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PREFACE

The present book consists of an introduction and six chapters. The introduction discusses basic notions and definitions of the traditional course of mathematical physics and also mathematical models of some phenomena in physics and engineering.

Chapters 1 and 2 are devoted to elliptic partial differential equations. Here much emphasis is placed on the Cauchy-Riemann system of partial differential equations, that is on fundamentals of the theory of analytic functions, which facilitates the understanding of the role played in mathematical physics by the theory of functions of a complex variable.

In Chapters 3 and 4 the structural properties of the solutions of hyperbolic and parabolic partial differential equations are studied and much attention is paid to basic problems of the theory of wave equation and heat conduction equation.

In Chapter 5 some elements of the theory of linear integral equations are given. A separate section of this chapter is devoted to singular integral equations which are frequently used in applications.

Chapter 6 is devoted to basic practical methods for the solution of partial differential equations. This chapter contains a number of typical examples demonstrating the essence of the Fourier method of separation of variables, the method of integral transformations, the finite-difference method, the method of asymptotic expansions and also the variational methods.

To study the book it is sufficient for the reader to be familiar with an ordinary classical course on mathematical analysis studied in colleges. Since such a course usually does not involve functional analysis, the embedding theorems for function spaces are not included in the present book.

A.V. Bitsadze

CONTENTS

INTRODUCTION

§ 1. Basic Notions and Definitions

1°. The Notion of a Partial Differential Equation and Its Solution. Let us denote by D a domain in the n-dimensional Euclidean space E_n of points x with orthogonal Cartesian coordinates $x_1, \ldots, x_n$ $(n \geqslant 2)$.

Let $F(x, \ldots, p_{i_1 \ldots i_n}, \ldots)$ be a given real function of the points x belonging to the domain D and of some real variables $p_{i_1 \ldots i_n}$ with nonnegative integral indices $i_1, \ldots$ $\ldots, i_n \left(\sum_{j=1}^{n} i_j = k;\ k = 0, \ldots, m;\ m \geqslant 1\right)$. We shall suppose that for $\sum_{j=1}^{n} i_j = m$ at least one of the derivatives $\frac{\partial F}{\partial p_{i_1 \ldots i_n}}$ of the function F is different from zero.

An equality of the form

$$F\left(x, \ldots, \frac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \ldots\right) = 0 \tag{1}$$

is called a *partial differential equation of the m-th order* with respect to the unknown function $u(x) = u(x_1, \ldots, x_n)$, $x \in D$; the left-hand member of this equality is called a *partial differential operator of the m-th order*.

A real function $u(x)$ defined in the domain D, where equation (1) is considered, which is continuous together with its partial derivatives contained in the equation and which turns the equation into an identity is called a *regular solution* of the equation.

Besides regular solutions, in the theory of partial differential equations an important role is played by certain solutions which are not regular at some isolated points or on some manifolds of a special type. The so-called *fundamental* (*elementary*) *solutions* which will be considered later belong to this class of solutions.

Equations encountered in the applications of the theory of partial differential equations possess solutions (as a rule, they possess not one solution but families of solutions). However, there also exist partial differential equations whose sets of solutions are rather poor; in some cases these sets may even be void. For instance, the set of the real solutions of the equation

$$\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 = 0$$

consists of the function $u(x) = \text{const}$ while the equation

$$\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 + 1 = 0$$

has no real solutions at all.

An equation of form (1) is said to be *linear* when F is a linear function with respect to all the variables $p_{i_1 \ldots i_n}$ $\left(\sum_{j=1}^{n} i_j = k;\ k = 0, \ldots, m\right)$. When the function F is linear with respect to the variables $p_{i_1 \ldots i_n}$ only for $\sum_{j=1}^{n} i_j = m$, equation (1) is called *quasi-linear*.

A linear partial differential equation $Lu = f(x)$ is said to be *homogeneous* or *non-homogeneous* depending on whether the right-hand member $f(x)$ is equal to zero for all $x \in D$ or is not identically equal to zero (here L symbolizes a *linear partial differential operator*).

It is evident that if two functions $u(x)$ and $v(x)$ are solutions of a non-homogeneous linear equation $Lu = f$ then their difference $w = u(x) - v(x)$ is a solution of the corresponding homogeneous equation $Lw = 0$. If $u_k(x)$ ($k = 1, \ldots, l$) are solutions of the homogeneous equation then so is the function $u = \sum_{k=1}^{l} c_k u_k(x)$ where c_k are real constants.

A linear partial differential equation of the second order can be written in the general form

$$\sum_{i,\,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{j=1}^{n} B_j\frac{\partial u}{\partial x_j}+Cu=f \qquad (2)$$

where A_{ij}, B_j, C and f are real functions of the variable point x in the domain D.

At those points $x \in D$ where all the coefficients A_{ij} $(i, j = 1, \ldots, n)$ turn into zero relation (2) is no longer a partial differential equation of the second order because at these points the order of equation (2) decreases (the equation degenerates). In what follows we shall always assume that the order of equation (2) is equal to two throughout its domain of definition.

2°. Characteristic Form of a Linear Partial Differential Equation. Classification of Linear Partial Differential Equations of the Second Order by Type. We shall assume that the function F possesses continuous partial derivatives of the first order with respect to the variables $p_{i_1 \ldots i_n}$, $\sum_{j=1}^{n} i_j = m$; under this assumption, we shall define the *characteristic form* corresponding to equation (1) (it plays an important role in the theory of equations (1)) which is the following multilinear form of order m with respect to real parameters $\lambda_1, \ldots, \lambda_n$:

$$K(\lambda_1, \ldots, \lambda_n)=\sum\frac{\partial F}{\partial p_{i_1\ldots i_n}}\lambda_1^{i_1}, \ldots \lambda_n^{i_n}, \quad \sum_{j=1}^{n} i_j=m \qquad (3)$$

In the case of a second-order partial differential equation of type (2) characteristic form (3) is a *quadratic form*:

$$Q(\lambda_1, \ldots, \lambda_n)=\sum_{i,\,j=1}^{n} A_{ij}(x)\,\lambda_i\lambda_j$$

For each point $x \in D$ the quadratic form Q can be brought to its *canonical form* with the aid of a *nonsingular* (*nondegenerate*) *affine transformation* of variables $\lambda_i = \lambda_i(\xi_1, \ldots$

$\ldots, \xi_n)$ $(i = 1, \ldots, n)$:

$$Q = \sum_{i=1}^{n} \alpha_i \xi_i^2$$

where the coefficients α_i $(i = 1, \ldots, n)$ assume the values 1, -1 and 0. As is known, the number of the nonzero coefficients (the *rank* of the quadratic form), the number of the positive coefficients (the *index* of the form) and the number of the positive coefficients diminished by the number of the negative ones (the *signature* of the form) are *invariants* of the non-degenerate affine transformations.

When all the coefficients α_i $(i = 1, \ldots, n)$ are equal to 1 or to -1, that is when the form Q is *positive definite* or *negative definite* respectively, equation (2) is said to be *elliptic* at the point $x \in D$. If one of the coefficients α_i is negative while all the others are positive (or vice versa), then equation (2) is said to be *hyperbolic* at the point x. In the case when l $(1 < l < n - 1)$ coefficients among α_i are positive while the other $n - l$ coefficients are negative, equation (2) is called *ultrahyperbolic*. Finally, if at least one of these coefficients is equal to zero (in the case under consideration all coefficients α_i cannot turn into zero simultaneously because we have excluded the decrease of the order of the equation), equation (2) is said to be *parabolic* at the point x.

We say that equation (2) is *elliptic*, *hyperbolic* or *parabolic in the domain D* where it is defined (or that (2) is of *elliptic*, *hyperbolic* or *parabolic type in D*) if this equation is elliptic, hyperbolic or parabolic, respectively, at each point of that domain.

An equation of form (2) is said to be *uniformly elliptic* in the domain D of its definition if there exist two nonzero real numbers k_0 and k_1 of one sign such that

$$k_0 \sum_{i=1}^{n} \lambda_i^2 \leqslant Q(\lambda_1, \ldots, \lambda_n) \leqslant k_1 \sum_{i=1}^{n} \lambda_i^2$$

for all $x \in D$ (in this case the partial differential operator

$$\sum_{i, j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{j=1}^{n} B_j \frac{\partial u}{\partial x_j} + Cu$$

is said to be *uniformly elliptic*).

The example of the equation

$$x_n \frac{\partial^2 u}{\partial x_1^2} + \sum_{i=2}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

shows that an equation which is elliptic in the domain of its definition must not necessarily be uniformly elliptic. The last equation is elliptic at every point belonging to the half-space $x_n > 0$ but it is not uniformly elliptic in that half-space.

In the case when equation (2) is of different type in different parts of the domain D we speak of (2) as a *mixed partial differential equation* in that domain. The above example demonstrates an equation which is mixed in any domain D of the space E_n whose intersection with the hyperplane $x_n = 0$ is not void.

In what follows, when speaking of a definite quadratic form, we shall always mean a positive definite form (because a negative definite quadratic form goes into a positive definite one after it has been multiplied by -1).

Without loss of generality, we can assume that the form Q is *symmetric*, that is $A_{ij} = A_{ji}$ $(i, j = 1, \ldots, n)$; then we can use the *Sylvester theorem* in order to determine the type of equation (2) in D without reducing the quadratic form Q to its canonical form; in particular, for the form Q to be positive definite (that is, for equation (2) to be elliptic in the domain D) it is necessary and sufficient that all *principal minors* of the matrix

$$A = \left\| \begin{matrix} A_{11}, \ldots, A_{1n} \\ \cdots\cdots \\ A_{n1}, \ldots, A_{nn} \end{matrix} \right\|$$

should be positive.

3°. Classification of Higher-Order Partial Differential Equations. In the case of a partial differential equation of order m having the form

$$\sum a_{i_1 \ldots i_n}(x) \frac{\partial^m u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} + L_1 u = f, \quad \sum_{=1}^{n} i_j = m \tag{4}$$

where L_1 is a partial differential operator of an order lower than m, characteristic form (3) is written as

$$K(\lambda_1, \ldots, \lambda_n) = \sum a_{i_1 \ldots i_n}(x) \lambda_1^{i_1} \ldots \lambda_n^{i_n}, \quad \sum_{j=1}^{n} i_j = m \qquad (5)$$

If for a fixed point $x \in D$ there exists an affine transformation of variables $\lambda_i = \lambda_i(\mu_1, \ldots, \mu_n)$ $(i = 1, \ldots, n)$ under which (5) reduces to a form containing only l, $0 < l < n$, variables μ_i we say that equation (4) is *parabolically degenerate* at the point x.

When there is no parabolic degeneration, that is when the manifold

$$K(\lambda_1, \ldots, \lambda_n) = 0 \qquad (6)$$

(which is a cone in the space of the variables $\lambda_1, \ldots, \lambda_n$), has no real points except the point $\lambda_1 = 0, \ldots, \lambda_n = 0$, equation (4) is said to be *elliptic* at the point x. Further, we say that equation (4) is *hyperbolic* at the point x if in the space of the variables $\lambda_1, \ldots, \lambda_n$ there exists a straight line such that when it is taken as one of the coordinate axes along which new variables $\mu_1, \ldots, \mu_n$ are reckoned (the new variables $\mu_1, \ldots, \mu_n$ are obtained from $\lambda_1, \ldots, \lambda_n$ by means of a non-degenerate affine transformation), then the transformed relation (6), considered as an equation with respect to the coordinate varying along that axis, has exactly m real roots (simple or multiple) for any choice of the values of the other coordinates μ.

In the case of a non-linear partial differential equation of form (1) the classification by type is carried out in an analogous manner depending on the properties of form (3) in which $p_{i_1 \ldots i_n}$ are replaced by $\dfrac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$, $\sum_{j=1}^{n} i_j = k$; $k = 0, \ldots, m$. Since in the non-linear case the coefficients of form (3) depend not only on the point x but also on the sought-for solution and on its derivatives, this classification by type makes sense only for that particular solution.

4°. Systems of Partial Differential Equations. When F is an N-dimensional vector $F = (F_1, \ldots, F_N)$ with components

$$F_i(x, \ldots, p_{i_1 \ldots i_n}, \ldots) \quad (i = 1, \ldots, N)$$

dependent on $x \in D$ and on the M-dimensional vectors

$$p_{i_1 \ldots i_n} = (p^1_{i_1 \ldots i_n}, \ldots, p^M_{i_1 \ldots i_n})$$

the vector equality of form (1) is called a *system of partial differential equations* with respect to the unknown functions $u_1, \ldots, u_M$ or, which is the same, with respect to the unknown vector $u = (u_1, \ldots, u_M)$. The highest order of the derivatives of the sought-for functions contained in a given equation belonging to the system is referred to as the *order* of that equation.

In the general definition of a system of partial differential equations it is not required that the number N of equations and the number M of the unknown functions should be equal or that all the equations belonging to a given system should be of the same order.

In the case when $M = N$ and the order of each of the equations forming system (1) is equal to m we can form the square matrices

$$\left\| \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} \right\| \quad \left(i, j = 1, \ldots, N; \quad \sum_{k=1}^{n} i_k = m\right)$$

The expression

$$K(\lambda_1, \ldots, \lambda_n) = \det \sum \left\| \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} \right\| \lambda_1^{i_1} \ldots \lambda_n^{i_n}, \quad \sum_{k=1}^{n} i_k = m \tag{7}$$

which is a form of order Nm with respect to the real scalar parameters $\lambda_1, \ldots, \lambda_n$, is called the *characteristic determinant* of system (1).

Systems of partial differential equations of form (1) are classified by type depending on the properties of form (7) in just the same way as in the case of one differential equation of order m considered above.

The variables on the left-hand side of equation (1) may be complex; then, for $x_k = y_k + iz_k$, by $\frac{\partial}{\partial x_k}$ is meant the operator $\frac{1}{2}\left(\frac{\partial}{\partial y_k} - i\frac{\partial}{\partial z_k}\right)$; under this convention equation (1), in the complex case, is obviously equivalent to a system of partial differential equations.

§ 2. Normal Form of Linear Partial Differential Equations of the Second Order in Two Independent Variables

1°. Characteristic Curves and Characteristic Directions. As was mentioned in Sec. 2°, § 1, the quadratic form Q can be reduced to its canonical form for every fixed point x of the domain D where partial differential equation (2) is defined. Therefore for any fixed point $x \in D$ there always exists a non-singular transformation of the independent variables $x_i = x_i(y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ under which equation (2) is reduced, for that fixed point, to its *normal form*

$$\sum_{i=1}^{n}\left(\alpha_i \frac{\partial^2 v}{\partial y_i^2} + \beta_i \frac{\partial v}{\partial y_i}\right) + \gamma v = \delta$$

where the constants α_i $(i = 1, \ldots, n)$ assume the values 1, -1 and 0,

$$v(y) = u[x(y)], \quad \delta(y) = f[x(y)]$$

and the functions β_i and γ are expressed in terms of the coefficients of equation (2).

It should be noted that it is by far not always possible to find a transformation of independent variables which reduces equation (2) to the normal form even in the neighbourhood of a given point of the domain in question. An exception to this general situation is the case of two independent variables, to whose investigation we now proceed.

In the case $n = 2$, using the notation

$$x_1 = x, \qquad x_2 = y$$

$$A_{11} = a(x, y), \quad A_{12} = A_{21} = b(x, y), \quad A_{22} = c(x, y)$$
$$B_1 = d(x, y), \ B_2 = e(x, y), \ C = g(x, y), \ f = f(x, y)$$

we can write equation (2) in the general form

$$a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x\, \partial y} + c \frac{\partial^2 u}{\partial y^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} + gu = f \qquad (8)$$

A curve described by an equation of the form $\varphi(x, y) = \text{const}$ where φ is a solution of the equation

$$a \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2b \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} + c \left(\frac{\partial \varphi}{\partial y} \right)^2 = 0 \qquad (9)$$

is called a *characteristic curve* (or, simply, a *characteristic*) of equation (8), and the direction determined by the infinitesimal vector (dx, dy) where dx and dy are specified by the equality

$$a\, dy^2 - 2b\, dy\, dx + c\, dx^2 = 0 \qquad (10)$$

is referred to as a *characteristic direction*. Equality (10) is an ordinary differential equation describing the characteristic curves.

As was stated in Sec. 2°, § 1, equation (8) is elliptic, hyperbolic or parabolic depending on whether the quadratic form $a\, dy^2 + 2b\, dy\, dx + c\, dx^2$ is definite (that is a positive definite or a negative definite form), indefinite (that is a non-degenerate form of alternating sign) or semi-definite (degenerate). Accordingly, equation (8) is elliptic, hyperbolic or parabolic depending on whether the discriminant $b^2 - ac$ of the quadratic form $a\, dy^2 + 2b\, dy\, dx + c\, dx^2$ is less than, greater than or equal to zero respectively. It follows that in the *domain of ellipticity* of equation (8) there are no real characteristic directions whereas for each point of hyperbolicity of the equation there exist two different real characteristic directions; as to the points of parabolicity, for each of them there exists exactly one real characteristic direction. Consequently, if the coefficients a, b and c are sufficiently smooth the domain of hyperbolicity of equation (8) is covered by a network consisting of two families of characteristic curves while the domain of parabolicity is covered by one such family.

As an example, let us consider the equation

$$y^m \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where m is an odd natural number. In this case equality (10) has the form $y^m\,dy^2 + dx^2 = 0$, whence it is seen that the above partial differential equation has no real characteristic directions in the half-plane $y > 0$ while at each point of the straight line $y = 0$ and at each point in the half-plane

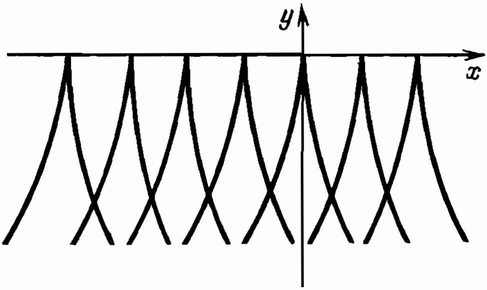

Fig. 1

$y < 0$ it has one and two characteristic directions respectively. On writing the equation of the characteristic curves $dx \pm (-y)^{m/2}\,dy = 0$ and integrating, we conclude that the half-plane $y < 0$ is covered by two families of characteristic curves (see Fig. 1) described by the equations

$$x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = \text{const}$$

and

$$x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = \text{const}$$

2°. Transformation of Partial Differential Equations of the Second Order in Two Independent Variables into the Normal Form. Under the assumption that the coefficients a, b and c of equation (8) are sufficiently smooth it is always possible to find a non-singular transformation $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ of the variables x and y which reduces this equation, in the given domain, to one of the following normal forms:

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + A\frac{\partial v}{\partial \xi} + B\frac{\partial v}{\partial \eta} + Cv = H \tag{11}$$

in the elliptic case,

$$\frac{\partial^2 v}{\partial \xi\, \partial \eta} + A\frac{\partial v}{\partial \xi} + B\frac{\partial v}{\partial \eta} + Cv = H \tag{12}$$

or

$$\frac{\partial_2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \eta^2} + A_1\frac{\partial v}{\partial \xi} + B_1\frac{\partial v}{\partial \eta} + C_1 v = H_1 \tag{12_1}$$

in the hyperbolic case and

$$\frac{\partial^2 v}{\partial \eta^2} + A\frac{\partial v}{\partial \xi} + B\frac{\partial v}{\partial \eta} + Cv = H \tag{13}$$

in the parabolic case.

It is rather difficult to prove that equation (8) can be brought into normal form (11), (12) or (13) throughout the domain D of the definition of the equation (or, as we say, that the equation can be brought into the normal form "in the large"). The argument showing that this possibility can be realized is considerably simplified if we limit ourselves to the consideration of a sufficiently small neighbourhood of an arbitrary point (x, y) of the domain D. Indeed, under the change of variables $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ the partial derivatives of the first and of the second order with respect to x and y are transformed in the following way:

$$\frac{\partial}{\partial x} = \xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial y} = \xi_y\frac{\partial}{\partial \xi} + \eta_y\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x^2} = \xi_x^2\frac{\partial^2}{\partial \xi^2} + 2\xi_x\eta_x\frac{\partial^2}{\partial \xi\,\partial \eta} + \eta_x^2\frac{\partial^2}{\partial \eta^2} + \xi_{xx}\frac{\partial}{\partial \xi} + \eta_{xx}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x\,\partial y} = \xi_x\xi_y\frac{\partial^2}{\partial \xi^2} + (\xi_x\eta_y + \xi_y\eta_x)\frac{\partial^2}{\partial \xi\,\partial \eta} + \\ + \eta_x\eta_y\frac{\partial^2}{\partial \eta^2} + \xi_{xy}\frac{\partial}{\partial \xi} + \eta_{xy}\frac{\partial}{\partial \eta}$$

and

$$\frac{\partial^2}{\partial y^2} = \xi_y^2\frac{\partial^2}{\partial \xi^2} + 2\xi_y\eta_y\frac{\partial^2}{\partial \xi\,\partial \eta} + \eta_y^2\frac{\partial^2}{\partial \eta^2} + \xi_{yy}\frac{\partial}{\partial \xi} + \eta_{yy}\frac{\partial}{\partial \eta}$$

where ξ and η with the indices x and y denote the corresponding derivatives, for instance, $\xi_x = \frac{\partial \xi}{\partial x}$, $\xi_{xx} = \frac{\partial^2 \xi}{\partial x^2}$, etc. Therefore, after the transformation of variables equation (8) takes the form

$$a_1 \frac{\partial^2 v}{\partial \xi^2} + 2b_1 \frac{\partial^2 v}{\partial \xi\, \partial \eta} + c_1 \frac{\partial^2 v}{\partial \eta^2} + d_1 \frac{\partial v}{\partial \xi} + e_1 \frac{\partial v}{\partial \eta} + g_1 v = f_1 \tag{14}$$

where

$$a_1 (\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \tag{15}$$

$$b_1 (\xi, \eta) = a\xi_x\eta_x + b (\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \tag{16}$$

$$c_1 (\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

$$v (\xi, \eta) = u [x (\xi, \eta), y (\xi, \eta)] \tag{17}$$

and $x = x (\xi, \eta)$, $y = y (\xi, \eta)$ is the transformation inverse to $\xi = \xi (x, y)$, $\eta = \eta (x, y)$. We do not write down the expressions for the other coefficients of equation (14) since they are of no interest for our present aims.

Now, let equation (8) be elliptic, that is let $b^2 - ac < 0$; then we take as $\xi (x, y)$ and $\eta (x, y)$ solutions of the system of partial differential equations of the first order

$$\begin{aligned} a\xi_x + b\xi_y + \sqrt{ac - b^2}\, \eta_y &= 0 \\ a\eta_x + b\eta_y - \sqrt{ac - b^2}\, \xi_y &= 0 \end{aligned} \tag{18}$$

for which the Jacobian is different from zero:

$$\frac{\partial (\xi, \eta)}{\partial (x, y)} \neq 0 \tag{19}$$

By virtue of (15), (16), (17), (18) and (19), we have

$$a_1 = c_1 = \frac{ac - b^2}{a} (\xi_y^2 + \eta_y^2) \neq 0 \quad \text{and} \quad b_1 = 0$$

On dividing all the terms of equation (14) by the expression

$$\frac{ac - b^2}{a} (\xi_y^2 + \eta_y^2)$$

which is different from zero we arrive at (11).

It should be noted that system (18) is equivalent to equation (9). This can readily be shown if we use the notation $\varphi = \xi + i\eta$ (where $i^2 = -1$).

Now let $b^2 - ac > 0$ and let the functions $\xi (x, y)$ and $\eta (x, y)$ be solutions of equation (9) satisfying condition (19). Let us suppose that $a \neq 0$ (if $a = 0$ and $c \neq 0$ the argument below can be adequately changed in an obvious manner). In this case, by virtue of (9), we obtain from (15), (16), (17) and (19) the relations $a_1 = c_1 = 0$ and $b_1 = 2\frac{ac-b^2}{a}\xi_y\eta_y \neq 0$, whence it follows that equation (14) assumes form (12) after it has been divided by the function $2b_1$. Further, the new change of variables $\alpha = \xi + \eta$, $\beta = \xi - \eta$ brings equation (12) to form (12_1).

It now remains to consider the case when $b^2 - ac = 0$. In this case we take as the function $\xi(x, y)$ a solution of equation (9) different from a constant while the function $\eta(x, y)$ should be chosen so that the condition $a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \neq 0$ holds. By virtue of the equality $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$, we conclude from (15) and (16) that $a_1 = b_1 = 0$. Consequently, after all the terms of equation (14) have been divided by $a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$ we obtain (13).

In the case when $b^2 - ac > 0$ equation (9) is equivalent to the system of two linear partial differential equations

$$a\varphi_x + \left(b + \sqrt{b^2 - ac}\right)\varphi_y = 0, \quad a\varphi_x + \left(b - \sqrt{b^2 - ac}\right)\varphi_y = 0$$

while in the case $b^2 - ac = 0$ it is equivalent to one equation

$$a\varphi_x + b\varphi_y = 0$$

Consequently, we can assume that for $b^2 - ac > 0$ the functions $\xi(x, y)$ and $\eta(x, y)$ are solutions of the equations

$$a\xi_x + \left(b + \sqrt{b^2 - ac}\right)\xi_y = 0, \quad a\eta_x + \left(b - \sqrt{b^2 - ac}\right)\eta_y = 0 \tag{20}$$

and that for $b^2 - ac = 0$ one of these functions, for instance $\xi(x, y)$, is a solution of the equation

$$a\xi_x + b\xi_y = 0 \tag{21}$$

The problem of the existence of solutions of linear partial differential equations of the first order is closely related to the theory of ordinary differential equations of the first order. As is known from the theory of ordinary differential equations, if the functions a, b and c are sufficiently smooth, the system of linear partial differential equations (18) and linear equations (20) and (21) possess solutions of the required type in a neighbourhood of every point (x, y) of the domain D of definition of equation (8). This proves the possibility of *reducing equation* (8) *to normal forms* (11), (12), (12_1) *and* (13) *in a neighbourhood of* (x, y) (as we say, the possibility of reducing to normal form "*in the small*").

§ 3. Simplest Examples of the Three Basic Types of Second-Order Partial Differential Equations

1°. The Laplace Equation. Let us denote by Δ the partial differential operator of the second order of the form

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

which can also be written as the scalar product by itself of the first-order vector partial differential operator

$$\nabla = \sum_{i=1}^{n} l_i \frac{\partial}{\partial x_i}$$

(called the *Hamiltonian operator* or *nabla* or *del*) where l_i $(i = 1, \ldots, n)$ are unit mutually orthogonal vectors along the coordinate axes x_i. The differential operator Δ is called *Laplace's operator* and the equation

$$\Delta u = 0 \tag{22}$$

is termed *Laplace's equation.*

Characteristic quadratic form (3) corresponding to equation (22) is

$$Q = \sum_{i=1}^{n} \lambda_i^2$$

The form Q is positive definite at all the points of the space E_n. Consequently, this equation is elliptic throughout E_n. Moreover, it is obviously uniformly elliptic in E_n.

A function $u(x)$ possessing continuous partial derivatives of the second order and satisfying Laplace's equation is called a *harmonic function.*

It can be checked directly that the function $E(x, \xi)$, dependent on the two points x and ξ, which is determined by the formula

$$E(x, \xi) = \begin{cases} \frac{1}{n-2} |\xi - x|^{2-n} & \text{for} \quad n > 2 \\ -\ln |\xi - x| & \text{for} \quad n = 2 \end{cases} \tag{23}$$

where $|\xi - x|$ is the distance between x and ξ, is a solution of Laplace's equation for $x \neq \xi$ both with respect to x and with respect to ξ. Indeed, for $x \neq \xi$ we derive from (23) the expressions

$$\frac{\partial^2 E}{\partial x_i^2} = -|\xi - x|^{-n} + n|\xi - x|^{-n-2}(\xi_i - x_i)^2 \quad (i = 1, \ldots, n) \tag{24}$$

On substituting the expressions of $\frac{\partial^2 E}{\partial x_i^2}$ $(i = 1, \ldots, n)$ given by formula (24) into the left-hand side of (22) we obtain

$$\Delta E = -n|\xi - x|^{-n} + n|\xi - x|^{-n-2} \sum_{i=1}^{n} (\xi_i - x_i)^2 = 0$$

Since the function $E(x, \xi)$ is symmetric with respect to x and ξ, we conclude that it satisfies Laplace's equation with respect to ξ $(\xi \neq x)$ as well.

The function $E(x, \xi)$ determined by formula (23) is referred to as the *fundamental* (or *elementary*) *solution* of Laplace's equation. For $n = 3$ this function represents the potential of unit charge localized at the point x (or at ξ).

Let S be a smooth hypersurface (closed or non-closed) in the space E_n and let $\mu(\xi)$ be a real continuous function defined on S.

The expression

$$u(x) = \int_S E(x, \xi)\, \mu(\xi)\, ds_\xi \tag{25}$$

where ds_ξ is the element of area of the hypersurface S whose position on the hypersurface is determined by the variable of integration ξ, is a harmonic function with respect to x for all the points x of the space E_n not lying on S.

