$\pi_i(G) = 0 \quad \text{for } i < 3 \quad \text{and} \quad \pi_3(G) \cong \mathbb{Z}. \quad (5.1)$$

Consider the classifying space BG of G . Since there is the universal bundle with EG ∞ -connected, we have a bijection

$$[K, G]_0 \xrightarrow{\Delta^{-1}} [SK, BG]_0 \cong [K, \Omega(BG)]_0 \quad \text{for a } CW\text{-complex } K$$

by Theorems 1.26 and 1.22 of Chapter IV. Let σ_G denote the map corresponding to 1_G for $K = G$, and we have

5.2. $\sigma_G: G \rightarrow \Omega(BG)$ is a homotopy equivalence.

This is seen from the fact that $\sigma_{G*}: [K, G]_0 \rightarrow [K, \Omega(BG)]_0$ coincides with the bijection above. As for the homotopy groups, since $\Delta: \pi_{i+1}(BG) \cong \pi_i(G)$ for $K = S^i$, we have

$$\pi_i(BG) = 0 \quad \text{for } i < 4 \quad \text{and} \quad \pi_4(BG) \cong \mathbb{Z}. \quad (5.2')$$

Now we construct the Eilenberg-Mac Lane space $K(\mathbb{Z}, 4)$ containing BG , killing the homotopy groups of dimension $n \geq 5$ by successively attaching cells. The inclusion $BG \hookrightarrow K(\mathbb{Z}, 4)$ corresponds to a generator of $H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$. Putting

$$P = P(K(\mathbb{Z}, 4), BG) \quad \text{and} \quad B\tilde{G} = \Omega(K(\mathbb{Z}, 4), BG),$$

we obtain a fibration $B\tilde{G} \rightarrow P \rightarrow K(\mathbb{Z}, 4)$, where BG is a deformation retract of P . Thus we obtain a fibration $B\tilde{G} \xrightarrow{Bf} BG \rightarrow K(\mathbb{Z}, 4)$ up to homotopy equivalence, where Bf assigns to a path in $B\tilde{G}$ the terminal point and $\Omega K(\mathbb{Z}, 4) = K(\mathbb{Z}, 3) \hookrightarrow B\tilde{G} \xrightarrow{Bf} BG$ is a fibration. In addition, we can obtain a fibration $\Omega(BG) \rightarrow K(\mathbb{Z}, 3) \rightarrow B\tilde{G}$ as in 2.4 of Chapter IV, where $\Omega(BG)$ can be replaced with G by 5.2. Putting

$$\tilde{G} = \Omega(B\tilde{G}) \quad \text{and} \quad f = \Omega(Bf): \tilde{G} \rightarrow \Omega(BG) \simeq G,$$

$\tilde{G} \xrightarrow{f} G \xrightarrow{j} K(\mathbb{Z}, 3)$ is a fibration and $G \xrightarrow{j} K(\mathbb{Z}, 3)$ corresponds to a generator of $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$. Repeating these processes one obtains the following sequence of fibrations:

5.3. (1) $B\tilde{G} \xrightarrow{Bf} BG \rightarrow K(\mathbb{Z}, 4)$, where $B\tilde{G}$ is 4-connected.

(2) $K(\mathbb{Z}, 3) \rightarrow B\tilde{G} \xrightarrow{Bf} BG$.

- (3) $G \xrightarrow{j} K(\mathbf{Z}, 3) \rightarrow B\tilde{G}$.
 (4) $\tilde{G} \xrightarrow{f} G \xrightarrow{j} K(\mathbf{Z}, 3)$, where $\tilde{G}$ is 3-connected.
 (5) $K(\mathbf{Z}, 2) \rightarrow \tilde{G} \xrightarrow{f} G$.
 (6) $\Omega G \rightarrow K(\mathbf{Z}, 2) \rightarrow \tilde{G}$.
 (7) $\Omega\tilde{G} \xrightarrow{\Omega f} \Omega G \rightarrow K(\mathbf{Z}, 2)$, where $\Omega\tilde{G}$ is 2-connected.
 (8) $K(\mathbf{Z}, 1) \rightarrow \Omega\tilde{G} \xrightarrow{\Omega f} \Omega G$.

Because they are given by the path fibrations and because all the fibres except (1) are loop spaces, we have

5.3'. (2) $\sim$ (8) of 5.3 are principal fibrations.

Hereafter, let G be a compact, 1-connected, simple Lie group whose (rational) type is

5.4. $(3 = 2m_1 + 1, 2m_2 + 1, \dots, 2m_l + 1)$, $m_1 = 1 < m_2 \leq \dots \leq m_l$.

Let p be a prime.

We first consider ΩG and $\Omega\tilde{G}$.

LEMMA 5.5. (1) $H^*(\Omega G; \mathbf{Z}_p) = \bigotimes_j \mathbf{Z}_p[y_{2n_j}]/(y_{2n_j}^{s_j})$ with $|y_{2n_j}| = 2n_j$, where s_j is the power of p (or ∞) and the following equality holds:

$$\prod_{i=1}^l (1 - t^{2m_i})^{-1} = \prod_j (1 - t^{2s_j n_j}) (1 - t^{2n_j})^{-1}.$$

(2) If we express $H^*(\Omega G; \mathbf{Z}_p) = \mathbf{Z}_p[y_2]/(y_2^{p^f}) \otimes B^*$ in (1), we have

$$H^*(\Omega\tilde{G}; \mathbf{Z}_p) \cong \Lambda(y_{2p^f-1}) \otimes B^* \quad \text{with } |y_{2p^f-1}| = 2p^f - 1$$

and $1 \otimes B^*$ corresponds to the image of $\Omega f^*: H^*(\Omega G; \mathbf{Z}_p) \rightarrow H^*(\Omega\tilde{G}; \mathbf{Z}_p)$.

PROOF. $H^*(\Omega G; \mathbf{Z}_p)$ consists only of elements of even degree by Theorem 4.16 of Chapter VI and its Poincaré series is $\prod (1 - t^{2m_i})^{-1}$. Applying Corollary 1.4 to the Hopf algebra $H^*(\Omega G; \mathbf{Z}_p)$ we see that it is expressed as in the form (1).

(2) Since $H^*(K(\mathbf{Z}, 1); \mathbf{Z}_p) = \Lambda(u_1)$ in the fibration of 5.3(8) and since $\Omega\tilde{G}$ is 2-connected, in the spectral sequence $\{E_r\}$:

$$E_2 = H^*(\Omega G; \mathbf{Z}_p) \otimes \Lambda(u_1) \Rightarrow H^*(\Omega\tilde{G}; \mathbf{Z}_p)$$

associated with (8) we have $d_2(1 \otimes u_1) = y_2 \otimes 1$ up to a nonzero coefficient. Computing d_2 , we have

$$E_3 = H(E_2) = H(\mathbf{Z}_p[y_2]/(y_2^{p^f}) \otimes \Lambda(u_1)) \otimes B^* = \Lambda(y_2^{p^f-1} \otimes u_1) \otimes B^*.$$

Obviously $E_3 = E_\infty$ and hence (2) is obtained. Q.E.D.

REMARK. The case $s_j = \infty$ does not actually occur.

We make use of Theorem 2.17 or Theorem 2.28 to obtain $H^*(\tilde{G}; \mathbb{Z}_p)$ from $H^*(\Omega\tilde{G}; \mathbb{Z}_p)$.

Mod p cohomology of G and $\tilde{G}$. When we know $H^*(G; \mathbb{Z}_p)$ up to a certain degree, there is a case where we can obtain the whole structure of it by Theorem 1.3. We put this in the form of a lemma.

LEMMA 5.6. *Let X be a 0-connected H -space of finite type and N be a positive integer.*

(1) *Let p be an odd prime. Suppose that there are elements x_i and y_j of $H^*(X; \mathbb{Z}_p)$ satisfying $|x_i| = 2a_i < N$ and $|y_j| = 2b_j + 1 < N$ and h'_i a power of p such that*

$$H^*(X; \mathbb{Z}_p) = \mathbb{Z}_p[x_i]/(x_i^{h'_i}) \otimes \Lambda(y_j) \text{ for degrees } < N.$$

Then by replacing x_i suitably modulo decomposables, we obtain a natural monomorphism

$$\mathbb{Z}_p[x_i]/(x_i^{h'_i}) \otimes \Lambda(y_j) \hookrightarrow H^*(X; \mathbb{Z}_p)$$

for some h_i a power of p satisfying $h'_i \leq h_i \leq \infty$. In addition, by adding suitable generators x_i and y_j of degrees $\geq N$, we have

$$H^*(X; \mathbb{Z}_p) = \mathbb{Z}_p[x_i]/(x_i^{h_i}) \otimes \Lambda(y_j).$$

In particular, if $X = G$ is a compact Lie group, the following equality holds:

$$\dim G = \sum_{k=1}^l (2m_k + 1) = \sum_i 2a_i(h_i - 1) + \sum_j (2b_j + 1).$$

(2) *If $p = 2$, the assumption is replaced by*

$$H^*(X; \mathbb{Z}_2) = \mathbb{Z}_2[x_i]/(x_i^{h'_i}) \text{ with } |x_i| = a_i < N \text{ in degrees } < N$$

for h'_i a power of 2, and the conclusion is

$$H^*(X; \mathbb{Z}_2) = \mathbb{Z}_2[x_i]/(x_i^{h_i})$$

by replacing x_i , adding x_i with $|x_i| = a_i \geq N$ and h_i the power of 2 ($h'_i \leq h_i \leq \infty$).

For a compact Lie group $X = G$

$$\dim G = \sum_{k=1}^l (2m_k + 1) = \sum_i a_i(h_i - 1).$$

PROOF. We consider the ring structure of $H^*(X; \mathbb{Z}_p)$ by Theorem 1.3 (Corollary 1.4) and can compare it with the assumption in degrees $< N$. $\dim G = \sum (2m_k + 1)$ coincides with the degree $\sum 2a_i(h_i - 1) + \sum (2b_j + 1)$

of the element $\prod_i x_i^{h_i-1} \prod_j y_j$ of maximal degree. Similarly for the case $p = 2$. Q.E.D.

In order to obtain $H^*(G; \mathbb{Z}_p)$ from the results on $H^*(\tilde{G}; \mathbb{Z}_p)$, we make use of the fibration 5.3(4) in addition to the above lemma.

LEMMA 5.7. Let $j: G \rightarrow K(\mathbb{Z}, 3)$ be the map in 5.3(4) corresponding to a generator of $H^3(G)$ and $j^*: H^*(\mathbb{Z}, 3; \mathbb{Z}_p) \rightarrow H^*(G; \mathbb{Z}_p)$ the homomorphism induced by j .

(1) If $p \neq 2$, denoting

$$H^*(\mathbb{Z}, 3; \mathbb{Z}_p) = \Lambda(u_3, u_{2p+1}, u_{2p^2+1}, \dots) \otimes \mathbb{Z}_p[u_{2p+2}, u_{2p^2+2}, \dots],$$

where $\varphi^{p^i} u_{2p^i+1} = u_{2p^{i+1}+1}$ and $\beta u_{2p^i+1} = u_{2p^i+2}$, we have that $\text{Ker } j^*$ is the ideal generated by a part of $\{u_{2p^f+1}, (u_{2p^f+1})^{p^f}; f \geq 0\}$.

(2) If $p = 2$, denoting

$$H^*(\mathbb{Z}, 3; \mathbb{Z}_p) = \mathbb{Z}_2[u_3, u_5, u_9, \dots, u_{2^i+1}, \dots],$$

where $Sq^{2^i} u_{2^i+1} = u_{2^{i+1}+1}$, then we have that $\text{Ker } j^*$ is the ideal generated by a part of $\{(U_{i+1})^{2^f}; f \geq 0\}$.

PROOF. Any two maps of a CW -complex to $K(\mathbb{Z}, 3)$ are homotopic if they are homotopic on the 3-skeleton of the complex, since $\pi_i(K(\mathbb{Z}, 3)) = 0$ for $i > 3$. It follows that j is an H -map and that j^* is a homomorphism of Hopf algebras. Therefore $\text{Im } j^*$ itself is a Hopf algebra, to which Theorem 1.3 can be applied. Checking the generators of $\text{Ker } j^*$ successively from the lowest degree, we see that the only possibilities for the degrees of the generators are $2p^i + 1$ and $p^f(2p^i + 2)$ ($2^f(2^i + 1)$ if $p = 2$). On the other hand, by considering the diagonal map we see that any monomial other than u_{2p^i+1} and $(u_{2p^i+2})^{p^f}$ ($(x_{2^i+1})^{2^f}$ if $p = 2$) can not be a generator of $\text{Ker } j^*$. Thus the lemma is obtained. Q.E.D.

We use the notation of Lemma 5.7 in the following theorem.

THEOREM 5.8. The following holds for the fibration 5.3(4):

(1) When $p \neq 2$; suppose that

$$H^*(\tilde{G}; \mathbb{Z}_p) = \Lambda(x_\alpha) \text{ with } |x_\alpha| \text{ odd } \leq N \text{ for degrees } \leq N.$$

Then we can choose a generator x_α such that x_α is transgressive, $\tau(x_\alpha) = (u_{2p^i+2})^{p^f}$ for $f \geq 0$ if $\tau(x_\alpha) \neq 0$, and there is only one x_α in this degree satisfying $\tau(x_\alpha) \neq 0$. Then if we put $\{x_{\alpha'}\} = \{x_\alpha \mid \tau(x_\alpha) = 0\}$, there exists $\bar{x}_{\alpha'}$ satisfying $f^*(\bar{x}_{\alpha'}) = x_{\alpha'}$ such that

$$H^*(G; \mathbb{Z}_p) = \text{Im } j^* \otimes \Lambda(\bar{x}_{\alpha'}),$$

$$\text{Im } j^* \cong H^*(\mathbb{Z}, 3; \mathbb{Z}_p) / (\tau(x_\alpha)) \text{ for degrees } \leq N.$$

(2) When $p = 2$; similar results hold, replacing $\Lambda(x_\alpha)$ and $\Lambda(\bar{x}_{\alpha'})$ with $\Delta(x_\alpha)$ and $\Delta(\bar{x}_{\alpha'})$ respectively and taking x_α so that $\tau(x_\alpha) = (u_{2'+1})^{2'}$.

PROOF. By induction on N we suppose that the theorem is true for degrees $< N$. Obviously $\{\tau(x_\alpha) \mid \tau(x_\alpha) \neq 0, |x_\alpha| < N\}$ is a regular sequence and so by Theorem 2.24 x_α with $|x_\alpha| = N$ can be chosen to be transgressive. If $\tau(x_\alpha) \neq 0$, it is a new generator of $\text{Ker } j^*$ and is of the form in the theorem up to a constant ($\neq 0$) multiple by Lemma 5.7. If there is another generator x_β in the same degree, we can choose it modulo x_α so that $\tau(x_\beta) = 0$. The theorem can be proved from these facts and the second half of Theorem 2.24. Q.E.D.

Mod p cohomology of the exceptional groups (the case without p -torsion)

THEOREM 5.9. If we suppose

$$m_1 < \text{Min}(pm_2, p^2 - 1)$$

for a compact, 1-connected, simple Lie group G of the type 5.4, then

$$H^*(G; \mathbf{Z}_p) = \Lambda(x_3, x_{2m_2+1}, \dots, x_{2m_l+1}).$$

Thus the (co)homology of G has no p -torsion.

PROOF. If $p = 2$, $SU(3)$ is the only G satisfying the assumption; then the theorem is obvious. So let $p > 2$.

We consider two cases depending on the generator y_2 of $H^2(\Omega G; \mathbf{Z}_2)$. The case $y_2^p \neq 0$. By the assumption of the theorem and by Lemma 5.5 we have

$$H^*(\Omega G; \mathbf{Z}_p) = \mathbf{Z}_p[y_2, y_{2m_2}, \dots, y_{2m_l}], \quad H^*(\Omega \tilde{G}; \mathbf{Z}_p) = \mathbf{Z}_p[y_{2m_2}, \dots, y_{2m_l}]$$

for degrees $s < 2m_l + 2$.

Moreover, applying Theorem 2.17 to $X = \tilde{G}$ we have for degrees $< 2m_l + 2$

$$H^*(\tilde{G}; \mathbf{Z}_p) = \Lambda(x_{2m_2+1}, \dots, x_{2m_l+1}).$$

Since $|u_{2p+2}^p| > |y_{2p+2}| = 2p^2 + 2 > 2m_l + 2$, we choose x_{2m_l+1} so as to be transgressive by Theorem 5.8; we have $\tau(x_{2m_l+1}) \neq 0$ only if $m_l = p$ and $\tau(x_{2m_l+1}) = u_{2p+2}$. In this case, using j^*u_{2p+1} in place of x'_{2m_l+1} , we obtain

$$H^*(G; \mathbf{Z}_p) = \Lambda(j^*u_3, x'_{2m_2+1}, \dots, x'_{2m_l+1}) \otimes B^*$$

for degrees $< 2m_l + 2$, where $B^* = \mathbf{Z}_p$ or $B^* = \mathbf{Z}_p[j^*u_{2p+2}] \otimes \Lambda(j^*u_{2p+1})$. Since $\dim G = \sum_{i=1}^l (2m_i + 1)$, by Lemma 5.6 (1)

$$H^*(G; \mathbf{Z}_p) = \Lambda(j^*u_3, x'_{2m_2+1}, \dots, x'_{2m_l+1})$$

holds for all degrees. Renaming the generators we obtain the theorem.