This assertion follows from the fact that, as was shown above, the function $E(x, \xi)$ is harmonic with respect to x for $x \neq \xi$ and that when the derivatives $\frac{\partial^2 u}{\partial x_i^2}$ $(i = 1, \ldots, n)$ are computed the operation of differentiation $\frac{\partial^2}{\partial x_i^2}$ on the right-hand side of (25) may be written under the integral sign.

An expression of the form

$$u(x) = \sum_{k \geqslant 0} (-1)^k \left[\frac{x_n^{2k}}{(2k)!} \Delta^k \tau (x_1, \ldots, x_{n-1}) + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^k \nu (x_1, \ldots, x_{n-1}) \right] \quad (26)$$

where the operator Δ^k *is defined by the relation* $\Delta^k = \Delta(\Delta^{k-1})$, and τ, ν *are arbitrary polynomials in the variables* $x_1, \ldots, x_{n-1}$, *is also a harmonic function (a harmonic polynomial) with respect to the variables* $x_1, \ldots, x_n$.

Indeed, since the sum on the right-hand side of (26) is finite (because, beginning with a certain value of k, all the expressions $\Delta^k \tau$ and $\Delta^k \nu$ under the summation sign are equal to zero), we have

$$\frac{\partial^2 u}{\partial x_i^2} = \sum_{k \geqslant 0} (-1)^k \left[\frac{x_n^{2k}}{(2k)!} \Delta^k \frac{\partial^2 \tau}{\partial x_i^2} + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^k \frac{\partial^2 \nu}{\partial x_i^2} \right]$$

$$i = 1, \ldots, n-1$$

and

$$\frac{\partial^2 u}{\partial x_n^2} = -\sum_{k \geqslant 0} (-1)^k \left[\frac{x_n^{2k}}{(2k)!} \Delta^{k+1} \tau + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^{k+1} \nu \right]$$

whence it follows that

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

It can be analogously proved that the sum $u(x)$ of the series on the right-hand side of (26) is a harmonic function in the case when the functions τ and ν are continuously differentiable, the series is convergent and it is legitimate to differentiate the series term-by-term twice with respect to x_i $(i = 1, \ldots, n)$.

2°. Wave Equation. The partial differential equation

$$\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial^2 u}{\partial x_n^2} = 0 \tag{27}$$

is known as the *wave equation*. This equation is frequently encountered in various applications. In the case $n = 4$ this equation can be written in the form

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{28}$$

if we denote $x_4 = t$ (it is meant that the unit of time t is chosen in the appropriate manner). Equation (28) describes the phenomenon of propagation of sound in the three-dimensional space E_3 of the variables x_1, x_2, x_3 (see below § 5, Sec. 2°).

Characteristic quadratic form (3) corresponding to equation (27) has the canonical form

$$Q = \sum_{i=1}^{n-1} \lambda_i^2 - \lambda_n^2$$

and, consequently, according to the definition stated in Sec. 2°, § 1, this equation is hyperbolic throughout the space E_n.

It can be verified directly that *the function*

$$u(x) = \sum_{k=0}^{\infty} \left[\frac{x_n^{2k}}{(2k)!} \Delta^k \tau (x_1, \ldots, x_{n-1}) + \right.$$
$$\left. + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^k \nu (x_1, \ldots, x_{n-1}) \right] \tag{29}$$

where τ *and* ν *are arbitrary infinitely differentiable functions, is a solution of equation* (27) *provided that series* (29) *is uniformly convergent and that the series obtained from* (29) *by differentiating it term-by-term twice with respect to* x_i $(i = 1, \ldots, n)$ *are also uniformly convergent.* In the case when τ and ν are polynomials the terms of the series on the right-hand side of

(29) are equal to zero beginning with some value of the index k; in this case the sum $u(x)$ of the series is a *polynomial solution* of equation (27).

Now let us consider the function

$$u(x, t)=\int_S \frac{\mu(y_1, y_2, y_3)}{|y-x|}\, ds_y \tag{30}$$

where $|y-x|$ is the distance between the points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, S is the sphere $|y-x|^2 = t^2$ and μ is an arbitrary real twice continuously differentiable function defined on S. We shall show that *function* (30) *is a solution of equation* (28).

Indeed, the change of variables $y_i - x_i = t\xi_i$ $(i = 1, 2, 3)$ brings expression (30) to the form

$$u(x, t)=t\int_\sigma \mu(x_1+t\xi_1, x_2+t\xi_2, x_3+t\xi_3)\, d\sigma_\xi \tag{31}$$

where σ is the unit sphere $|\xi|=1$ and $d\sigma_\xi=\frac{ds_y}{t^2}=\frac{ds_y}{|y-x|^2}$ is an element of area of the unit sphere. From (31) we obtain

$$\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}=t\int_\sigma \sum_{i=1}^{3}\frac{\partial^2\mu}{\partial y_i^2}\, d\sigma_\xi \tag{32}$$

Besides, we have

$$\frac{\partial u}{\partial t}=\int_\sigma \mu(x_1+t\xi_1, x_2+t\xi_2, x_3+t\xi_3)\, d\sigma_\xi + t\int_\sigma \sum_{i=1}^{3}\frac{\partial\mu}{\partial y_i}\xi_i\, d\sigma_\xi=\frac{u}{t}+\frac{1}{t}I \tag{33}$$

where

$$I=\int_S\left[\frac{\partial\mu}{\partial y_1}\nu_1+\frac{\partial\mu}{\partial y_2}\nu_2+\frac{\partial\mu}{\partial y_3}\nu_3\right]ds_y \tag{34}$$

and $\nu(y)=(\nu_1, \nu_2, \nu_3)$ is the outer normal to S at the point y.

On differentiating equality (33) we find

$$\frac{\partial^2 u}{\partial t^2}=-\frac{u}{t^2}+\frac{1}{t}\frac{\partial u}{\partial t}-\frac{1}{t^2}I+\frac{1}{t}\frac{\partial I}{\partial t}= -\frac{u}{t^2}+\frac{1}{t}\left(\frac{u}{t}+\frac{I}{t}\right)-\frac{I}{t^2}+\frac{1}{t}\frac{\partial I}{\partial t}=\frac{1}{t}\frac{\partial I}{\partial t} \tag{35}$$

From mathematical analysis it is known that *for real functions $A_i(x)$ $(i = 1, \ldots, n)$ continuous together with their partial derivatives of the first order in a closed domain $D \cup S$ with a smooth boundary S there holds the Gauss-Ostrogradsky formula*

$$\int_D \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i} d\tau_x = \int_S \sum_{i=1}^{n} A_i(y)\, \nu_i(y)\, ds_y \qquad (GO)$$

where $d\tau_x$ is an element of volume and $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal to S at the point $y \in S$.

The application of formula (GO) makes it possible to transform the right-hand member of (34) into an integral over the sphere $|y - x|^2 \leqslant t^2$:

$$I = \int \sum_{i=1}^{3} \frac{\partial^2 \mu}{\partial y_i^2} d\tau_y \qquad (36)$$

where $d\tau_y$ is the element of volume whose position is specified by the variable of integration y. On passing from the Cartesian coordinates y_1, y_2, y_3 to the spherical coordinates ρ, θ, φ, we bring expression (36) for I to the form

$$I = \int_0^t d\rho \int_0^\pi d\theta \int_0^{2\pi} \Delta\mu \rho^2 \sin\theta \, d\varphi$$

where $\rho^2 \sin\theta\, d\varphi\, d\theta\, d\rho = d\tau_y$. Now, since $\sin\theta\, d\theta\, d\varphi = d\sigma_\xi$, we find

$$\frac{\partial I}{\partial t} = t^2 \int_0^\pi d\theta \int_0^{2\pi} \Delta\mu \sin\theta\, d\varphi = t^2 \int_\sigma \sum_{i=1}^{3} \frac{\partial^2 \mu}{\partial y_i^2} d\sigma_\xi$$

Consequently, by virtue of (35), we can write

$$\frac{\partial^2 u}{\partial t^2} = t \int_\sigma \sum_{i=1}^{3} \frac{\partial^2 \mu}{\partial y_i^2} d\sigma_\xi \qquad (37)$$

By virtue of (32) and (37) we conclude that the function $u(x, t)$ represented by formula (30) is a solution of equation (28).

3°. Heat Conduction Equation. The equation

$$\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial x_n} = 0 \tag{38}$$

is an equation of parabolic type because characteristic form (3) corresponding to it is written as

$$Q = \sum_{i=1}^{n-1} \lambda_i^2$$

In the special case when $n = 4$, on condition that $t = x_4$ and that the unit of time is chosen in the appropriate way, we obtain from (38) the equation

$$\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = 0 \tag{39}$$

which describes the phenomenon of the conduction of heat in a body lying in the space E_3 of the variables x_1, x_2, x_3. That is why (39) is referred to as the *heat conduction equation* (see below § 5, Sec. 3°).

Like in the cases of equations (22) and (27), direct calculations readily show that *the expression*

$$u(x) = \sum_{k \geqslant 0} \frac{x_n^k}{k!} \Delta^k \tau(x_1, \ldots, x_{n-1}) \tag{40}$$

where τ is an arbitrary infinitely differentiable real function of the variables $x_1, \ldots, x_{n-1}$, satisfies equation (38) *provided that the series on the right-hand side of* (40) *and the series obtained from it by the termwise differentiation once with respect to x_n and twice with respect to x_i ($i = 1, \ldots, n-1$) are all uniformly convergent.* The function

$$E(x, \xi) = (x_n - \xi_n)^{\frac{1-n}{2}} \exp\left[-\frac{1}{4(x_n - \xi_n)} \sum_{i=1}^{n-1} (x_i - \xi_i)^2\right] \tag{41}$$

where $\xi_1, \ldots, \xi_n$ are real parameters and $x_n > \xi_n$, is also a solution of equation (38).

Indeed, we have

$$\frac{\partial^2 E}{\partial x_i^2} = -\frac{1}{2(x_n-\xi_n)} E(x, \xi) + \frac{(x_i-\xi_i)^2}{4(x_n-\xi_n)^2} E(x, \xi)$$
$$(i=1, \ldots, n-1)$$

and

$$\frac{\partial E}{\partial x_n} = -\frac{n-1}{2(x_n-\xi_n)} E(x, \xi) + \frac{E(x, \xi)}{4(x_n-\xi_n)^2} \sum_{i=1}^{n-1} (x_i-\xi_i)^2$$

whence it follows that

$$\sum_{i=1}^{n-1} \frac{\partial^2 E}{\partial x_i^2} - \frac{\partial E}{\partial x_n} = 0$$

The function $E(x, \xi)$ specified by formula (41) is called the *fundamental* (*elementary*) *solution* of equation (38).

4°. Statement of Some Problems for Partial Differential Equations. When we derive partial differential equations proceeding from general laws governing the natural phenomena in question, there arise some additional conditions imposed on the sought-for solutions. The proof of the *existence* and the *uniqueness* of the solutions satisfying the additional conditions plays an important role in the theory of partial differential equations. When it turns out that small variations of the data contained both in the equations and in the additional conditions produce small variations of the solutions satisfying them or, as we say, when the sought-for solutions are *stable*, we speak of *well-posed* (or *correctly set*) problems; if otherwise, the problems in question are referred to as *improperly posed* (*not well-posed*). It should also be noted that the conditions of the problems which must be satisfied by the sought-for solutions are essentially dependent on the type of the equations under consideration.

Let us state some problems whose well-posedness will be proved later.

Let the boundary S of a domain D in the space E_n be a *smooth* $(n-1)$-*dimensional hypersurface.* In what follows, by a *surface* in the space E_n we shall understand an $(n-1)$-dimensional hypersurface of that very kind.

Let us state the following problem: *it is required to determine the solution $u(x)$ of equation* (22) *regular in the domain D, continuous in the closed region $D \cup S$* and *satisfying the boundary condition*

$$\lim_{x \to y} u(x) = \varphi(y); \quad x \in D, \quad y \in S \tag{42}$$

where φ is *a given real continuous function defined on S.* The problem we have stated is referred to as the *first boundary-value problem* or the *Dirichlet problem.*

Let us denote by G a domain in the space E_{n-1} of the variables $x_1, \ldots, x_{n-1}$ and state the following problem: *it is required to find the regular solution $u(x)$ of equation* (27) *satisfying the conditions*

$$\begin{gathered} u(x_1, \ldots, x_{n-1}, 0) = \varphi(x) \\ \left. \frac{\partial u(x_1, \ldots, x_n)}{\partial x_n} \right|_{x_n = 0} = \psi(x) \end{gathered} \tag{43}$$

for $x \in G$ where φ and ψ are given real sufficiently smooth functions defined in G; this problem is referred to as the *Cauchy problem* (or the *initial-value problem*). Conditions (43) are known as the *Cauchy conditions* (φ and ψ are usually referred to as the *Cauchy data*) or the *initial conditions.*

Further, let D be a domain in the space E_n bounded by a cylindrical surface whose generators are parallel to the x_n-axis and by two planes $x_n = 0$ and $x_n = h$ $(h > 0)$. Let us denote by S the boundary of the domain D without the upper base $x_n = h$. We state the following problem: *it is required to find the solution $u(x)$ of equation* (38) *regular in the domain D and satisfying the boundary condition*

$$\lim_{x \to y} u(x) = \varphi(y); \quad x \in D, \quad y \in S \tag{44}$$

where φ is a given sufficiently smooth real function defined on S (the function φ specifies the prescribed (limiting) values of the solution $u(x)$ on S; we shall refer to φ as the *data prescribed on S*). This problem is also called the *first boundary-value problem* for equation (38).

§ 4. The Notion of an Integral Equation

1°. Notation and Basic Definitions. Let us denote by $\alpha(x)$ and $K_0(x, y, z)$ given real functions dependent on the points x and y, ranging in a domain D in the space E_n, and on a real (scalar) variable z. Let z be a function $z = \varphi(y)$ of the point $y \in D$; we shall suppose that the integral

$$\int_D K_0[x, y, \varphi(y)]\, d\tau_y$$

taken over the domain D exists. Then an equality of the form

$$\alpha(x)\varphi(x) + \int_D K_0[x, y, \varphi(y)]\, d\tau_y = 0, \quad x \in D \tag{45}$$

is referred to as an *integral equation* with respect to the *unknown function* $\varphi(x)$, $x \in D$.

Integral equation (45) is said to be *linear* when the function $K_0(x, y, z)$ depends linearly on z, that is when

$$K_0(x, y, z) = K(x, y)z + K^0(x, y)$$

A linear integral equation can be written in the form

$$\alpha(x)\varphi(x) + \int_D K(x, y)\varphi(y)\, d\tau_y = f(x), \quad x \in D \tag{46}$$

where

$$f(x) = -\int_D K^0(x, y)\, d\tau_y, \quad x \in D$$

is a given function.

Linear integral equation (46) is called *homogeneous* or *non-homogeneous* depending on whether $f(x) = 0$ for all $x \in D$ or $f(x)$ is not identically equal to zero.

The function $K(x, y)$ is called the *kernel* of integral equation (46) and the integral

$$\int_D K(x, y)\varphi(y)\, d\tau_y$$

on the left-hand side of (46) is referred to as an *integral operator* defined for the class of functions to which the unknown function $\varphi(x)$ belongs.

2°. Classification of Linear Integral Equations. When the domain D is bounded, the function $\alpha(x)$ is continuous and the kernel $K(x, y)$ is also a continuous function of the points $x, y \in D \cup S$ (or when the function $K(x, y)$ is bounded and integrable in the ordinary sense) where S is the boundary of the domain D, integral equation (46) is called the *Fredholm equation.*

Equation (46) is called a *Fredholm equation of the first, second* or *third kind* depending on whether the function $\alpha(x)$ is identically equal to zero, is identically equal to 1 or is neither identically equal to zero nor to 1, respectively.

In case $\alpha(x) \neq 0$ throughout $D \cup S$ Fredholm's integral equation of the third kind can be reduced to Fredholm's integral equation of the second kind by dividing all the terms of the former by $\alpha(x)$.

In what follows we shall only deal with Fredholm's integral equations of the second kind. The proof of the basic propositions of the theory of Fredholm's integral equations will be presented for the case when the domain D coincides with an interval (a, b) of the real number axis, the kernel $K(x, y)$ is a function continuous with respect to the point (x, y) for $a \leqslant x \leqslant b$, $a \leqslant y \leqslant b$ and $f(x)$ is a continuous function in the closed interval $a \leqslant x \leqslant b$. For this case we shall write Fredholm's integral equation of the second kind in the form

$$\varphi(x) - \lambda \int_a^b K(x, y)\, \varphi(y)\, dy = f(x), \quad a \leqslant x \leqslant b \tag{47}$$

where λ is a real parameter.

In the case when the function $K(x, y)$ tends to infinity for $|x - y| \to 0$ at the same rate as the function $|x - y|^{-n_0}$ where $n_0 < n$ and n is the dimension of the domain D, the integral equation

$$\varphi(x) - \lambda \int_D K(x, y)\, \varphi(y)\, d\tau_y = f(x)$$

can be reduced to an equivalent Fredholm integral equation of the second kind with a continuous kernel; therefore the above equation is also referred to as a Fredholm equation.

If the kernel $K(x, y)$ of Fredholm's integral equation of the second kind (47) is identically equal to zero for $y > x$ we arrive at an integral equation of the form

$$\varphi(x)-\lambda \int_a^x K(x, y)\, \varphi(y)\, dy = f(x) \tag{48}$$

which is a special case of Fredholm's equations. Equation (48) is called *Volterra's integral equation of the second kind.*

All the assertions concerning integral equation (48) remain valid for the class of the integral equations of the form

$$\varphi(x, y)-\lambda \int_a^x dt \int_b^y K(x, y;\ t, \tau)\, \varphi(t, \tau)\, d\tau - \mu \int_a^x K_1(x, y;\ t)\, \varphi(t, y)\, dt - \nu \int_b^y K_2(x, y;\ \tau)\, \varphi(x, \tau)\, d\tau = f(x, y)$$

which are also alcled the *Volterra integral equations of the second kind.*

By the *equations of mathematical physics* are usually meant not only partial differential equations but also integral equations.

Integral equations play an important role in various divisions of mathematics. In particular, the theory of integral equations is widely used in the investigation of differential equations, both ordinary and partial.

§ 5. Simplified Mathematical Models for Some Phenomena in Physics and Engineering

1°. Electrostatic Field. Here we shall limit ourselves to the consideration of a *plane electrostatic field* by which is meant a two-dimensional medium for whose every point P

a two-dimensional vector E (the *field intensity*) is defined. Let the coordinates of the variable point P be denoted as x and y and the components of the vector E along the coordinate axes as E_x and E_y respectively. We shall suppose that the functions E_x and E_y are continuous together with their partial derivatives of the first order at all the points P of the field.

Let D be an arbitrary domain lying within the electrostatic field under consideration, its boundary S being sufficiently smooth. By $|D|$, $\nu = (\cos \widehat{\nu x}, \cos \widehat{\nu y})$ and $s = (\cos \widehat{sx}, \cos \widehat{sy})$ we shall denote the area of the domain D, the outer normal to S and the unit tangential vector to S in the positive direction along S (as usual, the positive direction along S is defined as the one possessing the property that when S is described in that direction the interior of the domain D always remains on the left). The components of the vectors ν and s are connected by the obvious equalities

$$\cos \widehat{\nu x} = \cos \widehat{sy}, \quad \cos \widehat{\nu y} = -\cos \widehat{sx}$$

The expressions

$$N = \int_S E\nu\, ds \quad \text{and} \quad A = \int_S Es\, ds$$

where $E\nu$ and Es are the scalar products

$$E\nu = E_x \cos \widehat{\nu x} + E_y \cos \widehat{\nu y}$$

and

$$Es = E_x \cos \widehat{sx} + E_y \cos \widehat{sy} = E_y \cos \widehat{\nu x} - E_x \cos \widehat{\nu y}$$

are called the *flux of the vector E through the contour S* and the *circulation of E over S*.

According to formula (GO), we have

$$N = \int_D \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) dx\, dy$$

$$\text{and} \quad A = \int_D \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dx\, dy$$

On contracting the domain D to the point P, we obtain the limiting relations

$$\lim_{D\to P}\frac{N}{|D|}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=\operatorname{div} E$$

and

$$\lim_{D\to P}\frac{A}{|D|}=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=(\operatorname{rot} E)_z$$

where $(\operatorname{rot} E)_z$ is the projection of the vector rot E on the z-axis orthogonal to the xy-plane.

By definition, we have $N=4\pi e$ and $\lim\limits_{D\to P}\frac{N}{|D|}=4\pi\rho$ where e is the total charge lying within D and ρ is the surface charge density at the point P, and therefore we can write

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=4\pi\rho \tag{49}$$

Since A is equal to the work of the force E along the path S and the field in question is stationary, the law of conservation of energy implies $A=0$, whence

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0 \tag{50}$$

For the case when there are no charges in the field we obtain from formula (49) the equality

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0 \tag{51}$$

Equalities (50) and (51) show that the expressions $E_x\,dx+$ $+E_y\,dy$ and $E_y\,dx-E_x\,dy$ are total differentials. Let us introduce two scalar functions $v(x, y)$ and $u(x, y)$ defined by the formulas

$$dv=-E_x\,dx-E_y\,dy \quad \text{and} \quad du=-E_y\,dx+E_x\,dy$$

These formulas mean that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \tag{52}$$

The functions $u(x, y)$ and $v(x, y)$ are called the *force function* and the *potential* (the *potential function*) of the field respectively, and system of two equations (52) (satis fied by these functions) is referred to as the *Cauchy-Riemann system of partial differential equations.*

Thus, *the investigation of a plane electrostatic field can be reduced to the investigation of the system of partial differential equations* (52).

As will be shown later (see Chapter 2, § 2, Sec. 5°), *the functions* $u(x, y)$ *and* $v(x, y)$ *which are regular solutions of system* (52) *possess partial derivatives of all orders.* On differentiating the first equation of that system with respect to x and the second equation with respect to y and adding together the results, we see that $\Delta u = u_{xx} + u_{yy} = 0$, which means that $u(x, y)$ is a harmonic function. In a similar way we readily show that the function $v(x, y)$ is also harmonic.

2°. Oscillation of a Membrane. By a *membrane* is meant an elastic material surface which takes the form of a plane region G when it is at rest and whose potential energy E_p gained in the oscillation process is proportional to the increment of the area.

Let us suppose that the domain G lies in the plane of the variables x and y and that the transverse deflection $u(x, y, t)$ of the membrane, that is the vertical displacement of the point $(x, y) \in G$, is a sufficiently smooth function. We shall assume the oscillation of the membrane to be *small* in the sense that in the calculations it is legitimate, to within the appropriate accuracy, to neglect the powers of the quantities u, u_x, u_y and u_t higher than the second.

The area σ of the membrane at time instant t is given by the formula

$$\sigma = \int_G \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy \approx \int_G \left(1 + \frac{1}{2} u_x^2 + \frac{1}{2} u_y^2\right) dx\, dy$$

and the area of the membrane when it is at rest is equal to

$$|G| = \int_G dx\, dy$$

Therefore the potential energy E_p is expressed as

$$E_p = \frac{1}{2}\mu \int_G (u_x^2 + u_y^2)\, dx\, dy$$

where the proportionality factor μ is the *tension per unit length.*

The kinetic energy E_k of the membrane is expressed by the formula

$$E_k = \frac{1}{2}\int_G \rho u_t^2\, dx\, dy$$

where ρ is the *area (surface) mass density* of the membrane and u_t is the *speed of the displacement* of the particle of the membrane.

According to *Hamilton's principle*, the integral

$$\int_{t_1}^{t_2} (E_k - E_p)\, dt = \frac{1}{2}\int_{t_1}^{t_2} dt \int_G [\rho u_t^2 - \mu (u_x^2 + u_y^2)]\, dx\, dy \qquad (53)$$

where (t_1, t_2) is the time interval during which the oscillation process is observed, must assume a stationary value. Consequently, the function $u(x, y, t)$ must satisfy *Euler's equation* corresponding to the variational problem for integral (53):

$$\frac{\partial}{\partial t}(\rho u_t) - \frac{\partial}{\partial x}(\mu u_x) - \frac{\partial}{\partial y}(\mu u_y) = 0$$

If we assume that ρ and μ are constant then this equation takes the form

$$\frac{1}{a^2} u_{tt} - \Delta u = 0 \qquad (54)$$

where $a^2 = \mu/\rho$. The constant a is referred to as the *speed of sound.*

When investigating equation (54), we may assume, without loss of generality, that $a = 1$, because the simple change of the variable t according to the formula $\tau = at$ and the transformation $u(x, y, t) = u(x, y, \tau/a) = v(x, y, \tau)$ of the unknown function bring equation (54) to the form

$$v_{\tau\tau} - \Delta v = 0$$

Now let us consider the special case when u is independent of t, that is when the membrane, after being bent, is in a state of equilibrium described by an equation of the form $u = u(x, y)$. For this case equation (54) implies $\Delta u = 0$. We see that *in the case under consideration Laplace's equation we have obtained for the function u serves as Euler's equation of the variational problem for the so-called Dirichlet integral*

$$D(u) = \int\limits_{\tilde{G}} (u_x^2 + u_y^2)\, dx\, dy$$

The expression $D(u)$ describes the potential energy of the membrane when it is in the equilibrium state with the transverse deflection described by the function $u(x, y)$.

Indeed, let us suppose that the displacement of the boundary S of the domain G (of the edge of the membrane) is equal to a given function φ defined on S:

$$u(x, y) = \varphi, \quad (x, y) \in S \tag{55}$$

Let the function $u(x, y)$ receive a variation $\delta u = \varepsilon v$ where ε is an arbitrary real number and v is an arbitrary sufficiently smooth function satisfying the condition

$$v(x, y) = 0, \quad (x, y) \in S \tag{56}$$

Then the corresponding variation $\delta D = D(u + \varepsilon v) - D(u)$ of Dirichlet's integral is expressed by the formula

$$\delta D = 2\varepsilon \int\limits_G (u_x v_x + u_y v_y)\, dx\, dy + \varepsilon^2 \int\limits_G (v_x^2 + v_y^2)\, dx\, dy$$

Therefore the necessary condition for Dirichlet's integral to assume a minimum (stationary) value has the form

$$\int\limits_G (u_x v_x + u_y v_y)\, dx\, dy = 0 \tag{57}$$

Further, taking into account the relation

$$u_x v_x + u_y v_y = (u_x v)_x + (u_y v)_y - v\, \Delta u$$

and the fact that formula (GO) and conditions (56) imply the equalities

$$\int\limits_G [(u_x v)_x + (u_y v)_y]\, dx\, dy = \int\limits_S v \frac{\partial u}{\partial \nu}\, ds = 0$$

we obtain from (57) the equality

$$\int_G v\,\Delta u\,dx\,dy = 0 \tag{58}$$

Finally, since v is an arbitrary function defined in G, equality (58) implies that $\Delta u = 0$.

Hence, *the deflection* $u(x, y)$ *of the membrane in a state of equilibrium is the solution of Dirichlet problem* (55) *for the Laplace equation.*

3°. **Propagation of Heat.** The process of propagation of heat in a medium filled with a mass distributed with density ρ and having *specific heat* c and the *coefficient of thermal conductivity* k can be described mathematically in the following way. Let $u(x, t)$ be the temperature of the medium at the point x at time t and let D be an arbitrary domain within that medium containing the point x. By S we shall denote the boundary of the domain D. Let ds and ν be the element of area of S and the outer normal to S respectively; then, under the assumption that the function $u(x, t)$ is sufficiently smooth, the amount of heat Q flowing into D through S during a time interval (t_1, t_2) is given by the formula

$$Q = \int_{t_1}^{t_2} dt \int_S k\,\frac{\partial u}{\partial \nu}\,ds$$

expressing the *Fourier law of heat conduction.*

Due to the inflow of the amount Q of heat the temperature receives an increment $u(x, t+dt) - u(x, t) \approx u_t\,dt$, and therefore

$$Q = \int_{t_1}^{t_2} dt \int_D c\rho u_t\,d\tau$$

where $d\tau$ is an element of volume.

Consequently,

$$\int_{t_1}^{t_2} dt \int_S k\,\frac{\partial u}{\partial \nu}\,ds = \int_{t_1}^{t_2} dt \int_D c\rho u_t\,d\tau$$

We shall limit ourselves to the case when the quantities k, c and ρ are constant, then

$$k\int_{t_1}^{t_2} dt \int_S \frac{\partial u}{\partial \nu}\, ds = c\rho \int_{t_1}^{t_2} dt \int_D u_t\, d\tau \tag{59}$$

By virtue of formula (GO), we have

$$\int_S \frac{\partial u}{\partial \nu}\, ds = \int_D \Delta u\, d\tau$$

Therefore equality (59) can be written in the form

$$\int_{t_1}^{t_2} dt \int_D (c\rho u_t - k\,\Delta u)\, d\tau = 0$$

Now, since the time interval (t_1, t_2) and the volume of the domain D are quite arbitrary, we conclude that the equality

$$c\rho u_t - k\,\Delta u = 0$$

must hold; it follows that

$$\frac{1}{a} u_t - \Delta u = 0$$

where $a = \frac{k}{c\rho}$. It is evident that without loss of generality we may assume that in this case $a = 1$ as well.

If we prescribe the values of the function $u(x, t)$ at every point x of the medium at the initial instant $t = t_0$ (that is if we set the *initial condition*) and also the values of $u(x, t)$ at each point of the boundary of the medium for all values of t belonging to the interval $t_0 < t < T$ for a constant T (the *boundary condition*), we arrive at the corresponding boundary-value problem of type (44).

4°. **The Motion of a Material Point under the Action of the Force of Gravity.** Let us consider a vertical plane with orthogonal Cartesian coordinates x and y. Suppose that a material point $M(x, y)$ moves in that plane under the action of the force of gravity from a position (ξ, η), $\eta > 0$, to another position $(\xi_0, 0)$, $\xi_0 > \xi$, the time t of the motion being a given function $t = t(\eta)$ of the coordinate η reckoned along the vertical direction. It is required to determine the trajectory of motion of the point $M(x, y)$ (this is the so-called *tautochrone problem*).

As is known, the square of the absolute value v of the velocity vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ of the point $M(x, y)$ satisfies the equality

$$v^2 = 2g(\eta - y), \quad 0 \leqslant y \leqslant \eta \tag{60}$$

where g is the *acceleration of gravity*. Let us denote as $\alpha(x, y)$ the angle between the velocity vector and the positive direction of the x-axis, the angle α being reckoned counterclockwise. Then, by virtue of (60), we can write for the derivative $\frac{dy}{dt}$ the equality

$$\frac{dy}{dt} = v \sin\alpha = \sqrt{2g(\eta - y)} \sin\alpha \tag{61}$$

Since the trajectory $x = x(y)$ is unknown so is the quantity

$$\varphi(y) = \frac{1}{\sin\alpha[x(y), y]} \tag{62}$$

On the basis of (61) and (62) we obtain

$$t(\eta) = -\int_0^\eta \frac{\varphi(y)\, dy}{\sqrt{2g(\eta - y)}}$$

that is

$$\int_0^\eta \frac{\varphi(y)\, dy}{\sqrt{\eta - y}} = f(\eta) \tag{63}$$

where

$$f(\eta) = -\sqrt{2g}\, t(\eta)$$

Consequently, the function $\varphi(y)$ must be a solution of integral equation (63) known as *Abel's integral equation.*

Relation (62) shows that only those solutions $\varphi(y)$ of equation (63) have a physical meaning which satisfy the condition $|\varphi(y)| > 1$. If we manage to find the solution $\varphi(y)$ of equation (63) satisfying that condition, the geometrical equality

$$\frac{dx}{dy} = \cot\alpha = \sqrt{csc^2\alpha - 1} = \sqrt{\varphi^2(y) - 1}$$

makes it possible to immediately express in quadratures the equation of the sought-for trajectory:

$$x = \int_0^y \sqrt{\varphi^2(z) - 1}\, dz$$

CHAPTER 1

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

§ 1. Basic Properties of Harmonic Functions

1°. Definition of a Harmonic Function and Some of Its Basic Properties. According to the definition stated in Sec. 1°, § 3 of Introduction, *a function $u(x)$ is said to be harmonic in a domain D if it possesses continuous partial derivatives up to the second order inclusive in D and satisfies Laplace's equation in that domain.*

If $u(x)$ is a harmonic function in D then so is the function $u(\lambda Cx + h)$ where λ is a scalar constant, C is a constant real orthogonal matrix of order n and $h = (h_1, \ldots, h_n)$ is a constant real vector provided that the points x and $\lambda Cx + h$ belong to the domain D.

This assertion readily follows from the obvious equality

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(\lambda Cx + h) = \lambda^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} u(y)$$

where $y = \lambda Cx + h$.

For arbitrary constants c_k $(k = 1, \ldots, m)$ there holds the equality

$$\Delta \sum_{k=1}^{m} c_k u_k(x) = \sum_{k=1}^{m} c_k \Delta u_k(x)$$

and therefore if $u_k(x)$ $(k = 1, \ldots, m)$ are harmonic functions then so is the finite sum

$$u(x) = \sum_{k=1}^{m} c_k u_k(x)$$

We can also verify directly that if $u(x)$ is a harmonic function in a domain D then the function

$$v(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$$

is also harmonic at all the points where it is defined.

In the case when the domain D contains the point at infinity the definition of a harmonic function needs some additional stipulation because the notion of the derivative at the point at infinity does not make sense.

We shall say that a function $u(x)$ is *harmonic at infinity*, or, more precisely, in the neighbourhood of the point at infinity (by which is meant the exterior of a ball $|x| \leqslant R$ of a sufficiently large radius R), when the function

$$v(y) = |y|^{2-n} u\left(\frac{y}{|y|^2}\right)$$

(this expression makes sense for all $y \neq 0$) whose value at the point $y = 0$ is defined as $\lim_{y \to 0} v(y)$ is harmonic (in the ordinary sense) in the vicinity of the point $y = 0$.

The transformation $y = \frac{x}{|x|^2}$ of the variable y results in the formula

$$u(x) = |x|^{2-n} v\left(\frac{x}{|x|^2}\right)$$

Accordingly, *by a solution* $u(x)$ *of Laplace's equation regular at infinity we shall mean a function which is harmonic throughout the neighbourhood of the point at infinity except the point at infinity itself and which remains bounded for* $|x| \to \infty$ *in the case* $n = 2$ *and tends to zero not slower than* $|x|^{2-n}$ *in the case* $n > 2$.

Let D be a domain in the space E_n having a sufficiently smooth boundary S and let $u(x)$ and $v(x)$ be two real harmonic functions defined in D and continuous in $D \cup S$ together with their partial derivatives of the first order.

Integrating over the domain D the identities

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(v \frac{\partial u}{\partial x_i}\right) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}$$

and

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(v \frac{\partial u}{\partial x_i} - u \frac{\partial v}{\partial x_i} \right) = 0$$

and using formula (*GO*) (see Introduction), we obtain the formulas

$$\int_S v(y) \frac{\partial u(y)}{\partial \nu_y} ds_y = \sum_{i=1}^{n} \int_D \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} d\tau_x \tag{1}$$

and

$$\int_S \left[v(y) \frac{\partial u(y)}{\partial \nu_y} - u(y) \frac{\partial v(y)}{\partial \nu_y} \right] ds_y = 0 \tag{2}$$

respectively. In formulas (1) and (2) and henceforth we shall mean by ν the *outer* normal to S.

For formulas (1) and (2) to remain valid in the case when the domain D lying in the space E_n contains a point at infinity, it is natural to require that the integrands in these formulas should be absolutely integrable (or *summable* in the case when the expressions on the left-hand and the right-hand sides of formulas (1) and (2) are understood as the *Lebesgue integrals*).

Formulas (1) and (2) make it possible to readily establish a number of elementary properties of harmonic functions:

(1) *If a function $u(x)$ is harmonic in a domain D and continuous in $D \cup S$ together with its first-order partial derivatives and is equal to zero on the boundary S of the domain D, then $u(x) = 0$ for all $x \in D \cup S$* (this property is known as the *uniqueness theorem for harmonic functions*).

The indicated property follows from equality (1) if we put $u(x) = v(x)$ in it. Indeed, since $u(y) = 0$ for $y \in S$, formula (1) implies

$$\sum_{i=1}^{n} \int_D \left(\frac{\partial u}{\partial x_i} \right)^2 d\tau_x = \int_S u(y) \frac{\partial u(y)}{\partial \nu_y} ds_y \tag{3}$$

and therefore

$$\sum_{i=1}^{n} \int_D \left(\frac{\partial u}{\partial x_i} \right)^2 d\tau_x = 0$$

Consequently, $\frac{\partial u}{\partial x_i} = 0$ $(i = 1, \ldots, n)$, $x \in D$, that is $u(x) = \text{const}$ for all $x \in D$. Now, since $u(y) = 0$ for $y \in S$, by virtue of the continuity of $u(x)$ in the closed domain $D \cup S$, we conclude that $u(x) = 0$ for all $x \in D \cup S$.

(2) *Let $u(x)$ be a harmonic function in a domain D continuous throughout $D \cup S$ together with its partial derivatives of the first order; if the normal derivative $\frac{\partial u(y)}{\partial \nu_y}$ is equal to zero on the boundary S of the domain D, then $u(x) = \text{const}$ for all $x \in D$.*

This property of harmonic functions is proved in exactly the same way as the foregoing property; to this end it is sufficient to take into account that $\frac{\partial u(y)}{\partial \nu_y} = 0$ in (3) for all $y \in S$.

(3) *Again, let $u(x)$ be a harmonic function in a domain D continuous in $D \cup S$ together with its first-order partial derivatives; then the integral of the normal derivative $\frac{\partial u(y)}{\partial \nu_y}$ taken over the boundary S of D is equal to zero.*

Indeed, on putting $v(x) \equiv 1$ for all $x \in D$, we obtain

$$\int_S \frac{\partial u(y)}{\partial \nu_y} ds_y = 0 \tag{4}$$

2°. Integral Representation of Harmonic Functions. *For a function $u(x)$ harmonic in a domain D with boundary S and continuous in $D \cup S$ together with its partial derivatives of the first order there holds the integral representation*

$$u(x) = \frac{1}{\omega_n} \int_S E(x, y) \frac{\partial u(y)}{\partial \nu_y} ds_y - \frac{1}{\omega_n} \int_S u(y) \frac{\partial E(x, y)}{\partial \nu_y} ds_y \tag{5}$$

where $E(x, y)$ is the fundamental (elementary) solution of Laplace's equation considered in Sec. 1°, § 3 of Introduction, $\omega_n = \frac{1}{\Gamma\left(\frac{n}{2}\right)} 2\pi^{n/2}$ is the surface area of unit sphere in E_n and Γ is Euler's gamma function.

To derive formula (5) let us choose an arbitrary point x in the domain D and consider a closed ball $|y - x| \leqslant \varepsilon$ of radius $\varepsilon > 0$ such that it lies entirely inside D. The part of the domain D lying outside that ball will be denoted D_ε. On applying to the domain D_ε, bounded by the surface S and by the sphere $|y-x|=\varepsilon$, formula (2) in which we put $v(y)=E(x, y)$, we obtain

$$\int_S \left[E(x, y)\frac{\partial u(y)}{\partial \nu_y} - u(y)\frac{\partial E(x, y)}{\partial \nu_y}\right] ds_y =$$

$$= \int_{|y-x|=\varepsilon} \left[E(x, y)\frac{\partial u(y)}{\partial \nu_y} - u(y)\frac{\partial E(x, y)}{\partial \nu_y}\right] ds_y =$$

$$= \int_{|y-x|=\varepsilon} E(x, y)\frac{\partial u(y)}{\partial \nu_y}\, ds_y -$$

$$- \int_{|y-x|=\varepsilon} [u(y)-u(x)]\frac{\partial E(x, y)}{\partial \nu_y}\, ds_y -$$

$$- u(x)\int_{|y-x|=\varepsilon} \frac{\partial E(x, y)}{\partial \nu_y}\, ds_y \qquad (6)$$

Further, for the sphere $|y-x|=\varepsilon$ we have

$$E(x, y) = \begin{cases} \dfrac{1}{(n-2)\varepsilon^{n-2}} & \text{for } n>2 \\ -\ln\varepsilon & \text{for } n=2 \end{cases}$$

$$\frac{\partial E(x, y)}{\partial \nu_y} = \begin{cases} -\dfrac{1}{\varepsilon^{n-1}} & \text{for } n>2 \\ -\dfrac{1}{\varepsilon} & \text{for } n=2 \end{cases}$$

$$\lim_{\varepsilon\to 0} \int_{|y-x|=\varepsilon} [u(y)-u(x)]\frac{\partial E(x, y)}{\partial \nu_y}\, ds_y = 0$$

and

$$\int_{|y-x|=\varepsilon} \frac{ds_y}{\varepsilon^{n-1}} = \omega_n$$

Therefore, by virtue of (4), we obtain from formula (6) in the limit, for $\varepsilon\to 0$, integral representation (5) (see Fig. 2).

3°. **Mean-Value Formulas.** *If a ball* $|y-x|\leqslant R$ *lies entirely within the domain* D *of harmonicity of a function*

$u(x)$, then the value of that function at the centre of the ball is equal to the arithmetic mean of its values on the sphere $|y-x|=R$.

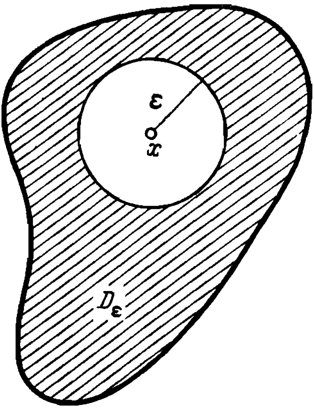

Fig. 2

Indeed, since for the sphere $|y-x|=R$ we have the equalities

$$E(x, y)=\begin{cases} \frac{1}{(n-2)R^{n-2}} & \text{for } n>2 \\ -\ln R & \text{for } n=2 \end{cases}$$

and

$$\frac{\partial E(x, y)}{\partial \nu_y}=\begin{cases} -\frac{1}{R^{n-1}} & \text{for } n>2 \\ -\frac{1}{R} & \text{for } n=2 \end{cases}$$

it is readily seen that, by virtue of (4), formula (5) results in

$$u(x)=\frac{1}{\omega_n R^{n-1}} \int\limits_{|y-x|=R} u(y)\, ds_y \tag{7}$$

On writing formula (7) for the sphere $|y-x|=\rho \leqslant R$ in the form

$$\rho^{n-1} u(x)=\frac{1}{\omega_n} \int\limits_{|y-x|=\rho} u(y)\, ds_y$$

and integrating the last equality with respect to ρ over the interval $0 \leqslant \rho \leqslant R$, we obtain

$$u(x) = \frac{n}{\omega_n R^n} \int\limits_{|y-x| \leqslant R} u(y)\, d\tau_y \qquad (8)$$

where $d\tau_y$ is the element of volume whose location within D is specified by the variable y and $\frac{\omega_n R^n}{n}$ is the volume of the ball $|y - x| < R$.

Formulas (7) and (8) are known as *mean-value formulas for harmonic functions, for a sphere and for a ball* respectively.

Using polar coordinates in the case $n = 2$ and spherical coordinates in the case $n = 3$ we can rewrite formula (7) in the forms

$$u(x_1, x_2) = \frac{1}{2\pi} \int\limits_0^{2\pi} u(x_1 + R\cos\theta,\ x_2 + R\sin\theta)\, d\theta \qquad (9)$$

and

$$u(x_1, x_2, x_3) = \frac{1}{4\pi} \int\limits_0^{\pi} d\theta \int\limits_0^{2\pi} u(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3) \sin\theta\, d\psi$$

where $y_1 = R\sin\theta\cos\psi$, $y_2 = R\sin\theta\sin\psi$ and $y_3 = R\cos\theta$.

4°. The Extremum Principle for the Dirichlet Problem. Uniqueness of the Solution. Given a harmonic function $u(x)$ in a domain D, we shall denote by M and m the supremum and the infimum of the values of the function respectively.

Proceeding from formula (8) we can readily establish the following property known as the *extremum principle for harmonic functions: a function $u(x)$ harmonic in a domain D and not identically equal to a constant can assume neither the value M nor the value m at any point $x \in D$.*

When $M = +\infty$ or $m = -\infty$ the assertion we have stated is evident because the function $u(x)$ can take on only finite values at every point of the domain D. Let us suppose that $M \neq +\infty$ and that $u(x_0) = M$ where $x_0 \in D$. We shall consider a ball $|x - x_0| < \varepsilon$ $(\varepsilon > 0)$ lying entirely inside D. Then for each point of that ball we must have $u(x) = M$. Indeed, if the inequality $u(y) < M$ held at a point y such that $|y - x_0| < \varepsilon$ then, by virtue of the continuity of $u(x)$, this inequality would hold throughout a neighbourhood $|\xi - y| < \delta$ $(\delta > 0)$ of the point y (the inequality $u(y) > M$ is impossible). Therefore the application of formula (8) to the ball $|x - x_0| < \varepsilon$ would result in the inconsistent inequality $M < M$. Hence, it follows that $u(x) = M$ throughout the ball $|x - x_0| < \varepsilon$. Now, let x be an arbitrary fixed point of the domain D and let l be a continuous curve lying within D and joining the points x and x_0. Let the number ε be less than the distance between the boundary S of the domain D and the curve l. Next, imagine that the centre y of the ball $|\eta - y| < \varepsilon$ is moved along the curve l from the point x_0 to the point x; using the fact that for every position of the point y we have the equality $u = M$ inside that ball, we conclude that $u(x) = M$. Consequently, $u(x) = M$ everywhere in the domain D. We have thus arrived at a contradiction, which proves the first part of the assertion we stated above. The other part of the assertion concerning m is proved analogously.

Further, *if it is additionally known that the function* $u(x)$ *harmonic in D is continuous in D* $\cup S$, *then this function must necessarily assume its maximum (minimum) value at some point* $x_0 \in D \cup S$. By virtue of the property of harmonic functions we proved above, it follows that *the point of extremum* x_0 *cannot belong to the interior of the domain* D, *and consequently* $x_0 \in S$.

The extremum principle for harmonic functions implies that *the Dirichlet problem stated in Sec. 4°, § 3 of Introduction cannot possess more than one solution.* Indeed, if we suppose that $u(x)$ and $v(x)$ are two solutions of that problem (see boundary condition (42) in Introduction), then their difference $w(x) = u(x) - v(x)$ is equal to zero on the boundary S of the domain D, and therefore, by virtue of the extremum principle, $w(x) \equiv 0$, that is $u(x) = v(x)$ everywhere in $D \cup S$.

§ 2. The Notion of Green's Function. Solution of the Dirichlet Problem for a Ball and for a Half-Space

1°. Green's Function of the Dirichlet Problem for the Laplace Equation. *By Green's function of the Dirichlet problem for the Laplace equation in a domain D is meant the function $G(x, \xi)$ dependent on two points $x \in D \cup S$ and $\xi \in D \cup S$ which possesses the following properties:* (1) *this function has the form*

$$G(x, \xi) = E(x, \xi) + g(x, \xi) \tag{10}$$

where $E(x, \xi)$ is the fundamental (elementary) solution of the Laplace equation and $g(x, \xi)$ is a harmonic function both with respect to $x \in D$ and with respect to $\xi \in D$; and (2) *when the point x or ξ lies on the boundary S of the domain D, the equality*

$$G(x, \xi) = 0 \tag{11}$$

is fulfilled.

It can easily be seen that $G(x, \xi) \geqslant 0$ throughout the domain D. Indeed, let us denote by D_δ the part of the domain D lying outside a ball $|y - \xi| \leqslant \delta$, $\xi \in D$ of a sufficiently small radius, $\delta > 0$. Since $\lim\limits_{x \to \xi} G(x, \xi) = +\infty$, we must have, for a sufficiently small δ, the inequality $G(x, \xi) > 0$, when $|x - \xi| < \delta$. Consequently, $G(x, \xi) \geqslant 0$ on the boundary of the domain D_δ and therefore, by the extremum principle, $G(x, \xi) \geqslant 0$ for all $x \in D_\delta$ whence we conclude that $G(x, \xi) \geqslant 0$ everywhere in $D \cup S$.

Next we state the *symmetry property of Green's function $G(x, y)$ with respect to the points x and y*:

$$G(x, y) \equiv G(y, x) \quad (x, y \in D)$$

To prove this property, let us remove the points x and y belonging to the domain D from that domain together with the closed balls d: $|z - x| \leqslant \delta$ and d': $|z - y| \leqslant \delta$ of a sufficiently small radius $\delta > 0$; the remaining part of the domain D will be denoted D_δ (see Fig. 3).

The functions $v(z) = G(z, y)$ and $u(z) = G(z, x)$ are harmonic within the domain D outside the balls d' and d

respectively. On applying formula (2) to the domain D_δ we obtain the equality

$$\int_S \left[G(z, y)\frac{\partial G(z, x)}{\partial \nu_z} - G(z, x)\frac{\partial G(z, y)}{\partial \nu_z}\right] ds_z =$$

$$= \int_C \left[G(z, y)\frac{\partial G(z, x)}{\partial \nu_z} - G(z, x)\frac{\partial G(z, y)}{\partial \nu_z}\right] ds_z +$$

$$+ \int_{C'} \left[G(z, y)\frac{\partial G(z, x)}{\partial \nu_z} - G(z, x)\frac{\partial G(z, y)}{\partial \nu_z}\right] ds_z$$

where ν_z denotes the outer normals to S and to the spheres C: $|z - x| = \delta$ and C': $|z - y| = \delta$ at the points z.

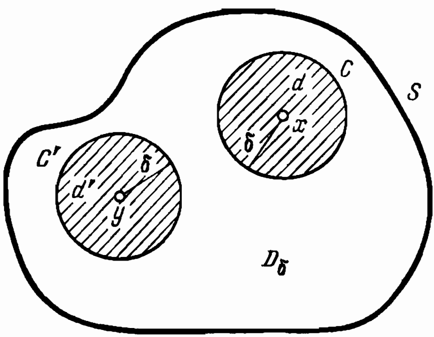

Fig. 3

By virtue of the equalities $G(z, x) = G(z, y) = 0$, $z \in S$, the last formula can be rewritten as

$$\int_C \left[G(z, y)\frac{\partial G(z, x)}{\partial \nu_z} - G(z, x)\frac{\partial G(z, y)}{\partial \nu_z}\right] ds_z =$$

$$= \int_{C'} \left[G(z, x)\frac{\partial G(z, y)}{\partial \nu_z} - G(z, y)\frac{\partial G(z, x)}{\partial \nu_z}\right] ds_z$$

Finally, using the relations

$$G(z, x) = E(z, x) + g(z, x)$$

and

$$G(z, y) = E(z, y) + g(z, y)$$

where $g(z, x)$ and $g(z, y)$ are harmonic functions, we obtain on passing to the limit for $\delta \to 0$ (like in the derivation of formula (5)), the equality $G(x, y) = G(y, x)$, which is what we intended to prove.

Now, let $u(x)$ in equality (5) be the solution of the Dirichlet problem for Laplace's equation and let us substitute $G(x, \xi)$ for $E(x, y)$; then the repetition of the argument used in the derivation of formula (5) and the application of (10) and (11) lead to the formula

$$u(x) = -\frac{1}{\omega_n} \int_S \frac{\partial G(x, \xi)}{\partial \nu_\xi} \varphi(\xi)\, ds_\xi \tag{12}$$

where φ is a given real continuous function.

When Green's function is known, *formula* (12) *expresses the solution of the Dirichlet problem stated in the following way: it is required to find the function* $u(x)$ *harmonic in the domain* D, *continuous in* $D \cup S$ *and satisfying the boundary condition*

$$\lim_{x \to x_0} u(x) = \varphi(x_0); \quad x \in D, \quad x_0 \in S \tag{13}$$

The harmonicity of the function $u(x)$ expressed by formula (12) follows from the fact that Green's function $G(x, \xi)$ is harmonic with respect to x for $x \neq \xi$. However, the fact that this function satisfies boundary condition (13) as well requires special proof.

2°. Solution of the Dirichlet Problem for a Ball. Poisson's Formula. In this section we shall construct explicitly Green's function for the case when the domain D is a ball; for this special case we shall prove that the harmonic function $u(x)$ represented by formula (12) does in fact satisfy boundary condition (13).

So, let the domain D be the ball $|x| < 1$ and let x and ξ be two interior points of that ball. The point $\xi' = \xi/|\xi|^2$ is symmetric to the point ξ with respect to the sphere S: $|x| = 1$. Let us show that *Green's function* $G(x, \xi)$ *of the Dirichlet problem for the ball* $|x| < 1$ *has the form*

$$G(x, \xi) = E(x, \xi) - E\left(|x|\, \xi, \frac{x}{|x|}\right) \tag{14}$$

Indeed, since

$$\left| |x|\xi - \frac{x}{|x|} \right| = [|x|^2 |\xi|^2 - 2x\xi + 1]^{1/2} = \left| |\xi| x - \frac{\xi}{|\xi|} \right| =$$
$$= |\xi| \left| x - \frac{\xi}{|\xi|^2} \right| = |x| \left| \xi - \frac{x}{|x|^2} \right| \quad (15)$$

we conclude that the function $g(x, \xi) = -E\left(|x|\xi, \frac{x}{|x|}\right)$ is harmonic for $|x| < 1$, $|\xi| < 1$ both with respect to x and with respect to ξ. Further, for $|\xi| = 1$ we have

$$|\xi - x| = [|x|^2 - 2x\xi + 1]^{1/2} =$$
$$= \left| |\xi| x - \frac{\xi}{|\xi|} \right| = \left| |x|\xi - \frac{x}{|x|} \right| \quad (16)$$

Consequently, the function $G(x, \xi)$ expressed by formula (14) satisfies all the conditions enumerated in the definition of Green's function.

By virtue of (16), for $|\xi| = 1$ we obtain

$$\frac{\partial G(x, \xi)}{\partial \nu_\xi} = -\sum_{i=1}^{n} \left\{ \frac{\xi_i (\xi_i - x_i)}{|\xi - x|^n} - |x| \frac{\xi_i \left(|x| \xi_i - \frac{x_i}{|x|} \right)}{\left| |x|\xi - \frac{x}{|x|} \right|^n} \right\} =$$
$$= -\frac{1 - |x|^2}{|\xi - x|^n}$$

whence it follows that in the case under consideration formula (12) takes the following form:

$$u(x) = \frac{1}{\omega_n} \int_{|\xi|=1} \frac{1 - |x|^2}{|\xi - x|^n} \varphi(\xi)\, ds_\xi \quad (17)$$

Formula (17) is known as *Poisson's formula.*

Using spherical coordinates in the case $n = 3$ and polar coordinates in the case $n = 2$ we can write Poisson's formula in the form

$$u(x_1, x_2, x_3) = \frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} \frac{1 - |x|^2}{(1 - 2|x| \cos\gamma + |x|^2)^{3/2}} \varphi \sin\theta\, d\psi$$

$$\varphi = \varphi(\xi_1, \xi_2, \xi_3)$$

$$|x| \cos\gamma = x\xi, \quad \xi_1 = \sin\theta \cos\psi$$

$$\xi_2 = \sin\theta \sin\psi, \quad \xi_3 = \cos\theta$$

and

$$u(x_1, x_2) = = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|x|^2}{1-2|x|\cos(\theta-\psi)+|x|^2} \varphi(\cos\psi, \sin\psi)\, d\psi \quad (18)$$

$$x_1 = |x|\cos\theta, \quad x_2 = |x|\sin\theta, \quad \xi_1 = \cos\psi, \quad \xi_2 = \sin\psi$$

respectively.

We have derived formula (17) for the unit ball with centre at the point $x = 0$. Now let us consider a more general case when the function $u(x)$ is harmonic in a ball $|x| < R$, continuous in the closed ball $|x| \leqslant R$ (where $R > 0$ is an arbitrary number) and satisfies the boundary condition $\lim\limits_{x \to y} u(x) = \varphi(y)$, $|x| < R$, $|y| = R$. In this case the function $v(z) = u(Rz)$ is harmonic in the ball $|z| < 1$, continuous for $|z| \leqslant 1$ and satisfies the boundary condition

$$\lim_{z \to t} v(z) = \varphi(Rt), \quad |z| < 1, \quad |t| = 1$$

Therefore, by virtue of formula (17), we have

$$v(z) = \frac{1}{\omega_n} \int_{|\xi|=1} \frac{1-|z|^2}{|\xi - z|^n} \varphi(R\xi)\, ds_\xi$$

whence

$$u(x) = v\left(\frac{x}{R}\right) = \frac{1}{\omega_n R} \int_{|\xi|=1} \frac{R^2-|x|^2}{|R\xi - x|^n} R^{n-1} \varphi(R\xi)\, ds_\xi$$

On making the change $y = R\xi$, we obtain the formula

$$u(x) = \frac{1}{\omega_n R} \int_{|y|=R} \frac{R^2-|x|^2}{|y-x|^n} \varphi(y)\, ds_y \quad (19)$$

Now let us proceed to the case of an arbitrary ball $|x - x_0| < R$. Let the function $u(x)$ be harmonic in that ball $|x - x_0| < R$, continuous in the closed ball $|x - x_0| \leqslant R$ and satisfy the boundary condition $\lim\limits_{x \to y} u(x) = \varphi(y)$, $|x - x_0| < R$, $|y - x_0| = R$. In this case the function $w(z) = u(z + x_0)$ is harmonic in the ball $|z| < R$, continuous for $|z| \leqslant R$ and satisfies the boundary condition $\lim\limits_{z \to t} w(z) = \varphi(t + x_0)$, $|z| < R$, $|t| = R$.