The case $y_2^p = 0$. Similarly one has

$$H^*(\Omega G; \mathbf{Z}_p) = \mathbf{Z}_p[y_2]/(y_2^p) \otimes \mathbf{Z}_p[y_{2p}, y_{2m_2}, \dots, y_{2m_l}],$$

$$H^*(\Omega \tilde{G}; \mathbf{Z}_p) = \Lambda(y_{2p-1}) \otimes \mathbf{Z}_p[y_{2p}, y_{2m_2}, \dots, y_{2m_l}],$$

$$H^*(\tilde{G}; \mathbf{Z}_p) = \mathbf{Z}_p[x_{2p}] \otimes \Lambda(x_{2p+1}, x_{2m_2+1}, \dots, x_{2m_l+1}),$$

for degrees $< 2m_l + 2$. Here if $p > m_l$, $\mathbf{Z}_p[x_{2p}] \otimes \Lambda(x_{2p+1})$ can be excluded and the rest is similar to the above. If $p \leq m_l$, we need to generalize Theorem 5.8(1) a little bit. That is, we also consider the case $H^*(\tilde{G}; \mathbf{Z}_p) = \mathbf{Z}_p[x_{2p}] \otimes \Lambda(x_\alpha)$. By Theorem 2.24 we can take x_{2p} to be transgressive. If $\tau(x_{2p}) = 0$, then there is $\bar{x}_{2p}$ such that $j^*(\bar{x}_{2p}) = x_{2p}$ and so by Theorem 2.25 we have the form $H^*(G; \mathbf{Z}_p) = \text{Im } j^* \otimes \mathbf{Z}_p[\bar{x}_{2p}] \otimes \Lambda(x'_\alpha)$, which does not fit in with $\dim G$. Hence we have $\tau(x_{2p}) \neq 0$. Up to a nonzero coefficient, $\tau(x_{2p}) = u_{2p+1}$, which implies $\tau(\beta x_{2p}) = u_{2p+2}$, and hence by Theorem 2.25 $H^*(G; \mathbf{Z}_p)$ is of the form in the theorem for degrees $< 2m_l + 2$, where $\text{Ker } j^*$ contains u_{2p+1} and u_{2p+2} . In any case, it is the same as the case $y_2^p \neq 0$ in the result. Q.E.D.

REMARK 5.9'. Since $u_{2p+1} = \wp^1 u_3$ in the proof of the theorem,

(1) $y_2^p \neq 0$ and $p \leq m_l$ implies $\wp^1 x_3 = x_{2p+1} \neq 0$,

(2) $y_2^p = 0$ and $p \leq m_l$ implies $\wp^1 x_3 = 0$.

COROLLARY 5.10. Under the assumption of Theorem 5.9, x_{2m_i+1} for $i = 1, 2, \dots, l$ can be chosen to be universally transgressive so that

$$H^*(BG; \mathbf{Z}_p) = \mathbf{Z}_p[y_4, y_{2m_2+2}, \dots, y_{2m_l+2}],$$

where $y_{2m_i+2} = \tau(x_{2m_i+1})$ is the transgression image of the x_{2m_i+1} . Hence the (co)homology of BG is p -torsion free.

PROOF. Use the transgression theorem (Theorem 2.9). Q.E.D.

Using the classification of the simple groups, one obtains the following theorem.

THEOREM 5.11. The (co)homology of a compact, 1-connected, simple group G and its classifying space BG are p -torsion free in the following cases:

$$p \geq 2, \quad G = SU(n), Sp(n);$$

$$p \geq 3, \quad G = G_2, Spin(n);$$

$$p \geq 5, \quad G = F_4, E_6, E_7;$$

$$p \geq 7, \quad G = E_8.$$

PROOF. The cases of classical type are already known. For the case of the exceptional groups; the value of (m_2, m_l) is (5, 5) for G_2 , (5, 11) for F_4 , (4, 11) for E_6 , (5, 17) for E_7 , and (7, 29) for E_8 and hence the values p in the above satisfy the assumption of Theorem 5.9. Q.E.D.

Mod p cohomology of the exceptional groups (the case with p -torsion). Of the cases left out in Theorem 5.11, the case G of the classical type ($p = 2$, $G = \text{Spin}(n)$ for $n \geq 7$) is known to have p -torsion. The same is true for the exceptional groups.

THEOREM 5.12. *The (co)homology of a 1-connected exceptional group G has p -torsion in the following cases:*

$$G = G_2 \ (p = 2); \quad G = F_4, E_6, E_7 \ (p = 2, 3); \quad G = E_8 \ (p = 2, 3, 5).$$

PROOF. By Theorems 7.12, 7.19, and 7.15 of Chapter VI, $f_*: \pi_i(G) \rightarrow \pi_i(K(\mathbf{Z}, 3))$ is an isomorphism for $i \leq 2p + 1$ and an epimorphism for $i = 2p + 2$ for the above G and the values of p . Therefore $f_*: H_{2p+1}(G) \cong H_{2p+1}(K(\mathbf{Z}, 3))$ by the theorem of J. H. C. Whitehead. On the other hand we have $\beta \wp^1 x_3 \neq 0$ for $p \neq 2$ and $Sq^1 Sq^2 x_3 = x_3^2 \neq 0$ for $p = 2$ in $H^*(\mathbf{Z}, 3; \mathbf{Z}_p)$. This shows that $H_{2p+1}(K(\mathbf{Z}, 3))$ and hence $H_{2p+1}(G)$ has p -torsions. Q.E.D.

Let us study the cases $p = 3$ and 5 of the above

By Theorems 7.11, 7.18, and 7.14 of Chapter VI we have

5.13. *For the generator y_2 of $H^*(\Omega G; \mathbf{Z}_p)$*

(1) $y_2^p \neq 0$ if $(G, p) = (F_4, 3), (E_6, 3), (E_8, 5)$.

(2) $y_2^9 \neq 0$ if $(G, p) = (E_7, 3), (E_8, 3)$.

THEOREM 5.14. $H^*(F_4; \mathbf{Z}_3) = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}),$

$$H^*(E_6; \mathbf{Z}_3) = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}),$$

where $\wp^1 x_3 = x_7$ and $\beta x_7 = x_8$.

PROOF. Let $G = F_4$ or $G = E_6$. Their types are $(3, 11, 15, 23)$ and $(3, 9, 11, 15, 17, 23)$. Using 5.13(1), similarly to the first half of the proof of Theorem 5.9, one obtains

$$H^*(\tilde{F}_4; \mathbf{Z}_3) = \Lambda(x_{11}, x_{15}), \quad H^*(\tilde{E}_6; \mathbf{Z}_3) = \Lambda(x_9, x_{11}, x_{15}, x_{17})$$

for degrees < 18 . Since

$$H^*(\mathbf{Z}, 3; \mathbf{Z}_3) = \Lambda(u_3, u_7, u_{19}, \dots) \otimes \mathbf{Z}_3[u_8, u_{20}, \dots],$$

we have $\tau(x_i) = 0$ in Theorem 5.8, and hence

$$H^*(G; \mathbf{Z}_3) \cong \mathbf{Z}_3[x_8] \otimes \Lambda(x_3, x_7) \otimes H^*(\tilde{G}; \mathbf{Z}_3), \quad \wp^1 x_3 = x_7 \text{ and } \beta x_7 = x_8$$

holds for degrees < 18 . We obtain the theorem from Lemma 5.6(1), as $\dim G = 3 + \sum |x_i| + 23 = 3 + 7 + 8 \times 2 + \sum |x_i|$ (where $i = 11, 15$ or $i = 9, 11, 15, 17$). Q.E.D.

THEOREM 5.15.

$$H^*(E_7; \mathbf{Z}_3) = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}),$$

$$H^*(E_8; \mathbf{Z}_3)$$

$$= \mathbf{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}),$$

where $\wp^1 x_3 = x_7$, $\beta x_7 = x_8$, $\wp^3 x_7 = x_{19}$, and $\beta^1 x_{19} = x_{20}$ ($x_{20} = 0$ in E_7).

PROOF. First, consider E_7 . Since its type is $(3, 11, 15, 19, 23, 27, 35)$, one can compute, for degrees < 30 , as follows:

$$H^*(\Omega E_7; \mathbf{Z}_3) = \mathbf{Z}_3[y_2, y_{10}, y_{14}, y_{18}, y_{22}, y_{26}],$$

$$H^*(\Omega \tilde{E}_7; \mathbf{Z}_3) = \mathbf{Z}_3[y_{10}, y_{14}, y_{18}, y_{22}, y_{26}],$$

$$H^*(\tilde{E}_7; \mathbf{Z}_3) = \Lambda(x_{11}, x_{15}, x_{19}, x_{23}, x_{27}),$$

using $y_2^9 \neq 0$ in 5.13 (2), Lemma 5.5, and Theorem 2.17. Since $H^*(\mathbf{Z}, 3; \mathbf{Z}_3) = \Lambda(u_3, u_7, u_{19}, u_{35}, \dots) \otimes \mathbf{Z}_3[u_8, u_{20}, u_{36}, \dots]$, the only possibilities for $\tau(x_i) \neq 0$ are $\tau(x_{19}) = u_{20}$ and $\tau(x_{23}) = u_8^3$. Furthermore, taking Lemma 5.6 into account, one sees that it is of the following form:

$$H^*(E_7; \mathbf{Z}_3) = \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}) \otimes B^* \otimes C^* \otimes A^*,$$

$$B^* = \mathbf{Z}_3[x_8]/(x_8^3) \quad \text{or} \quad = \Lambda(x_{23}) \otimes \mathbf{Z}_3[x_8]/(x_8^{3'}) \quad \text{with } f \geq 2,$$

$$C^* = \mathbf{Z}_3 \quad \text{or} \quad = \Lambda(x_{19}) \otimes \mathbf{Z}_3[x_{20}]/(x_{20}^{3'}) \quad \text{with } g \geq 1,$$

$$A^* = \Lambda(x_\alpha) \otimes \mathbf{Z}_3[x_\beta]/(x_\beta^{h_\beta}) \quad (|x_\alpha|, |x_\beta| \geq 30).$$

Here, since $\dim E_7 = 133 = |x_3 x_7 x_{11} x_{15} x_{19} x_{27} x_{35}| + 35$, we must have $B^* \otimes C^* \otimes A^* = \mathbf{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_{35})$ in order for the maximal degree to be 133. We have $\wp^1 x_3 = x_7$, $\beta x_7 = x_8$, and $\wp^3 x_7 = x_{19}$ since $x_i = j^*(u_i)$ for $i = 3, 7, 8, 19$ in $H^*(E_7; \mathbf{Z}_3)$. And, $\beta x_{19} = j^*(u_{20}) = j^*\tau(x_{19}) = 0$.

Similarly considering for the type $(3, 15, 23, 27, 35, 39, 47, 59)$ of E_8 , one has

$$H^*(\tilde{E}_8; \mathbf{Z}_3) = \Lambda(x_{15}, x_{23}, x_{27}, x_{35}, x_{39}) \quad \text{for degrees } < 42$$

$$\text{and } H^*(E_8; \mathbf{Z}_3) = \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}) \otimes B^* \otimes C^* \otimes A^*,$$

$$B^* = \mathbf{Z}_3[x_8]/(x_8^3) \quad \text{or} \quad = \Lambda(x_{23}) \otimes \mathbf{Z}_3[x_8]/(x_8^{3'}) \quad \text{with } f \geq 2,$$

$$C^* = \mathbf{Z}_3[x_{20}]/(x_{20}^{3'}) \quad \text{with } g \geq 1,$$

$$A^* = \Lambda(x_\alpha) \otimes \mathbf{Z}_3[x_\beta]/(x_\beta^{h_\beta}) \quad (|x_\alpha|, |x_\beta| \geq 42).$$

Since $\dim E_8 = 248 = |x_3 x_7 x_{15} x_{19} x_{27} x_{35} x_{39} x_{47} x_{59}| + 47$, we see that $B^* \otimes C^* \otimes A^* = \mathbf{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_{47})$.

The results on cohomology operations are similar to E_7 . Q.E.D.

THEOREM 5.16.

$$H^*(E_8; \mathbf{Z}_5) = \mathbf{Z}_5[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}),$$

where $\wp^1 x_3 = x_{11}$, $\beta x_{11} = x_{12}$.

PROOF. Considering the type of E_8 , similarly to the proof of Theorem 5.15, we have

$$H^*(\tilde{E}_8; \mathbf{Z}_5) = \Lambda(x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}),$$

$$H^*(\mathbf{Z}, 3; \mathbf{Z}_5) = \mathbf{Z}_5[u_{12}] \otimes \Lambda(u_3, u_{11})$$

for degrees < 50 . Here there is no possibility for $\tau(x_i) \neq 0$, and we have $H^*(\mathbf{Z}, 3; \mathbf{Z}_5) \otimes H^*(\tilde{E}_8; \mathbf{Z}_5) \subset H^*(E_8; \mathbf{Z}_5)$ for degrees < 50 , hence the theorem follows from Lemma 5.6(1). Q.E.D.

REMARK 5.17. $H^*(\text{Ad}(E_6); \mathbf{Z}_p) \cong H^*(E_6; \mathbf{Z}_p)$ for $p \neq 3$.

$$H^*(\text{Ad}(E_6); \mathbf{Z}_3) \cong \mathbf{Z}_3[x_2, x_8]/(x_2^9, x_8^3) \otimes \Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{15}).$$

$$H^*(\text{Ad}(E_7); \mathbf{Z}_p) \cong H^*(E_7; \mathbf{Z}_p) \text{ for } p \neq 2.$$

Cohomology operations. We consider the case where G is an exceptional group in Theorem 5.9 (Theorem 5.11).

THEOREM 5.18. *We can choose suitably a universal transgressive generator x_{2m_i+1} so that it satisfies the following:*

$$(\alpha) \quad \wp^1 x_{2m_i+1} = x_{2m_j+1} \text{ if } m_i \neq p \text{ and } m_j = m_i + p - 1.$$

$$(\beta) \quad \wp^1 x_{2m_i+1} = 0 \text{ if } m_i = p \text{ or there is no } m_j \text{ such that } m_j = m_i + p - 1.$$

PROOF. $x_{2m_i+1} = \sigma y_{2m_i+2}$ is universally transgressive and hence is primitive. Thus $\wp^1 x_{2m_i+1}$ is also primitive. Since a primitive element of

$$H^*(G; \mathbf{Z}_p) = \Lambda(x_{2m_i+1})$$

is a constant multiple of x_{2m_i+1} , the second half of (β) is obtained. Also for the first half of (β) ; if $m_i = p$, $\wp^1(x_3) = x_{2p+1}$ by (α) , hence $\wp^1 x_{2p+1} = \wp^1 \wp^1 x_3 = 2\wp^2 x_3 = 0$ ((4.23) and Theorem 4.17 (4)). As for (α) , operate σ after applying Theorem 4.24 to $H^*(G; \mathbf{Z}_p)$ for each G . We illustrate this with the case $G = E_8$ and $p = 7$. By Theorem 4.24 (1) we have $\wp^1 y_{48} = y_{60}$, $\wp^1 y_{36} = y_{48}$, and $\wp^1 y_{28} = y_{40}$. Furthermore by Theorem 4.24 (2), we have $\wp^1 y_{24} = y_{36}$, and moreover $\wp^1 y_4 = y_{16}$. Applying σ to them, we obtain (α) . The other cases are similar and so are omitted. Q.E.D.

We have the following result for the case of the classical type.

THEOREM 5.19. *For G of the classical type, (β) of Theorem 5.18 and the formula below are satisfied:*

$$\wp^k x_{2m_i+1} = \binom{m_i}{k} x_{2m_i+2k(p-1)+1} \quad \text{where } p \text{ is an odd prime.}$$

PROOF. First, the case $G = SU(l+1)$; the map $f = \bar{\beta}_{1,l}: SP_l(\mathbb{C}) \rightarrow SU(l+1)$ induces an epimorphism

$$f^*: H^*(SU(l+1); \mathbb{Z}_p) \rightarrow H^*(SP_l(\mathbb{C}); \mathbb{Z}_p)$$

such that $\text{Ker } f^*$ is the set of the decomposables. Since $\wp^k x_{2m_i+1}$ is a constant multiple of $x_{2m_i+2k(p-1)+1}$ as in the proof of Theorem 5.18, it suffices to show the formula in the theorem for the image of f^* . Further, it suffices to show $\wp^k(u^r) = \binom{m}{k} u^{m+k(p-1)}$ for the generator u of $H^2(P_l(\mathbb{C}); \mathbb{Z}_p)$, since $S: H^*(P_l(\mathbb{C}); \mathbb{Z}_p) \cong H^{*+1}(SP(\mathbb{C}); \mathbb{Z}_p)$ is an isomorphism commuting with $\wp^k$. The formula, however, is easily obtained from the Cartan formula and $\wp^1(u) = u^p$ and $\wp^i(u) = 0$ for $i > 1$.

For the case $G = Sp(n)$ use the fact that the generators correspond through the inclusion (Theorem 6.7 of Chapter III).

For the case $G = \text{Spin}(n)$ G can be replaced with $SO(n)$ since $p \neq 2$. Next reduce to the case SU , using the decomposition in Theorem 6.23 of Chapter IV. Q.E.D.

The following corollary is obtained by studying the types of the simple Lie groups in each case.

COROLLARY 5.20. *Let p be an odd prime. If $(G, p) \neq (G_2, 3)$ and $p \leq m_1$, then there is m_i satisfying $m_i = p$ such that $\wp^1 x_3 = x_{2p+1} \neq 0$. Thus the case (2) in Remark 5.9' is $G = G_2$ and $p = 3$.*

REMARK. For the case with p -torsion; we can choose the generators x_{2m_i+1} such that $\bar{\phi}(x_{11}) = x_8 \otimes x_3$ for $p = 3$ and $\bar{\phi}(x_{15}) = x_{12} \otimes x_3$ for $p = 5$. The images of $\bar{\phi}$ of the generators are also determined, and by using them, the cohomology operations in the case with p -torsion are determined.

§6. Mod 2 cohomology of the exceptional groups

Mod 2 cohomology of G_2 and F_4

LEMMA 6.1. *Let $G = G_2$ or F_4 . Then*

$$j^*: H^*(\mathbb{Z}, 3; \mathbb{Z}_2) \rightarrow H^*(G; \mathbb{Z}_2)$$

is an isomorphism for degrees < 8 .

PROOF. By Theorem 7.11 of Chapter VI we have $y_2^2 \neq 0$ for the generator y_2 of $H^2(\Omega G; \mathbb{Z}_2)$. Since the type of G_2 and F_4 are $(3, 11)$ and

(3, 11, 15, 23) respectively, we obtain by Lemma 5.5

$$\begin{aligned} H^*(\Omega G; \mathbf{Z}_2) &= \mathbf{Z}_2[y_2, y_8, y_{10}]/(y_2^4) \quad \text{or} \quad = \mathbf{Z}_2[y_2, y_{10}] \\ H^*(\Omega \tilde{G}; \mathbf{Z}_2) &= \Lambda(y_7) \otimes \mathbf{Z}_2[y_8, y_{10}] \quad \text{or} \quad = \mathbf{Z}_2[y_{10}] \end{aligned}$$

for degrees < 14 .

Using Lemma 5.6 or the suspension isomorphism, one obtains

$$H^*(\tilde{G}; \mathbf{Z}_2) = \Delta(x_8, x_9, x_{11}) \quad \text{or} \quad \Delta(x_{11}) \quad \text{for degrees} < 15. \quad (6.1')$$

The lemma is now obtained from the cohomology exact sequence associated with 5.3(4). Q.E.D.

THEOREM 6.2. $H^*(G_2; \mathbf{Z}_2) = \mathbf{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5)$, where $x_5 = Sq^2 x_3$.

PROOF. Since $H^*(Z, 3; \mathbf{Z}_2) = \mathbf{Z}_2[u_3, u_5, u_9, \dots]$ where $u_5 = Sq^2 u_3$, the theorem is true for degrees < 8 by Lemma 6.1. The theorem is obtained by Lemma 5.6(2), as $\dim G_2 = 3 + 11 = |x_3^3 x_5|$. Q.E.D.

COROLLARY 6.3. $H^*(BG_2; \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_{10}]$, where $y_6 = Sq^2 y_4$ and $y_7 = Sq^3 y_4 = Sq^1 y_6$.

PROOF. We have $H^*(G_2; \mathbf{Z}_2) = \Delta(x_3, x_5, x_3^2)$, where x_3 is transgressive so $x_5 = Sq^2 x_3$ and $x_3^2 = Sq^3 x_3$ are also transgressive. The corollary follows from Corollary 2.8(1). Q.E.D.

As for F_4 , consider its subgroups $\text{Spin}(9)$ and $SO(9) = \text{Spin}(9)/\mathbf{Z}_2$. By Corollary 5.10 and Theorem 5.12 of Chapter III one has

6.4. (1) $H^*(BSO(n); \mathbf{Z}_2) = \mathbf{Z}_2[w_2, w_3, \dots, w_{n-1}, w_n]$, where $w_k = 0$ for $k > n$.

(2)

$$Sq^j w_k = \sum_{i=0}^j \binom{k-i-1}{j-i} w_{k+j-i} w_i \quad \text{for } 0 \leq j \leq k.$$

LEMMA 6.5. (1) $H^*(B \text{Spin}(9); \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_7, y_8, y_{16}]$, where $y_6 = Sq^2 y_4$ and $y_7 = Sq^3 y_4 = Sq^1 y_6$.

(2) $H^*(\text{Spin}(9); \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_7, x_{15})$, where $x_i = \sigma(y_{i+1})$.

PROOF. We calculate using the spectral sequence of the fibration $B\mathbf{Z}_2 = K(\mathbf{Z}_2, 1) \xrightarrow{i} B \text{Spin}(9) \rightarrow BSO(9)$:

$$E_2 = H^*(BSO(9); \mathbf{Z}_2) \otimes H^*(\mathbf{Z}_2, 1; \mathbf{Z}_2) \Rightarrow H^*(B \text{Spin}(9); \mathbf{Z}_2).$$

Here we have $H^*(\mathbf{Z}_2, 1; \mathbf{Z}_2) = \mathbf{Z}_2[u_1]$ and $B \text{Spin}(9)$ is 4-connected, and so $d_2(1 \otimes u_1) = w_2 \otimes 1$. Computing E_3 , one has

$$\begin{aligned} E_3 &= H(E_2) = H(\mathbf{Z}_2[w_2] \otimes \Delta(u_1)) \otimes \mathbf{Z}_2[w_3, \dots, w_9] \otimes \mathbf{Z}_2[u_1^2] \\ &= \mathbf{Z}_2[w_3, \dots, w_9] \otimes \mathbf{Z}_2[u_1^2]. \end{aligned}$$

By the commutativity of the transgression with Sq^1 and Sq^2 , we have

$$d_3(1 \otimes u_1^2) = d_3(1 \otimes Sq^1 u_1) = Sq^1 w_2 \otimes 1 = w_3 \otimes 1$$

and

$$d_3(1 \otimes u_1^4) = d_3(1 \otimes Sq^2(u_1^2)) = Sq^2 w_3 \otimes 1 = (w_5 + w_3 w_2) \otimes 1 = w_5 \otimes 1.$$

Thus

$$E_4 = H(E_3) = \mathbf{Z}_2[w_4, w_5, \dots, w_9] \otimes \mathbf{Z}_2[u_1^4], \quad E_5 = E_4,$$

$$E_6 = H(E_5) = \mathbf{Z}_2[w_4, w_6, w_7, w_8, w_9] \otimes \mathbf{Z}_2[u_1^8].$$

Further using $E_9 = E_6$ and $d_9(1 \otimes u_1^8) = d_9(1 \otimes Sq^4(u_1^4)) = Sq^4 w_5 \otimes 1 = (w_9 + w_7 w_2 + w_6 w_3 + w_5 w_4) \otimes 1 = w_9 \otimes 1$, we have

$$E_{10} = H(E_9) = \mathbf{Z}_2[w_4, w_6, w_7, w_8] \otimes \mathbf{Z}_2[u_1^{16}], \quad E_{10} = E_{16}.$$

Finally, since u_1^{16} is also transgressive with the transgression image $\tau(u_1^{16}) = \tau(Sq^8(u_1^8)) = Sq^8 w_9 = w_9 w_8 = 0$, $d_{16}(1 \otimes u_1^{16}) = 0$, which implies $E_{16} = E_\infty$, and hence there exists y_{16} with $i^*(y_{16}) = u_1^{16}$. Putting $y_i = \rho^*(w_i)$ for $i = 4, 6, 7, 8$, we have $H^*(B\text{Spin}(9); \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_7, y_8, y_{16}]$. As we have $Sq^2 w_4 = w_6$ and $Sq^3 w_4 = Sq^1 w_6 = w_7$, (1) is proved.

(2) is obtained by applying Corollary 2.8(2) to (1). Q.E.D.

REMARK 6.5'. $H^*(B\text{Spin}(n); \mathbf{Z}_2)$ is not a polynomial ring for $n > 9$. For example,

$$H^*(B\text{Spin}(10); \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10}).$$

One can also show that $H^*(\text{Spin}(n); \mathbf{Z}_2)$ is not primitively generated and $H_*(\text{Spin}(n); \mathbf{Z}_2)$ is not commutative for $n > 9$ and $n \neq 2^f + 1$.

THEOREM 6.6. (1) $H^*(F_4; \mathbf{Z}_2) = \mathbf{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_{15}, x_{23})$, where $x_5 = Sq^2 x_3$, $x_3^2 = Sq^2 x_3 = Sq^1 x_5$, $x_{23} = Sq^8 x_{15}$, and each x_i is universally transgressive and hence primitive.

(2) $H^*(BF_4; \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_7, y_{16}, y_{24}]$, where $y_6 = Sq^2 y_4$, $y_7 = Sq^3 y_4 = Sq^1 y_6$, and $y_{24} = Sq^8 y_{16}$.

PROOF. (1) We have $H^*(F_4/\text{Spin}(9); \mathbf{Z}_2) = \mathbf{Z}_2[x_8]/(x_8^3)$ (Theorem 7.8 (1) of Chapter VI) and $H^*(\text{Spin}(9); \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_7, x_{15})$ in the spectral sequence of the fibre bundle $\text{Spin}(9) \xrightarrow{i} F_4 \xrightarrow{p} F_4/\text{Spin}(9)$:

$$E_2 = H^*(F_4/\text{Spin}(9); \mathbf{Z}_2) \otimes H^*(\text{Spin}(9); \mathbf{Z}_2) \Rightarrow H^*(F_4; \mathbf{Z}_2).$$

By Theorem 6.5 (2), the generators of $H^*(\text{Spin}(9); \mathbf{Z}_2)$ are suspension images and hence are universally transgressive. Therefore they are also transgressive in the above spectral sequence.

For degree reasons, we have $\tau(x_i) = 0$ for $i = 3, 5, 6$. As for x_7 , one has

$$(\alpha) \quad \tau(x_7) = x_8 \quad \text{or} \quad (\beta) \quad \tau(x_7) = 0.$$

For case (β) ; $H^*(F_4; \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_7)$ for degrees < 15 , which contradicts Lemma 6.1 as we have

$$H^*(\mathbf{Z}, 3; \mathbf{Z}_2) = \Delta(u_3, u_5, u_6, u_9, u_{10}, \dots).$$

Thus case (α) is true, and we have

$$H^*(F_4; \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_{15}, x_{23})$$

since $H(\mathbf{Z}_2[x_8]/(x_8^3) \otimes \Delta(x_7)) = \Delta(x_8^2 \otimes x_7)$. We see that $Sq^2 x_3 = x_5$, $x_3^2 = Sq^3 x_3 = Sq^1 x_5 = x_6$ by Lemma 6.1. The rest is shown by using Corollary 2.8(2) after (2) is proved

(2) Consider the spectral sequence

$$\begin{aligned} E_2 &= H^*(BF_4; \mathbf{Z}_2) \otimes \mathbf{Z}_2[x_8]/(x_8^3) \Rightarrow H^*(B \operatorname{Spin}(9); \mathbf{Z}_2) \\ &= \mathbf{Z}_2[y_4, y_6, y_7, y_8, y_{16}] \end{aligned}$$

of the fibre bundle $F_4/\operatorname{Spin}(9) \xrightarrow{i} B \operatorname{Spin}(9) \xrightarrow{\rho} BF_4$. The result (α) in (1) shows $i^*(y_8) = x_8$. Thus i^* is an epimorphism, and the spectral sequence is trivial: $E_2 = E_\infty$. Therefore, we see that ρ^* is a monomorphism so that we have

$$H^*(BF_4; \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_7, y_{16}, y_{24}]$$

where $\rho^* y_{16} = y_{16} + \dots$ and $\rho^* y_{24} = y_8^3 + \dots$. We see for degree reasons and by the Cartan formula that the ideal (y_4, y_6, y_7) is closed under Sq^i and hence we calculate modulo the ideal. By the Adem relation $Sq^8 Sq^{16} = Sq^{24} + Sq^{20} Sq^4 + Sq^{22} Sq^2 + Sq^{23} Sq^1$ we have $0 \neq y_{24}^2 = Sq^{24} y_{24} \equiv Sq^8 Sq^{16} y_{24}$ so $Sq^{16} y_{24} \neq 0$ and hence $Sq^{16} y_{24} \equiv y_{16} y_{24}$.

By the Cartan formula, $y_{24}^2 \equiv Sq^8 (y_{16} y_{24}) \equiv (Sq^8 y_{16}) y_{24} + y_{16} (Sq^8 y_{24})$. Hence $y_{24} \equiv Sq^8 y_{16} \pmod{(y_4, y_6, y_7)}$. Replacing y_{24} with $y_{24} = Sq^8 y_{16}$, we obtain (2). Q.E.D.

REMARK 6.6'. The $Sq^k x_i$ for $k > 0$ other than the ones stated in Theorem 6.6(1) are 0. For $Sq^i x_k$ must be x_{i+k} (except $x_6 = x_3^2$) if not 0 since it is primitive.

Closed subgroups of E_8 . Consider the root system of the compact, 1-connected exceptional Lie group E_8 as in 6.26 of Chapter V:

$$\left\{ \pm e_i \pm e_j \text{ for } 1 \leq i < j \leq 8, \frac{1}{2} \sum_{i=1}^8 (-1)^{\nu(i)} e_i \text{ with } \sum \nu(i) \text{ even} \right\}.$$

Since E_8 is 1-connected, the integer lattice $\Gamma(E_8) = \Gamma_0(E_8) = \exp^{-1}(e) \subset L(T^8) = \mathbf{R}^8$ is generated by the inverse root θ_r^* . Therefore we see

$$\Gamma(E_8) = \left\{ \frac{1}{2}(n_1, \dots, n_8) \mid n_i \in \mathbf{Z}, n_1 \equiv \dots \equiv n_8 \pmod{2}, \right. \\ \left. \sum_{i=1}^8 n_i \equiv 0 \pmod{4} \right\}. \quad (6.7)$$

THEOREM 6.8. *The compact, 1-connected groups E_7 and E_6 as well as $A_8 = SU(9)/\mathbf{Z}_3$, $SU(7)$, and $SU(6)$ are contained in E_8 as subgroups so that*

$$E_7 \supset E_6, \quad A_8 \cap E_7 \supset SU(7), \quad SU(7) \cap E_6 \supset SU(6).$$

In addition, the inclusions of these groups induce the isomorphisms of π_3 , H^3 , and H_3 .

PROOF. Let A_8 be given by $N(X_2)$ in Theorem 7.16 of Chapter V. A_8 has $\{\alpha_i \text{ for } i \neq 2, -\bar{\alpha}\} = \{\frac{1}{2}(e_1 - e_2 - \dots - e_7 + e_8), e_{i+1} - e_i \text{ for } 1 \leq i \leq 6, -e_7 - e_8\}$ as simple roots, and $\exp^{-1}(Z(A_8))$ is the intersection of the inverse images of $\mathbf{Z}$ by these simple roots and hence the following is obtained:

$$\exp^{-1}(Z(A_8)) = \left\{ \frac{1}{6}(n_1, \dots, n_7, -n_8) \mid n_i \in \mathbf{Z}, n_1 \equiv \dots \equiv n_8 \pmod{6}, \right. \\ \left. \sum_{i=1}^8 n_i \equiv 0 \pmod{12} \right\}.$$

Comparing with (6.7), we have $Z(A_8) \cong \mathbf{Z}_3$. $Z(SU(9)) \cong \mathbf{Z}_9$ implies $A_8 = SU(9)/\mathbf{Z}_3$.

Let T' be the image by $\exp$ of the line $L' = \{(0, \dots, 0, t, t)\}$ containing the inverse root $\bar{\alpha}^*$ of $\bar{\alpha}$ and consider $Z(T')$. The root system of $Z(T')$ is the same as the root system of E_7 in 6.26 of Chapter V. Therefore $Z(T')$ is locally isomorphic with $E_7 \times T'$ and there is a homomorphism $f: E_7 \rightarrow Z(T') \subset E_8$ such that $f(E_7) \cdot T' = Z(T')$. $\exp^{-1}(Z(f(E_7)))$ is given by $\bigcap_{i=1}^7 \alpha_i^{-1}(\mathbf{Z})$ in $T^7 = T^8|_{x_7 = -x_8}$. In particular, the point $(\frac{1}{2}, \dots, \frac{1}{2}, 0, 0)$ belongs not to $\Gamma(E_8)$ but to $\exp^{-1}(Z(f(E_7)))$ and hence $Z(f(E_7)) \supset \mathbf{Z}_2$, which shows $f: E_7 \cong f(E_7)$ since $Z(E_7) \cong \mathbf{Z}_2$.