Therefore, according to (19), we can write

$$w(z)=\frac{1}{\omega_n R}\int\limits_{|t|=R}\frac{R^2-|z|^2}{|t-z|^n}\varphi(t+x_0)\,ds_t$$

whence readily follows Poisson's formula for the ball $|x-x_0|<R$:

$$u(x)=w(x-x_0)=\frac{1}{\omega_n R}\int\limits_{|\xi-x_0|=R}\frac{R^2-|x-x_0|^2}{|\xi-x|^n}\varphi(\xi)\,ds_\xi \quad (20)$$

For $x=x_0$ we obtain from (20) formula (7) expressing the mean-value theorem.

3°. Verification of Boundary Conditions. Now we shall show that *the function* $u(x)$ *specified by Poisson's formula satisfies boundary condition* (13); *this will mean that Poisson's formula expresses the solution of the Dirichlet problem whose statement was given at the end of Sec. 2°, § 2.*

For the sake of simplicity we shall limit ourselves to the case $n=2$ (that is we shall investigate Poisson's formula for the circle). Since $u(x)\equiv 1$ is a harmonic function satisfying the boundary condition $\lim\limits_{x\to x_0} u(x)=1$, $|x|<1$ where x_0 is an arbitrary fixed point on the circumference $|x|=1$ of the circle $|x|<1$, formula (17) implies that for all the points x lying in the circle $|x|<1$ the equality

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1-|x|^2}{|\xi-x|^2}d\psi=1,\quad \xi_1=\cos\psi,\quad \xi_2=\sin\psi \quad (21)$$

holds. On the basis of formulas (17) and (21) we can write

$$u(x)-\varphi(x_0)=\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1-|x|^2}{|\xi-x|^2}[\varphi(\xi)-\varphi(x_0)]\,d\psi,$$

$$|x|<1 \quad (22)$$

Since the function φ is uniformly continuous on the circumference $|x|=1$ of the circle $|x|\leqslant 1$, for any given $\varepsilon>0$ there exists a number $\delta(\varepsilon)>0$ such that for all ψ and ψ_0 satisfying the condition $|\psi-\psi_0|<\delta$ the inequality

$$|\varphi(\xi)-\varphi(x_0)|<\varepsilon \quad (23)$$

is fulfilled where $\xi_1=\cos\psi$, $\xi_2=\sin\psi$, $x_{10}=\cos\psi_0$ and $x_{20}=\sin\psi_0$ (here $x_0=(x_{10}, x_{20})$ and $\xi=(\xi_1, \xi_2)$).

Let us rewrite expression (22) in the form

$$u(x)-\varphi(x_0)=I_1+I_2$$

where

$$I_1=\frac{1}{2\pi}\int\limits_{\psi_0-\delta}^{\psi_0+\delta}\frac{1-|x|^2}{|\xi-x|^2}[\varphi(\xi)-\varphi(x_0)]\,d\psi$$

and

$$I_2=\frac{1}{2\pi}\int\limits_{0}^{\psi_0-\delta}\frac{1-|x|^2}{|\xi-x|^2}[\varphi(\xi)-\varphi(x_0)]\,d\psi+ +\frac{1}{2\pi}\int\limits_{\psi_0+\delta}^{2\pi}\frac{1-|x|^2}{|\xi-x|^2}[\varphi(\xi)-\varphi(x_0)]\,d\psi$$

From (21) and (23) we conclude that $|I_1|<\varepsilon$.

After the number $\delta(\varepsilon)$ has been chosen we can take a point x lying so close to x_0 that the inequality

$$\int\limits_{0}^{\psi_0-\delta}\frac{1-|x|^2}{|\xi-x|^2}\,d\psi+ +\int\limits_{\psi_0+\delta}^{2\pi}\frac{1-|x|^2}{|\xi-x|^2}\,d\psi<\frac{\pi\varepsilon}{M}\quad\left(M=\max_{0\leqslant\psi\leqslant2\pi}|\varphi(\xi)|\right)$$

holds, that is $|I_2|<\varepsilon$. Consequently, $|u(x)-\varphi(x_0)|< <2\varepsilon$, and hence

$$\lim_{x\to x_0}u(x)=\varphi(x_0),\quad |x|<1,\quad |x_0|=1$$

4°. Solution of the Dirichlet Problem for a Half-Space. Let us consider the case when the domain D is a half-space; for definiteness, let D be the half-space $x_n>0$. Here we shall require that the sought-for solution of the Dirichlet problem should be bounded. Let x and ξ be two points belonging to that half-space and let us take the point $\xi'= =(\xi_1,\ldots,\xi_{n-1},-\xi_n)$ symmetric to the point ξ about the plane $\xi_n=0$. Since the function $g(x,\xi)=-E(x,\xi')$ is harmonic both with respect to x and with respect to ξ for $x_n>0$, $\xi_n>0$, and, besides, $E(x,\xi)-E(x,\xi')=0$ for $\xi_n=0$, the expression

$$G(x,\xi)=E(x,\xi)-E(x,\xi')\tag{24}$$

is Green's function for the half-space in question.

We shall assume that in the case under consideration the sought-for solution $u(x)$ of the Dirichlet problem can be represented in form (12). This assumption is sure to be fulfilled if the inequalities

$$|u(x)| \leqslant \frac{A}{|x|^h} \quad \text{and} \quad \left|\frac{\partial u}{\partial x_i}\right| \leqslant \frac{A}{|x|^{h+1}} \quad (i = 1, \ldots, n)$$

hold for all $x \in D$ when $|x| \to \infty$, where A and h are some positive constants. Accordingly, the function $\varphi(y_1, \ldots, y_{n-1})$ defined on the plane $y_n = 0$ must satisfy the condition

$$|\varphi| < \frac{A}{\delta^n}$$

for sufficiently large values of $\delta = \left(\sum_{i=1}^{n-1} y_i^2\right)^{1/2}$.

On substituting the expression of $G(x, \xi)$ specified by formula (24) into the right-hand side of formula (12) and taking into account the fact that

$$\frac{\partial G(x, \xi)}{\partial \nu_\xi} = -\frac{\partial G(x, \xi)}{\partial \xi_n} = \frac{\xi_n - x_n}{|\xi - x|^n} - \frac{\xi_n + x_n}{|\xi' - x|^n} = -\frac{2x_n}{\left[\sum_{i=1}^{n-1} (\xi_i - x_i)^2 + x_n^2\right]^{n/2}}$$

for $\xi_n = 0$, we arrive at the formula

$$u(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} x_n \int_{\xi_n = 0} \frac{\varphi(\xi_1, \ldots, \xi_{n-1})}{\left[\sum_{i=1}^{n-1} (\xi_i - x_i)^2 + x_n^2\right]^{n/2}} d\xi_1 \ldots d\xi_{n-1} \tag{25}$$

Formula (25) expresses the solution of the Dirichlet problem with the boundary condition

$$\lim_{x \to y} u(x) = \varphi(y_1, \ldots, y_{n-1}); \quad x_n > 0, \quad y_n = 0 \tag{26}$$

for the half-space $x_n > 0$; this formula is also called *Poisson's formula*.

The fact that the function $u(x)$ determined by formula (25) satisfies boundary condition (26) can be proved in exactly the same way as it was done above in the case of the Dirichlet problem for a circle.

The solution of the Dirichlet problem for an arbitrary half-space specified by a general relation $\sum_{k=1}^{n} a_k x_k - b > 0$ reduces to the special case considered above; to perform the reduction one should take into account that if $u(x)$ is a harmonic function then so is the function $u(\lambda Cx + h)$ where λ is a scalar constant, C is a constant orthogonal matrix and h is a constant vector (see Sec. 1°, § 1 of the present chapter).

5°. **Some Important Consequences of Poisson's Formula. Theorems of Liouville and Harnack.** Formula (19) implies the following proposition: *if a function $u(x)$ harmonic throughout the space E_n is nonnegative (or nonpositive) everywhere in E_n then it is identically equal to a constant.*

Indeed, if $u(x) \geqslant 0$ then, since for $|x| < R$ and $|y| = R$ the inequalities $R - |x| \leqslant |y - x| \leqslant R + |x|$ hold, formula (19) implies, by virtue of (7), that

$$R^{n-2} \frac{R-|x|}{(R+|x|)^{n-1}} u(0) \leqslant u(x) \leqslant R^{n-2} \frac{R+|x|}{(R-|x|)^{n-1}} u(0) \qquad (27)$$

for any $R > 0$. Now, fixing an arbitrary point $x \in E_n$ and making R tend to infinity, we see that the function $u(x)$ satisfies the equality $u(x) = u(0)$ for every point x of the space E_n.

Formula (27) directly implies the following proposition known as *Liouville's theorem*: *if a function $u(x)$ is harmonic throughout E_n and is bounded above (or below) then it is identically equal to a constant.*

Indeed, let, for definiteness, the function $u(x)$ satisfy the inequality $u(x) \leqslant M$ for all $x \in E_n$ where M is a constant. Since the function $M - u(x)$ is harmonic in E_n and is nonnegative, it follows, according to what was proved above, that $M - u(x) = M - u(0)$, that is $u(x) = u(0)$.

Liouville's theorem implies the following property: *the Dirichlet problem for the half-space $x_n > 0$ which was considered in the foregoing section cannot have more than one solution in the class of bounded functions.*

Indeed, if $u_1(x)$ and $u_2(x)$ are any two solutions of that problem, then their difference $v(x) = u_1(x) - u_2(x)$ satisfies the boundary condition $v(x) = 0$ for $x_n = 0$. Let us construct the function

$$w(x) = \begin{cases} v(x_1, \ldots, x_n) & \text{for} \quad x_n \geqslant 0 \\ -v(x_1, \ldots, -x_n) & \text{for} \quad x_n \leqslant 0 \end{cases}$$

The function $w(x)$ is harmonic both for $x_n > 0$ and for $x_n < 0$. Moreover, the function $w(x)$ is harmonic throughout the space E_n because, for any $R > 0$, it coincides within the ball $|x| < R$ with the harmonic function $w^*(x)$ satisfying the boundary condition $w^*(x) = w(x)$ for $|x| = R$. By the hypothesis, the function $w(x)$ is bounded, and therefore Liouville's theorem implies that it is identically equal to a constant. Finally, we have $w(x) = 0$ for $x_n = 0$, whence it follows that $w(x) = 0$ everywhere in E_n, and consequently, $u_1(x) = u_2(x)$.

Using the extremum principle for harmonic functions and Poisson's formula (20) we can easily prove the following proposition (*Harnack's theorem*): *if* $u_k(x)$ $(k = 1, 2, \ldots)$ *are harmonic functions in a domain* D *which are continuous in* $D \cup S$ *and if the series* $\sum_{k=1}^{\infty} u_k(x)$ *is uniformly convergent on the boundary* S *of the domain* D *then this series is uniformly convergent in* $D \cup S$, *and its sum* $u(x) = \sum_{k=1}^{\infty} u_k(x)$ *is a harmonic function in* D.

Indeed, the uniform convergence of the series $\sum_{k=1}^{\infty} u_k(y)$ for $y \in S$ implies that, given an arbitrary $\varepsilon > 0$, there exists an index $N(\varepsilon)$ such that the inequality $\left|\sum_{i=1}^{p} u_{N+i}(y)\right| < \varepsilon$ holds for all $p \geqslant 1$.

Since the finite sum $\sum_{i=1}^{p} u_{N+i}(x)$ is harmonic in D and continuous in $D \cup S$, it follows, by the extremum principle, that $\left|\sum_{i=1}^{p} u_{N+i}(x)\right| < \varepsilon$ for all x belonging to $D \cup S$. As is known from the course of mathematical analysis, the last inequality is a necessary and sufficient condition for the series $\sum_{k=1}^{\infty} u_k(x)$ to be uniformly convergent in $D \cup S$.

Let x_0 be an arbitrary point of the domain D and let $|y-x_0|<R$ be a ball lying entirely inside D. Each harmonic function $u_k(x)$ $(k=1, 2, \ldots)$ can be represented in that ball using Poisson's formula (20):

$$u_k(x)=\frac{1}{\omega_n R}\int\limits_{|y-x_0|=R}\frac{R^2-|x-x_0|^2}{|y-x|^n}u_k(y)\,ds_y$$

Consequently, since a uniformly convergent series can be integrated term-by-term, we can write

$$u(x)=\sum_{k=1}^{\infty}\frac{1}{\omega_n R}\int\limits_{|y-x_0|=R}\frac{R^2-|x-x_0|^2}{|y-x|^n}u_k(y)\,ds_y=$$

$$=\frac{1}{\omega_n R}\int\limits_{|y-x_0|=R}\frac{R^2-|x-x_0|^2}{|y-x|^n}u(y)\,ds_y$$

whence follows the harmonicity of the function $u(x)$ in the ball $|x-x_0|<R$. Since x_0 is an arbitrary point belonging to the domain D, we conclude that $u(x)$ is harmonic everywhere in D.

§ 3. Potential Function for a Volume Distribution of Mass

1°. Continuity of Volume Potential and Its Derivatives of the First Order. Let us consider the expression

$$u(x)=\int\limits_{D}E(x,\xi)\,\mu(\xi)\,d\tau_\xi \tag{28}$$

If the integral on the right-hand side of formula (28) *is convergent, the function* $u(x)$ *determined by that formula is called the potential function for a volume distribution of mass* (or, simply, a *volume potential*) *with* (volume) *mass density* μ *in the domain* D.

In what follows we shall assume that D is a bounded domain.

Since $E(x,\xi)$ is a harmonic function for $x\neq\xi$, the volume potential $u(x)$ is a harmonic function for the points x lying outside $D\cup S$ where S is the boundary of the domain D. Besides, in the case $n>2$ the function $u(x)$ tends to zero for $|x|\to\infty$.

Let us prove the following proposition: *if the function* μ *is continuous and bounded in* D, *the volume potential* $u(x)$ *is a continuous function possessing continuous partial derivatives of the first order everywhere in* E_n *which are expressed by the formulas*

$$\frac{\partial u}{\partial x_i} = \int_D \frac{\partial}{\partial x_i} E(x, \xi)\, \mu(\xi)\, d\tau_\xi \quad (i = 1, \ldots, n) \tag{29}$$

Let $\varepsilon > 0$ be a given sufficiently small arbitrary positive number. Let us consider the function

$$u_\varepsilon(x) = \int_D E_\varepsilon(x, \xi)\, \mu(\xi)\, d\tau_\xi \tag{30}$$

where $E_\varepsilon(x, \xi)$ is a continuously differentiable function in E_n coinciding with $E(x, \xi)$ outside the closed ball $|\xi - x| \leqslant \varepsilon$. For $n > 2$ (here we limit ourselves to the consideration of this very case) we can, for instance, define $E_\varepsilon(x, \xi)$ in the ball $|\xi - x| < \varepsilon$ by means of the formula

$$E_\varepsilon(x, \xi) = \frac{1}{2(n-2)\varepsilon^{n-2}} \left[n - (n-2) \frac{|\xi - x|^2}{\varepsilon^2} \right] \tag{31}$$

It is evident that the function $u_\varepsilon(x)$ expressed by formula (30) is continuously differentiable throughout E_n. By virtue of (28), (30) and (31), there holds the equality

$$|u_\varepsilon(x) - u(x)| \leqslant \omega_n M \int_0^\varepsilon [E_\varepsilon(x, \xi) + E(x, \xi)]\, \rho^{n-1} d\rho =$$

$$= \omega_n M \frac{n+6}{2(n^2-4)} \varepsilon^2$$

where $M = \sup\limits_{\xi \in D} |\mu(\xi)|$. Therefore the function $u_\varepsilon(x)$ tends to $u(x)$, for $\varepsilon \to 0$, uniformly with respect to x, whence it follows that the function $u(x)$ is continuous in E_n.

Further, since

$$\frac{\partial u_\varepsilon(x)}{\partial x_i} = \int_D \frac{\partial E_\varepsilon(x, \xi)}{\partial x_i} \mu(\xi)\, d\tau_\xi \quad (i = 1, \ldots, n)$$

and since the improper integral

$$v_i(x) = \int_D \frac{\partial E(x, \xi)}{\partial x_i} \mu(\xi)\, d\tau_\xi \quad (i = 1, \ldots, n)$$

is uniformly convergent, the difference

$$\frac{\partial u_\varepsilon(x)}{\partial x_i} - v_i(x) = \int_D \frac{\partial}{\partial x_i}[E_\varepsilon(x, \xi) - E(x, \xi)]\,\mu(\xi)\,d\tau_\xi$$

satisfies the inequality

$$\left|\frac{\partial u_\varepsilon(x)}{\partial x_i} - v_i(x)\right| \leqslant \omega_n M \int_0^\varepsilon \left(\frac{\rho}{\varepsilon^n} + \frac{1}{\rho^{n-1}}\right)\rho^{n-1}d\rho = \omega_n M \frac{n+2}{n+1}\varepsilon$$

which holds uniformly with respect to x. Now, as above, taking into account the continuity of the functions $\frac{\partial u_\varepsilon(x)}{\partial x_i}$, we conclude that *the function $u(x)$ possesses continuous partial derivatives of the first order for all $x \in E_n$, and they can be computed using formula* (29).

2°. **Existence of the Derivatives of the Second Order of Volume Potential.** Now we can easily prove that *if the mass density μ possesses continuous partial derivatives of the first order which are bounded in D, then volume potential* (28) *possesses partial derivatives of the second order in D.*

Indeed, using the equality $\frac{\partial E(x, \xi)}{\partial x_i} = -\frac{\partial E(x, \xi)}{\partial \xi_i}$ we can rewrite formula (29) in the form

$$\frac{\partial u}{\partial x_i} = -\int_D \mu(\xi)\frac{\partial}{\partial \xi_i}E(x, \xi)\,d\tau_\xi$$

On performing integration by parts, we obtain

$$\frac{\partial u}{\partial x_i} = -\int_S E(x, \xi)\,\mu(\xi)\cos\widehat{\nu_\xi \xi_i}\,ds_\xi + \int_D \frac{\partial \mu}{\partial \xi_i}E(x, \xi)\,d\tau_\xi \qquad (32)$$

where ν_ξ is the outer normal to S at the point ξ.

The first summand on the right-hand side of (32) possesses continuous derivatives with respect to x_i for $x \in D$, and these derivatives can be found by performing the differentiation under the integral sign. Since the derivatives $\frac{\partial \mu}{\partial \xi_i}$ are bounded and continuous in the domain D, the second summand on the right-hand side of formula (32) also possesses continuous partial derivatives of the first order:

$$\frac{\partial}{\partial x_i}\int_D \frac{\partial \mu}{\partial \xi_i}E(x, \xi)\,d\tau_\xi = \int_D \frac{\partial \mu}{\partial \xi_i}\frac{\partial E(x, \xi)}{\partial x_i}\,d\tau_\xi; \quad i = 1, \ldots, n$$

We have thus proved the existence of continuous partial derivatives of the second order of the function $u(x)$ for $x \in D$, and they are expressed by the formulas

$$\frac{\partial^2 u}{\partial x_i^2} = -\int_S \frac{\partial E(x,\xi)}{\partial x_i} \mu(\xi) \cos \widehat{\nu_\xi \xi_i}\, ds_\xi + \int_D \frac{\partial \mu}{\partial \xi_i} \frac{\partial E(x,\xi)}{\partial x_i}\, d\tau_\xi =$$

$$= \int_S \frac{\partial E(x,\xi)}{\partial \xi_i} \mu(\xi) \cos \widehat{\nu_\xi \xi_i}\, ds_\xi -$$

$$- \int_D \frac{\partial \mu}{\partial \xi_i} \frac{\partial E(x,\xi)}{\partial \xi_i}\, d\tau_\xi \quad (i = 1, \ldots, n)$$

Consequently, for $x \in D$ we have

$$\Delta u = \int_S \sum_{i=1}^n \frac{\partial E(x,\xi)}{\partial \xi_i} \mu(\xi) \cos \widehat{\nu_\xi \xi_i}\, ds_\xi - \int_D \sum_{i=1}^n \frac{\partial \mu}{\partial \xi_i} \frac{\partial E(x,\xi)}{\partial \xi_i}\, d\tau_\xi =$$

$$= \int_S \frac{\partial E(x,\xi)}{\partial \nu_\xi} \mu(\xi)\, ds_\xi - \lim_{\varepsilon \to 0} \int_{D_\varepsilon} \sum_{i=1}^n \frac{\partial \mu}{\partial \xi_i} \frac{\partial E(x,\xi)}{\partial \xi_i}\, d\tau_\xi \quad (33)$$

where D_ε is the part of the domain D lying outside the closed ball $|\xi - x| \leqslant \varepsilon$.

Since $\sum_{i=1}^n \frac{\partial^2 E(x,\xi)}{\partial \xi_i^2} = 0$ for $\xi \neq x$, we have $\sum_{i=1}^n \frac{\partial \mu}{\partial \xi_i} \frac{\partial E}{\partial \xi_i} = \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\left(\mu \frac{\partial E}{\partial \xi_i}\right)$, and therefore, using formula (GO), we obtain

$$\int_{D_\varepsilon} \sum_{i=1}^n \frac{\partial \mu}{\partial \xi_i} \frac{\partial E(x,\xi)}{\partial \xi_i}\, d\tau_\xi =$$

$$= \int_S \mu(\xi) \frac{\partial E(x,\xi)}{\partial \nu_\xi}\, ds_\xi - \int_{|\xi - x| = \varepsilon} \mu(\xi) \frac{\partial E(x,\xi)}{\partial \nu_\xi}\, ds_\xi \quad (34)$$

On the basis of (33) and (34) we derive for $x \in D$ the relation

$$\Delta u = \lim_{\varepsilon \to 0} \int_{|\xi - x| = \varepsilon} \mu(\xi) \frac{\partial E(x, \xi)}{\partial \nu_\xi} ds_\xi =$$

$$= -\lim_{\varepsilon \to 0} \int_{|\xi - x| = \varepsilon} \frac{\mu(\xi)\, ds_\xi}{|\xi - x|^{n-1}} =$$

$$= -\lim_{\varepsilon \to 0} \int_{|\xi - x| = \varepsilon} \frac{\mu(\xi) - \mu(x)}{\varepsilon^{n-1}} ds_\xi - \lim_{\varepsilon \to 0} \int_{|\xi - x| = \varepsilon} \frac{\mu(x)\, ds_\xi}{\varepsilon^{n-1}} =$$

$$= -\omega_n \mu(x) \quad (35)$$

In these calculations we have assumed that the boundary S of the domain D is a smooth surface. However, this additional assumption may in fact be dropped. To this end we represent the function $u(x)$ in the form

$$u(x) = \int_{D_R} E(x, \xi) \mu(\xi)\, d\tau_\xi + \int_{|\xi - x| \leqslant R} E(x, \xi) \mu(\xi)\, d\tau_\xi, \; x \in D$$

where $|\xi - x| \leqslant R$ is a ball lying inside the domain D and D_R is the part of D lying outside that ball $|\xi - x| \leqslant R$. The first summand on the right-hand side of the last formula is a harmonic function inside the ball $|\xi - x| < R$, while the second summand is obviously such that the above argument may be applied to it.

In the case $n = 2$ we can analogously derive the formula $\Delta u = -2\pi\mu(x)$.

3°. **The Poisson Equation.** On the basis of formula (35) we conclude that *the function $u(x)$ specified by the formula*

$$u(x) = -\frac{1}{\omega_n} \int_D G(x, \xi) f(\xi)\, d\tau_\xi \quad (36)$$

where $G(x, \xi)$ is Green's function of the Dirichlet problem for harmonic functions in the domain D and $f(x)$ is a bounded function possessing continuous first-order partial derivatives, bounded in D, is a regular solution of Poisson's equation

$$\Delta u = f(x), \quad x \in D \quad (37)$$

Let us show that the function $u(x)$ satisfies the boundary condition

$$\lim_{x \to x_0} u(x) = 0, \quad x \in D, \quad x_0 \in S \tag{38}$$

It should be noted that the passage to the limit under the integral sign on the right-hand side of formula (36) is not legitimate here because, although Green's function

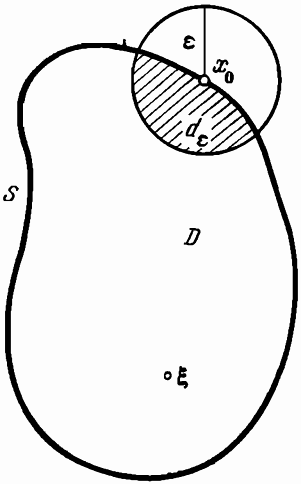

Fig. 4

$G(x, \xi)$ tends to zero when $x \to x_0 \in S$, this passage to the limit is not uniform with respect to $\xi \in D$. That is why we shall use another technique.

Let us represent the function $u(x)$ in the form

$$u(x) = -\frac{1}{\omega_n} \int_{D_\varepsilon} G(x, \xi) f(\xi) \, d\tau_\xi - \frac{1}{\omega_n} \int_{d_\varepsilon} G(x, \xi) f(\xi) \, d\tau_\xi$$

where $d_\varepsilon = D \cap \{ | x - x_0 | < \varepsilon \}$ and D_ε is the part of D lying outside the ball $| \xi - x_0 | \leqslant \varepsilon$ (see Fig. 4).

It is evident that

$$\lim_{x \to x_0} \int_{D_\varepsilon} G(x, \xi) f(\xi) \, d\tau_\xi = \int_{D_\varepsilon} \lim_{x \to x_0} G(x, \xi) f(\xi) \, d\tau_\xi = 0$$

If we manage to show that for all $x \in d_\varepsilon$ there holds the inequality

$$\int_{d_\varepsilon} G(x, \xi)\, d\tau_\xi < N(\varepsilon)$$

where $\lim_{\varepsilon \to 0} N(\varepsilon) = 0$, then, since the function $f(x)$ is bounded, the validity of equality (38) will be proved.

Let us denote by S_R a sphere $|y - \xi| = R$ with centre at a point $\xi \in D$ whose radius R is so large that this sphere contains entirely the domain D for any $\xi \in D$. Let us denote $\Omega(x, \xi) = E(x, \xi) - E(y, \xi)$, $|y - \xi| = R$. It is obvious that the function $\Omega(x, \xi)$ is nonnegative within the ball $|\xi - x| < R$ and that on the boundary S of the domain D we have

$$G(x, \xi) - \Omega(x, \xi) \leqslant 0$$

Now, taking into account the harmonicity of the difference $G(x, \xi) - \Omega(x, \xi)$ in the domain D, we conclude, on the basis of the extremum principle, that

$$\Omega(x, \xi) \geqslant G(x, \xi) \geqslant 0$$

everywhere in D. From the inequalities we have derived immediately follow the inequalities

$$\int_{d_\varepsilon} G(x, \xi)\, d\tau_\xi \leqslant \int_{d_\varepsilon} \Omega(x, \xi)\, d\tau_\xi \leqslant N(\varepsilon), \quad x \in d_\varepsilon$$

we are interested in because the integral of the function $\Omega(x, \xi)$ over the domain d_ε is uniformly convergent.

Thus, if Green's function $G(x, \xi)$ is known, the volume potential $u(x)$ determined by formula (36) in the domain D gives the solution of homogeneous Dirichlet problem (38) for Poisson's equation (37).

On substituting expression (14) of Green's function $G(x, \xi)$ into the right-hand side of formula (36), we obtain the expression in quadratures for the solution of the homogeneous Dirichlet problem (38) for equation (37) in the case of the unit ball.

Now, instead of a homogeneous boundary condition of type (38), let us consider a non-homogeneous boundary condition of the form

$$\lim_{x \to x_0} u(x) = \varphi(x_0) \quad (x \in D, \quad x_0 \in S) \tag{39}$$

If $v(x)$ is a harmonic function in D satisfying boundary condition (39), that is

$$\lim_{x \to x_0} v(x) = \varphi(x_0) \quad (x \in D, \quad x_0 \in S)$$

and if $u(x)$ is the sought-for solution of non-homogeneous Dirichlet problem (39) for equation (37), then the difference $u(x) - v(x) = w(x)$ is a regular solution of the equation

$$\Delta w = f(x), \ x \in D$$

and $w(x)$ satisfies the homogeneous boundary condition

$$\lim_{x \to x_0} w(x) = 0 \qquad (x \in D, \quad x_0 \in S) \tag{40}$$

Thus, *the problem of the determination of the solution* $u(x)$ *of non-homogeneous Dirichlet problem* (39) *for equation* (37) *reduces to the determination of the solution* $w(x)$ *of the same equation satisfying homogeneous boundary condition* (40).

4°. **Gauss Formula.** For our further aims it is necessary to prove the formula

$$\int_\sigma \frac{\partial}{\partial \nu_x} E(x, \xi)\, ds_x = \begin{cases} -\omega_n & \text{for} \quad \xi \in d \\ -\frac{1}{2}\omega_n & \text{for} \quad \xi \in \sigma \\ 0 & \text{for} \quad \xi \in C(d \cup \sigma) \end{cases} \tag{41}$$

where d is an arbitrary bounded domain with a sufficiently smooth boundary σ and $C(d \cup \sigma)$ is the complement of $d \cup \sigma$ with respect to the whole space E_n (see Fig. 5).