Also for E_6 there is a homomorphism $f: E_6 \rightarrow E_7$ such that $f(E_6) \cdot T' = Z(T')$ for $T' = \exp\{(0, \dots, 0, 2t, t, -t)\}$ and that $\exp^{-1}(Z(f(E_6))) = \bigcap_{i=1}^6 \alpha_i^{-1}(\mathbf{Z})$ contains the point $(0, \dots, 0, \frac{2}{3}, \frac{2}{3}, -\frac{2}{3})$ in $T^6 = T^7|_{x_6 = x_7}$ and $Z(f(E_6)) \supset \mathbf{Z}_3$. We obtain $f: E_6 \cong f(E_6)$, since $Z(E_6) \cong \mathbf{Z}_3$.

Similarly one obtains $E_7 \supset A_6$ having $\{\alpha_1, \alpha_3, \alpha_4, \dots, \alpha_7\}$ as simple roots by $T' = \exp\{(t, \dots, t, -2t, 2t)\}$ and $E_6 \supset A_5$ having $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ as simple roots by $T' = \exp\{(t, \dots, t, -t, t)\}$. The image

$\exp(\frac{2}{7}, \dots, \frac{2}{7}, \frac{3}{7}, -\frac{3}{7})$ is an element of order 7 in $Z(A_6)$ and $Z(SU(7)) \cong \mathbf{Z}_7$ and hence $A_6 \cong SU(7)$. Similarly $\exp(\frac{1}{4}, \dots, \frac{1}{4}, \frac{5}{12}, \frac{5}{12}, -\frac{5}{12})$ is an element of order 6 in $Z(A_5)$ and $Z(SU(6)) \cong \mathbf{Z}_6$ and hence $A_5 \cong SU(6)$. We see that $A_5 \subset A_6 \subset A_8$ by the correspondences of the roots.

The subgroup S^3 corresponding to α_1 is a common subgroup to E_8 and all the subgroups in the theorem, and S^3 gives a generator of π_3 by Corollary 6.25 of Chapter VI. Hence the second half is obtained. Q.E.D.

COROLLARY 6.9. *The inclusion $SU(6) \subset SU(7) \subset A_8$ can be identified by a suitable isomorphism with the composition of the standard inclusion and the projection $SU(9) \rightarrow A_8 = SU(9)/\mathbf{Z}_3$.*

We quote the results on the special unitary group $SU(n)$ from Chapter III:

- 6.10. (1) $H^*(BSU(n); \mathbf{Z}_2) = \mathbf{Z}_2[c_2, c_3, c_4, \dots, c_n] \mid |c_i| = 2i$.
 (2) $H^*(SU(n); \mathbf{Z}) = \Lambda(e_3, e_5, e_7, \dots, e_{2n-1}), e_{2i-1} = \sigma(c_i)$.
 (3) (The Wu formula)

$$Sq^{2j}c_i = \sum_{k=0}^j \binom{i-k-1}{j-k} c_{i+j-k} c_k \quad \text{for } 0 \leq j \leq i,$$

where $c_0 = 1, c_a = 0$ ($a = 1, a > n$), $\binom{-1}{0} = 1$.

- (4) $Sq^{2j}e_{2i-1} = \binom{i-1}{j} e_{2i+2j-1}$ ($e_{2a-1} = 0$ for $a > n$).

The following special cases of the Wu formula are quite useful.

- 6.11. (1) $Sq^2c_2 = c_3, Sq^2c_4 = c_5, Sq^2c_6 = c_7, Sq^2c_8 = c_9, Sq^2c_{2i+1} = 0$.
 (2) $Sq^4c_3 = c_5 + c_3c_2, Sq^4c_4 = c_6 + c_4c_2, \dots$
 (3) $Sq^6c_4 = c_7 + c_5c_2 + c_4c_3, \dots$
 (4) $Sq^8c_5 = c_9 + c_7c_2 + c_6c_3 + c_5c_4$.
 (5) $Sq^{2i}c_8 = c_8c_i + i \cdot c_9c_{i-1}$ for $2 \leq i \leq 8$, if $n = 9$.

The mod 2 cohomology of E_6 . To begin with, we will determine the rough form of $H^*(E_6; \mathbf{Z}_2)$.

LEMMA 6.12. $H^*(E_6; \mathbf{Z}_2) = \mathbf{Z}_2[x_3]/(x_3^4) \otimes \Delta(x_5, x_9, x_{15}, x_{17}, x_{23})$, where $x_5 = Sq^2x_3, x_9 = Sq^4x_5, x_5^2 = x_9^2 = 0$. For the generator $y_2 \in H^2(\Omega E_6; \mathbf{Z}_2)$ we have

- (α) $Sq^8x_9 = x_{17}$ if $y_2^8 \neq 0$.
 (β) $Sq^8x_9 = 0$ if $y_2^8 = 0$.

PROOF. (α) Let $y_2^8 \neq 0$. Since the type of E_6 is (3, 9, 11, 15, 17, 23), by Lemma 5.5 we have

$$H^*(\Omega E_6; \mathbf{Z}_2) = \mathbf{Z}_2[y_2] \otimes \Delta(y_8, y_{10}, y_{14}, y_{16}, y'_{16}, y'_{20}, y_{22}), \\ H^*(\Omega \tilde{E}_6; \mathbf{Z}_2) = \Delta(y_8, y_{10}, y_{14}, y_{16}, y'_{16}, y'_{20}, y_{22}),$$

for degrees < 28 , where y_8 , y_{10} , and y_{14} are transgressive.

If we suppose that y_{16} and y'_{16} are also transgressive, by Theorem 2.7 we would have

$$H^*(\tilde{E}_6; \mathbf{Z}_2) = \mathbf{Z}_2[x_9, x_{11}, x_{15}, x_{17}, x'_{17}] \text{ for degrees } < 21.$$

Using the Adem relation $Sq^9 = Sq^2Sq^7 + Sq^8Sq^1$, we would have $x_9^2 = Sq^9x_9 = 0$, which is a contradiction. Therefore we may suppose that at least one of y_{16} and y'_{16} is not transgressive and hence $d_9(1 \otimes y_{16}) = x_9 \otimes y_8$. If y'_{16} is not transgressive either, $y'_{16} + y_{16}$ is transgressive. Choosing this element as y'_{16} , we have

$$H^*(\tilde{E}_6; \mathbf{Z}_2) = \Lambda(x_9) \otimes \mathbf{Z}_2[x_{11}, x_{15}, x'_{17}] \text{ for degrees } < 21,$$

as in Theorem 2.28 (2). We will calculate $H^*(E_6; \mathbf{Z}_2)$ using Theorem 5.8 (2). Since we have

$$H^*(\mathbf{Z}, 3; \mathbf{Z}_2) = \mathbf{Z}_2[u_3, u_5, u_9, u_{17}, u_{33}, \dots], \quad (6.13)$$

where $u_5 = Sq^2u_3$, $u_9 = Sq^4u_5$, $u_{17} = Sq^8u_9$, there is the possibility that $\tau(x_9) = u_5^2$, $\tau(x_{11}) = u_3^4$, and $\tau(x'_{17}) = u_9^2$, but we have $\tau(x_{15}) = 0$. Therefore there is $x_{15} \in H^{15}(E_6; \mathbf{Z}_2)$ such that $f^*(x_{15}) = x_{15}$ and hence

$$H^*(E_6; \mathbf{Z}_2) = \Delta(x_3, x_5, x_3^2, x_9, x_{15}, x_{17}) \otimes \Delta(x_\alpha), \\ x_i = j^*(u_i) \text{ for } i = 3, 5, 9, 17,$$

where either $\{x_\alpha\}$ contains some of $\{x'_9, x_5^2\}$, $\{x_{11}, x_3^4\}$, $\{x'_{17}, x_9^2\}$, or $|x_\alpha| \geq 20$. By Lemma 5.8 we have $\Delta(x_\alpha) = \Delta(x_{11}, x_3^4)$ or $\Delta(x_\alpha) = \Delta(x_{23})$, since $\dim E_6 = |x_3^3x_5x_{15}x_{17}| + 23$.

That $\{x'_9, x_5^2\}$ does not appear shows that $\tau(x_9) = u_5^2$, and hence $x_5^2 = 0$. By (6.13) and the commutativity of j^* and Sq^i , we have $x_5 = Sq^2x_3$, $x_9 = Sq^4x_5$, and $x_{17} = Sq^8x_9$.

Again using the Cartan formula and the Adem relation $Sq^1Sq^{2i} = Sq^{2i+1}$ one has

$$x_3^4 = (Sq^3x_3)^2 = (Sq^1Sq^2x_3)^2 = (Sq^1x_5)^2 = Sq^2x_5^2 = 0, \\ x_9^2 = (Sq^4x_5)^2 = Sq^8x_5^2 = 0.$$

Therefore $\Delta(x_\alpha)$ is not $\Delta(x_{11}, x_3^4)$ but $\Delta(x_\alpha) = \Delta(x_{23})$.

(β) For the case $y_8^2 = 0$, it is similarly shown that replacing $\mathbf{Z}_2[y_2]$ by $\mathbf{Z}_2[y_2]/(y_2^8) \otimes \Delta(y''_{16})$ and adding $\Delta(y_{15}, y''_{16})$ to $H^*(\Omega\tilde{E}_6; \mathbf{Z}_2)$ in (α), one can obtain

$$H^*(\tilde{E}_6; \mathbf{Z}_2) = \Lambda(x_9) \otimes \mathbf{Z}_2[x_{11}, x_{15}, x_{16}, x'_{17}, x''_{17}] \text{ for degrees } < 21.$$

By a similar argument from that on, one has

$$\begin{aligned}\tau(x_9) &= u_3^2, & \tau(x_{11}) &= u_3^4, & \tau(x_{15}) &= 0, \\ \tau(x_{16}) &= u_{17}, & \tau(x'_{17}) &= u_9^2, & \tau(x''_{17}) &= 0,\end{aligned}$$

and the form of $H^*(E_6; \mathbf{Z}_2)$ is the same in the result, where x_{17} is such that

$$f^*(x_{17}) = x''_{17} \quad \text{and} \quad Sq^8 x_9 = j^*(Sq^8 u_9) = j^*(u_{17}) = j^*(\tau(x_{16})) = 0.$$

Q.E.D.

LEMMA 6.14. (1) *The case (β) does not occur in Lemma 6.12, that is,*

$$Sq^8 x_9 = x_{17} \quad \text{and} \quad y_2^8 \neq 0.$$

(2) *For the transgression τ associated with the fibration $E_6 \rightarrow E_6/SU(6) \rightarrow BSU(6)$, x_{15} is transgressive and $\tau(x_{15}) = c_4^2$. In addition x_{15} is chosen so that $x_{23} = Sq^8 x_{15}$ and $\tau(x_{23}) = c_6^2$.*

PROOF. (1) Since $x_3, x_5 = Sq^2 x_3, x_6 = x_3^2 = Sq^3 x_3$ and $x_9 = Sq^4 x_5$ are transgressive in $H^*(E_6; \mathbf{Z}_2) = \Delta(x_3, x_5, x_6, x_9, x_{15}, x_{17}, x_{23})$, by Theorem 2.7 we have

$$H^*(BE_6; \mathbf{Z}_2) = \mathbf{Z}_2[y_4, y_6, y_7, y_{10}], \quad \text{for degrees} < 16$$

where $y_i = \tau(x_{i-1})$ for $i = 4, 6, 7, 10$. Supposing $Sq^8 x_9 = 0$, we deduce a contradiction. We have $Sq^8 y_{10} \equiv Sq^8 \tau(x_9) \equiv \tau(Sq^8 x_9) \equiv 0$ modulo decomposables. Thus there is a polynomial P such that $Sq^8 y_{10} = P(y_4, y_6, y_7, y_{10})$. We have $\rho^*(y_4) = c_2$ since $\rho: BSU(6) \rightarrow BE_6$ induced by the inclusion of Theorem 6.8 gives an isomorphism of π_4 and so of H^4 . Applying Sq^2, Sq^4 , and Sq^8 successively, we obtain

$$\begin{aligned}\rho^*(y_6) &= c_3, & \rho^*(y_{10}) &= c_5 + c_3 c_2, \\ \rho^*(Sq^8 y_{10}) &= c_6 c_3 + c_5 c_4 + (\text{the terms of orders } \geq 3)\end{aligned}$$

from 6.10. Obviously $\rho^*(y_7) = 0$ and hence

$$\begin{aligned}P(c_2, c_3, 0, c_5 + c_3 c_2) &= \rho^* P(y_4, \dots) = \rho^*(Sq^8 y_{10}) \\ &= c_6 c_3 + c_5 c_4 + (\text{the terms of orders } \geq 3),\end{aligned}$$

which is a contradiction. Thus $Sq^8 y_9 \neq 0$. (1) is obtained from Lemma 6.12.

(2) By the naturality of the transgression, one has $\tau(x_3) = \rho^*(y_4) = c_2$. Applying Sq^i , one has

$$\tau(x_5) = c_3, \quad \tau(x_9) \equiv c_5 \pmod{(c_2, c_3)}, \quad \tau(x_{17}) \equiv 0 \pmod{(c_2, c_3, c_5)}.$$

Since $\{c_2, c_3, c_5\}$ is a regular sequence of $H^*(BSU(6); \mathbf{Z}_2)$, x_{15} is transgressive by Theorem 2.24. $\tau(x_{15}) \neq 0$ implies that $\{c_2, c_3, c_5, \tau(x_{15})\}$ is also

a regular sequence. Including the case $\tau(x_{15}) \equiv 0$, x_{23} is transgressive, and hence by the calculation of the spectral sequence we have

$$H^*(E_6/SU(6); \mathbf{Z}_2) \cong \mathbf{Z}_2[c_2, c_4]/(\tau(x_{15}), \tau(x_{23})) \otimes \Delta(x_6, (x_{15}), x_{17}, (x_{23})),$$

where (x_{15}) appears if $\tau(x_{15}) = 0$ and similarly for (x_{23}) . Since $E_6/SU(6)$ is of finite dimension, both $\tau(x_{15})$ and $\tau(x_{23})$ are not 0. Obviously for degree reasons $\tau(x_{15}) = c_4^2$, $\tau(x_{23}) = c_6^2$, $Sq^8(c_4^2) = (Sq^4 c_4)^2 \equiv c_6^2 \pmod{(c_2, c_3, c_5)}$, and hence $Sq^8 x_{15} = x_{23}$. Q.E.D.

COROLLARY 6.15. $H^*(E_6/SU(6); \mathbf{Z}_2) = \Lambda(x_6, x_8, x_{12}, x_{17})$.

PROOF. We have $x_6^2 = Sq^6 x_6 = Sq^2 Sq^4 x_6 = 0$. It is easy to show $x_8^2 = x_{12}^2 = x_{17}^2 = 0$. Q.E.D.

COROLLARY 6.16. $H^*(\tilde{E}_6; \mathbf{Z}_2) = \mathbf{Z}_2[x_{32}] \otimes \Delta(x_9, x_{11}, x_{15}, x_{17}, x_{23}, x_{33})$.

PROOF. We calculate using the spectral sequence

$$E_2 = H^*(E_6; \mathbf{Z}_2) \otimes H^*(\mathbf{Z}, 2; \mathbf{Z}_2) \Rightarrow H^*(\tilde{E}_6; \mathbf{Z}_2)$$

associated with 5.3(5) for $G = E_6$. We have

$$\tau(u_2) = x_3, \tau(u_2^2) = x_5, \tau(u_2^4) = x_9 \quad \text{and} \quad \tau(u_2^8) = x_{17}$$

in

$$H^*(\mathbf{Z}, 2; \mathbf{Z}_2) = \mathbf{Z}_2[u_2] = \Delta(u_2, u_2^2, u_2^4, u_2^8) \otimes \mathbf{Z}_2[u_2^{16}].$$

One can see for the degree reasons that $\tau(u_2^{16}) = Sq^{16} x_{17} = 0$. The following result follows from this:

$$E_\infty = \Delta(x_3^3 \otimes u_2, x_5 \otimes u_2^2, x_9 \otimes u_2^4, x_{17} \otimes u_2^8, x_{15} \otimes 1, x_{23} \otimes 1) \otimes \mathbf{Z}_2[u_2^{16}],$$

and hence the corollary is obtained. Q.E.D.

The following theorem is a fundamental tool for the study of more delicate structure.

THEOREM 6.17. (1) The generator x_{15} of $H^{15}(E_6; \mathbf{Z}_2)$ is not universally transgressive.

(2) $\bar{\phi}(x_{15}) = x_3^2 \otimes x_9$, if x_{15} is suitably chosen.

(3) $Sq^2 x_9 = x_{11}$, $Sq^4 x_{11} = x_{15}$, $Sq^8 x_9 = x_{17}$, $Sq^8 x_{15} = x_{23}$ for generators x_i in Corollary 6.16.