The validity of the equality $\int_\sigma \frac{\partial}{\partial \nu_x} E(x, \xi)\, ds_x$ for $\xi \in$ $\in C(d \cup \sigma)$ (the third part of (41)) follows from formula (4) because the function $E(x, \xi)$ is harmonic with respect to x when $x \neq \xi$. Next let us consider the case when $\xi \in d \cup \sigma$; we shall denote by d_ε the part of the domain d lying outside the intersection of the closed ball $|\xi - x| \leqslant \varepsilon$ with the union $d \cup \sigma$. For a sufficiently small $\varepsilon > 0$ the set d_ε is a domain whose boundary consists of the following parts;

(1) the surface σ for $\xi \in d$ or its part σ_1 lying outside the ball $|\xi - x| < \varepsilon$ for $\xi \in \sigma$ and (2) the sphere $|\xi - x| = \varepsilon$ for $\xi \in d$ or the part σ_2 of that sphere lying within d for $\xi \in \sigma$. In the case under consideration, by virtue of the

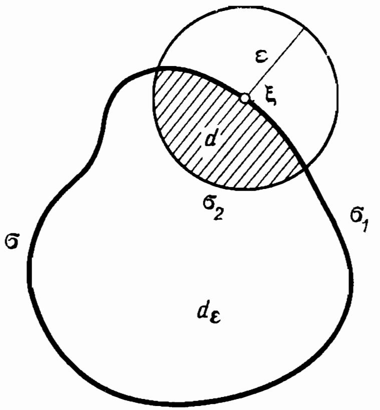

Fig. 5

harmonicity of the function $E(x, \xi)$ for $x \neq \xi$ and on the basis of (4), we can again write

$$\int_{\sigma} \frac{\partial E(x, \xi)}{\partial \nu_x} ds_x = - \int_{|x-\xi|=\varepsilon} \frac{ds_x}{\varepsilon^{n-1}} = -\omega_n, \quad \xi \in d \qquad (42)$$

and

$$\int_{\sigma_1} \frac{\partial E(x, \xi)}{\partial \nu_x} ds_x = - \int_{\sigma_2} \frac{ds_x}{\varepsilon^{n-1}}, \quad \xi \in \sigma \qquad (43)$$

When writing the last equality we have taken into account the fact that formula (4) remains valid in the case when the boundary of the domain is a piecewise smooth surface. Equality (42) is the first part of formula (41), and the passage to the limit in equality (43) for $\varepsilon \to 0$ leads to the second part of (41). This completes the proof of (41).

From formula (41) we easily derive the *Gauss formula* for the potential function $u(x)$ for a volume distribution of mass with density μ over a domain D:

$$\int_{\sigma} \frac{\partial u(x)}{\partial \nu_x} ds_x = -\omega_n \int_{D \cap d} \mu(\xi) d\tau_\xi \tag{44}$$

where d is an arbitrary domain with a sufficiently smooth boundary σ lying in the space E_n.

Indeed, since

$$\frac{\partial u}{\partial \nu_x} = \int_D \mu(\xi) \frac{\partial E(x, \xi)}{\partial \nu_x} d\tau_\xi$$

we have

$$\int_{\sigma} \frac{\partial u(x)}{\partial \nu_x} ds_x = \int_{\sigma} ds_x \int_D \mu(\xi) \frac{\partial E(x, \xi)}{\partial \nu_x} d\tau_\xi =$$

$$= \int_D \mu(\xi) d\tau_\xi \int_{\sigma} \frac{\partial E(x, \xi)}{\partial \nu_x} ds_x =$$

$$= \int_{D \cap d} \mu(\xi) d\tau_\xi \int_{\sigma} \frac{\partial E(x, \xi)}{\partial \nu x} ds_x +$$

$$+ \int_{d_1} \mu(\xi) d\tau_\xi \int_{\sigma} \frac{\partial E(x, \xi)}{\partial \nu_x} ds_x \tag{45}$$

where d_1 is the part of D lying outside $d \cup \sigma$.

By virtue of (41), the second summand on the right-hand side of (45) is equal to zero while the first summand coincides with the right-hand member of formula (44).

§ 4. Double-Layer and Single-Layer Potentials

1°. Definition of a Double-Layer Potential. Let D be a bounded domain in the space E_n with a sufficiently smooth boundary S and let μ be a real continuous function defined on S.

By a potential function for a double layer of distribution of dipoles on the surface S (or, simply, a *double-layer potential) is meant the function*

$$u(x) = \frac{1}{\omega_n} \int_S \mu(\xi) \frac{\partial E(x, \xi)}{\partial \nu_\xi} ds_\xi \tag{46}$$

where μ is the moment per unit area of the dipole distribution.

Since $\xi \in S$, the function $E(x, \xi)$ is harmonic for $x \neq \xi$, and $\frac{\partial E(x, \xi)}{\partial \nu_\xi}$ tends to zero for $|x| \to \infty$, we see that *the double-layer potential* $u(x)$ *specified by formula* (46) *is a harmonic function throughout the space* E_n *except the points belonging to the surface* S, *and it tends to zero for* $|x| \to \infty$.

The domain D and the complement $C(D \cup S)$ of $D \cup S$ with respect to the whole space E_n will be denoted as D^+ and D^- respectively. The integrand expression on the right-hand side of formula (46) has a simple physical meaning in the case $n = 3$. Namely, let ξ' and ξ'' be two points on the normal ν_ξ to the surface S at the point ξ which are located symmetrically with respect to ξ, and $\xi' \in D^+$, $\xi'' \in D^-$. Let us suppose that at the points ξ' and ξ'' there are (concentrated) electric charges $-\mu_0$ and μ_0 respectively such that when $|\xi'' - \xi'| \to 0$ the equality

$$\mu_0 |\xi'' - \xi'| = \mu(\xi)$$

permanently holds. The potential of the field generated by these charges at a point $x \neq \xi$ has the form

$$\frac{\mu_0}{|\xi'' - x|} - \frac{\mu_0}{|\xi' - x|}$$

The limiting configuration of the charges (for $|\xi'' - \xi'| \to 0$) is called a *dipole*, and μ and ν_ξ are referred to as the *polarization* (the *dipole moment*) and the *axis of the dipole* respectively.

On the basis of the definition of a directional derivative along a given direction, we readily conclude that

$$\lim_{|\xi''-\xi'| \to 0} \mu_0 \left(\frac{1}{|\xi'' - x|} - \frac{1}{|\xi' - x|} \right) =$$

$$= \mu \lim_{|\xi''-\xi'| \to 0} \frac{1}{|\xi'' - \xi'|} \left(\frac{1}{|\xi'' - x|} - \frac{1}{|\xi' - x|} \right) = \mu \frac{\partial}{\partial \nu_\xi} E(x, \xi)$$

Let us investigate the behaviour of the function $u(x)$ when the variable point x passes from D^+ to D^-. We shall limit ourselves to the consideration of the case of two independent variables, that is the case when

$$u(x) = -\frac{1}{2\pi} \int_S \mu(\xi) \frac{\partial}{\partial \nu_\xi} \ln |\xi - x| \, ds_\xi \qquad (47)$$

We shall assume that S in (47) is a simple closed *Jordan curve* possessing continuous curvature and that μ is a twice continuously differentiable function.

The curvilinear coordinates of two points x^0 and ξ on S (which are the arc lengths reckoned along S in the counterclockwise direction from a fixed point belonging to S to the points x^0 and ξ respectively) will be denoted as s and t.

Let us show that *formula* (47) *expressing the double-layer potential makes sense for* $x = x^0$ *as well.* Indeed, for the function

$$\pi K(s, t) = \frac{\partial}{\partial \nu_\xi} \ln |\xi - x^0|$$

we have

$$\pi K(s, t) = \frac{1}{|\xi - x^0|^2} \sum_{i=1}^{2} (\xi_i - x_i^0) \frac{\partial \xi_i}{\partial \nu_\xi} = \frac{\cos \varphi}{|\xi - x^0|} = \frac{\partial}{\partial t} \theta(s, t) \tag{48}$$

where

$$\cos \varphi = \frac{(\xi - x^0)\, \nu_\xi}{|\xi - x^0|}$$

and

$$\theta(s, t) = \arctan \frac{\xi_2 - x_2^0}{\xi_1 - x_1^0}$$

It can easily be seen that $K(s, t)$, *considered as a function of the two variables for* $s \in S$ *and* $t \in S$, *is continuous with respect to the point* (s, t).

Indeed, let us denote

$$\alpha(s, t) = \frac{\xi_2(t) - x_2^0(s)}{t - s} \quad \text{and} \quad \beta(s, t) = \frac{\xi_1(t) - x_1^0(s)}{t - s}$$

It is evident that

$$\xi_1(t) - x_1^0(s) = (t - s) \int_0^1 y_1'[t + \tau(s - t)]\, d\tau$$

and

$$\xi_2(t) - x_2^0(s) = (t - s) \int_0^1 y_2'[t + \tau(s - t)]\, d\tau$$

where

$$y_1 = y_1(z), \quad y_2 = y_2(z), \quad 0 \leqslant z \leqslant l$$

are parametric equations of the curve S and l is its arc length.

By virtue of (48), we have

$$K(s, t) = \frac{1}{\pi} \frac{\beta\alpha'_t - \alpha\beta'_t}{\alpha^2 + \beta^2}$$

whence it follows that

$$\lim_{t \to s} K(s, t) = \frac{1}{2\pi} \frac{x_1^{0\prime}(s)\, x_2^{0\prime\prime}(s) - x_2^{0\prime}(s)\, x_1^{0\prime\prime}(s)}{x_1^{0\prime 2} + x_2^{0\prime 2}} = \frac{K(s)}{2\pi}$$

where $K(s)$ is the curvature of the curve S.

Let us define the value of $K(s, t)$ for $t = s$ by putting it equal to

$$\lim_{t \to s} K(s, t) = \frac{K(s)}{2\pi}$$

Since the curvature of the curve S is continuous, the functions α, β, α'_t and β'_t are continuous with respect to (s, t); besides $\alpha^2 + \beta^2 \neq 0$ for all the values of s and t on S. Therefore the assertion we have stated is in fact true.

From the continuity of the function $K(s, t)$ it follows that the expression

$$u(x^0) = -\frac{1}{2\pi} \int_S \frac{\cos\varphi}{|\xi - x^0|} \mu(\xi)\, ds_\xi \qquad (49)$$

of double-layer potential (47) makes sense for $x^0 \in S$ and that $u(x)$ is a continuous function on S at every point $x = x^0$, $x^0 \in S$.

2°. Formula for the Jump of a Double-Layer Potential. Reduction of the Dirichlet Problem to an Integral Equation. Let us denote by d a circle $|x - x^0| < \varepsilon$ (with centre at a point $x^0 \in S$) of a sufficiently small radius $\varepsilon > 0$ and by S' the part of S lying inside d (see Fig. 6). By $v(x)$ we shall denote a function which is continuous together with its partial derivatives of the first and of the second order in the domain $d' = d \cap D^+$ including its boundary and satisfies the conditions

$$v(x) = \mu(x), \quad \frac{\partial v(x)}{\partial \nu_x} = 0, \quad x \in S' \qquad (50)$$

Let σ be the part of the contour $|x - x^0| = \varepsilon$ lying in the domain D^+ (Fig. 6).

Let us integrate the identity

$$\sum_{i=1}^{2} \frac{\partial}{\partial \xi_i}\left(\ln|\xi-x|\frac{\partial v}{\partial \xi_i} - v\frac{\partial}{\partial \xi_i}\ln|\xi-x|\right) = \\ = \ln|\xi-x|\,\Delta v - v\Delta\ln|\xi-x|$$

over the domain d' (in the case when x belongs to $d' \cup S'$ the point x should be deleted from $d' \cup S'$ together with a closed circle $|\xi-x| \leqslant \delta$ of a sufficiently small radius $\delta > 0$, the integral should be taken over the remaining

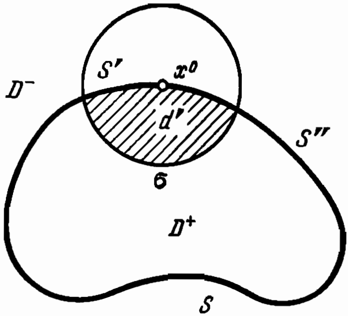

Fig. 6

part of the domain d' and then δ should be made to tend to zero). On performing this integration and taking into account equalities (50), we can write

$$-\int_{S'} \mu \frac{\partial}{\partial \nu_\xi}\ln|\xi-x|\,ds_\xi + \\ + \int_{\sigma}\left(\ln|\xi-x|\frac{\partial v}{\partial \nu_\xi} - v\frac{\partial}{\partial \nu_\xi}\ln|\xi-x|\right)ds_\xi + \\ + q(x)\,v(x) = \int_{d'} \ln|\xi-x|\,\Delta v\,d\tau_\xi \tag{51}$$

where

$$q(x) = \begin{cases} 2\pi & \text{for } x \in d' \\ \pi & \text{for } x \in S' \\ 0 & \text{for } x \in D^- \end{cases} \tag{52}$$

Next we write expression (47) of the double-layer potential in the form

$$u(x) = -\frac{1}{2\pi}\int_{S''} \mu \frac{\partial}{\partial \nu_\xi} \ln|\xi - x|\, ds_\xi - \\ -\frac{1}{2\pi}\int_{S'} \mu \frac{\partial}{\partial \nu_\xi} \ln|\xi - x|\, ds_\xi$$

where S'' is the part of S lying outside the circle d, and use equality (51), which yields

$$u(x) = -\frac{1}{2\pi}\int_{S''} \mu \frac{\partial}{\partial \nu_\xi} \ln|\xi - x|\, ds_\xi + \\ +\frac{1}{2\pi}\int_{\sigma} \left(v \frac{\partial}{\partial \nu_\xi} \ln|\xi - x| - \ln|\xi - x| \frac{\partial v}{\partial \nu_\xi} \right) ds_\xi + \\ +\frac{1}{2\pi}\int_{d'} \ln|\xi - x|\, \Delta v\, d\tau_\xi - \frac{1}{2\pi} q(x)\, v(x) \qquad (53)$$

The terms on the right-hand side of (53) involving integration vary continuously when the point x passes from the domain D^+ to the domain D^- through the point x^0. Taking into account what has been said and equality (52) we conclude that the expressions

$$u(x^0), \qquad u^+(x^0) = \lim_{\substack{x \to x^0 \\ x \in D^+}} u(x) \quad \text{and} \quad u^-(x^0) = \\ = \lim_{\substack{x \to x^0 \\ x \in D^-}} u(x) \qquad (x^0 \in S)$$

satisfy the relations

$$u^+(x^0) - u(x^0) = -\frac{1}{2}\mu(x^0) \qquad (54)$$

and

$$u^-(x^0) - u(x^0) = \frac{1}{2}\mu(x^0) \qquad (55)$$

Thus, we have come to the conclusion that *double-layer potential* (47) *suffers jumps expressed by formulas* (54) and (55) *for* $x \to x^0 \in S$.

Since the integral terms on the right-hand side of (53) are continuously differentiable when the point x passes from D^+ to D^- through the point x^0, we conclude, on the basis of (50) and (52), that *there exist the limits*

$$\lim_{\substack{x \to x^0 \\ x \in D^+}} \frac{\partial u}{\partial \nu_x} = \left(\frac{\partial u}{\partial \nu_{x^0}} \right)^+ \quad \text{and} \quad \lim_{\substack{x \to x^0 \\ x \in D^-}} \frac{\partial u}{\partial \nu_x} = \left(\frac{\partial u}{\partial \nu_{x^0}} \right)^-$$

and that

$$\left(\frac{\partial u}{\partial \nu_{x^0}} \right)^+ = \left(\frac{\partial u}{\partial \nu_{x^0}} \right)^-$$

(that is the normal derivative of $u(x)$ is continuous on passing through x^0).

The properties of the double-layer potential we have established also remain valid when the boundary S of the domain D and the function μ satisfy some weaker requirements; for instance, when the boundary is not very smooth and the function μ satisfies only the condition of continuity.

Now let us construct the solution $u(x)$ of the Dirichlet problem for Laplace's equation in the domain D^+ with the boundary condition

$$u^+(x^0) = g(x^0), \quad x^0 \in S \tag{56}$$

under the assumption that the curvature of the curve S and the function $g(x^0)$ are continuous) in the form of a double-layer potential (47) with the unknown moment per unit area μ.

According to (48), (49) and (54), for the function $u(x)$ expressed by formula (47) (this function is harmonic in the domain D^+) to satisfy boundary condition (56), the equality

$$\mu(s) + \int_S K(s, t)\, \mu(t)\, dt = -2g(s) \tag{57}$$

must hold. Equality (57) is a Fredholm linear integral equation of the second kind with respect to the unknown function μ. Thus, *the Dirichlet problem reduces to integral equation* (57).

In Sec. 1°, § 3 of Chapter 5 we shall prove that *integral equation* (57) *possesses a single solution* μ. This means that *double-layer potential* (47) *with the function* μ *satisfying in-*

tegral equation (57) *is the solution of the Dirichlet problem with boundary condition* (56); *this proves the existence of the solution of that problem.*

3°. Single-Layer Potential. The Neumann Problem. Let us consider the expression

$$u(x)=\frac{1}{\omega_n}\int_S E(x, \xi)\,\mu(\xi)\,ds_\xi \tag{58}$$

The function $u(x)$ *determined by formula* (58) *is referred to as a potential function for a surface distribution of mass* (or, simply, a *single-layer potential*) *with surface density of mass* μ. *This function is harmonic at all the points* x *of the space* E_n *not belonging to* S, *and in the case* $n>2$ *it tends to zero as* $|x|\to\infty$. In the case $n=2$ the single-layer potential has the form

$$u(x)=\frac{1}{2\pi}\int_S \ln\frac{1}{|\xi-x|}\,\mu(\xi)\,ds_\xi \tag{59}$$

that is

$$u(x)=-\frac{\ln|x|}{2\pi}\int_S \mu(\xi)\,ds_\xi+\frac{1}{2\pi}\int_S \ln\frac{|x|}{|\xi-x|}\,\mu(\xi)\,ds_\xi$$

whence it follows that $\lim\limits_{|x|\to\infty} u(x)=0$ only when the condition

$$\int_S \mu(\xi)\,ds_\xi=0$$

is fulfilled.

We shall limit ourselves to the investigation of the properties of a single-layer potential only for the case $n=2$, under the assumption that the curvature of the curve S is continuous and that the function μ is twice continuously differentiable.

Let us repeat the procedure used above in the derivation of formula (53) with the only distinction that in the case under consideration the function $u(x)$ is specified by formula (59) and that the function $v(x)$ satisfies, instead of (50), the conditions

$$v(x)=0 \quad\text{and}\quad \frac{\partial}{\partial\nu_x}v(x)=\mu(x), \quad x\in S' \tag{60}$$

This results in

$$\int_{S'} \ln|\xi-x|\,\mu(\xi)\,ds_\xi + \int_\sigma \left(\ln|\xi-x|\frac{\partial v}{\partial \nu_\xi} - v\frac{\partial}{\partial \nu_\xi}\ln|\xi-x|\right) ds_\xi + q(x)\,v(x) = \int_{d'} \ln|\xi-x|\,\Delta v\,d\tau_\xi \qquad (61)$$

From formula (61) we obtain the following expression for single-layer potential (59):

$$u(x) = -\frac{1}{2\pi}\int_{S''} \ln|\xi-x|\,\mu(\xi)\,ds_\xi - \frac{1}{2\pi}\int_\sigma \left(v\frac{\partial \ln|\xi-x|}{\partial \nu_\xi} - \ln|\xi-x|\frac{\partial v}{\partial \nu_\xi}\right) ds_\xi + \frac{1}{2\pi}q(x)\,v(x) - \frac{1}{2\pi}\int_{d'} \ln|\xi-x|\,\Delta v\,d\tau_\xi \qquad (62)$$

Since the integral terms on the right-hand side of (62) are continuously differentiable everywhere in E_2 except the points belonging to the arcs S'' and σ, we conclude, taking into account equalities (52) and (60), that *when the point x passes from the domain D^+ to the domain D^- through the point $x^0 \in S$, single-layer potential* (59) *remains continuous whereas its normal derivative $\frac{\partial u}{\partial \nu_x}$ suffers a jump so that*

$$\left(\frac{\partial u}{\partial \nu}\right)^+ - \frac{\partial u(x^0)}{\partial \nu_{x_0}} = \frac{1}{2}\mu(x^0) \qquad (63)$$

and

$$\left(\frac{\partial u}{\partial \nu}\right)^- - \frac{\partial u(x^0)}{\partial \nu_{x0}} = -\frac{1}{2}\mu(x^0) \qquad (64)$$

In formulas (63) and (64) the directional derivative $\frac{\partial u(x^0)}{\partial \nu_{x0}}$ is expressed by the formula

$$\frac{\partial u(x^0)}{\partial \nu_{x0}} = \frac{1}{2\pi}\int_S \frac{(\xi-x^0)\,\nu_{x0}}{|\xi-x^0|^2}\mu(\xi)\,ds_\xi = \frac{1}{2}\int_S K^*(s,t)\,\mu(t)\,dt \qquad (65)$$

where

$$K^* (s, t) = \frac{1}{\pi} \frac{\partial}{\partial s} \arctan \frac{\xi_2 - x_2^0}{\xi_1 - x_1^0} \tag{66}$$

Let us define the value of $K^* (s, t)$ for $s = t$ as being equal to $\lim_{s \to t} K^* (s, t)$; in the same way as it was done in Sec. 1° where the function $K (s, t)$ was considered we can readily check that the function $K^* (s, t)$ thus extended is continuous with respect to the point (s, t) $(x \in S,\ t \in S)$.

The *Neumann problem* (also called the *second boundary-value problem*) of the theory of harmonic functions is stated in the following way: *it is required to find the function $u (x)$ harmonic in the domain D^+ which is continuous together with its partial derivatives of the first order in $D^+ \cup S$ and satisfies the boundary condition*

$$\left(\frac{\partial u}{\partial \nu}\right)^+ = g (x^0), \quad x^0 \in S \tag{67}$$

where $g (x^0)$ is a given function defined on S.

If $u (x)$ and $u_1 (x)$ are two solutions of the Neumann problem, their difference $u (x) - u_1 (x) = w (x)$ satisfies the condition $\frac{\partial w}{\partial \nu_x} = 0$, $x \in S$ (this follows directly from (67)). From this condition, by virtue of Property (2) of harmonic functions proved in Sec. 1°, § 1 of the present chapter, it follows that $w (x) = \text{const}$, that is $u_1 (x) = u (x) + C$.

If a function $u (x)$ is a solution of the Neumann problem then so is the function $u (x) + C$ where C is an arbitrary real constant.

From formula (4) expressing property (3) of harmonic functions, by virtue of (67), it follows that *for the Neumann problem with boundary condition* (67) *to be solvable it is necessary that the condition*

$$\int_S g (s)\, ds = 0 \tag{68}$$

should be fulfilled.

Now let us represent the solution $u (x)$ of the Neumann problem in the form of single-layer potential (59) with an unknown density μ; then, by virtue of (63), (65) and (67), we obtain for the determination of the function μ Fredholm's integral equation of the second kind

$$\mu(s)+\int_S K^*(s,\ t)\,\mu(t)\,dt=2g(s) \tag{69}$$

whose kernel $K^*(s,\ t)$ is expressed by formula (66). Thus, *the Neumann problem reduces to integral equation* (69).

As will be shown later (see Chapter 5, § 3, Sec. 1°), *condition* (68) *is not only necessary but also sufficient for the solution of the Neumann problem to exist.*

4°. The Dirichlet Problem and the Neumann Problem for Unbounded Domains. Boundary-value problems can be stated not only for bounded domains but also for unbounded ones. For instance, the Dirichlet problem for the unbounded domain D^- with boundary S which was considered in Sec. 1°, § 4 of the present chapter is stated as follows: *it is required to find a regular harmonic function $u(x)$ in D^- which satisfies the boundary condition*

$$u^-(y) = \varphi(y), \quad y \in S \tag{70}$$

where φ is a given real continuous function.

The boundary-value problem stated in this way is called the *exterior Dirichlet problem* (for the unbounded domain D^-) in contradistinction to the Dirichlet problem for the bounded domain D^+ which is referred to as the *interior Dirichlet problem.*

As was already mentioned in Sec. 1°, § 1 of the present chapter, the regularity of a harmonic function $u(x)$ in a domain of the type of D^- is understood in the sense that in the case $n>2$ this function tends to zero not slower than $|x|^{2-n}$ for $|x|\to\infty$ and in the case $n=2$ it tends to a finite limit for $|x|\to\infty$.

Now we note that under the transformation of *inversion* expressed by the formula $x' = x/|x|^2$ the domain D^- with boundary S goes into a bounded domain D' with a boundary S' lying in the space E'_n of the variables $x'_1, \ldots, x'_n$. Without loss of generality we can assume that the point $x=0$ is contained in the domain D^+.

Let us consider the function

$$v(x')=|x'|^{2-n}u\left(\frac{x'}{|x'|^2}\right)$$

which is harmonic in the domain D'. According to (70), the function $v(x')$ satisfies the boundary condition

$$v(y') = |y'|^{2-n} \varphi\left(\frac{y'}{|y'|^2}\right), \quad y' \in S' \tag{71}$$

We see that *if the function* $v(x')$ *is the solution of the Dirichlet problem in the domain* D' *satisfying boundary condition* (71), *then the function*

$$u(x) = |x|^{2-n} v\left(\frac{x}{|x|^2}\right)$$

is the solution of the exterior Dirichlet problem satisfying boundary condition (70).

From what has been said it follows that *in the case when* D^- *is the exterior of the closed ball* $|x| \leqslant 1$ *the solution of the exterior Dirichlet problem satisfying boundary condition* (70) *on the sphere* $|y| = 1$ *is given by the formula*

$$u(x) = \frac{1}{\omega_n} \int\limits_{|\xi|=1} \frac{|x|^2 - 1}{|\xi - x|^n} \varphi(\xi)\, ds_\xi$$

It is obvious that the Dirichlet problem stated above cannot possess more than one solution.

If we seek the solution of the exterior Dirichlet problem in the form of double-layer potential (47), this results, by virtue of (55), in a Fredholm integral equation of the second kind for the determination of the unknown function μ.

The Neumann problem for an unbounded domain (the *exterior Neumann problem*) can be stated thus: *it is required to find a regular harmonic function* $u(x)$ *in the domain* D^- *satisfying the boundary condition*

$$\left(\frac{\partial u}{\partial \nu}\right)^- = \varphi(x), \quad x \in S$$

where ν *is the normal to* S *and* φ *is a given real continuous function defined on* S.

It should be noted that the exterior Neumann problem cannot be reduced to an analogous problem for a bounded domain (in contrast to the exterior Dirichlet problem for which such a reduction was demonstrated above). However, if we try to find the solution of the exterior Neumann problem in the form of a single-layer potential of type (59), this will result in a Fredholm integral equation of the second kind for the determination of the unknown density μ (this follows from (64)).

§ 5. Elements of the General Theory of Elliptic Linear Partial Differential Equations of the Second Order

1°. Adjoint Operator. Green's Theorem. Let us consider a linear differential operator of the second order

$$Lu = \sum_{i,\,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} B_i(x) \frac{\partial u}{\partial x_i} + C(x)\, u, \quad A_{ij} = A_{ji}$$

defined in a domain D of the space E_n.

If the coefficients A_{ij} possess partial derivatives of the first order, the operator L can be rewritten in the form

$$Lu = \sum_{i,\,j=1}^{n} \frac{\partial}{\partial x_j} \left(A_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} e_i \frac{\partial u}{\partial x_i} + Cu \tag{72}$$

where

$$e_i(x) = B_i - \sum_{j=1}^{n} \frac{\partial A_{ij}}{\partial x_j} \quad (i = 1, \ldots, n) \tag{73}$$

If the functions $e_i(x)$ $(i = 1, \ldots, n)$ possess partial derivatives of the first order, the notion of the *adjoint operator* L^* may be introduced:

$$L^* v = \sum_{i,\,j=1}^{n} \frac{\partial}{\partial x_j} \left(A_{ij} \frac{\partial v}{\partial x_i} \right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (e_i v) + Cv \tag{74}$$

An operator L is said to be *self-adjoint* if the equality $Lu = L^* u$ is identically fulfilled.

From (72), (73) and (74) it is obvious that an operator L of form (72) is self-adjoint if and only if the conditions

$$\sum_{j=1}^{n} \frac{\partial A_{ij}}{\partial x} = B_i(x) \quad (i = 1, \ldots, n)$$

hold throughout the domain D.