PROOF. (1) Consider the spectral sequence

$$E_2 = H^*(\mathbf{Z}, 4; \mathbf{Z}_2) \otimes H^*(B\tilde{E}_6; \mathbf{Z}_2) \Rightarrow H^*(BE_6; \mathbf{Z}_2)$$

associated with 5.3(1). First we have

$$H^*(\mathbf{Z}, 4; \mathbf{Z}_2) = \mathbf{Z}_2[u_4, u_6, u_7, u_{10}, u_{11}, u_{13}, u_{18}, u_{19}, \dots],$$

where $u_6 = Sq^2 u_4$, $u_7 = Sq^3 u_4$, $u_{10} = Sq^4 u_6$, $u_{11} = Sq^5 u_6$, $u_{13} = Sq^6 u_7, \dots$

Applying Theorems 2.28(1) and 2.7 to Corollary 6.16, we have

$$H^*(B\tilde{E}_6; \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{y}_{10}, \tilde{y}_{12}, \tilde{y}_{16}, \tilde{y}_{18}] \quad \text{for degrees} < 24. \quad (6.17')$$

And, x_3 and hence $x_5 = Sq^2 x_3$, $x_6 = x_3^2 = Sq^3 x_3$, $x_9 = Sq^4 x_5$, $x_{17} = Sq^8 x_9 \in H^*(E_6; \mathbb{Z}_2)$ are universally transgressive, and so by Lemma 6.12 (α) and Theorem 2.7 the natural homomorphism

$$\mathbb{Z}_2[y_4, y_6, y_7, y_{10}, y_{18}] \rightarrow H^*(BE_6; \mathbb{Z}_2), \quad \text{where } y_i = \tau(x_{i-1})$$

is an isomorphism for degrees < 16 , and a monomorphism for degrees < 24 if x_{15} is universally transgressive.

By comparing these and by using the naturality of Sq^i , it is clear that $Bj^*: H^*(\mathbb{Z}, 4; \mathbb{Z}_2) \rightarrow H^*(BE_6, \mathbb{Z}_2)$ satisfies $Bj^*(u_j) = y_j$ for $j = 4, 6, 7, 10$, and 18 . The element of the lowest degree in $\text{Ker } Bj^*$ is $u_{11} + au_4 u_7$ with $a \in \mathbb{Z}_2$, which coincides with $\tau(\tilde{y}_{10})$.

Using the Adem relation and the Cartan formula one can calculate as follows:

$$\begin{aligned} \tau(Sq^2 \tilde{y}_{10}) &= Sq^2 \tau(\tilde{y}_{10}) = Sq^2(u_{11} + au_4 u_7) \\ &= Sq^2 Sq^5 Sq^2 u_4 + a(Sq^2 u_4 \cdot u_7 + u_4 Sq^2 Sq^3 u_4) \\ &= Sq^6 Sq^3 u_4 + au_6 u_7 = u_{13} + au_6 u_7. \\ \tau(Sq^1 Sq^2 \tilde{y}_{10}) &= Sq^1(u_{13} + au_6 u_7) = Sq^1 Sq^6 u_7 + a(Sq^1 u_6 \cdot u_7) \\ &= Sq^7 u_7 + au_7^2 = (1+a)u_7^2. \\ \tau(Sq^4 Sq^2 \tilde{y}_{10}) &= Sq^4(u_{13} + au_6 u_7) \\ &= Sq^4 Sq^6 u_7 + a(Sq^4 u_6 \cdot u_7 + Sq^1 u_6 Sq^3 u_7 + u_6 Sq^4 u_7) \\ &= (Sq^{10} + Sq^9 Sq^1 + Sq^8 Sq^2)u_7 \\ &\quad + a(u_{10} u_7 + u_7 Sq^1 Sq^2 u_7 + u_6 Sq^5 Sq^2 u_4) \\ &= a(u_7 u_{10} + u_6 u_{11}) \equiv au_7(u_{10} + u_6 u_4). \end{aligned}$$

Therefore $Sq^2 \tilde{y}_{10} \neq 0$, which implies $Sq^2 \tilde{y}_{10} = \tilde{y}_{12}$. $Sq^1 Sq^2 \tilde{y}_{10} = 0$ implies $a = 1$ so $\tau(Sq^4 Sq^2 \tilde{y}_{10}) = u_7(u_{10} + u_6 u_4) \in \text{Ker } Bj^*$. It follows that $r_{17} = y_7(y_{10} + y_6 y_4) = 0$. This shows that x_{15} is not universally transgressive.

(2) x_3 and hence $x_5 = Sq^2 x_3$, x_3^2 and $x_9 = Sq^4 x_5$ are primitive in $H^*(E_6; \mathbb{Z}_2) = \Delta(x_3, x_5, x_3^2, x_9, x_{15}, x_{17}, x_{23})$. By Lemma 1.11 we can replace the generator x_{15} by x'_{15} so that

$$\bar{\phi}(x'_{15}) = ax_9 \otimes x_3^2 \quad \text{with } a \in \mathbb{Z}_2.$$

If we compute applying Theorem 2.28(2) to $E_6 = \Omega BE_6$, we have

$$H^*(BE_6; \mathbb{Z}_2) = \mathbb{Z}_2[y_4, y_6, y_7, y_{10}, y_{18}]/(r_{17}) \quad \text{for degrees} < 24,$$

where we have $r_{17} \equiv ay_{10}y_7 \pmod{(y_4, y_6)}$, but $r_{17} = y_7(y_{10} + y_6y_4)$ in the proof of (1), which implies $a = 1$. Putting $x_{15} = x'_{15} + x_3^2x_9$, (2) is proved.

(3) From the proof of (1) we have $Sq^2\tilde{y}_{10} = \tilde{y}_{12}$ and $Sq^4\tilde{y}_{12} = Sq^4Sq^2\tilde{y}_{10} \neq 0$, which implies $Sq^4\tilde{y}_{12} = \tilde{y}_{16}$. Also, $Sq^8\tau(\tilde{y}_{10}) = Sq^8Sq^3u_6 + Sq^8(u_4u_7) = Sq^9Sq^4u_6 + u_4^2Sq^4u_7 + Sq^2u_4 \cdot Sq^6u_7 = u_{19} + u_4^2u_7 + u_6u_{13} \neq 0$ implies $Sq^8\tilde{y}_{10} = \tilde{y}_{18}$. Applying σ to them, the first three equalities of (3) are obtained. The last equality follows from Lemma 6.14. Q.E.D.

REMARK 6.17". Corollary 6.16 and Theorem 6.17(3) are utilized to compute the 2-primary component of the homotopy group of E_6 .

THEOREM 6.18. *There is a generator $x_i \in H^i(E_6; \mathbb{Z}_2)$ satisfying*

$$H^*(E_6; \mathbb{Z}_2) = \mathbb{Z}_2[x_3]/(x_3^4) \otimes \Delta(x_5, x_9, x_{15}, x_{17}, x_{23}),$$

where $Sq^2x_3 = x_5$, $Sq^4x_5 = x_9$, $Sq^8x_9 = x_{17}$, $Sq^8x_{15} = x_{23}$, $Sq^1x_i = 0$ for $i \neq 5$, and

$$\bar{\phi}(x_{15}) = x_3^2 \otimes x_9, \quad \bar{\phi}(x_{23}) = x_3^2 \otimes x_{17}.$$

PROOF. Applying Sq^8 to the formula for $\bar{\phi}(x_{15})$ we obtain the one for $\bar{\phi}(x_{23})$. Since $x_{15}^2, x_{17}^2, x_{23}^2$, and Sq^1x_i for $i \neq 5$ are primitive, they are 0 by Corollary 1.17. The other relations have already been obtained. Q.E.D.

The mod 2 cohomology of E_7 . Let H be a closed subgroup of G and consider $BH \subset BG \subset K(\mathbb{Z}, 4)$, where we regard the map $Bi: BH \rightarrow BG$ induced by the inclusion to be an inclusion using mapping cylinder argument. If one puts $B\tilde{H} = \Omega(K(\mathbb{Z}, 4), BH)$ similarly to $B\tilde{G} = \Omega(K(\mathbb{Z}, 4), BG)$, one obtains a natural map $B\tilde{H} \rightarrow B\tilde{G}$. For each fibration in 5.3 one obtains maps between fibrations induced by the above map, that is, one obtains the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \rightarrow & \tilde{H} & \xrightarrow{f} & H & \xrightarrow{j} & K(\mathbb{Z}, 3) \rightarrow B\tilde{H} \xrightarrow{Bf} BH \rightarrow K(\mathbb{Z}, 4) \\ & & \downarrow i & & \downarrow i & & \downarrow \text{id} & & \downarrow Bi & & \downarrow \text{id} \\ \cdots & \rightarrow & \tilde{G} & \xrightarrow{f} & G & \xrightarrow{j} & K(\mathbb{Z}, 3) \rightarrow B\tilde{G} \xrightarrow{Bf} BG \rightarrow K(\mathbb{Z}, 4) \end{array} \quad (6.19)$$

where, to regard the upper sequence as the fibrations in 5.3, one needs

$$i_*: \pi_3(H) \cong \pi_3(G) \cong \mathbb{Z}. \quad (6.19')$$

We remark that the following sequence of fibrations is also obtained (cf. §1 of Chapter IV):

$$\tilde{H} \xrightarrow{i} \tilde{G} \rightarrow G/H \rightarrow B\tilde{H} \xrightarrow{Bi} B\tilde{G}. \quad (6.20)$$

LEMMA 6.21.

$$H^*(E_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Delta(x_{15}, x_{17}, x_{23}, x_{27}),$$

where $x_5 = Sq^2x_3$, $x_9 = Sq^4x_5$ and $x_{17} = Sq^8x_9$.

PROOF. Since $\Omega i: \Omega E_6 \rightarrow \Omega E_7$ induced by the inclusion $i: E_6 \rightarrow E_7$ in Theorem 6.8 gives an isomorphism of π_2 , the image $\Omega i^* y_2$ of the generator y_2 of $H^2(\Omega E_7; \mathbb{Z}_2)$ by $\Omega i^*: H^*(\Omega E_7; \mathbb{Z}_2) \rightarrow H^*(\Omega E_6; \mathbb{Z}_2)$ is a generator of $H^2(\Omega E_6; \mathbb{Z}_2)$. By Lemma 6.14(1), $\Omega i^*(y_2^8) = (\Omega i^* y_2)^8 \neq 0$ and hence $y_2^8 \neq 0$. Since the type of E_7 is $(3, 11, 15, 19, 23, 27, 35)$, by Lemma 5.5 we have for degrees $< 3L$

$$\begin{aligned} H^*(\Omega E_7; \mathbb{Z}_2) &= \mathbb{Z}_2[y_2] \otimes \Delta(y_{10}, y_{14}, y_{18}, y'_{20}, y_{22}, y_{26}, y'_{28}), \\ H^*(\Omega \tilde{E}_7; \mathbb{Z}_2) &= \Delta(y_{10}, y_{14}, y_{18}, y'_{20}, y_{22}, y_{26}, y'_{28}). \end{aligned}$$

By Theorem 2.28 there are the following two possibilities for degrees < 29 :

$$H^*(\tilde{E}_7; \mathbb{Z}_2) = \begin{cases} \Lambda(x_{11}) \otimes \mathbb{Z}_2[x_{15}, x_{19}, x_{23}, x_{27}] & \text{if } x_{11}^2 = 0, \\ \mathbb{Z}_2[x_{11}, x_{15}, x_{19}, x'_{21}, x_{23}, x_{27}] & \text{if } x_{11}^2 \neq 0. \end{cases}$$

Using the Adem relation $Sq^4 Sq^7 = Sq^{11} + Sq^9 Sq^2$ we have $x_{11}^2 = Sq^{11} x_{11} = Sq^4 Sq^7 x_{11} + Sq^9 Sq^2 x_{11} = 0$, hence the second case does not occur. If we apply Theorem 5.8(2) to $G = E_7$, the possibilities for $\tau(x_i) \neq 0$ are $\tau(x_{11}) = u_3^4$, $\tau(x_{19}) = u_5^4$, and $\tau(x_{23}) = u_8^3$, where $\tau(x_{11}) = u_3^4$ implies $\tau(x_{23}) = 0$. Recalling $\dim E_7 = 133 = |x_3^3 x_5^3 x_9^3 x_{17} x_{15} x_{27}| + 23$, by Theorem 5.8(2) we see that

(α) $H^*(E_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Delta(x_{15}, x_{17}, x_{23}, x_{27})$ or

(β) $H^*(E_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^8, x_5^4, x_9^4) \otimes \Delta(x_{11}, x_{15}, x_{17}, x_{27})$.

Thus it suffices to show that (β) can not occur.

Consider $i^*: H^*(E_7; \mathbb{Z}_2) \rightarrow H^*(E_6; \mathbb{Z}_2)$. i^* is an isomorphism of H^3 so $i^*(x_3) = x_3$. Applying Sq^i , we have $i^*(x_i) = x_i$ for $i = 3, 5, 9, 17$. Then $i^*(x_3^4) = i^*(x_5^2) = i^*(x_9^2) = 0$ (Theorem 6.18). For the case (β), $Sq^8 x_{15}$ is decomposable in E_7 but not in E_6 . This shows that $i^* x_{15}$ is decomposable, and hence x_{15} can be chosen so that $i^* x_{15} = 0$.

Consider the spectral sequence associated with the fibration $E_6 \xrightarrow{i} E_7 \xrightarrow{p} E_7/E_6$, then it is easy to see that $x_5^2, x_3^4 = Sq^2 x_5^2$ and $x_9^2 = Sq^8 x_5^2$, as well as x_{11} and x_{15} in the case of (β), are in the image of $p^*: H^*(E_7/E_6; \mathbb{Z}_2) \rightarrow H^*(E_7; \mathbb{Z}_2)$ and hence the element $x_3^4 x_5^2 x_9^2 x_{11} x_{15} \neq 0$ of degree 66 is also in the image of p^* . This contradicts $\dim(E_7/E_6) = \dim E_7 - \dim E_6 = 133 - 78 = 55$. Thus the case (β) is eliminated and the lemma is proved. Q.E.D.

COROLLARY 6.22.

$$H^*(\tilde{E}_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_{32}] \otimes \Delta(x_{11}, x_{15}, x_{19}, x_{23}, x_{27}, x_{33}, x_{35}).$$

PROOF. Similarly to Corollary 6.16; x_{19} and x_{35} correspond to $x_5^3 \otimes u_2^2$ and $x_9^3 \otimes u_2^4$ respectively. Q.E.D.

LEMMA 6.23. (1) $\tilde{i}^*: H^*(\tilde{E}_7; \mathbb{Z}_2) \rightarrow H^*(\tilde{E}_6; \mathbb{Z}_2)$ satisfies $\tilde{i}^*(x_j) = x_j$ for $j = 11, 15$, and 23 . Also $Sq^4 x_{11} = x_{15}$ and $Sq^8 x_{15} = x_{23}$ in Corollary 6.22.

(2) $i^*: H^*(E_7; \mathbb{Z}_2) \rightarrow H^*(E_6; \mathbb{Z}_2)$ is an epimorphism, and in Lemma 6.21 x_i can be chosen so that $i^*(x_i) = x_i$ for $i = 3, 5, 9, 15, 17$ and 23 . We have $H^*(E_7/E_6; \mathbb{Z}_2) = \Lambda(x_{10}, Sq^8 x_{10}, x_{27})$.

PROOF. (1) By the commutativity of (6.19), we have $\tau' = \tau \circ \tilde{i}^*$ for the transgressions $\tau: H^{11}(\tilde{E}_7; \mathbb{Z}_2) \rightarrow H^{12}(\mathbb{Z}, 3; \mathbb{Z}_2)$ and $\tau': H^{11}(\tilde{E}_6; \mathbb{Z}_2) \rightarrow H^{12}(\mathbb{Z}, 3; \mathbb{Z}_2)$. It is known that $\tau(x_{11}) = u_3^4 = \tau'(x_{11})$ in the proofs of Lemmas 6.12 and 6.21. Hence $\tilde{i}^*(x_{11}) = x_{11}$. The second half is obtained from the commutativity of $\tilde{i}^*$ and Sq^4, Sq^8 .

(2) By the commutativity of (6.19), $f^* i^* x_{15} = \tilde{i}^* f^* x_{15} \neq 0$ for $x_{15} \in H^{15}(E_7; \mathbb{Z}_2)$ hence $i^* x_{15}$ is indecomposable, from which (2) easily follows. Q.E.D.