Let us suppose that the differential operator L is *uniformly elliptic* and that the boundary S of the domain D is sufficiently smooth. If $u(x)$ and $v(x)$ are two sufficiently smooth functions, then, by virtue of formula (GO), the integration of the identity

$$vLu-uL^*v=\sum_{i,\,j=1}^{n}\frac{\partial}{\partial x_j}\left[A_{ij}\left(\frac{\partial u}{\partial x_i}v-\frac{\partial v}{\partial x_i}u\right)\right]+ \\ +\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(e_i uv)$$

over the domain D results in the formula

$$\int_D (vLu-uL^*v)\,d\tau_x=\int_S\left[av\frac{du}{dN}-uQ_\xi v\right]ds_\xi$$

where

$$Q_\xi v=a\frac{dv}{dN}-bv \tag{75}$$

N is the unit vector (the *conormal*) at the point $\xi\in S$ with direction cosines

$$\cos\widehat{N\xi_i}=\frac{1}{a}\sum_{j=1}^{n}A_{ij}\cos\widehat{\nu\xi_j}\quad(i=1,\ldots,n)$$

and ν is the outer normal to S at the point ξ, the expressions a and b being given by the equalities

$$a^2=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}A_{ij}\cos\widehat{\nu\xi_j}\right)^2\quad\text{and}\quad b=\sum_{i=1}^{n}e_i\cos\widehat{\nu\xi_i}$$

The formula

$$\int_D (vLu-uL^*v)\,d\tau_x=\int_S\left[av\frac{du}{dN}-uQ_\xi v\right]ds_\xi$$

we have derived expresses *Green's theorem*.

Since the operator L is uniformly elliptic, the vector N does not lie in the tangent plane to S at any point $\xi\in S$, and $a\neq 0$.

2°. Existence of Solutions of Elliptic Linear Partial Differential Equations of the Second Order. We shall denote by a_{ij} $(i, j = 1, \ldots, n)$ the ratio of the cofactor of the element A_{ij} of the determinant $A = \det \| A_{ij} \|$ of the matrix $\| A_{ij} \|$ to that determinant. Let us consider the function

$$\sigma(x, \xi) = \sum_{i, j=1}^{n} a_{ij}(x)(x_i - \xi_i)(x_j - \xi_j)$$

where x and ξ are two arbitrary points of the domain D. We shall assume, without loss of generality, that $\sigma(x, \xi) \geqslant 0$.

By the uniform ellipticity of the operator L, there exist two positive constants k_0 and k_1 such that

$$k_0 \mid x - \xi \mid^2 \leqslant \sigma(x, \xi) \leqslant k_1 \mid x - \xi \mid^2$$

We shall suppose that the functions A_{ij} and B_j possess continuous partial derivatives of the third order in the domain $D \cup S$ and that the function C possesses continuous partial derivatives of the first order in this domain.

Let us construct the function

$$\psi(x, \xi) = \begin{cases} \dfrac{1}{n-2} \sigma_0(\xi) \sigma^{\frac{2-n}{2}} & \text{for} \quad n > 2 \\ -\sigma_0(\xi) \ln \sigma^{\frac{1}{2}} & \text{for} \quad n = 2 \end{cases} \tag{76}$$

where

$$\sigma_0(\xi) = \frac{1}{\omega_n \sqrt{A(\xi)}}$$

For $x \neq \xi$ the partial derivatives of the first and the second order of the function $\psi(x, \xi)$ specified by (76) are given by the formulas

$$\frac{\partial \psi}{\partial x_i} = -\sigma_0(\xi)(n-2)\sigma^{-\frac{n}{2}} \sum_{j=1}^{n} a_{ij}(x)(x_j - \xi_i) + P_i(x, \xi) \qquad (n > 2) \tag{77}$$

and

$$\frac{\partial^2\psi}{\partial x_i\,\partial x_j}=\sigma_0(\xi)(n-2)\,\sigma^{-\frac{n+2}{2}}\Big[-a_{ij}(x)\,\sigma(x,\ \xi)+$$

$$+n\sum_{k,\ l=1}^{n}a_{ik}(x)\,a_{jl}(x)\,(x_k-\xi_k)(x_l-\xi_l)\Big]+P_{ij}(x,\ \xi)\qquad(78)$$

where the expressions $P_i(x,\ \xi)$ and $P_{ij}(x,\ \xi)$ tend to infinity for $|x-\xi|\to 0$ at the same rate as $|x-\xi|^{2-n}$ and $|x-\xi|^{1-n}$ respectively.

From (78) it follows that when $n>2$ and $x\neq\xi$ we have the equality

$$\sum_{i,\ j=1}^{n}A_{ij}(x)\frac{\partial^2\psi}{\partial x_i\,\partial x_j}=\sum_{i,\ j=1}^{n}A_{ij}(x)\,P_{ij}(x,\ \xi)\qquad(79)$$

From (76), (77), (78) and (79) we conclude that when $|x-\xi|\to 0$ the function $L\psi(x,\ \xi)$ tends to infinity at the same rate as $|x-\xi|^{1-n}$.

For the case $n=2$ we can assume, without loss of generality, that $a_{ij}=0$ for $i\neq j$ and $a_{ii}=1$ $(i,\ j=1,\ 2)$, and then, for $x\neq\xi$, we obtain the equality

$$\sum_{i,\ j=1}^{2}A_{ij}(x)\frac{\partial^2\psi}{\partial x_i\,\partial x_j}=0$$

Let us introduce the function

$$w(x)=\int_{D_0}\psi(x,\ \xi)\,\mu(\xi)\,d\tau_\xi$$

where D_0 is a subdomain of the domain D with a boundary S_0; the function $w(x)$ is referred to as a *generalized potential function of a volume distribution of mass over the domain* D_0 *with density* μ.

Under the assumption that the function $\mu(\xi)$ is continuously differentiable in $D_0\cup S_0$ we can repeat the arguments given in Secs. 1° and 2° in § 3 of the present chapter, which leads to the conclusion that

$$Lw(x)=-\mu(x)+\int_{D_0}L\psi(x,\ \xi)\,\mu(\xi)\,d\tau_\xi\qquad(80)$$

where the second summand on the right-hand side is an ordinary improper integral.

Let us construct the solution $u(x)$ of the equation

$$Lu = f(x) \tag{81}$$

in the form

$$u(x) = \omega(x) + \int_{D_0} \psi(x, \xi) \mu(\xi) \, d\tau_\xi \tag{82}$$

where $\omega(x)$ is an arbitrary real function continuous in $D_0 \cup S_0$ together with its partial derivatives up to the third order inclusive and μ is a real function yet unknown.

According to formula (80), the function $u(x)$ expressed by formula (82) is a solution of equation (81) if and only if

$$\mu(x) + \int_{D_0} K(x, \xi) \mu(\xi) \, d\tau_\xi = F(x) \tag{83}$$

where

$$K(x, \xi) = -L\psi(x, \xi) \quad \text{and} \quad F(x) = L\omega(x) - f(x)$$

Equality (83) is Fredholm's integral equation of the second kind with respect to the unknown function μ; as will be shown in Chapter 5, *at least in the case of a domain* D_0 *of a sufficiently small diameter integral equation* (83) *always possesses a solution.*

Since $\omega(x)$ is an arbitrary function, it follows that *equation* (81) *possesses a family of regular solutions in a sufficiently small neighbourhood of each point of the domain where the equation is defined.*

In the case when $\omega = \psi(x, y)$ the right-hand member of integral equation (83) tends to infinity for $|y - x| \to 0$ at the same rate as the expression $|y - x|^{1-n}$; nevertheless, according to the remark in Sec. 2°, § 2 of Chapter 5, we can repeat the above argument and thus conclude that in this case formula (82) gives the *fundamental* (*elementary*) solution of equation (81):

$$E(x, y) = \psi(x, y) + \int_{D_0} \psi(x, \xi) \mu(\xi) \, d\tau_\xi$$

3°. **Boundary-Value Problems.** Let $p_i(x)$ $(i = 1, \ldots, n)$, $q(x)$ and $r(x)$ be real functions defined on the boundary S

of a domain D. The following *Poincaré linear boundary-value problem* embraces a wide class of problems for equation (81): *it is required to find a regular solution $u(x)$ of equation (81) in the domain D which satisfies the boundary condition*

$$\sum_{i=1}^{n} p_i(x) \frac{\partial u(x)}{\partial x_i} + q(x) u(x) = r(x), \quad x \in S \tag{84}$$

where by the values of $\frac{\partial u(x)}{\partial x_i}$ and $u(x)$ *at the points* $x \in S$ *are meant the limits of these functions for the case when the variable point approaches the boundary S from the interior of the domain D.*

In the case when $p_i(x) = 0$ $(i = 1, \ldots, n)$ and $q(x) \neq 0$ everywhere on S, boundary condition (84) can be rewritten in the form

$$u(x) = g(x) \tag{85}$$

where

$$g(x) = r(x)/q(x)$$

Problem (81), (85) is referred to as *the Dirichlet problem* or *the first boundary-value problem* (for equation (81)).

When $q(x) = 0$ on S the Poincaré problem reduces to its special case known as the *problem with oblique derivative boundary condition*:

$$\sum_{i=1}^{n} p_i(x) \frac{\partial u}{\partial x_i} = r(x), \quad x \in S \tag{86}$$

In case we have

$$p_i(x) = \cos \widehat{Nx_i} \quad (i = 1, \ldots, n)$$

everywhere on S in boundary condition (86), we arrive at the special case of the problem with oblique derivative boundary condition which is referred to as the *Neumann problem* or the *second boundary-value problem* (for equation (81))*.

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* When $p_i(x) = \cos \widehat{Nx_i}$ $(i = 1, \ldots, n)$ and $q \not\equiv 0$ on S in boundary condition (84), the Poincaré problem reduces to the so-called *third boundary-value problem* or the *mixed boundary-value problem*.— *Tr.*

4°. The Extremum Principle. The Uniqueness of the Solution of the Dirichlet Problem. In the theory of elliptic partial differential equation (81) an important role is played by the following *extremum principle*: *if the inequality*

$$c(x) < 0 \tag{87}$$

holds everywhere in the domain D, then a solution $u(x)$ of the homogeneous equation

$$Lu = 0 \tag{88}$$

which is regular in that domain can attain at any point $x \in D$ neither a negative relative minimum nor a positive relative maximum.

Let us prove this extremum principle. If we suppose that the function $u(x)$ attains a negative relative minimum at a point $x \in D$ then we can write

$$\frac{\partial u}{\partial x_i} = 0 \qquad (i = 1, \ldots, n) \tag{89}$$

and

$$\sum_{i, j=1}^{n} \frac{\partial^2 u}{\partial x_i \, \partial x_j} \lambda_i \lambda_j \geqslant 0 \tag{90}$$

where $\lambda_1, \ldots, \lambda_n$ are arbitrary real parameters.

The quadratic form $\sum_{i, j=1}^{n} A_{ij} \lambda_i \lambda_j$ is positive definite, and therefore it can be written in the form

$$\sum_{i, j=1}^{n} A_{ij} \lambda_i \lambda_j = \sum_{k=1}^{n} \Big(\sum_{l=1}^{n} g_{kl} \lambda_l\Big)^2$$

for every point $x \in D$; hence the coefficients A_{ij} can be represented as

$$A_{ij} = \sum_{s=1}^{n} g_{si} g_{sj} \quad (i, j = 1, \ldots, n) \tag{91}$$

From (90) and (91) we derive

$$\sum_{i, j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} = \sum_{i, j, s=1}^{n} \frac{\partial^2 u}{\partial x_i \, \partial x_j} g_{si} g_{sj} \geqslant 0 \tag{92}$$

Finally, taking into account the inequality $u(x) < 0$, we obtain, by virtue of (87), (89), (90) and (92), the inequality $Lu > 0$, which contradicts equality (88). We have thus arrived at a contradiction, and hence the assumption that the function $u(x)$ attains a negative relative minimum at a point $x \in D$ is wrong.

The fact that a regular solution $u(x)$ of equation (88) cannot attain a positive relative maximum at a point $x \in D$ is proved quite analogously.

The extremum principle implies that *Dirichlet problem* (81), (85) *cannot have more than one solution when condition* (87) *is fulfilled.*

Indeed, the difference $u_1(x) - u_2(x) = u(x)$ of any two solutions $u_1(x)$ and $u_2(x)$ of problem (81), (85) satisfies the conditions

$$Lu(x) = 0 \quad \text{for} \quad x \in D \quad \text{and} \quad u(y) = 0 \quad \text{for} \quad y \in S$$

Since we have $\max |u(y)| = 0$ on S, the extremum principle implies that $u(x) = 0$, that is $u_1(x) = u_2(x)$ everywhere in the domain D.

It can also be easily proved that *if the condition*

$$\sum_{i=1}^{n} \frac{\partial e_i}{\partial x_i} - 2C \geqslant 0 \quad \textit{everywhere in } D \tag{93}$$

is fulfilled, Dirichlet problem (81), (85) *cannot have more than one solution.*

Indeed, for the difference $u_1(x) - u_2(x) = u(x)$ of any two solutions $u_1(x)$ and $u_2(x)$ of equation (81) there holds the equality

$$\sum_{i, j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} \left(\sum_{i=1}^{n} \frac{\partial e_i}{\partial x_i} - 2C \right) u^2 =$$

$$= \sum_{i, j=1}^{n} \frac{\partial}{\partial x_i} \left(A_{ij} u \frac{\partial u}{\partial x_j} \right) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (e_i u^2)$$

On integrating this equality over the domain D and using formula (GO), we obtain, by virtue of the equality $u(x) = 0$

$(x \in D)$, the relation

$$\int\limits_D \left[\sum_{i,\,j=1}^{n}{}' A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} \left(\sum_{i=1}^{n}{}' \frac{\partial e_i}{\partial x_i} - 2C \right) u^2 \right] d\tau_x = 0$$

Taking into account that the quadratic form $\sum_{i,j=1}^{n}{}' A_{ij}\lambda_i\lambda_j$ is positive definite and using condition (93), we conclude from the last relation that $u(x) = 0$ throughout D, that is $u_1(x) = u_2(x)$.

5°. Generalized Single-Layer and Double-Layer Potentials. In Sec. 2°, § 5 we proved the existence of the elementary solution of equation (81) for a domain of a sufficiently small diameter.

Let us consider the case when the coefficients of equation (88) are sufficiently smooth functions defined throughout the whole space E_n; it can be proved that if the conditions (a) $C(x) \leqslant 0$ in the domain D where the solution of the equation is sought, and (b) $C(x) < -k^2$ outside a bounded domain containing the domain D where k is a nonzero constant are fulfilled, then *equation* (88) *possesses the so-called principal elementary* (*fundamental*) *solution* $E^*(x, \xi)$ *defined for all points* x *and* ξ *belonging to the space* E_n. The distinction between the elementary solution considered in Sec. 2° and the principal elementary solution $E^*(x, \xi)$ is that the latter possesses the following two additional properties: (α) $E^*(x, \xi)$ is a solution, with respect to ξ for $\xi \neq x$, of the adjoint equation $L^*E^*(x, \xi) = 0$ and (β) for $|x - \xi| \to +\infty$ the function $E^*(x, \xi)$ and its partial derivatives $\frac{\partial E^*}{\partial x_i}$ $(i = 1, \ldots, n)$ decrease like $e^{-R|x-\xi|}$ where R is a positive number.

It can be verified directly that the principal elementary solution of the *Helmholtz equation*

$$\Delta u - \lambda^2 u = 0, \quad \lambda = \text{const} \neq 0$$

is the function $E^*(x, \xi) = \lambda^{n-2}\varphi_n(\lambda r)$ where $r = |\xi - x|$ and $\varphi_n(r)$ is a solution of the ordinary linear differential equation

$$r\varphi_n'' + (n-1)\varphi_n' - r\varphi_n = 0$$

In particular, for $n = 2$ and $n = 3$ we have

$$\varphi_2(r) = \frac{1}{2\pi} \int_{-\infty}^{-1} \frac{e^{rt}\,dt}{\sqrt{t^2-1}}$$

and

$$\varphi_3(r) = \frac{1}{4\pi} \frac{e^{-r}}{r}$$

respectively.

Let $u_1(x)$ be a regular solution in a domain D of nonhomogeneous equation (81) with a sufficiently smooth bounded right-hand member $f(x)$ and let $u_2(x)$ be the *generalized volume potential with density function* $f(\xi)$:

$$u_2(x) = \int_D E(x, \xi) f(\xi)\, d\tau_\xi$$

Then, by virtue of (80), the sum $u_1(x) + u_2(x) = u(x)$ satisfies the equation

$$Lu = Lu_1 + Lu_2 = f(x) - f(x) = 0$$

Consequently, *in the theory of equation* (81) *we can assume, without loss of generality, that* $f(x) = 0$ *everywhere in the domain of definition of that equation.*

When the coefficients of equation (88) are sufficiently smooth functions in the domain D, and $C(x) \leqslant 0$ everywhere in D, it is always possible to define these coefficients outside D so that they remain smooth functions in such a way that outside a wider domain containing D the condition $C(x) < -k^2$ will hold. Consequently, we can assume that in this case the *principal elementary solution of equation* (88) *exists.*

Let S be a sufficiently smooth surface bounding a domain D^+. The functions

$$u(x) = \int_S E^*(x, \xi) \mu(\xi)\, ds_\xi \tag{94}$$

and

$$v(x) = \int_S Q_\xi E^*(x, \xi) \lambda(\xi)\, ds_\xi \tag{95}$$

where the operator Q_ξ has form (75), and μ and λ are sufficiently smooth real functions defined on S, are called *the* *generalized potential function for a surface distribution of mass* and *the generalized potential function for a double layer of distribution of dipoles on* S respectively. Both these potentials are regular solutions of equation (88) at any point x of the space E_n not belonging to S, and when the point x passes from the domain D^+ to the domain $D^- = C\,(D^+ \cup S)$ through the surface S the potentials behave in just the same way as the harmonic single-layer and double-layer potentials.

The generalized double-layer and single-layer potentials make it possible to reduce the Dirichlet and the Neumann problems for equation (88) to Fredholm's integral equations of the second kind.

CHAPTER 2

CAUCHY-RIEMANN SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS. ELEMENTS OF THE THEORY OF ANALYTIC FUNCTIONS

§ 1. The Notion of an Analytic Function of a Complex Variable

1°. Cauchy-Riemann System of Partial Differential Equations. As was already mentioned in Sec. 1°, § 5 of Introduction, the system of linear partial differential equations of the first order in two independent variables of the form

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0, \quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0 \qquad (CR)$$

is referred to as the *Cauchy-Riemann system of partial differential equations.*

According to the classification of partial differential equations (see Sec. 4°, § 1 of Introduction) a system of the form

$$A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}=0$$

where

$$A=\begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix}, \quad B=\begin{Vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{Vmatrix} \quad \text{and} \quad u=(u_1, u_2)$$

is called elliptic when the quadratic form

$$Q(\lambda_1, \lambda_2)=\det\begin{Vmatrix} A_{11}\lambda_1+B_{11}\lambda_2 & A_{12}\lambda_1+B_{12}\lambda_2 \\ A_{21}\lambda_1+B_{21}\lambda_2 & A_{22}\lambda_1+B_{22}\lambda_2 \end{Vmatrix}$$

is positive (or negative) definite.

In the case of system (CR) we have $u=u_1$, $v=u_2$, $A_{11}=A_{22}=-B_{12}=B_{21}=1$ and $B_{11}=A_{12}=A_{21}=$ $=B_{22}=0$, and the corresponding quadratic form

$$Q(\lambda_1, \lambda_2)=\det\begin{Vmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{Vmatrix}=\lambda_1^2+\lambda_2^2$$

is positive definite, therefore system (CR) is elliptic.

If $\omega(x, y)$ *is an arbitrary harmonic function in two variables* x *and* y *then the pair of the functions* $u = \frac{\partial \omega}{\partial x}$, $v = -\frac{\partial \omega}{\partial y}$ *is a solution of system (CR).*

Indeed, we have

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0$$

and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 \omega}{\partial x\, \partial y} - \frac{\partial^2 \omega}{\partial y\, \partial x} = 0$$

The last equality is written on the basis of the well-known fact that the mixed partial derivatives are independent of the order of differentiation provided they are continuous: $\frac{\partial^2 \omega}{\partial x\, \partial y} = \frac{\partial^2 \omega}{\partial y\, \partial x}$

2°. The Notion of an Analytic Function. An expression of the form $u(x, y) + iv(x, y) = f(z)$ where $u(x, y)$ and $v(x, y)$ are real functions of the real variables x, y and i *is* the "imaginary unit" (that is $i^2 = -1$), is called a *function of the complex variable* $z = x + iy$.

Let us regard the real variables x and y as the orthogonal Cartesian coordinates of the point (x, y) on the Euclidean plane E_2; the point (x, y) will be considered as representing the complex variable $z = x + iy$ (and the plane E_2 will be referred to as the *complex plane*). The domain D where the functions $u(x, y)$ and $v(x, y)$ are defined serves as the *domain of definition* of the function $f(z)$.

Using the representation of the complex numbers with the aid of the Riemann sphere and postulating the existence of the single point at infinity ∞, we shall refer to the complex plane to which the point ∞ is added as the *extended complex plane.*

When speaking of a function $w = f(z)$ of the complex variable z we shall usually mean (provided the contrary is not stipulated) that to each point z belonging to the domain of definition D of the function there corresponds a definite point w on the plane over which the values of $f(z)$ range, that is, generally speaking, we shall consider *one-valued* functions. (However, it is sometimes necessary to consider many-valued functions as well.)

We say that a number $w_0 = u_0 + iv_0$ is the *limit* of a function $f(z) = u(x, y) + iv(x, y)$ for $z \to z_0 = x_0 + iy_0$, $z \neq z_0$, if

$$\lim_{(x, y)\to(x_0, y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x, y)\to(x_0, y_0)} v(x, y) = v_0$$

A function $f(z) = u(x, y) + iv(x, y)$ is said to be *continuous* at a point $z \in D$ if the functions $u(x, y)$ and $v(x, y)$ are continuous at the point $(x, y) \in D$ $(z = x + iy)$. For the modulus of the difference $f(z) - f(z_0)$ we can write the expression

$$|f(z) - f(z_0)| =$$
$$= \{[u(x, y) - u(x_0, y_0)]^2 + [v(x, y) - v(x_0, y_0)]^2\}^{1/2}$$

and therefore the definition of the continuity stated above is equivalent to the following definition: *the function $f(z)$ is said to be continuous at a point $z_0 \in D$ if, given an arbitrary $\varepsilon > 0$, there exists a number $\delta > 0$ such that the inequality $|z - z_0| < \delta$ implies the inequality $|f(z) - f(z_0)| < \varepsilon$.*

Let $\Delta w = \Delta u + i\Delta v$ be the increment $f(z + \Delta z) - f(z)$ $(z + \Delta z, z \in D)$ of the function $f(z)$ corresponding to the increment $\Delta z = (z + \Delta z) - z$ of the independent variable z.

If the limit $\lim\limits_{\Delta z\to 0} \dfrac{\Delta w}{\Delta z} = w'(z) = f'(z)$ exists and is independent of the path along which Δz tends to zero, $f(z)$ is called a *monogenic function of the complex variable at the point* z.

Let us first put $\Delta z = \Delta x$ and then $\Delta z = i\,\Delta y$; according to the definition, for a function $f(z)$ monogenic at a point z we have

$$w'(z) = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x\to 0} \frac{\Delta u + i\,\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} =$$
$$= \lim_{\Delta y\to 0} \frac{\Delta u + i\,\Delta v}{i\,\Delta y} = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

that is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

We see that equality (1) expresses a *necessary condition* for the function $f(z)$ to be monogenic at the point $z = x + iy$.

These equalities are nothing other than the Cauchy-Riemann system of partial differential equations.

A function $f(z)$ monogenic at each point $x \in D$ is said to be *analytic in the domain D*.

Now we remind the reader that a pair of functions $u(x, y)$, $v(x, y)$ is a *regular* solution of system (CR) if they are continuous together with their partial derivatives of the first order in their domain of definition D and satisfy that system in D. Consequently, *if $u(x, y)$, $v(x, y)$ is a pair forming a regular solution of system (CR), then the function $w = f(z) = u(x, y) + iv(x, y)$ is continuous in the domain D*. Moreover, the continuity of the functions $u(x, y)$ and $v(x, y)$ and of their first-order partial derivatives implies the existence of the *total differentials*

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{and} \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} \partial y$$

that is

$$\begin{aligned} \Delta u &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + o(\Delta z) \\ \Delta v &= \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + o(\Delta z) \end{aligned} \qquad (2)$$

where $o(\Delta z)$ denotes an infinitesimal of order higher than Δz.

According to (2), we can write for $\Delta w = \Delta u + i\,\Delta v$ the expression

$$\Delta w = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + i\frac{\partial v}{\partial x} \Delta x + i\frac{\partial v}{\partial y} \Delta y + o(\Delta z) \qquad (3)$$

Further, using (1) and (3) we can write

$$w' = f'(z) = \lim_{\Delta z \to 0} \frac{\frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y - i\frac{\partial u}{\partial y} \Delta x + i\frac{\partial u}{\partial x} \Delta y + o(\Delta z)}{\Delta x + i\,\Delta y} =$$

$$= \lim_{\Delta z \to 0} \left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + \frac{o(\Delta z)}{\Delta z} \right] = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad (4)$$

This means that the function $f(z)$ is monogenic at each point $z \in D$.

Thus, we have arrived at the conclusion that *a function $f(z)$ whose real part $u(x, y)$ and imaginary part $v(x, y)$*

form a regular solution of system (CR) in a domain D is analytic in that domain.

Let us sum up what has been established: *conditions (CR) are necessary for the function* $f(z) = u(x, y) + iv(x, y)$ *to be analytic in the domain D; if the additional requirement that the partial derivatives of the first order of the functions* $u(x, y)$ *and* $v(x, y)$ *should be continuous is fulfilled then conditions (CR) are also sufficient for the analyticity of the function* $f(z) = u(x, y) + iv(x, y)$ *in the domain D.*

Expression (4) obtained for $w' = f'(z)$ is called the *derivative of the function* $f(z)$ *analytic in the domain D*, and the principal linear part of Δw (with respect to Δx and Δy), that is the expression

$$dw = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\frac{\partial v}{\partial x}\Delta x + i\frac{\partial v}{\partial y}\Delta y = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\Delta z = f'(z)\,\Delta z \qquad (5)$$

is called the *differential of the analytic function* $f(z)$ *at the point z.*

In the case when $f(z) = w = z$ we have $f'(z) = \lim\limits_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$, and therefore, in accordance with (5), $dz = \Delta z$. Using the equality we have obtained we can write differential (5) in the form $dw = f'(z)\,dz$, which accounts for the notation

$$\frac{dw}{dz} = f'(z) \qquad (6)$$

On denoting

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (7)$$

we can rewrite system (CR) in the form

$$\frac{\partial}{\partial \bar{z}}\, w(z) = 0 \qquad (8)$$

Consequently, *analytic functions* $w = f(z)$ *of the complex variable z always satisfy equation* (8). Taking into account equality (8), we see that notation (6) for the derivative $f'(z)$ of an analytic function $f(z)$ in a domain D is in a complete agreement with notation (7).

3°. Examples of Analytic Functions. The class of analytic functions is rather wide. In particular, if $\omega(x, y)$ is a harmonic function of the real variables x and y in a domain D then the function $w = f(z) = \frac{\partial \omega}{\partial x} - i\frac{\partial \omega}{\partial y}$ is analytic in that domain.

Let $f(z)$ and $\varphi(z)$ be two analytic functions defined in D. Using the usual technique applied to the investigation of differentiable functions of one real variable in the course of mathematical analysis, we can easily prove that, for such functions $f(z)$ and $\varphi(z)$, the functions $f(z) \pm \varphi(z)$, $f(z)\varphi(z)$ and $f(z)/\varphi(z)$ $(\varphi(z) \neq 0)$ are also analytic in D, and

$$(f \pm \varphi)' = f' \pm \varphi', \quad (f \cdot \varphi)' = f'\varphi + f\varphi', \quad \left(\frac{f}{\varphi}\right)' = \frac{f'\varphi - f\varphi'}{\varphi^2} \tag{9}$$

Since $\frac{dz}{dz} = 1$, formulas (9) imply that *every polynomial*

$$P_n(z) = \sum_{k=0}^{n} a_k z^k$$

is an analytic function throughout the whole complex plane of the variable z, *and*

$$P_n' = \sum_{k=1}^{n} k a_k z^{k-1}$$

It also follows that *every linear-fractional function*

$$w(z) = \frac{az+b}{cz+d} \quad \left(z \neq -\frac{d}{c}\right) \tag{10}$$

where a, b, c *and* d *are some constants, is analytic everywhere in the complex plane of the variable* z *except the point* $z = -d/c$, *and*

$$w' = \frac{ad-bc}{(cz+d)^2}, \quad z \neq -\frac{d}{c}$$

Let us consider a power series

$$S(z) = \sum_{k=0}^{\infty} a_k z^k \tag{11}$$

which is convergent in its *circle of convergence* $|z| < R$ $(R > 0)$, the function $S(z)$ being the sum of series (11). We shall show that *the sum $S(z)$ of the power series is an analytic function in the circle of convergence $|z| < R$.*

First of all we note that if R is the radius of convergence of series (11) then the radius of convergence of the series

$$S_0(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad (12)$$

is also equal to R. Indeed, according to the well-known *Cauchy-Hadamard formula* proved in the course of mathematical analysis, the radii of convergence R and R_1 of series (11) and (12) are expressed by the formulas $R = 1/l$ and $R_1 = 1/l_1$ respectively where

$$l = \overline{\lim}\, |a_k|^{\frac{1}{k}} \quad \text{and} \quad l_1 = \overline{\lim}\, (k\,|a_k|)^{\frac{1}{k-1}}$$

Therefore the equality $R = R_1$ follows from the obvious relations

$$\overline{\lim_{k\to\infty}}\, (k\,|a_k|)^{\frac{1}{k-1}} = \overline{\lim_{k\to\infty}}\, k^{\frac{1}{k-1}} \left(|a_k|^{\frac{1}{k}}\right)^{\frac{k}{k-1}} = \overline{\lim_{k\to\infty}}\, |a_k|^{\frac{1}{k}}$$

Here $\overline{\lim}$ symbolizes the *limit superior*.

As is known, a power series is absolutely convergent inside its circle of convergence. The absolute convergence of series (12) for $|z| < R$ implies that for any $\varepsilon > 0$ there exists a natural number N such that

$$\sum_{k=n+1}^{\infty} k\,|a_k|\, r^{k-1} < \frac{\varepsilon}{3} \qquad (13)$$

for all $n \geqslant N$ provided that $r < R$.

It is obvious that

$$\left|\frac{S(z+\Delta z) - S(z)}{\Delta z} - S_0(z)\right| \leqslant$$

$$\leqslant \left|\sum_{k=1}^{n} a_k\left[(z+\Delta z)^{k-1} + \ldots + z^{k-1} - kz^{k-1}\right]\right| +$$

$$+ \left|\sum_{k=n+1}^{\infty} a_k\left[(z+\Delta z)^{k-1} + \ldots + z^{k-1}\right]\right| + \left|\sum_{k=n+1}^{\infty} k a_k z^{k-1}\right| \qquad (14)$$

If the increment Δz is sufficiently small and if $|z| < r$ and $|z + \Delta z| < r$, then, since the first term on the right-hand side of (14) is continuous and inequality (13) holds, we can write the relation

$$\left| \frac{S(z+\Delta z) - S(z)}{\Delta z} - S_0(z) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

The last inequality means that

$$\lim_{\Delta z \to 0} \frac{S(z+\Delta z) - S(z)}{\Delta z} = S'(z) = S_0(z)$$

Consequently, *power series* (11) *can be differentiated termwise inside its circle of convergence, and the sum of the differentiated series is equal to the derivative of the sum of the original series.*

From this assertion it readily follows that *the elementary functions* e^z, $\cos z$, $\sin z$, $\cosh z$ *and* $\sinh z$ *are analytic throughout the complex plane of the variable* z, *and the following formulas hold:*

$$(e^z)' = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \right)' = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$(\cos z)' = \left(\frac{e^{iz} + e^{-iz}}{2} \right)' = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

$$(\sin z)' = \left(\frac{e^{iz} - e^{-iz}}{2i} \right)' = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$(\cosh z)' = \left(\frac{e^z + e^{-z}}{2} \right)' = \frac{e^z - e^{-z}}{2} = \sinh z$$

and

$$(\sinh z)' = \left(\frac{e^z - e^{-z}}{2} \right)' = \frac{e^z + e^{-z}}{2} = \cosh z$$

4°. **Conformal Mapping.** Given an analytic function $w = f(z)$ in a domain D, to each point $z \in D$ there corresponds a definite point on the complex plane of the variable w. If this correspondence between the points $z \in D$ and $w \in D_1$ (where D_1 is the range of the function $w = f(z)$) is one-to-one, the function $w = f(z)$ is called *one-sheeted* (or *univalent*). In the case of such correspondence between the points of the domains D and D_1 determined by the function $w = f(z)$ we say that there is a *mapping of the domain D onto the domain* D_1. The point $w \in D_1$ is called the *image* of the point $z \in D$; and the point z is called the *pre-image* (or the *inverse image* or the *original*) of the point w.

In this section we shall study mappings specified by analytic functions.

Let

$$w = f(z) = u(x, y) + iv(x, y) \tag{15}$$

be an analytic function in a domain D satisfying the condition

$$f'(z_0) \neq 0 \tag{16}$$

at a point $z_0 \in D$. Condition (16) is equivalent to the requirement that the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ should be different from zero at the point z_0:

$$\frac{\partial(u, v)}{\partial(x, y)} = \left\| \begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} \right\| = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = |f'(z)|^2 \neq 0$$

$$\text{for} \quad z = z_0$$

Consequently, by virtue of the well-known theorem on implicit functions, we can assert that *the system of equalities* $u = u(x, y)$, $v = v(x, y)$ *can be uniquely resolved with respect to* x *and* y *in some neighbourhood of the point* $z_0 \in D$ *provided that the partial derivatives of the first order of the functions* $u(x, y)$ *and* $v(x, y)$ *are continuous* (as will be shown later, the real and the imaginary parts of an analytic function possess partial derivatives of all orders). In other words, every point $z_0 \in D$ at which condition (16) is fulfilled has a neighbourhood in which the function $w = f(z)$ is one-sheeted, and the inverse function $z = f^{-1}(w)$ is analytic in some neighbourhood of the point $w_0 = f(z_0)$; besides, it can easily be seen that

$$\lim_{\Delta w \to 0} \frac{\Delta z}{\Delta w} = \lim_{\Delta z \to 0} \frac{1}{\frac{\Delta w}{\Delta z}} = \frac{1}{f'(z)} = [f^{-1}(w)]'$$

Let γ be a smooth Jordan curve passing through the point z_0 and specified by an equation $z = z(t)$; this means that

$x = x(t)$ and $y = y(t)$ for $\alpha \leqslant t \leqslant \beta$. The curve γ possesses a tangent at the point $z_0 = z(t_0)$, that is

$$z'(t_0) \neq 0 \tag{17}$$

The image of γ under mapping (15) is an arc $\Gamma = f(\gamma)$ passing through the point $w_0 = f(z_0)$, the equation specifying that arc being $w = f[z(t)]$. Besides, we have $w'(t) = f'(z) z'(t)$, and, by virtue of (16) and (17), the condition

$$w'(t_0) = f'(z_0) z'(t_0) \neq 0 \tag{18}$$

holds. Condition (18) means that the arc Γ also has a tangent at the point w_0.

From (18) we conclude that, to within a summand of the form $2k\pi$ where $k = 0, \pm 1, \ldots$, the relation

$$\arg f'(z_0) = \arg w'(t_0) - \arg z'(t_0) \tag{19}$$

is fulfilled, that is the argument of the number $f'(z_0)$ is equal to the angle through which the arc γ passing through the point z_0 is turned under mapping (15).

Now let us consider another smooth Jordan curve γ_1 (different from γ) specified by an equation $z_1 = z_1(\tau)$ $(\alpha_1 \leqslant \tau \leqslant \beta_1)$ and passing through the point z_0, and let $\Gamma_1 = f(\gamma_1)$ be the image of γ_1 specified by the equation $w_1 = f[z_1(\tau)]$ $(z_1(\tau_0) = z_0)$.

On repeating the argument presented above, we obtain

$$\arg f'(z_0) = \arg w_1'(\tau_0) - \arg z_1'(\tau_0) \tag{20}$$

Equalities (19) and (20) imply

$$\arg w'(t_0) - \arg w_1'(\tau_0) = \arg z'(t_0) - \arg z_1'(\tau_0)$$

This means that the angle between the curves γ and γ_1 at the point z_0 is equal to the angle between their images Γ and Γ_1 at the point $w_0 = f(z_0)$. In other words: *for each point z_0 at which the condition $f'(z_0) \neq 0$ is fulfilled, mapping* (15) *possesses the angle-preserving property* (the mapping preserves both the magnitude of the angles and the directions in which they are reckoned; see Fig. 7). Since

$$|dz_0| = \sqrt{(x_0')^2 + (y_0')^2}\, dt = ds_\gamma$$

and

$$|dw_0| = \sqrt{(u_0')^2 + (v_0')^2}\, dt = d\sigma_0$$

are the elements of lengths of the curves γ and Γ at the points z_0 and w_0 respectively, and $\left(\frac{dw}{dz}\right)_{z=z_0} = f'(z_0)$, we have

$$|f'(z_0)| = \frac{|dw_0|}{|dz_0|} = \frac{d\sigma_0}{ds_0}$$

This means that *the modulus of the derivative of an analytic function* $w = f(z)$ *at a point* z_0 *for which condition* (16) *is fulfilled is equal to the magnification factor* (*of the element of length*) *under mapping* (15), *the magnification being uniform, that is one and the same for all directions passing through the point* z_0*.

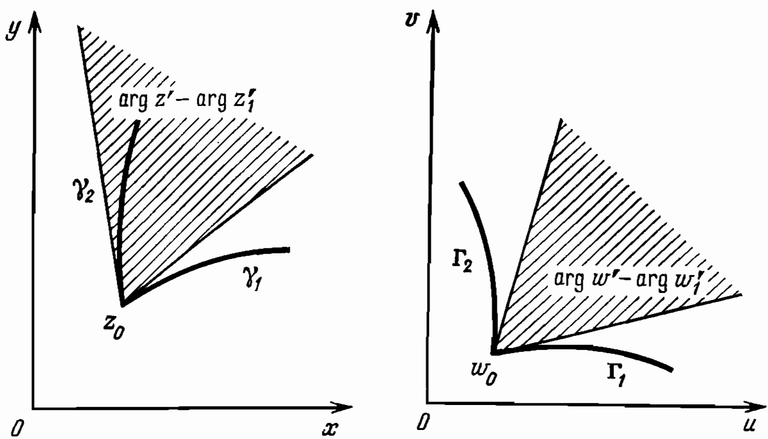

Fig. 7

A one-to-one mapping of form (15) of a domain D lying in the complex plane of the variable z onto a domain D_1 lying in the complex plane of the variable w under which at each point z_0 of the domain D the angle-preserving property

* Here the word *magnification* is understood in a general sense and can correspond to stretching if $|f'(z_0)| > 1$ or shrinking if $|f'(z_0)| < 1$ (or neither if $|f'(z_0)| = 1$).— *Tr.*

holds and the magnification factor is one and the same for all directions is called a *conformal* mapping.

What was established above shows that a *mapping specified by an analytic function* $w = f(z)$ *is conformal in a sufficiently small neighbourhood of each point* $z \in D$ *for which the condition* $f'(z) \neq 0$ *holds.*

In the theory of conformal mappings a fundamental role is played by the following four theorems which we state below without proof.

Theorem on conformal mapping determined by a univalent function: *if* $f(z)$ *is a univalent (one-sheeted) analytic function in a domain* D *then its derivative* $f'(z)$ *does not turn into zero at any point of the domain* D, *and the function* $f(z)$ *specifies a conformal mapping of its domain of definition* D *onto a definite part* D_1 *of the complex plane of the variable* w, *the inverse function* $z = f^{-1}(w)$ *being analytic in* D_1.

The repetition of the well-known argument on the differentiability of a composite function used in the course of mathematical analysis leads to the following conclusion: *if a function* $w = f(z)$ *analytic in a domain* D *specifies a conformal mapping of the domain* D *onto a domain* D_1 *of the complex plane of the variable* w *and if* $\zeta = \varphi(w)$ *is an analytic function defined in* D_1, *then the composite function* $\zeta = \varphi[f(z)]$ *is analytic in the domain* D *and*

$$\frac{d\zeta}{dz} = \varphi'[f(z)]\, f'(z)$$

Riemann's conformal mapping theorem: *for any simply connected domain* D *in the complex plane of the variable* z *whose boundary consists of more than one point there exists an analytic function* $w = f(z)$ *specifying a conformal mapping of the domain* D *onto the interior of unit circle* D_1 *lying in the plane of the complex variable* w; *under the additional requirement that this mapping should transform a given point* $z_0 \in D$ *and a given direction passing through that point into a given point* $w_0 \in D_1$ *and a given direction passing through* w_0, *the function* $f(z)$ *is determined uniquely.*

Theorem on the correspondence of boundaries: *let a function* $w = f(z)$ *specify a conformal mapping of a domain* D *onto a domain* D_1, *the boundaries of these domains being two* *closed Jordan curves* Γ *and* Γ_1 *respectively; then the function* $w=f(z)$ *determines a one-to-one and continuous correspondence between* $D \cup \Gamma$ *and* $D_1 \cup \Gamma_1$, *and under this correspondence the direction in which* Γ *and* Γ_1 *are described is preserved.*

Theorem on one-to-one correspondence: *let the boundaries* Γ *and* Γ_1 *of two simply connected domains* D *and* D_1 *be closed piecewise-smooth Jordan curves and let a function* $w = f(z)$ *analytic in* D *specify a one-to-one and continuous mapping of* Γ onto Γ_1 *with the preservation of the direction in which these curves are described; then this function determines a conformal mapping of the domain* D *onto the domain* D_1.

5°. Conformal Mappings Determined by Some Elementary Functions. Inverse Functions. The Notion of a Riemann Surface. As was already mentioned in Sec. 3°, § 1, linear-fractional function (10) is analytic in the plane of the complex variable z everywhere except the point $z = -d/c$.

It can easily be seen that *for linear-fractional function* (10) *which is not identically equal to a constant to be univalent* (*one-sheeted*) *it is necessary and sufficient that the condition*

$$ad - bc \neq 0 \tag{21}$$

should hold.

Indeed, when considering linear-fractional function (10), we naturally exclude the case in which the constants c and d are simultaneously equal to zero. For all the other cases the violation of condition (21) means that the function w is identically equal to a constant, which is impossible by the hypothesis.

Under the mapping specified by (10) the points $z = -d/c$ and $w = \infty$ and also $z = \infty$ and $w = a/c$ are in one-to-one correspondence respectively. Further, if $z_1 \neq -d/c$ and $z_2 \neq -d/c$ are two different values of the variable z, the difference between the corresponding values w_1 and w_2 of the function w is equal, by virtue of (10), to the expression

$$w_2 - w_1 = \frac{(ad-bc)(z_2-z_1)}{(cz_1+d)(cz_2+d)}$$

If follows that condition (21) guarantees the univalence of mapping (10) on the extended complex plane of the variable z, and

$$z = \frac{dw-b}{-cw+a}$$

In particular, for $c = 0$ and $d = 1$ formula (21) implies the condition $a \neq 0$ which guarantees the univalence of the linear function $w = az + b$ whose inverse function is

$$z = \frac{w}{a} - \frac{b}{a}$$

In the case when the coefficients a, b, c and d satisfying condition (21) are real, linear-fractional function (10) determines a one-to-one mapping of the real axis $\operatorname{Im} z = 0$ onto the real axis $\operatorname{Im} w = 0$; under this mapping the direction in which the axis is traced is preserved when $ad - bc > 0$ and changes to the opposite when $ad - bc < 0$. Indeed, for the real values of z, by virtue of the formula

$$\frac{dw}{dz} = \frac{ad - bc}{(cz + d)^2}$$

we have $\frac{dw}{dz} > 0$ for $ad - bc > 0$ and $\frac{dw}{dz} < 0$ for $ad - bc < 0$. By the above theorem on the one-to-one correspondence, it follows that *in the case under consideration linear-fractional function* (10) *specifies a conformal mapping of the upper half-plane* z^+: $\operatorname{Im} z > 0$ *(of the lower half-plane* z^-: $\operatorname{Im} z < 0$) *onto the upper half-plane* w^+: $\operatorname{Im} w > 0$ *(onto the lower half-plane* w^-: $\operatorname{Im} w < 0$) *for* $ad - bc > 0$ *and of the upper half-plane* z^+ *(the lower half-plane* z^-) *onto the lower half-plane* w^- *(onto the upper half-plane* w^+) *for* $ad - bc < 0$.

The linear-fractional function

$$w = e^{i\theta} \frac{z - z_0}{z - \bar{z}_0} \tag{22}$$

where θ is a real constant and z_0 is a complex constant with $\operatorname{Im} z_0 > 0$, possesses the property that

$$|w| = |e^{i\theta}| \left| \frac{z - z_0}{z - \bar{z}_0} \right| = \frac{|z - z_0|}{|z - \bar{z}_0|} = 1$$

for $\operatorname{Im} z = 0$ and that the point $z = z_0$ goes into the point $w = 0$ under the mapping specified by (22). This means that the function w defined by formula (22) specifies a mapping under which the points of the real axis $\operatorname{Im} z = 0$ are in one-to-one correspondence with the points of the contour $|w| = 1$, and consequently, by virtue of the theorem on the one-to-one correspondence, *this function performs a con-*

formal mapping of the upper half-plane z^+ *(of the lower half-plane* z^-*) onto the interior of the circle* $|w| < 1$ *(onto the exterior of the circle* $|w| \leqslant 1$*). In the case* $\operatorname{Im} z_0 < 0$ *the function* w *determined by formula* (22) *specifies a conformal mapping of the half-plane* z^+ (z^-) *onto the exterior of the circle* $|w| \leqslant 1$ *(onto the interior of the circle* $|w| < 1$*).*

It can readily be verified that *under mapping* (22) *any two points* z and $\bar{z}$ *which are symmetric about the real axis* $\operatorname{Im} z = 0$ *go into two points* w and $w_* = 1/w$ *which are symmetric with respect to the circle* $|w| = 1$.

Now let us consider the linear-fractional function

$$w = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad |z_0| < 1 \tag{23}$$

Since for $|z| = 1$ and $0 \leqslant \varphi < 2\pi$ we have

$$|w| = |e^{i\theta}| \left| \frac{e^{i\varphi} - z_0}{1 - \bar{z}_0 e^{i\varphi}} \right| = \frac{|e^{i\varphi} - z_0|}{|e^{-i\varphi} - \bar{z}_0|} = 1, \quad z = e^{i\varphi}$$

the repetition of the argument presented above shows that *function* (23) *specifies a conformal mapping for the circle* $|z| < 1$ *onto the circle* $|w| < 1$.

In the case $|z_0| > 1$, *formula* (23) *determines a conformal mapping of the exterior of the circle* $|z| \leqslant 1$ *onto the interior of the circle* $|w| < 1$.

Each of the linear-fractional functions we considered above (they specify conformal mappings of the upper half-plane $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$, of the upper half-plane $\operatorname{Im} z > 0$ onto the circle $|w| < 1$, and of the circle $|z| < 1$ onto the circle $|w| < 1$ respectively) involves three real parameters; these parameters are determined uniquely when one of the following conditions is fulfilled: (1) three given boundary points z_1, z_2 and z_3 go into three given boundary points w_1, w_2 and w_3; (2) an interior point z_1 and a boundary point z_2 go into an interior point w_1 and a boundary point w_2 respectively; (3) an interior point z_1 and a given direction passing through it go into an interior point w_1 and a direction passing through it respectively.

Let us consider the *power function*

$$w = z^n \tag{24}$$

where n is a natural number. As a domain of univalence of function (24) *we can take any angular region D, with vertex at the point $z = 0$ enclosing an angle of $2\pi/n$.* Indeed, for any two different values $z_1 = re^{i\varphi_1}$ and $z_2 = re^{i\varphi_2}$ of the independent variable z we have $w_1 = r^n e^{in\varphi_1}$ and $w_2 = r^n e^{in\varphi_2}$ whence it follows that $w_2 - w_1 = r^n (e^{in\varphi_2} - e^{in\varphi_1}) = r^n e^{in\varphi_1} (e^{in(\varphi_2-\varphi_1)}) - 1 \neq 0$ provided that $n(\varphi_2 - \varphi_1) \neq 2\pi k$ $(k = 0, \pm 1, \ldots)$, that is $w_2 \neq w_1$ for $|\varphi_2 - \varphi_1| < 2\pi/n$. The function $z = f^{-1}(w)$ inverse to w inside the angle D is denoted as

$$z = \sqrt[n]{w} = w^{\frac{1}{n}}$$

and

$$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}} = \frac{1}{nz^{n-1}} = \frac{1}{n} w^{\frac{1}{n}-1} \tag{25}$$

Under the mapping specified by the function w defined by formula (24) the rays $\arg z = \varphi$ go into the rays $\arg w = n\varphi$; in particular, the ray $\arg z = 2k\pi/n$ $(k \geqslant 0)$ goes into the positive real axis $\operatorname{Im} w = 0$, $\operatorname{Re} w > 0$, and the ray $\arg z = 2(k+1)\pi/n$ also goes into the positive real axis $\operatorname{Im} w = 0$, $\operatorname{Re} w > 0$. Therefore this function specifies a conformal mapping of the angular domain $\frac{2k\pi}{n} < \arg z < \frac{2(k+1)\pi}{n}$ onto the complex plane of the variable w with a cut made along the real semi-axis $0 \leqslant u < \infty$.

On denoting the variables z and w as $z = re^{i\varphi}$ and $w = \rho e^{i\psi}$ we obtain from (24) the equality $\rho e^{i\psi} = r^n e^{in\varphi}$, whence $r^n = \rho$ and $n\varphi = \psi + 2k\pi$ that is

$$r = \rho^{\frac{1}{n}}, \quad \varphi = \frac{\psi + 2k\pi}{n}; \quad k = 0, \pm 1, \ldots, 0 < \psi < 2\pi$$

Thus, for each of the angular domains

$$D_k: \frac{2k\pi}{n} < \arg z < \frac{2(k+1)\pi}{n} \quad (k = 0, \ldots, n-1)$$

we can write the equality

$$z_k = \rho^{\frac{1}{n}} e^{i\frac{\psi + 2k\pi}{n}} \quad (k = 0, \ldots, n-1) \tag{26}$$

for the inverse function (whose derivative is expressed by formula (25)).

It is inconvenient to consider each of the quantities z_k as a separate function of the variable w because, for instance, in the domain D: $\pi/n < \arg z < 3\pi/n$ the inverse function $z = w^{1/n}$ coincides with z_0 for $\pi/n < \arg z < 2\pi/n$ and with z_1 for $\frac{2\pi}{n} < \arg z < \frac{3\pi}{n}$. Therefore we speak of the *many-valued function inverse to* (24), and each function z_k ($k = 0, \ldots, n - 1$) is referred to as a *branch of the many-valued function* $z = w^{1/n}$.

Consequently, *the inverse function* $z = w^{1/n}$ *whose derivative is given by formula* (25) *is not one-valued*. When the point z runs once throughout the complex plane z, the point w representing the inverse function runs through the complex plane w not once but n times. Therefore function (24) is said to be *n-sheeted*, and the inverse function (whose branches are determined by formula (26)) is called *n-valued*.

A mapping specified by a many-sheeted function can be interpreted as a one-to-one mapping of the domain of definition of that function onto the so-called *Riemann surface*. We shall demonstrate what has been said by the example of the function

$$w = z^2 \tag{27}$$

To this end let us consider two replicas E^+ and E^- of the complex plane w with cuts made along their positive real semi-axes. When the point z runs through the half-plane z^+ the point w runs through E^+. When the point z passes from z^+ to z^- across the negative real axis $\operatorname{Im} z = 0$, $\operatorname{Re} z < 0$, the point w passes from the lower edge of the cut made in E^+ to the upper edge of the cut in E^-. Let us "stick together" the lower edge of the cut in E^+ and the upper edge of the cut in E^-. Further, when z runs throughout z^- and approaches the positive real axis, the point w runs through E^- and approaches the lower edge of the cut made in E^-. Let us identify the points belonging to the lower edge of the cut in E^- with the corresponding points belonging to the upper edge of the cut in E^+ (in the real three-dimensional space in which the planes E^+ and E^- lie it is impossible to realize the sticking of these edges); this will result in a *two-sheeted Rie-*

mann surface (see Fig. 8) onto which function (27) maps the complex plane z in a one-to-one manner.

Relation (27) specifies a one-to-one correspondence between the points of the extended complex plane of the variable z and the points of the Riemann surface on which the function $z = w^{1/2}$ is defined, the mapping determined

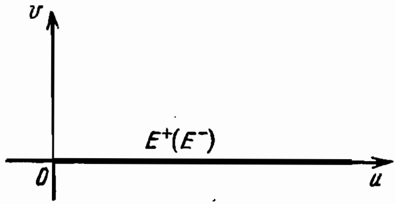

Fig. 8

by (27) being conformal everywhere except the points $z = 0$ and $z = \infty$. The points $w = 0$ and $w = \infty$ corresponding to the points $z = 0$ and $z = \infty$ possess the following property: if we describe closed contours about these points in the counterclockwise direction once, starting with a fixed point w, then the corresponding point z passes from one branch z_0 to the other branch z_1 when the point w returns to its original position. That is why the points $w = 0$ and $w = \infty$ are referred to as the *branch points* of the function $z = w^{1/2}$.

In just the same way the notion of the branch points of the function $z = w^{1/n}$ is introduced; these branch points are $w = 0$ and $w = \infty$.

Now let us consider the *exponential function*

$$w = e^z \tag{28}$$

As its domain of univalence can serve any strip D of width 2π *parallel to the real axis* $\operatorname{Im} z = 0$. This follows from the fact that for two different points z_1 and z_2 $(z_1 \neq z_2)$ the equality $e^{z_1} = e^{z_2}$ is only possible when $z_2 - z_1 = 2k\pi i$ $(k = 0, \pm 1, \ldots)$.

The inverse function $z = f^{-1}(w)$ of (28) considered in the strip D is called the *logarithmic function* and is denoted as

$$z = \ln w \tag{29}$$

The derivative of (29) is given by the formula

$$\frac{dz}{dw}=\frac{1}{\frac{dw}{dz}}=\frac{1}{e^z}=\frac{1}{w}$$

From the equality $w = u + iv = \rho e^{i\psi} = e^z = e^x e^{iy}$ we find that $\rho = e^x$ and $iy = i\psi + 2k\pi i$, that is $x = \ln \rho$ and $y = \psi + 2k\pi$. Consequently, in each strip D_k: $2k\pi < \operatorname{Im} z < 2(k+1)\pi$ we have $z_k = \ln |w| + i \arg w + 2k\pi i$, $0 \leqslant \arg w < 2\pi$ $(k = 0, \pm 1, \ldots)$ for function (29). The value $z_0 = \ln |w| + i \arg w$, $0 \leqslant \arg w < 2\pi$ of z is called the *principal value* of logarithmic function (29). Since under the mapping specified by function (28) the straight lines $\operatorname{Im} z = \text{const}$ go into the rays $\arg w = \text{const}$, this function specifies a conformal mapping of each of the strips D_k $(k = 0, \pm 1, \ldots)$ onto the complex plane of the variable w with a cut made along the nonnegative real axis. Consequently, *exponential function* (28) *is infinite-sheeted, and logarithmic function* (29) *is infinite-valued.*

The logarithmic function makes it possible to define the power function $w = z^\alpha$ for any exponent α by putting $w = e^{\ln z^\alpha} = e^{\alpha \ln z}$ and the exponential function $w = \alpha^z$ for any base α by putting $w = e^{\ln \alpha^z} = e^{z \ln \alpha}$.

In order to state the definition of the function $z = \arcsin w$ inverse to

$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

let us rewrite the last relation in the form of a quadratic equation with respect to e^{iz}:

$$e^{2iz} - 2iwe^{iz} - 1 = 0$$

The solution of this equation is the function

$$e^{iz} = iw \pm \sqrt{1 - w^2}$$

It follows that, since $\ln e^{iz} = iz = \ln\left(iw \pm \sqrt{1 - w^2}\right)$, the variable z is expressed as

$$z = \arcsin w = \frac{1}{i} \ln\left(iw \pm \sqrt{1 - w^2}\right)$$

and the derivative of z with respect to w is

$$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}} = \frac{1}{\cos z} = \frac{1}{\sqrt{1 - \sin^2 z}} = \frac{1}{\sqrt{1 - w^2}}$$

§ 2. Complex Integrals

1°. Integration Along a Curve in the Complex Plane. Let S be a piecewise smooth (closed or non-closed) Jordan curve, and let $f(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$ be a continuous function of the variable $\zeta = \xi + i\eta$ defined on S.

Since the line integrals

$$\int_S u\,d\xi - v\,d\eta \quad \text{and} \quad \int_S u\,d\eta + v\,d\xi \tag{30}$$

exist, and since

$$f(\zeta)\,d\zeta = (u+iv)(d\xi + i\,d\eta) = u\,d\xi - v\,d\eta + i(u\,d\eta + v\,d\xi)$$

the expression of the form

$$\int_S u\,d\xi - v\,d\eta + i\int_S u\,d\eta + v\,d\xi = \int_S f(\zeta)\,d\zeta \tag{31}$$

is naturally called the *complex integral of the function* $f(\zeta)$ *along the curve* S.

In the case when the curve S is a closed contour we say, when the point ζ (ζ is the variable of integration along S) traces the contour S, that S is described in positive (negative) direction if the finite domain D whose boundary is the contour S always remains on the left (on the right). Accordingly, we speak of a *positively oriented* contour $S^+ = S$ and of the *negatively oriented* contour S^-. When the integration goes in the negative direction along S we put, by definition,

$$\int_{S^-} f(\zeta)\,d\zeta = -\int_S f(\zeta)\,d\zeta \tag{32}$$

When the curve S is non-closed and is specified by a parametric equation of the form $\zeta = \zeta(t)$ where t is a real parameter, the positive direction along S is the one corresponding to the increase of t.