THEOREM 6.24. There is a generator $x_i \in H^i(E_7; \mathbb{Z}_2)$ that satisfies the following

$$H^*(E_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27}),$$

where $Sq^2 x_3 = x_5$, $Sq^4 x_5 = x_9$, $Sq^8 x_9 = x_{17}$, $Sq^8 x_{15} = x_{23}$, $Sq^4 x_{23} = x_{27}$, and

$$\begin{aligned}\bar{\phi}(x_{15}) &= x_3^2 \otimes x_9 + x_5^2 \otimes x_5, & \bar{\phi}(x_{23}) &= x_3^2 \otimes x_{17} + x_9^2 \otimes x_5, \\ \bar{\phi}(x_{27}) &= x_5^2 \otimes x_{17} + x_9^2 \otimes x_9.\end{aligned}$$

PROOF. We put $x_{27} = Sq^4 x_{23}$ in addition to the generators in Lemma 6.23. By the fact that i^* is a homomorphism of Hopf algebras and by Theorem 6.18, $\bar{\phi}(x_{15}) - x_3^2 \otimes x_9$ is in the kernel of $i^* \otimes i^*$. Considering the kernel of i^* , one sees that one can write $\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + ax_5^2 \otimes x_5 + bx_5 \otimes x_5^2$ with $a, b \in \mathbb{Z}_2$. Using the Cartan formula one obtains

$$\bar{\phi}(Sq^4 x_{15}) = x_5^2 \otimes x_9 + ax_5^2 \otimes x_9 + bx_9 \otimes x_5^2.$$

Since $Sq^4 x_{15}$ is decomposable and is expressed as a polynomial of x_3, x_5 , and x_9 , $\bar{\phi}(Sq^4 x_{15})$ is symmetric and so $1 + a = b$. When $b = 1$, replacing x_{15} with $x_{15} + x_3^2 x_9$, we have $\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5$. Applying Sq^8 and $Sq^4 Sq^8$ on it, we obtain the equalities for $\bar{\phi}(x_{23})$ and $\bar{\phi}(x_{27})$. Since x_{15}^2, x_{23}^2 , and x_{27}^2 are primitive, they are 0 by Corollary 1.17. That x_{27} is not decomposable follows from the fact that there are no monomials other than x_{27} and $x_5^2 x_{17}$ that have an image under $\bar{\phi}$ such as $\bar{\phi}(x_5^2 x_{17}) = x_5^2 \otimes x_{17} + x_{17} \otimes x_5^2$. Q.E.D.

COROLLARY 6.25. We have $Sq^4 x_{11} = x_{15}$, $Sq^8 x_{15} = x_{23}$, $Sq^8 x_{11} = x_{19}$ and $Sq^4 x_{23} = x_{27}$ in Corollary 6.22.

REMARK 6.26.

$$H^*(\text{Ad}(E_7); \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4) \otimes \Lambda(x_6, x_{15}, x_{17}, x_{23}, x_{27}).$$

The mod 2 cohomology of E_8

LEMMA 6.27. (1) There are the following two possibilities for degrees < 32 :

$$(\alpha) \quad H^*(\tilde{E}_8; \mathbb{Z}_2) = \mathbb{Z}_2[x_{15}, x_{23}, x_{27}, x_{29}].$$

$$(\beta) \quad H^*(\tilde{E}_8; \mathbb{Z}_2) = \Lambda(x_{15}, x_{23}, x_{27}).$$

$$(2) \quad H^*(E_8; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9, x_{15}, x_{17}],$$

for degrees < 23 where $x_5 = Sq^2 x_3$, $x_9 = Sq^4 x_5$, and $x_{17} = Sq^8 x_9$.

PROOF. The type of E_8 is (3, 15, 23, 27, 35, 39, 47, 59). Using the inclusion $E_7 \rightarrow E_8$ in Theorem 6.8 and Lemma 5.5 one obtains

$$H^*(\Omega \tilde{E}_8; \mathbb{Z}_2) = \Delta(y_{14}, y_{22}, y_{26}, y'_{28}) \quad \text{for degrees } < 31$$

and (1) as in the proof of the first half of Lemma 6.20. (2) is easily obtained from Theorem 5.8. Q.E.D.

To obtain results for higher degrees we consider the subgroup $A_8 = SU(9)/\mathbb{Z}_3$ of E_8 . Let T^8 be the maximal torus of diagonal matrices of $SU(9)$ and then $T = T^8/\mathbb{Z}_3$ is a maximal torus of A_8 and hence of E_8 . Since $H^*(B\mathbb{Z}_3; \mathbb{Z}_2)$ is trivial, we obtain isomorphisms $H^*(BA_8; \mathbb{Z}_2) \cong H^*(BSU(9); \mathbb{Z}_2)$ and $H^*(BT; \mathbb{Z}_2) \cong H^*(BT^8; \mathbb{Z}_2)$ by considering the fibrations

$$B\mathbb{Z}_3 \rightarrow BSU(9) \rightarrow BA_8, \quad B\mathbb{Z}_3 \rightarrow BT^8 \rightarrow BT.$$

So we identify them respectively. Also we have $SU(9)/T^8 \cong A_8/T$ (a homeomorphism). From these and the results in Chapter III we have

$$6.28. (1) \quad H^*(BA_8; \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3, c_4, \dots, c_9].$$

(2) The map $A_8/T \rightarrow BT$, which induces the T -bundle $A_8 \rightarrow A_8/T$, gives an epimorphism $H^*(BT; \mathbb{Z}_2) \rightarrow H^*(A_8/T; \mathbb{Z}_2)$.

LEMMA 6.29. The projection $E_8/T \rightarrow E_8/A_8$ induces a monomorphism $H^*(E_8/A_8; \mathbb{Z}_2) \rightarrow H^*(E_8/T; \mathbb{Z}_2)$, and hence $H^*(E_8/A_8; \mathbb{Z}_2)$ consists only of elements of even degrees.

PROOF. Since the map of 6.28(2) is decomposed as $A_8/T \xrightarrow{i} E_8/T \rightarrow BT$, $i^*: H^*(E_8/T; \mathbb{Z}_2) \rightarrow H^*(A_8/T; \mathbb{Z}_2)$ is an epimorphism, that is, A_8/T is mod 2 totally nonhomologous to zero in E_8/T . Therefore the first half is obtained, and the second half is clear from Theorem 4.21 of Chapter VI. Q.E.D.

THEOREM 6.30. E_7 is mod 2 totally nonhomologous to zero in E_8 . That is, $i^*: H^*(E_8; \mathbb{Z}_2) \rightarrow H^*(E_7; \mathbb{Z}_2)$ is an epimorphism, and $p^*: H^*(E_8/E_7; \mathbb{Z}_2) \rightarrow H^*(E_8; \mathbb{Z}_2)$ is a monomorphism and $H^*(E_7; \mathbb{Z}_2) \cong H^*(E_8; \mathbb{Z}_2)/(\text{Im } p^+)$.

PROOF. By Theorem 6.24 the generators of $H^*(E_7; \mathbb{Z}_2)$ are obtained by operating with Sq^i several times on x_3 and x_{15} . Therefore it suffices to show that x_3 and x_{15} are in the image of i^* . Since it is obvious for x_3 , it suffices to show that i^*x_{15} is not decomposable.

Consider the spectral sequence

$$E_2 = H^*(BA_8; \mathbb{Z}_2) \otimes H^*(E_8; \mathbb{Z}_2) \Rightarrow H^*(E_8/A_8; \mathbb{Z}_2)$$

associated with the fibration $E_8 \rightarrow E_8/A_8 \rightarrow BA_8$ for degrees < 23 . We see $\tau(x_3) = c_2$ from $\pi_3(A_8) \cong \pi_3(E_8)$. Since τ and Sq^i commute, we have

$$\tau(x_5) = Sq^2 c_2 = c_3, \quad \tau(x_9) = Sq^4 c_3 = c_5, \quad \tau(x_{17}) = Sq^8 c_5 = c_9.$$

Since $H^*(BA_8; \mathbb{Z}_2)$ consists only of elements of even degrees, we have $\tau(x_3^2) = \tau(x_5^2) = \tau(x_9^2) = 0$. Also, by Theorem 2.24, x_{15} is transgressive and one can write

$$\tau(x_{15}) \equiv ac_8 + bc_4^2 \pmod{(c_2, c_3, c_5)} \quad \text{where } a, b \in \mathbb{Z}_2.$$

Here if $a = b = 0$, then $1 \otimes x_{15} \in E_\infty^{0,15}$ gives a nontrivial element in $H^{15}(E_8/A_8; \mathbb{Z}_2)$, which contradicts Lemma 6.29. Thus $a \neq 0$ or $b \neq 0$.

If $a \neq 0$, then $\tau(Sq^2 x_{15}) = Sq^2(c_8 + bc_4^2) = c_9 = \tau(x_{17})$, where $Sq^2 x_{15} = x_{17}$, so $Sq^2(i^*x_{15}) = i^*x_{17}$. If i^*x_{15} is decomposable, so is $i^*x_{17} = x_{17} \in H^{17}(E_7; \mathbb{Z}_2)$, which contradicts Theorem 6.24. Thus x_{15} is indecomposable.

When $a = 0$ and $b \neq 0$, the commutative diagram

$$\begin{array}{ccccc} E_6 & \longrightarrow & E_6/SU(6) & \longrightarrow & BSU(6) \\ \downarrow i' & & \downarrow & & \downarrow \\ E_8 & \longrightarrow & E_8/A_8 & \longrightarrow & BA_8 \end{array}$$

gives the naturality of τ , and hence we have

$$\tau(i'^*x_{15}) = c_4^2 \in H^{16}(BSU(6); \mathbb{Z}_2).$$

By Lemma 6.14(2) we see that i'^*x_{15} and hence x_{15} is not decomposable. Q.E.D.

REMARK 6.31. Using a result that is given later, one sees that $\tau(x_{15}) = c_8 + c_4^2$.

THEOREM 6.32. If a generator $x_{15} \in H^{15}(E_8; \mathbb{Z}_2)$ is suitably chosen, then

$$\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + x_3^4 \otimes x_3.$$

If we put $x_{23} = Sq^8 x_{15}$, $x_{27} = Sq^4 x_{23}$, and $x_{29} = Sq^2 x_{27}$, then we have $x_{15}^2 = Sq^1 x_{29}$ and the following are satisfied:

$$\begin{aligned}\bar{\phi}(x_{23}) &= x_3^2 \otimes x_{17} + x_9^2 \otimes x_5 + x_3^4 \otimes x_3, \\ \bar{\phi}(x_{27}) &= x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 + x_3^8 \otimes x_3, \\ \bar{\phi}(x_{29}) &= x_3^4 \otimes x_{17} + x_5^4 \otimes x_9 + x_3^8 \otimes x_5, \\ \bar{\phi}(x_{15}^2) &= x_3^4 \otimes x_9^2 + x_5^4 \otimes x_5^2 + x_3^8 \otimes x_3^2.\end{aligned}$$

PROOF. The kernel of $i^*: H^*(E_8; \mathbb{Z}_2) \rightarrow H^*(E_7; \mathbb{Z}_2)$ is generated by x_3^4 for degrees ≤ 15 . If we take x_{15} so that $i^*(x_{15}) = x_{15} \in H^{15}(E_7; \mathbb{Z}_2)$, we see by Theorem 6.24 that we can write

$$\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + ax_3^4 \otimes x_3 + bx_3 \otimes x_3^4 \quad (a, b \in \mathbb{Z}_2).$$

Since $Sq^2 x_{15} = P(x_3, x_5, x_9, x_{17})$, $\bar{\phi}(Sq^2 x_{15})$ is symmetric. On the other hand, since we have

$$Sq^2 \bar{\phi}(x_{15}) = \bar{\phi}(Sq^2 x_{15}) = x_3^4 \otimes x_5 + ax_3^4 \otimes x_5 + bx_5 \otimes x_3^4,$$

it follows that $a + 1 = b$. If $b = 1$, by replacing x_{15} with $x_{15} + x_3^5$ we have $a = 1$ and $b = 0$. The remaining formulas for ϕ are obtained by applying Sq^8 , Sq^4 , Sq^2 , and Sq^1 . By using the Adem relation $Sq^{15} = Sq^1 Sq^2 Sq^4 Sq^8$ one obtains $x_{15}^2 = Sq^{15} x_{15} = Sq^1 x_{29}$. Q.E.D.

THEOREM 6.33. The following holds for the generators x_i in Lemma 6.27(2) and Theorem 6.32:

$$H^*(E_8; \mathbb{Z}_2) = \mathbb{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29}),$$

where

$$\begin{aligned}Sq^1 x_3 &= 0, \quad Sq^1 x_{2i-1} = x_i^2 \quad (i = 3, 5, 9, 15), \\ Sq^1 x_{15} &= x_5^2 x_3^2, \quad Sq^1 x_{23} = x_9^2 x_3^2, \quad Sq^1 x_{27} = x_9^2 x_5^2, \\ Sq^2 x_3 &= x_5, \quad Sq^2 x_{15} = x_{17}, \quad Sq^2 x_{27} = x_{29}, \quad Sq^2 x_i = 0 \quad (i = 5, 9, 17, 23, 29), \\ Sq^4 x_5 &= x_9, \quad Sq^4 x_{23} = x_{27}, \quad Sq^8 x_9 = x_{17}, \quad Sq^8 x_{15} = x_{23}.\end{aligned}$$

The other $Sq^4 x_i$, $Sq^8 x_i$, and $Sq^{16} x_i$ are trivial.

PROOF. When we calculate $H^*(E_8; \mathbb{Z}_2)$ from Lemma 6.27(1) according to Theorem 5.8(2), we have $\tau(x_{15}) = \tau(x_{27}) = 0$ ($= \tau(x_{29})$) and the problem is whether $\tau(x_{23}) = 0$ or u_3^3 . If $\tau(x_{23}) = u_3^3$, then there is no generator of degree 23 and hence $Sq^8 x_{15}$ is decomposable and of the form $ax_3 x_5 x_{15} + P(x_3, x_5, x_9, x_{17})$. But, if we calculate $\bar{\phi}(x_{23})$, we see that it can not be of the form in Theorem 6.32. Therefore we have $\tau(x_{23}) = 0$, and by using

Theorem 5.8(2) and Lemma 5.7(2) we have

$$(\alpha) H^*(E_8; \mathbb{Z}_2)$$

$$= \mathbb{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Delta(x_{17}, x_{23}, x_{27}, x_{29}),$$

$$(\beta) H^*(E_8; \mathbb{Z}_2)$$

$$= \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^{16}, x_5^8, x_9^4) \otimes \Delta(x_{15}, x_{17}, x_{23}, x_{27}, x_{29}),$$

according to the cases (α) and (β) respectively in Lemma 6.27 (1). For the case (β) , $x_{29} = Sq^2 x_{27}$ is decomposable but this is seen to be a contradiction by considering $\bar{\phi}$. Therefore (α) holds. By calculating $\bar{\phi}$ we see that $x_{17}^2, x_{23}^2, x_{27}^2, x_{29}^2$ and any $Sq^{2^i} x_i$ other than those listed in the theorem are primitive, and hence they are 0 by Corollary 1.17. Since $Sq^1 x_{15} - x_5^2 x_3^2$ is also primitive, we have $Sq^1 x_{15} = x_5^2 x_3^2$. Similarly for $Sq^1 x_{23}$ and $Sq^1 x_{27}$.

Looking back at the results so far, the only thing remaining is the proof that $Sq^2 x_{15} = x_{17}$. To that end it suffices to show $\tau(x_{15}) = c_8 + bc_4^2$ as observed in the proof of Theorem 6.30. If it is not so, we have $\tau(x_{15}) = c_4^2$, where this is the case (α) also in Lemma 6.27, and $x_{15}^2 = Sq^1 Sq^{14} x_{15} \neq 0$ implies $Sq^{14} x_{15} = x_{29}$ from which we obtain $\tau(x_{29}) = Sq^{14} c_4^2 = (Sq^7 c_4)^2 = 0$, that is, x_{29} is the image from $H^{29}(E_8/A_8; \mathbb{Z}_2)$, contradicting Lemma 6.29. Thus $\tau(x_{15}) = c_8 + bc_4^2$ and hence $Sq^2 x_{15} = x_{17}$. Q.E.D.

COROLLARY 6.34. *The relations in Theorem 6.33 also hold in $H^*(E_7; \mathbb{Z}_2)$ by putting $x_3^4 = x_5^4 = x_9^4 = x_{29} = 0$. Furthermore, they also hold in $H^*(E_6; \mathbb{Z}_2)$ by further putting $x_5^2 = x_9^2 = x_{27} = 0$.*

REMARK 6.35. Using the Adem relations and the Cartan formula, etc., one sees that the relations in Theorem 6.33 perfectly describe the behavior of Sq^i in $H^*(E_8; \mathbb{Z}_2)$. The same things can be said for $H^*(E_7; \mathbb{Z}_2)$ and $H^*(E_6; \mathbb{Z}_2)$ by Corollary 6.34.

$$\text{COROLLARY 6.36. } H^*(E_8/E_7; \mathbb{Z}_2) = \mathbb{Z}_2[x_{12}]/(x_{12}^4) \otimes \Delta(x_{20}, x_{29}, x_{30}).$$

COROLLARY 6.37.

$$H^*(\tilde{E}_8; \mathbb{Z}_2) = \mathbb{Z}_2[x_{32}, x_{15}]/(x_{15}^4) \otimes \Delta(x_{23}, x_{27}, x_{29}, x_{33}, x_{39}, x_{47}),$$

where $x_{23} = Sq^8 x_{15}$, $x_{27} = Sq^4 x_{23}$, and $x_{29} = Sq^2 x_{27}$.

REMARK 6.38. There is a sequence of the following subgroups such that every subgroup is mod 2 totally nonhomologous to zero in the group containing it:

$$S^3 \subset SU(3) \subset G_2 \subset \text{Spin}(7) \subset \text{Spin}(9) \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

In particular, for $E_6/F_4 = EIV$, we obtain

$$H^*(E_6/F_4; \mathbb{Z}_2) = \Delta(x_9, Sq^8 x_9).$$

Postscript

We describe briefly some results about which we do not have space to write in detail.

The (complex) representation ring of a compact, connected Lie group. Let G be a compact, connected Lie group and let T^l be its maximal torus with $l = \text{rank } G$. Here the ground field is $\mathbb{C}$. Thus a representation space V of G is over $\mathbb{C}$ and is given the structure of a G -module of V by a continuous homomorphism $\theta: G \rightarrow \text{Aut } V$.

DEFINITION. $R(G)$ = (The free abelian group generated by the equivalence classes of irreducible G -modules).

By definition an element of $R(G)$ is a sum $\sum n_i V_i$, where V_i is (an equivalence class of) an irreducible G -module and $n_i \in \mathbb{Z}$ are 0 except for a finite number of values. By the following two properties

1. Any G -module is a direct sum of irreducible G -modules.

2. $\sum m_i V_i \cong \sum n_i V_i$ implies $m_i = n_i$ for all i ,

one sees that there is a one-to-one correspondence

$$\{\text{equivalence classes of } G\text{-modules}\} \leftrightarrow \left\{ \sum n_i V_i \in R(G) \mid n_i \geq 0 \right\}.$$

One can introduce a product in $R(G)$ given by the tensor product of representations so that $R(G)$ is a ring and it is then called the *representation ring* of G .

We define a *character* $\chi_V: G \rightarrow \mathbb{C}$ of a G -module V by

$$\chi_V(g) = \text{tr } \theta(g), \quad \text{for } g \in G$$

and then χ_V is continuous. One can see the following by the properties of tr :

3. $\chi_V(hgh^{-1}) = \chi_V(g)$, for $h, g \in G$;

4. $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$, for $g \in G$;

5. $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$, for $g \in G$.

Putting $\mathcal{E}l(G) = \{f: G \rightarrow \mathbb{C} \mid f(hgh^{-1}) = f(g), \text{ for } h, g \in G\}$, we define a homomorphism of abelian groups

$$\chi: R(G) \rightarrow \mathcal{E}l(G) \quad \text{by} \quad \chi\left(\sum n_i V_i\right) = \sum n_i \chi_i,$$

where χ_i is the character of V_i . Then $\chi(V) = \chi_V$ for any G -module V and χ is a ring homomorphism. Moreover, one can prove the following

THEOREM 6. $\chi: R(G) \rightarrow \mathcal{E}l(G)$ is a monomorphism

Then putting $\text{Im } \chi = \mathcal{E}h(G)$, we call it the *characteristic ring* of G .

By 3 and Theorem 3.15 of Chapter V one sees

7. χ_V is determined by the restriction to T^l and is invariant under the action of the Weyl group $W(G)$.

Therefore, for the inclusion $i: T^l \rightarrow G$ we have

THEOREM 8. $i^*: R(G) \rightarrow R(T^l)$ is monomorphic and

$\text{Im } i^* \subset R(T^l)^{W(G)} = (\text{the subring of } R(T^l) \text{ invariant under } W(G)).$

Now, if $A = G$ is an abelian group and V is an irreducible A -module, then $\theta(a): V \rightarrow V$, $a \in A$, is an (iso)morphism of A -modules. So for $\theta(a)$, by applying the Schur lemma, there exists $\lambda(a) \in \mathbb{C}$ so that

$$\theta(a)v = \lambda(a)v, \quad v \in V.$$

Thus the irreducibility of V implies $\dim V = 1$, since all subspaces of V are A -invariant. Therefore $\theta = \lambda: A \rightarrow \text{Aut } \mathbb{C} = \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$ (a multiplicative group). In particular, one can assume $\lambda(A) \subset S^1 \subset \mathbb{C}^*$.

Since any homomorphism $\alpha': T^1 \rightarrow T^1$ is of the form $\alpha'(x) \equiv nx \pmod{1}$, a homomorphism $\alpha: T^l \rightarrow T^1$ is of the form $\alpha(x_1, \dots, x_l) \equiv \sum n_i x_i \pmod{1}$ with $n_i \in \mathbb{Z}$. As $T^1 \cong S^1$ by the correspondence $x \mapsto \exp(2\pi i x)$, one sees that λ has an action of the form

$$\lambda(x_1, \dots, x_l) = \exp\left(2\pi i \sum n_j x_j\right) \in \mathbb{C}^*$$

in the irreducible T^l -module. Now if α_j , $1 \leq j \leq l$ is the T^l -module with the action defined by

$$\lambda(x_1, \dots, x_l) = \exp(2\pi i x_j), \quad \text{for } i \leq j \leq l,$$

then $\alpha_1^{n_1} \dots \alpha_l^{n_l}$ given by

$$\lambda(x_1, \dots, x_l) = \exp(2\pi i (n_1 x_1 + \dots + n_l x_l))$$

for any $n_j \in \mathbb{Z}$ is also a T^l -module and so we obtain the following theorem.

THEOREM 9. $R(T^l) \cong \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_l, \alpha_l^{-1}]$.

Recall that the Weyl group $W(G)$ acts on $L(T^l)^*$. If we define the elementary symmetric sum $S(h) \in \mathbb{C}$ for a weight $h \in L(T^l)^*$ by

$$S(h) = \sum \exp(2\pi i \phi h) \quad (\text{the sum runs over all } \phi \in W(G)),$$

then obviously the set $\{S(h) \mid h \text{ is a representative of the } W(G)\text{-orbit}\}$ is a basis (over $\mathbb{Z}$) of the set of all the $W(G)$ -invariant elements of $R(T^l)$. Then by carefully inspecting the characters, one sees that there is a one-to-one correspondence

$$\begin{aligned} & \{\text{irreducible representations of } G\} \\ & \leftrightarrow \{\text{elementary symmetric sums in } R(T^l)\}. \end{aligned}$$

THEOREM 10. *If G is a compact, connected Lie group, then $R(G) \cong R(T^l)^{W(G)}$.*

If G is simply connected, one can get more by introducing a partial order on the weights in $L(T^l)^*$:

THEOREM 11. *If $\pi_1(G) = 0$, then $R(G) \cong \mathbb{Z}[\rho_1, \dots, \rho_l]$.*

Here if $\pi_1(G) = 0$, then G has l simple roots $\alpha_1, \dots, \alpha_l$ and fundamental weights $w_1, \dots, w_l$ such that $(2(\alpha_r, w_i))/(\alpha_r, \alpha_r) = \delta_{ri}$. Then $\rho_1, \dots, \rho_l$ are irreducible representations of G such that

$$\chi(\rho_r)|T^l = S(w_r) + (\text{terms of lower order}).$$

See the following references for more details on the above facts:

1. J. F. Adams, *Lectures on Lie groups*, W. A. Benjamin, 1969.
2. D. Husemoller, *Fibre bundles*, Springer-Verlag, 1975.
3. I. Yokota, *Groups and representations*, Shōkabō, 1973 (in Japanese).

$K^*(G)$ (**The Hodgkin theorem**). Denote by X^n the n -skeleton of a finite CW -complex X and consider a generalized cohomology theory h^* . Putting

$$\begin{aligned} Z_r^{p,q} &= \text{Im}(h^{p+q}(X^{p+r-1}, X^{p-1}) \rightarrow h^{p+q}(X^p, X^{p-1})), \\ B_r^{p,q} &= \text{Ker}(h^{p+q}(X^p, X^{p-1}) \rightarrow h^{p+q}(X^p, X^{p-r})), \\ E_r^{p,q} &= Z_r^{p,q} / B_r^{p,q} \end{aligned}$$

one obtains a spectral sequence $\{E_r(X), d_r\}$ satisfying the following properties:

$$\begin{aligned} E_2^{p,q} &\cong H^p(X, h^q(x_0)), \quad x_0 \in X, \\ E_\infty^{p,q} &\cong \mathcal{G}r(h^{p+q}(X)) = h_p^{p+q}(X)/h_{p+1}^{p+q}(X), \\ d_r: E_r^{p,q} &\rightarrow E_r^{p+r, q-r+1}, \end{aligned}$$

where $h_p^n(X) = \text{Ker}(h^n(X) \rightarrow h^n(X^{p-1}))$. The spectral sequence is called the *Atiyah-Hirzebruch spectral sequence* [6]. Let p be a prime and apply the spectral sequence in particular with $h^*() = K^*(; \mathbb{Z}_p)$, where $K^*(X, A; \mathbb{Z}_p) = K^*((X, A) \times (M_p, *))$ with $M_p = S^1 \bigcup_p e^2$. From now on we denote this

spectral sequence by $\{E_r(X; \mathbb{Z}_p), d_r\}$. One sees that $d_r = 0$ for r even.

Now if we suppose $d_i = 0$ for $1 < i < r$ ($\leq 2p - 1$) and r is odd, then $d_r: H^q(X; \mathbb{Z}_p) \rightarrow H^{q+r}(X; \mathbb{Z}_p)$ is an element of the Steenrod algebra $\mathcal{A}_p^r$ of dimension r , since it is a stable cohomology operation of degree r . Since $\mathcal{A}_p^r = 0$ for $2 \leq r < 2p - 2$,

12. $d_r = 0$ for $2 \leq r \leq 2p - 2$ and so $H^q(X; \mathbb{Z}_p) \cong E_r^{q,0}(X; \mathbb{Z}_p)$ for $2 \leq r \leq 2p - 1$.

Thus the first nontrivial differential operator is $d_{2p-1} \in \mathcal{A}_p^{2p-1}$, which has a basis consisting of $\varphi^1 \beta$ and $\beta \varphi^1$ ($\varphi^1 = Sq^2$, $\beta = Sq^1$ for $p = 2$) and hence

$$d_{2p-1} = a\varphi^1 \beta + b\beta \varphi^1 \quad \text{with } a, b \in \mathbb{Z}_p.$$

The relation $d_{2p-1} \circ d_{2p-1} = 0$ implies $a + b = 0$. Moreover, by evaluating d_{2p-1} on a particular space, one sees $d_{2p-1} \neq 0$ and so

$$d_{2p-1} = Q_1 = \varphi^1 \beta - \beta \varphi^1. \quad (13)$$

LEMMA 14. $\dim_p K^*(X; \mathbb{Z}_p) \geq \dim_Q K^*(X; \mathbb{Q})$, where $K^*(X; \mathbb{Q}) = K^*(X) \otimes \mathbb{Q}$. In particular,

equality holds iff $K^*(X)$ is free of p -torsion.

PROOF. Put $K^*(X) = F \oplus P \oplus T$, where $F =$ (the maximal free abelian group), $P =$ (the subgroup of elements with order the power of p) and $T =$ (the subgroup of elements with order prime to p). Then

$$K^*(X; \mathbb{Z}_p) = (F \otimes \mathbb{Z}_p) \oplus (P \otimes \mathbb{Z}_p) \oplus \text{Tor}(P, \mathbb{Z}_p), \quad K^*(X; \mathbb{Q}) = F \otimes \mathbb{Q},$$

where $\dim_Q F \otimes \mathbb{Q} = \dim_p F \otimes \mathbb{Z}_p$. On the other hand, $P \otimes \mathbb{Z}_p = \text{Tor}(P, \mathbb{Z}_p) = 0$ iff $P = 0$, which holds iff $K^*(X)$ is free of p -torsion.

By using the fact that the Atiyah-Hirzebruch spectral sequence is trivial for $K^*(; \mathbb{Q})$ one obtains

LEMMA 15. If $K^*(X)$ has p -torsion then $H^*(X)$ has p -torsion.

THEOREM 16. If G is a simple, simply connected Lie group, $K^*(G)$ has no torsion.

NOTATION. $\theta = \{(G, p) \mid H^*(G) \text{ has } p\text{-torsion}\}$.

By the result in Chapter VI, θ is given by the following list:

$$\begin{aligned} p = 2, & \quad G = \text{Spin}(n) \text{ for } n \geq 7, G_2, F_4, E_6, E_7, E_8; \\ p = 3, & \quad G = F_4, E_6, E_7, E_8; \\ p = 5, & \quad G = E_8. \end{aligned}$$

In order to prove Theorem 16 it suffices to show the following since $K^*(G)$ has no p -torsion for $(G, p) \notin \theta$ by Lemma 15.

17. $K^*(G)$ has no p -torsion for $(G, p) \in \theta$.

Since $\dim_{\mathbb{Q}} K^*(G; \mathbb{Q}) = \dim_{\mathbb{Q}} H^*(G; \mathbb{Q}) = 2^l$ with $l = \text{rank } G$, 17 is equivalent to the following by Lemma 14:

$$\dim_p K^*(G; \mathbb{Z}_p) = 2^l \text{ for } (G, p) \in \theta. \quad (17')$$

LEMMA 18. (1) If $\theta \ni (G, p) \neq (E_7, 2), (E_8, 2)$ then $\dim_p E_{2p}(G; \mathbb{Z}_p) = 2^l$.

(2) $\dim_2 E_6(G; \mathbb{Z}_2) = 2^l$ for $G = E_7, E_8$.

In this proof the results on $H^*(G; \mathbb{Z}_p)$ over $\mathcal{A}_p$ and (13) play a useful role. Theorem 16 is proved in this manner; see [7] for the details.

If G is a semisimple, 1-connected Lie group, G is decomposable as $G_1 \times \cdots \times G_k$ by Theorem 5.30 of Chapter V where G_i is simple. Since $K^*(G_i)$ is free by Theorem 16 one can apply the Künneth formula and obtain $K^*(G) = K^*(G_1) \otimes \cdots \otimes K^*(G_k)$. Thus

THEOREM 16'. If G is semisimple and 1-connected, then $K^*(G)$ has no torsion.

Above we used individual properties of a Lie group G to prove Theorem 16 but S. Araki gave a proof without using the classification theorem in [4].

By regarding a representation $\rho: G \rightarrow U(n)$ of G simply as a continuous map, we obtain an element of $K^{-1}(G)$, which we denote by $\beta(\rho)$. Let $\rho_1, \dots, \rho_l$ be the irreducible representations corresponding to the fundamental weights $\lambda_1, \dots, \lambda_l$, which form a basis of the character group $\hat{T}$ of a maximal torus T of G , and put $a = \prod \beta(\rho_i) \in K^*(G)$. The following holds for the homomorphism $\pi^*: K^*(G) \rightarrow K^*((G/T) \times T) \cong K^*(G/T) \otimes K^*(T)$ induced by the map $\pi: (G/T) \times G \rightarrow G$ (where $\pi(gT, u) = gug^{-1}$):

$$\text{ch}\pi^*(a)[(G/T) \times T] = |W(G)| \quad (\text{the order of } W(G)), \quad (19)$$

where the left-hand side is the Chern character of $\pi^*(a)$ evaluated on the fundamental homology class of $(G/T) \times T$. Since the mapping degree of π is $|W(G)|$, one obtains

THEOREM 20. $(\text{cha})[G] = 1$.

Since $H^*(G; \mathbb{Q}) \cong \Lambda(x_1, \dots, x_l)$ with $|x_i|$ odd by the Hopf theorem (Theorem 6.26 of Chapter IV) and since with $\mathbb{Q}$ -coefficients the Chern character ch is an isomorphism, we have

$$\text{rank } K^*(G) = \text{rank } H^*(G) = 2^l.$$

Now if we define $j: \Lambda = \Lambda(e_1, \dots, e_l) \rightarrow K^*(G)$ by $j(e_i) = \beta(\rho_i)$, j is monomorphic by Theorem 20. On the other hand, $[K^*(G): \Lambda]$ is finite, since $\text{rank } \Lambda = 2^l = \text{rank } K^*(G)$. Since G is parallelizable, one can apply the Riemann-Roch theorem [6] and see that there exists a homomorphism

$f_i: K^*(G) \rightarrow \mathbb{Z}$ satisfying $f_i x = (chx)[G]$ for $x \in K^*(G)$. Therefore one can define a homomorphism

$$k: K^*(G) \rightarrow \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \text{ by } k(x)y = f_i(xj(y))$$

for $x \in K^*(G)$, $y \in \Lambda$. Then by Theorem 20, $k \circ j: \Lambda \cong \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Hence $j(\Lambda)$ is a direct summand of $K^*(G)$, which implies $j(\Lambda) = K^*(G)$, and so $j: \Lambda \cong K^*(G)$. Thus

THEOREM 21. *If G is a semisimple 1-connected, compact Lie group with $l = \text{rank } G$, then*

$$K^*(G) = \Lambda(\beta(\rho_1), \dots, \beta(\rho_l)).$$

See the following references for the details of these facts:

4. S. Araki, *Hopf structures attached to K -theory: Hodgkin's theorem*, Ann. Math. **85** (1967), 508–525.

5. M. F. Atiyah, *On the K -theory of compact Lie groups*, Topology **4** (1965), 65–99.

6. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III Amer. Math. Soc., Providence, R.I., 1961, 7–38.

7. L. Hodgkin, *On the K -theory of Lie groups*, Topology **6** (1967), 1–36.

K -groups of homogeneous spaces. Let G be a compact Lie group. A topological space X is called a G -space if there is a continuous map $G \times X \rightarrow X$ satisfying $ex = x$, $g_1(g_2x) = (g_1g_2)x$ for e the unit element of G , $g_i \in G$ and $x \in X$. A map between G -spaces is called a G -map if it commutes with the action of G . A G -vector bundle over a G -space X consists of a triple (E, p, X) satisfying the following properties:

- (1) E is a G -space;
- (2) $p: E \rightarrow X$ is a G -map;
- (3) $g: p^{-1}(x) \rightarrow p^{-1}(gx)$ is a linear map for $g \in G$.

Let X be a compact G -space. The set of equivalence classes of G -vector bundles over X forms an abelian semi-group $E_G(X)$ under the direct sum. Then we define $K_G(X) = E_G(X)/\{E_1 \oplus E_2 - E_1 - E_2\}$ and call it the *equivariant K -group* of X . Then $K_G(X)$ is a commutative ring, since the tensor product of two G -vector bundles is again a G -vector bundle where G acts diagonally. Similarly to the case of the ordinary K -group, $K'_G(X)$ can be defined for an arbitrary integer r . So we put $K^*_G(X) = \sum K'_G(X)$. Then it is easily verified that the following hold:

$$K^*_G(X) = K^*(X), \text{ for } G = \{e\} \quad (22.1)$$

$$\text{If } G \supset H \text{ is closed then } K^*_G(G/H) \cong R(H). \quad (22.2)$$

$$\text{In particular, } K^*_G(G) \cong \mathbb{Z} \text{ and } K^*_G(\text{a point}) \cong R(G).$$

$$\text{If } X \text{ is a free } G\text{-space then } K^*_G(X) = K^*(X/G). \quad (22.3)$$

$$\text{In particular, if } G \text{ acts on } X \times G \text{ diagonally, } K^*_G(X \times G) \cong K^*(X).$$

Hodgkin [8, 9] constructs a spectral sequence $\{E_r^{*,*}\}$ satisfying the following property for G -spaces X and Y

$$E_2^{*,*}(X, Y) = \text{Tor}_{R(G)}^{*,*}(K_G^*(X), K_G^*(Y)) \Rightarrow F_G^*(X, Y),$$

where there exists a natural transformation $\lambda: F_G^*(X, Y) \rightarrow K_G^*(X \times Y)$ such that λ is an isomorphism if either X or Y is a free G -space. In particular, if $X = G$ and $Y = G/H$, the spectral sequence is of the form:

$$E_2^{*,0} = \text{Tor}_{R(G)}^{*,0}(\mathbb{Z}, R(H)) \Rightarrow K^*(G/H).$$

Let G be a compact, connected, Lie group with $\pi_1(G)$ torsion free, H a connected, closed subgroup of G , T a maximal torus of H and $i: T \rightarrow H$ an inclusion. The natural projection $G/T \rightarrow G/H$ induces a morphism $\{\varphi_r\}: \{E_r(G, G/H)\} \rightarrow \{E_r(G, G/T)\}$ of spectral sequences. Here since $R(H)$ is a direct summand of $R(T)$ as an $R(G)$ -module, $i^*: R(H) \rightarrow R(T)$ is a monomorphism so $\varphi_2 = \text{Tor}_{R(G)}^{*,*}(1, i^*)$ is a monomorphism. Hence

LEMMA 23. If $E_2^{*,0} = \text{Tor}_{R(G)}^{*,0}(\mathbb{Z}, R(T)) \Rightarrow K^*(G/T)$ is trivial, so is $E_2^{*,0} = \text{Tor}_{R(G)}^{*,0}(\mathbb{Z}, R(H)) \Rightarrow K^*(G/H)$.

By choosing suitable representations $\rho_i: G \rightarrow U(n_i)$ of G , one can embed G into $U = \prod U(n_i)$ so that $j^*: R(U) \rightarrow R(G)$ is an epimorphism for the inclusion $j: G \rightarrow U$.

LEMMA 24. If $E_2^{*,0} = \text{Tor}_{R(U)}^{*,0}(\mathbb{Z}, R(T)) \Rightarrow K^*(U/T)$ is trivial, then so is $E_2^{*,0} = \text{Tor}_{R(G)}^{*,0}(\mathbb{Z}, R(T)) \Rightarrow K^*(G/T)$.

Here since both $H^*(BU)$ and $H^*(BT)$ are polynomial rings, one can apply the Snaith result [12, II, Theorem 5.5] and see that $E_2^{*,0} = \text{Tor}_{R(G)}^{*,0}(\mathbb{Z}, R(T)) \Rightarrow K^*(U/T)$ is trivial. Thus

THEOREM 25. If G is a compact, connected, Lie group with $\pi_1(G)$ torsion free and H is a connected, closed subgroup, then the following spectral sequence is trivial:

$$E_2^{*,0} = \text{Tor}_{R(G)}^{*,0}(\mathbb{Z}, R(H)) \Rightarrow K^*(G/H).$$

Let η be the standard H -bundle of G/H . For a given H -bundle V , if we denote by $\alpha(V)$ the element of $K^0(G/H)$ represented by a vector bundle with fibre V associated with η , then α gives rise to a homomorphism of rings $\alpha: R(H) \rightarrow K^0(G/H)$. Obviously α decomposes as

$$\alpha: R(H) \rightarrow \mathbb{Z} \bigotimes_{R(G)} R(H) \rightarrow K^0(G/H).$$

Then we denote $\mathbb{Z} \bigotimes_{R(G)} R(H) \rightarrow K^0(G/H)$ by the same letter α . If $\rho_i: G \rightarrow U(n)$ for $i = 1, 2$ is a representation of G such that $\rho_1|_H = \rho_2|_H$, the map

$$f: G/H \rightarrow U(n) \text{ given by } f(gH) = \rho_1(g)\rho_2(g)^{-1} \quad \triangle$$

defines an element of $K^{-1}(G/H)$, which we denote by $\beta(\rho_1 - \rho_2)$.

With this notation the results of Minami [15] are stated as follows:

THEOREM 26. *If G is a compact, 1-connected Lie group with involution σ and K a subgroup that consists of the fixed points of σ , that is, G/K is a symmetric space, then*

(1) *there exist elements $\rho_1, \dots, \rho_l$ of $R(G)$ satisfying the following condition: there exists r such that $\sigma^*(\rho_k) = \rho_k$ for $r+1 \leq k \leq l$ and such that*

$$R(G) = \mathbb{Z}[\rho_1, \dots, \rho_r, \sigma^*(\rho_1), \dots, \sigma^*(\rho_r), \rho_{r+1}, \dots, \rho_l].$$

(2) $\alpha: \mathbb{Z} \otimes_{R(G)} R(H) \rightarrow K^0(G/H)$ *is a monomorphism and*

$$K^*(G/K) = \Lambda(\beta(\rho_1 - \sigma^*(\rho_1)), \dots, \beta(\rho_r - \sigma^*(\rho_r))) \otimes \left(\mathbb{Z} \otimes_{R(G)} R(K) \right).$$

(3) $K^*(G/K)$ *is torsion free.*

Similar results are obtained also by Pittie [11] but the proof is algebraic.

THEOREM 27. *If G is a compact, connected Lie group and U is a connected, closed subgroup of maximal rank, then $\alpha: R(U) \rightarrow K^0(G/K)$ is an epimorphism and gives an isomorphism $K^*(G/U) \cong \mathbb{Z} \otimes_{R(G)} R(U)$.*

See the following references for the details:

8. L. Hodgkin, *An equivariant Künneth formula in K-theory*, Univ. Warwick, 1968 (mimeographed notes).

9. —, *Künneth formula spectral sequence*, Univ. London College, 1972 (mimeographed notes).

10. H. Minami, *K-groups of symmetric spaces*. I, II, Osaka J. Math., **12** (1975), 623–624; **13** (1976), 271–287.

11. H. V. Pittie, *Homogeneous vector bundles on homogeneous spaces*, Topology **11** (1972), 199–203.

12. V. P. Snaith, *Massey products in K-theory*. I, II, Proc. Cambridge Philos. Soc. **68** (1970), 303–320; **69** (1971), 259–289.

These topics were originally planned as Chapter VIII but we were obliged to omit them for lack of space.

We add some further references on homotopy, cohomology, and (co)bordism.

Homotopy. Let G be a 1-connected, simple Lie group. See the following papers for the quasi- p -regularity, which is a generalization of the p -regularity in §6 of Chapter IV:

13. M. Mimura and H. Toda, *Cohomology operations and the homotopy groups of compact Lie groups*. I, Topology **9** (1970), 317–336.

14. S. Oka, *On the homotopy groups of sphere bundles over spheres*, J. Sci. Hiroshima Univ. **33** (1966), 161–195.

More generally we have the following definitions.

G is called p -decomposable into r spaces if there exist 1-connected complexes X_i with $\tilde{H}^*(X_i; \mathbb{Z}_p) \neq 0$ for $1 \leq i \leq r$ and a p -equivalence $f: \prod X_i \rightarrow G$.

Furthermore, we call the decomposition an *irreducible p -decomposition* if each X_i is not p -decomposable. The following theorem is proved in [15].

THEOREM 28. *Suppose that $H_*(G)$ has no p -torsion, the type of G is $(2m_1+1, \dots, 2m_l+1)$, and $r(G) = (\text{the number of distinct } m_i \bmod (p-1))$. If $G \neq \text{Spin}(2n)$, then G is irreducibly p -decomposable into $r(G)$ spaces. If $G = \text{Spin}(2n)$, then G is irreducibly p -decomposable into $r(G) + 1$ spaces.*

15. M. Mimura, G. Nishida, and H. Toda, *Mod p decomposition of compact Lie groups*, Publ. RIMS of Kyoto Univ. **13** (1977), 627–680.

As for the metastable homotopy groups of G there are the following papers:

16. M. G. Barratt and M. E. Mahowald, *The metastable homotopy of $O(n)$* , Bull. Amer. Math. Soc. **69** (1963), 531–533,

17. H. Imanishi, *Unstable homotopy groups of classical groups (odd primary components)*, J. Math. Kyoto Univ. **7** (1967), 221–243.

18. M. A. Kervaire, *Some nonstable homotopy groups of Lie groups*, Illinois J. Math. **4** (1960), 161–169.

This result is mainly deduced from Paechter's calculation:

19. G. F. Paechter, *The groups $\pi_r(V_{n,m})$* , I, II, III, IV, V, Quart. J. Math. Oxford Ser. (2) **7** (1956), 249–268; **9** (1958), 8–27; **10** (1959), 17–37, 241–260; **11** (1960), 1–16.

Incidentally, we add here a series of papers by Matsunaga computing $\pi_{2n+i}(U(n))$ for $3 \leq i \leq 7$, which appeared in Mem. Fac. Sci. Kyushu Univ. Ser. A **15**, **16**, **17**.

For unstable homotopy groups see Chapter 10 of M. Mimura and H. Toda, *Homotopy theory*, Kinokuniya, 1975 (in Japanese). (Some references are also listed). Furthermore, for the exceptional groups see the following

20. H. Kachi, *The homotopy groups of compact Lie groups E_6 , E_7 and E_8* , Nagoya Math. J. **32** (1968), 109–139.

21. H. Imanishi, H. Kachi, M. Mimura, and H. Toda, *Homotopy groups of Lie groups*, RIMS of Kyoto Univ. Lecture series, vol. 21 (1967) (in Japanese).

(Generalized) cohomology. For the cohomology of homogeneous spaces of (exceptional) Lie groups, see the following:

22. K. Ishitoya, *Integral cohomology ring of irreducible symmetric space EII*, J. Math. Kyoto Univ. **17** (1977), 375–397.

23. K. Ishitoya and H. Toda, *On the cohomology of irreducible symmetric spaces of exceptional type*, J. Math. Kyoto Univ. **17** (1977), 225–243.

24. H. Toda, *On the cohomology ring of some homogeneous spaces*, J. Math. Kyoto Univ. **15** (1975), 185–199.

25. H. Toda and T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T* , J. Math. Kyoto Univ. **14** (1974), 257–286.

26. T. Watanabe, *The integral cohomology ring of symmetric space EVII*, J. Math. Kyoto Univ. **15** (1975), 363–385.

For the Hopf algebra structure of the cohomology of Lie groups see the following:

27. P. F. Baum and W. Browder, *The cohomology quotients of classical groups*, Topology **3** (1965), 305–336.

28. K. Ishitoya, A. Kono, and H. Toda, *Hopf algebra structure of mod 2 cohomology of simple Lie groups*, Publ. RIMS of Kyoto Univ. **12** (1976), 141–167.

29. A. Kono, *Hopf algebra structure of simple Lie groups*, J. Math. Kyoto Univ. **17** (1977), 259–298.

30. A. Kono and M. Mimura, *Cohomology mod 2 of the classifying space of the compact connected Lie groups of type E_6* , J. Pure Appl. Algebra **6** (1975), 61–81.

31. —, *Cohomology operations and the Hopf algebra structure of compact, exceptional Lie groups E_7 and E_8* , Proc. London Math. Soc. (3) **35** (1977), 345–358.

32. A. Kono, M. Mimura, and N. Shimada, *On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie groups E_7* , J. Pure Appl. Algebra **8** (1976), 267–283.

As for the cohomology of the classifying spaces of the exceptional Lie groups there are [30], [32] and the following:

33. H. Toda, *Cohomology mod 3 of the classifying space of BF_4 of the exceptional group F_4* , J. Math. Kyoto Univ. **13** (1973), 97–115.

See the following paper for the (co)bordism of Lie groups:

34. T. Petrie, *The weakly complex bordism of Lie groups*, Ann. Math. **88** (1968), 371–402.

We now list the references for each chapter in the second volume, although the story lacks sequence.

Chapter V. There is a concise summary on integration over G , representations of G and so on for compact G in

35. G. F. Bredon, *Introduction to compact transformation groups*, Academic Press, 1972.

But see the following for the details:

36. C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton, 1946.

The following is also worth reading:

37. L. S. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, 1939.

Arguments similar to those in this chapter about maximal tori, roots, and Weyl groups and so on for Lie groups G are written in an easy and simple way in [1]. We also add that the table of the root systems (positive roots, simple roots, dominant roots, fundamental roots, ...) for simple groups G is listed at the end of the following

38. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4, 5, et 6, Elem. Math., Hermann, 1968.

The classification of Dynkin diagrams is detailed in various books on Lie algebras. See for example

39. G. D. Mostov, *Lectures on Lie groups and Lie algebras*, Yale Univ. (mimeographed notes).

The proof of the relation between the integer lattice $\Gamma(G)$ and the fundamental group $\pi_1(G)$ in this chapter is given more or less along the lines of

40. O. Loos, *Symmetric spaces II: Compact spaces and classification*, W. A. Benjamin, 1969.

The argument used to study subgroups of maximal rank in §7 is due to

41. A. Borel and J. de Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221.

Chapter VI. For Morse theory see the following:

42. M. Morse, *The calculus of variations in the large*, Amer. Math. Soc. (1934).

43. H. Seifert and W. Threlfall, *Variationsrechnung im Grossen*, Teubner, 1938.

Among expository books on the applications of the Morse theory to the periodicity of classical groups there are the following:

44. J. W. Milnor, *Morse theory*, Ann. of Math. Stud., vol. 51, Princeton, N.J., 1963.

45. T. Nagano, *Calculus of variations in the large*, Kyôritu-Kôza Modern Math. **17** (1971) (in Japanese).

It would be beneficial to read the papers by Bott (the third one is expository):

46. R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281.

47. —, *The stable homotopy of the classical groups*, Ann. Math. **70** (1959), 313–337.

48. —, *The periodicity theorem for the classical groups and some of its applications*, Adv. in Math. **4** (1970), 353–411.

For Bott-Samelson theory and its applications see the following.

49. R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029.

Chapter VII. For the various facts on Hopf algebras and the cohomology of Lie groups (or H -spaces) see the following:

50. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. Math. **57** (1954), 115–207.

51. —, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. **76** (1954), 273–342.

52. —, *Topics in homology theory of fibre bundles*, Lecture Notes in Math., vol. 36, Springer-Verlag, 1967.

For a proof of the Lefschetz fixed point theorem which we quote in §3 of this chapter and §3 of Chapter V without proof see for example:

53. M. Nakaoka, *Topology (the homology theory)*, Kyôritu-Kôza Modern Math. **15** (1970) (in Japanese).

For cohomology operations and the cohomology of Eilenberg–Mac Lane spaces it might be interesting to read

54. Seminaire de Cartan 7 (1954/55), *Algèbre d'Eilenberg–Mac Lane et homotopie*.

55. J. W. Milnor, *The Steenrod algebra and its dual*, Ann. Math., **67** (1958), 150–171.

For the history of the determination of the cohomology of exceptional groups see:

56. S. Araki, *Mod p cohomology of compact exceptional groups*, Sûgaku **14** (1961), 219–235 (in Japanese),

where a method of computation different from the one in this chapter is clearly stated together with an abundance of references at the end.

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