Below we enumerate the basic properties of integral (31) which are direct consequences of the corresponding properties of real line integrals (30).

(1) If $f_k(\zeta)$ $(k = 1, \ldots, m)$ are continuous functions defined on S and c_k are given constants then

$$\int_S \sum_{k=1}^{m} c_k f_k(\zeta)\, d\zeta = \sum_{k=1}^{m} c_k \int_S f_k(\zeta)\, d\zeta \tag{33}$$

(2) If S is a curve consisting of m arcs $S_1, \ldots, S_m$ then

$$\int_S f(\zeta)\, d\zeta = \sum_{k=1}^{m} \int_{S_k} f(\zeta)\, d\zeta \tag{34}$$

where we assume that the integration along each arc S_k is in the direction generated by the direction of integration along S.

Let S be a collection of pairwise disjoint closed curves

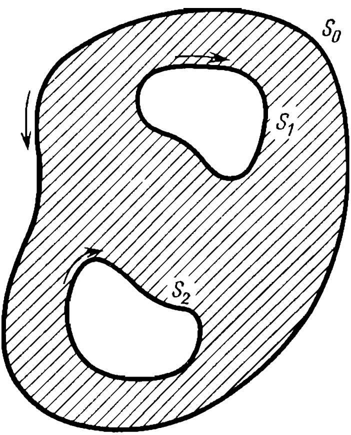

Fig. 9

$S_0, S_1, \ldots, S_m$ which form the boundary of an $(m + 1)$-connected bounded domain D and let $S_1, \ldots, S_m$ lie inside the bounded domain D_0 whose boundary is S_0 (see Fig. 9); then

$$\int_S f(\zeta)\, d\zeta = \int_{S_0} f(\zeta)\, d\zeta - \sum_{k=1}^{m} \int_{S_k} f(\zeta)\, d\zeta \tag{35}$$

(3) If a continuous function $f(\zeta)$ is integrable over S then so is the function $|f(\zeta)|$, and

$$\left| \int_S f(\zeta)\, d\zeta \right| \leqslant \int_S |f(\zeta)|\,|d\zeta| \leqslant l \max_{\zeta \in S} |f(\zeta)| \tag{36}$$

where l is the length of S.

(4) If $f_k(\zeta)$ $(k = 1, 2, \ldots)$ are continuous functions defined on S and if the series $\sum_{k=1}^{\infty} f_k(\zeta)$ (the sequence $\{f_k(\zeta)\}$) is uniformly convergent on S, then the sum $f(\zeta)$ of that series (the limit $f(\zeta)$ of that sequence) is continuous on S, and we have

$$\int_S \sum_{k=1}^{\infty} f_k(\zeta)\, d\zeta = \sum_{k=1}^{\infty} \int_S f_k(\zeta)\, d\zeta \tag{37}$$

in the case of the series and

$$\lim_{k \to \infty} \int_S f_k(\zeta)\, d\zeta = \int_S f(\zeta)\, d\zeta \tag{38}$$

in the case of the sequence.

(5) The change of the variable of integration in complex integral (31) is performed according to the same ordinary rules for the change of variable in the real line integrals on the left-hand side of formula (31).

2°. ***Cauchy's Theorem.*** *If $f(z) = u(x, y) + iv(x, y)$ is an analytic function in a bounded domain D with a piecewise smooth boundary S and the functions $u(x, y)$ and $v(x, y)$ are continuous in $D \cup S$ together with their partial derivatives of the first order then*

$$\int_S f(\zeta)\, d\zeta = 0 \tag{39}$$

Indeed, on transforming line integrals (30) with the aid of formula (*GO*), we obtain

$$\int_S f(\zeta)\, d\zeta = -\int_D \left[\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \eta} + i \left(\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \xi} \right) \right] d\xi\, d\eta$$

whence, since the functions $u(\xi, \eta)$ and $v(\xi, \eta)$ satisfy system (CR) in the domain D, we see that equality (39) does in fact hold.

It should also be noted (this remark will be used in what follows) that *Cauchy's theorem also remains true when S and $f(\zeta)$ satisfy some general (weaker) requirements.* In particular, equality (39) also holds when it is only required that the function $f(z)$ analytic in the domain D should be *continuous* in $D \cup S$.

Cauchy's theorem directly implies that *if a function $f(z)$ is analytic in a simply connected domain D, and z_0 and z are two points belonging to D, then the integral*

$$\int_{z_0}^{z} f(\zeta)\, d\zeta \tag{40}$$

assumes one and the same value for all the curves joining z_0 and z and lying within D, that is the value of integral (40) *does not depend on the path of integration* (it is meant that integral (40) is taken with respect to ζ in the direction from z_0 to z).

Indeed, let S and S_1 be two curves lying in D and joining the points z_0 and z. By virtue of formulas (32) and (39), we have

$$\int_{S_1^-} f(\zeta)\, d\zeta = -\int_{S_1} f(\zeta)\, d\zeta$$

and

$$\int_{S} f(\zeta)\, d\zeta + \int_{S_1^-} f(\zeta)\, d\zeta = \int_{S} f(\zeta)\, d\zeta - \int_{S_1} f(\zeta)\, d\zeta = 0$$

whence it follows that

$$\int_{S} f(\zeta)\, d\zeta = \int_{S_1} f(\zeta)\, d\zeta$$

Let S be the circle $|\zeta| = R$; then for the integral exponents n of the power function ζ^n we have

$$\int_{|\zeta|=R} \zeta^n\, d\zeta = \begin{cases} 0 & \text{for} \quad n \neq -1 \\ 2\pi i & \text{for} \quad n = -1 \end{cases} \tag{41}$$

Indeed,

$$\int\limits_{|\zeta|=R} \zeta^n\,d\zeta = \int\limits_0^{2\pi} (Re^{i\varphi})^n\, iRe^{i\varphi}\,d\varphi = iR^{n+1}\int\limits_0^{2\pi} \cos(n+1)\varphi\,d\varphi -$$

$$- R^{n+1}\int\limits_0^{2\pi} \sin(n+1)\varphi\,d\varphi = \begin{cases} 0 & \text{for} \quad n \neq -1 \\ 2\pi i & \text{for} \quad n = -1 \end{cases}$$

Now let us consider the integral

$$\int\limits_S \zeta^n\,d\zeta$$

where n is an integral number and S is a piecewise smooth contour bounding a (finite) domain D.

In the case $n \geqslant 0$, by virtue of the analyticity of the function z^n and on the basis of (39), we have

$$\int\limits_S \zeta^n\,d\zeta = 0 \tag{42}$$

irrespective of whether the point $z = 0$ belongs or does not belong to $D \cup S$; in the case $n < 0$ we have the same equality provided that the point $z = 0$ does not belong to $D \cup S$.

Now let $n < 0$ and let $z = 0 \in D$. On deleting the point $z = 0$ together with a closed circle $|z| \leqslant \varepsilon$ of a sufficiently small radius $\varepsilon > 0$ lying within D from the domain D, we can write, by virtue of formulas (36), (39) and (41), the relation

$$\int\limits_S \zeta^n\,d\zeta = \int\limits_{|\zeta|=\varepsilon} \zeta^n\,d\zeta = \begin{cases} 0 & \text{for} \quad n < -1 \\ 2\pi i & \text{for} \quad n = -1 \end{cases} \tag{43}$$

From (42) and (43) it follows that for all integral exponents $n \neq -1$ the value of the integral

$$\int\limits_{z_0}^{z} \zeta^n\,d\zeta \tag{44}$$

is one and the same for all the curves connecting the points z_0 and z and not passing through the point $z = 0$ when $n < -1$.

To compute integral (44) for $n \geqslant 0$, let us take as the path of integration the line segment $\zeta = z_0 + (z - z_0) t$, $0 \leqslant t \leqslant 1$. Then, by virtue of (31) and (33), we obtain

$$\int_{z_0}^{z} \zeta^n \, d\zeta = \int_0^1 [z_0 + (z - z_0) t]^n (z - z_0) \, dt =$$

$$= \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^{k+1} \int_0^1 t^k \, dt =$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} z_0^{n-k} (z - z_0)^{k+1} =$$

$$= \frac{1}{n+1} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} z_0^{n-k+1} (z - z_0)^k - z_0^{n+1} \right] =$$

$$= \frac{z^{n+1}}{n+1} - \frac{z_0^{n+1}}{n+1} \qquad (45)$$

3°. Cauchy's Integral Formula. In this section we shall prove *Cauchy's integral formula*

$$\frac{1}{2\pi i} \int_S \frac{f(\zeta) \, d\zeta}{\zeta - z} = \begin{cases} 0 & \text{for} \quad z \in D^- \\ f(z) & \text{for} \quad z \in D^+ \end{cases} \qquad (46)$$

where D^+ is a finite domain bounded by a piecewise smooth contour S, D^- is the complement of $D^+ \cup S$ with respect to the whole complex plane and $f(z)$ is a function analytic in D^+ and continuous in $D^+ \cup S$.

Let $z \in D^-$; in this case, by the analyticity of the function $f(\zeta)/(\zeta - z)$ with respect to ζ in D^+ and by its continuity in $D^+ \cup S$, the first equality (46) is nothing other than equality (39). If $z \in D^+$ then we remove the point z from the domain D^+ together with a closed circle $|\zeta - z| \leqslant \varepsilon$ of a sufficiently small radius $\varepsilon > 0$ lying in D^+; taking into account the analyticity of the function $f(\zeta)/(\zeta - z)$ with respect to ζ in the remaining part D_ε of the domain D^+ and the continuity of this function in the domain D^+ including its boundary, we obtain, by (39) and (35) (see

Fig. 10), the equality

$$\frac{1}{2\pi i}\int_S \frac{f(\zeta)\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\frac{f(\zeta)\,d\zeta}{\zeta-z}=$$

$$=\frac{1}{2\pi i}\int_{|\zeta-z|=\varepsilon}\frac{f(\zeta)-f(z)}{\zeta-z}\,d\zeta+\frac{1}{2\pi i}f(z)\int_{|\zeta-z|=\varepsilon}\frac{d\zeta}{\zeta-z}$$

From this equality, on the basis of (41) and the obvious equality

$$\lim_{\varepsilon\to 0}\int_{|\zeta-z|=\varepsilon}\frac{f(\zeta)-f(z)}{\zeta-z}\,d\zeta=0$$

we derive, on passing to the limit for $\varepsilon\to 0$, the relation

$$\frac{1}{2\pi i}\int_S \frac{f(\zeta)\,d\zeta}{\zeta-z}=f(z),\quad z\in D^+ \tag{47}$$

It is evident that *formula* (47) *also remains valid when* D^+

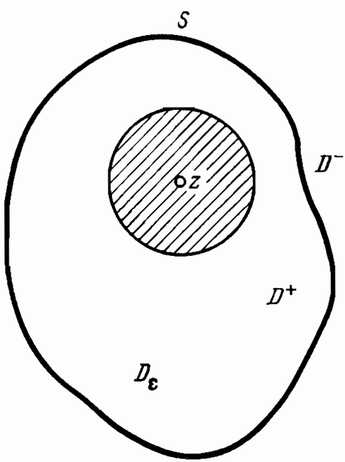

Fig. 10

is an $(m+1)$*-connected finite domain whose boundary* S *consists of closed pairwise disjoint piecewise smooth curves* S_0, $S_1, \ldots, S_m$. In particular, if the bounded domain D_0 with boundary S_0 contains the curves $S_1, \ldots, S_m$, we can write, by (35), the formula

$$f(z)=\frac{1}{2\pi i}\int_{S_0}\frac{f(\zeta)\,d\zeta}{\zeta-z}-\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{S_k}\frac{f(\zeta)\,d\zeta}{\zeta-z},\quad z\in D^{+}\tag{48}$$

Let $f(z)$ be an analytic function in a simply connected domain D; then, as was already shown above, integral (40) depends solely on the position of the points z_0 and z inside D, whence it follows that *for a fixed* z_0 *the expression*

$$F(z)=\int_{z_0}^{z} f(\zeta)\,d\zeta$$

is a one-valued function of the complex variable z in the domain D. Moreover, for the ratio $\Delta F/\Delta z$ we can write the expression

$$\frac{\Delta F}{\Delta z}=\frac{1}{\Delta z}\left(\int_{z_0}^{z+\Delta z} f(\zeta)\,d\zeta-\int_{z_0}^{z} f(\zeta)\,d\zeta\right)=\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)\,d\zeta$$

where, for sufficiently small nonzero values of $|\Delta z|$, we can assume that the integral is taken along the line segment δ joining the points z and $z+\Delta z$.

According to formula (45), we have

$$\int_{z}^{z+\Delta z} d\zeta=\Delta z$$

and therefore

$$\left|\frac{\Delta F}{\Delta z}-f(z)\right|=\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(\zeta)-f(z)]\,d\zeta\right|\leqslant\max_{\zeta\in\delta}|f(\zeta)-f(z)|$$

Since $\max|f(\zeta)-f(z)|\to 0$ for $|\zeta-z|\to 0$, the last inequality implies that

$$\lim_{\Delta z\to 0}\frac{\Delta F}{\Delta z}=F'(z)=f(z)$$

The family $\Phi(z)$ of analytic functions in the domain D whose derivatives are $\Phi'(z)=f(z)$ is called the *indefinite* *integral of* $f(z)$ (each of the functions $\Phi(z)$ is called an *antiderivative* of $f(z)$).

For the difference $\Phi(z) - F(z) = \Psi(z) = u(x, y) + iv(x, y)$ we have the equality $\Psi'(z) = \Phi'(z) - F'(z) = f(z) - f(z) = 0$ everywhere in the domain D, and therefore $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, whence it follows that $u(x, y) = \text{const}$ and $v(x, y) = \text{const}$, and consequently $\Phi(z) = F(z) + C$ where C is an arbitrary (complex) constant. Since $\Phi(z_0) = F(z_0) + C = C$, we obtain the formula

$$\int_{z_0}^{z} f(\zeta)\, d\zeta = \Phi(z) - \Phi(z_0)$$

for the computation of integral (40) which is known as the *Newton-Leibniz formula.*

4°. **The Cauchy-Type Integral.** Let S be a closed or non-closed piecewise smooth Jordan curve and let $f(\zeta)$ be a continuous function defined on S.

For a fixed value of z not belonging to S the expression $\varphi(z, \zeta) = \frac{f(\zeta)}{\zeta - z}$ considered as function of ζ is continuous, and therefore the integral

$$F(z) = \frac{1}{2\pi i} \int_S \frac{f(\zeta)\, d\zeta}{\zeta - z} \tag{49}$$

exists and is a one-valued function of z. Expression (49) is called the *Cauchy-type integral.*

If S is a closed contour bounding a finite domain D and if the function $f(z)$ is analytic in D and continuous in $D \cup S$, the right-hand member of (49) coincides with $f(z)$ at each point $z \in D$, and in this case (49) is nothing other than Cauchy's integral formula (47).

Since for $\zeta \in S$ the expression $\varphi(z, \zeta) = f(\zeta)/(\zeta - z)$ considered as function of z is analytic at each point not belonging to S, that is $\frac{\partial}{\partial \bar{z}} \varphi(z, \zeta) = 0$, and since the operation $\frac{\partial}{\partial \bar{z}}$ can be written under the integral sign on the right-hand side of (49), we conclude that $\frac{\partial F(z)}{\partial \bar{z}} = 0$. Consequently, *the function $F(z)$ is analytic on the complex plane of the variable z everywhere except the points belonging to the curve S.* Further, since for $\zeta \in S$ the expression $\varphi(z, \zeta)$ considered as function of z possesses the derivatives

$$\frac{d^n}{dz^n}\varphi(z, \zeta) = n!\,\frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

of any order n at each point not lying on S and since the operation d^n/dz^n can be written under the integral sign on the right-hand side of (49), we see that *the function $F(z)$ represented by Cauchy-type integral* (49) *possesses derivatives of all orders at the points z not belonging to S, and*

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_S \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}; \quad n = 1, 2, \ldots \tag{50}$$

From the property of the Cauchy-type integral we have just established it follows that *if $f(z)$ is an analytic function in a domain D then it possesses derivatives of all orders in that domain.* Indeed, let z_0 be an arbitrary point in the domain D and let $|z - z_0| \leqslant \varepsilon$ be a circle of a sufficiently small radius $\varepsilon > 0$ lying inside D; then, by virtue of Cauchy's integral formula (46), we have

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = \varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z}, \quad |z - z_0| < \varepsilon \tag{51}$$

The right-hand member of (51) is a special case of the Cauchy-type integral, and therefore the function $f(z)$ possesses derivatives of all orders in the circle $|z - z_0| < \varepsilon$; since the point z_0 is quite arbitrary, we see that the assertion stated above is in fact true.

5°. Conjugate Harmonic Functions. Morera's Theorem. If $f(z)$ is an analytic function, then its derivative

$$f'(z) = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}$$

is itself an analytic function in the domain D where $f(z)$ is defined, and therefore the functions $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ possess the partial derivatives of the first order $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\,\partial y}$, $\frac{\partial^2 u}{\partial y\,\partial x}$

and $\frac{\partial^2 u}{\partial y^2}$. Further, according to condition (CR), we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

and

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

We have thus proved *the existence of partial derivatives of the second order of the functions* $u(x, y)$ *and* $v(x, y)$ *and also the harmonicity of these functions in the domain* D. The repetition of this argument shows that *the functions* $u(x, y)$ *and* $v(x, y)$ *possess derivatives of all orders in the domain* D *of analyticity of the function* $f(z)$.

The real and the imaginary parts $u(x, y)$ and $v(x, y)$ of a function $f(z)$ analytic in a domain D are called *conjugate harmonic functions.*

Now let us suppose that the real part $u(x, y)$ of an analytic function $f(z) = u(x, y) + iv(x, y)$ in a simply connected domain D is known. Then we can easily reconstruct the function $f(z)$ itself. Indeed, we have

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \tag{52}$$

The expression for dv we have obtained (see (52)) is a total differential because of the condition

$$-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

which holds since the function $u(x, y)$ is harmonic in the domain D. Consequently,

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + C$$

where C is an arbitrary real constant, and the integral on the right-hand side is independent of the path of integration connecting the points (x_0, y_0) and (x, y) and lying in the domain D.

Thus, we arrive at the conclusion that *a function* $f(z)$ *analytic in a simply connected domain* D *can be reconstructed to within an arbitrary pure imaginary constant* iC *from its* *given real part* $u(x, y)$:

$$f(z) = u(x, y) + i \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + iC \qquad (53)$$

Now let us prove the converse of Cauchy's integral theorem known as ***Morera's theorem***: *if $f(z)$ is a continuous function in a simply connected domain D and if the integral of this function along any closed piecewise smooth Jordan curve lying in D is equal to zero, then the function $f(z)$ is analytic in D.*

Indeed, by the condition of the theorem, integral (40) of the function $f(\zeta)$ depends solely on the position of the points z_0 and z and is independent of the shape of the path of integration joining z_0 and z. Further, as was seen, for a continuous function $f(\zeta)$ this property of integral (40) is sufficient for the function $F(z)$ defined by the formula

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$

to be analytic in D and for the equality $F'(z) = f(z)$ to hold. Finally, since the derivative of an analytic function is itself an analytic function, the equality $F'(z) = f(z)$ implies the analyticity of the function $f(z)$.

§ 3. Some Important Consequences of Cauchy's Integral Formula

1°. Maximum Modulus Principle for Analytic Functions. Let $f(z)$ be an analytic function in a domain D, and M be the supremum of $|f(z)|$ for $z \in D$.

The maximum modulus principle is stated thus: *if the function $f(z)$ analytic in the domain D is not identically equal to a constant in D, then the modulus of this function cannot assume the value equal to M at any point in that domain.*

If $M = \infty$, this assertion is obviously true because at any point $z \in D$ the function $f(z)$ can only assume a finite value. The case $M = 0$ can be excluded from the consideration since this means that $f(z) \equiv 0$. Now let us suppose that the number M is finite and that there is a point $z_0 \in D$ for which the equality $|f(z_0)| = M$ takes place. Let us consider a closed circle $|\zeta - z_0| \leqslant \delta$ lying within D. By Cauchy's integral formula (47), we have

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = \delta} \frac{f(\zeta)\, d\zeta}{\zeta - z_0} = \frac{1}{2\pi} \int\limits_0^{2\pi} f(z_0 + \delta e^{i\theta})\, d\theta$$

$$\zeta - z_0 = \delta e^{i\theta}$$

whence, according to the hypothesis, it follows that

$$M \leqslant \frac{1}{2\pi} \int\limits_0^{2\pi} |f(z_0 + \delta e^{i\theta})|\, d\theta \qquad (54)$$

Inequality (54) implies that $|f(\zeta)| = M$ everywhere on the circumference $|\zeta - z_0| = \delta$ of that circle. Indeed, suppose that there is a point $\zeta_0 = \delta e^{i\theta_0}$ for which the inequality $|f(\zeta_0)| < M$ holds (the opposite inequality $|f(\zeta_0)| > M$ is impossible). Then, since the function $|f(\zeta)|$ is continuous, the inequality $|f(\zeta)| < M$ must also hold for an interval $\theta_0 - \varepsilon < \theta < \theta_0 + \varepsilon$, and therefore (54) leads to the inconsistent inequality $M < M$. Consequently, $|f(\zeta)| = M$ for all the values of δ $(0 \leqslant \delta \leqslant \delta_0)$, that is this relation holds throughout the neighbourhood $|z - z_0| < \delta_0$ of the point z_0. Since $\ln |f(z)| = \frac{1}{2} \ln f(z) \overline{f(z)}$, we have

$$\frac{\partial}{\partial z} \ln |f(z)| = \frac{1}{2} \frac{f'(z)}{f(z)} = \frac{\partial}{\partial z} \ln M = 0$$

for all the values of z in the circle $|z - z_0| < \delta_0$, that is $f'(z) = 0$ everywhere in this circle, and consequently $f(z) = \text{const}$. Now we can repeat the corresponding part of the argument used in Sec. 4°, § 1 of Chapter 1 for proving the extremum principle for harmonic functions, and thus come to the conclusion that $f(z) = \text{const}$ everywhere in D, which is impossible. Hence, the assumption that $|f(z_0)| = M$ for $z_0 \in D$ leads to a contradiction, which proves the maximum modulus principle.

If $f(z) \neq 0$ everywhere in D and m is the infimum of $|f(z)|$ for $z \in D$, then we can apply the maximum modulus principle to the function $1/f(z)$ which is analytic in the domain D, whence it follows that the function $|f(z)|$ cannot assume the value equal to m at any point belonging to the domain D. Consequently, *in the case under consideration the modulus of the function $f(z)$ analytic in the domain D can attain its extremum only on the boundary of the domain D.*

2°. **Weierstrass' Theorems.** ***Weierstrass' first theorem:*** *if a series of the form*

$$\sum_{k=1}^{\infty} f_k(z) \tag{55}$$

where $f_k(z)$ ($k = 1, 2, \ldots$) are analytic functions in a domain D, is uniformly convergent on any closed subset of the domain D, then the sum $f(z)$ of this series is an analytic function in D, and for every natural number p the series $\sum_{k=1}^{\infty} f_k^{(p)}(z)$ whose terms are the derivatives of the p-th order of the functions $f_k(z)$ is uniformly convergent on any closed subset of the domain D and

$$f^{(p)}(z) = \sum_{k=1}^{\infty} f_k^{(p)}(z) \tag{56}$$

To begin with, we shall prove that the function $f(z)$ is continuous in the domain D. Let z_0 be an arbitrary fixed point belonging to the domain D. Since series (55) is uniformly convergent, given an arbitrary $\varepsilon > 0$, there is a natural number $N(\varepsilon)$ such that $|f(z) - S_N(z)| < \varepsilon/3$ for all the points z belonging to a circle $|z - z_0| \leqslant \delta_1$ lying within the domain D; here $S_N(z) = \sum_{k=1}^{N} f_k(z)$. Since the finite sum $S_N(z)$ is continuous, there exists a number $\delta_2(\varepsilon) > 0$ such that $|S_N(z) - S_N(z_0)| < \varepsilon/3$ for all the points z belonging to the circle $|z - z_0| < \delta_2$. Consequently, for all z belonging to the circle $|z - z_0| < \delta$ where $\delta = \min(\delta_1, \delta_2)$ we can write, on the basis of the inequalities we have derived, the following inequalities:

$$|f(z)-f(z_0)|=$$
$$=|f(z)-f(z_0)+S_N(z)-S_N(z)+S_N(z_0)-S_N(z_0)|\leqslant$$
$$\leqslant|f(z)-S_N(z)|+|f(z_0)-S_N(z_0)|+|S_N(z)-S_N(z_0)|<$$
$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

The last inequality means that the function $f(z)$ is continuous at the point z_0; since z_0 is quite arbitrary, it follows that $f(z)$ is continuous throughout the domain D.

Now let us consider a closed circle $|z-z_0|\leqslant\delta$ lying inside the domain D. On integrating the equality

$$\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta-z}=\frac{1}{2\pi i}\frac{1}{\zeta-z}\sum_{k=1}^{\infty}f_k(\zeta),\quad |z-z_0|<\delta$$

over the circumference $|\zeta-z_0|=\delta$ of that circle and using Property (4) of complex integrals proved in Sec. 1°, § 2 (which is expressed by formula (37)) and Cauchy's integral formula (47), we obtain the equality

$$\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=\delta}\frac{f(\zeta)\,d\zeta}{\zeta-z}=\sum_{k=1}^{\infty}f_k(z)=f(z)$$

for all the points z of the circle $|z-z_0|<\delta$. Now, taking into account the analyticity of the Cauchy-type integral, we conclude that the function $f(z)$ is analytic in the circle $|z-z_0|<\delta$ and, in particular, at the point z_0. Since z_0 is an arbitrary point of the domain D, we see that $f(z)$ is analytic everywhere in D.

On integrating the uniformly convergent series

$$\frac{p!}{2\pi i}\sum_{k=1}^{\infty}f_k(\zeta)\frac{1}{(\zeta-z)^{p+1}}=\frac{p!}{2\pi i}\frac{f(\zeta)}{(\zeta-z)^{p+1}}$$

over the contour $|\zeta-z_0|=\delta$, we conclude, by virtue of (50), that $\sum\limits_{k=1}^{\infty}f_k^{(p)}(z)=f^{(p)}(z)$ for the points of the circle $|z-z_0|<\delta$, and hence the last equality holds throughout the domain D.

From the uniform convergence of series (55) on the contour $|\zeta - z_0| = \delta$ it follows that, given an arbitrary $\varepsilon > 0$, there is a natural number $N(\varepsilon)$ such that $\left|\sum_{k=1}^{\infty} f_{n+k}(\zeta)\right| < \varepsilon$ for all $n \geqslant N$. Therefore for all the points z of the circle $|z - z_0| < \delta/2$ the inequality

$$\left|\sum_{k=1}^{\infty} f_{n+k}^{(p)}(z)\right| = \left|\frac{p!}{2\pi i} \int_{|\zeta - z_0| = \delta} \sum_{k=1}^{\infty} f_{n+k}(\zeta) \frac{d\zeta}{(\zeta - z)^{p+1}}\right| < \frac{2^{p+1} p! \varepsilon}{\delta^p}$$

holds. This means that the series on the right-hand side of (56) is uniformly convergent in the circle $|z - z_0| < \delta/2$. From this fact, using the well-known *finite covering theorem* proved in mathematical analysis, we conclude that series (56) is uniformly convergent on every closed subset of the domain D (for our aims the following statement of the finite covering theorem is sufficient: *every covering of a closed set by open circles contains a finite subcovering of that set*).

Weierstrass' second theorem: *if series* (55) *consisting of functions* $f_k(z)$, *which are analytic in the domain* D *and continuous in* $D \cup S$, *is uniformly convergent on the boundary* S *of the domain* D, *then this series is uniformly convergent in* $D \cup S$.

This theorem can be proved in exactly the same way as Harnack's theorem (see Sec. 5°, § 2 in Chapter 1). Indeed, from the uniform convergence of series (55) on S it follows that, given any $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that $\left|\sum_{k=1}^{p} f_{N+k}(\zeta)\right| < \varepsilon$ for any $p \geqslant 1$ and for all $\zeta \in S$. Since the modulus of the finite sum $\sum_{k=1}^{p} f_{N+k}(z)$ of the functions $f_k(z)$ analytic in the domain D attains its maximum on the boundary S of D, we have $\left|\sum_{k=1}^{p} f_{N+k}(z)\right| < \varepsilon$, for all $z \in D \cup S$, which means that series (55) is uniformly convergent in $D \cup S$.

3°. Taylor's Series. As was already shown in Sec. 3°, § 1 of the present chapter, the sum $S(z)$ of a power series of the form

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k \tag{57}$$

is an analytic function inside the circle of convergence $|z-z_0|<R$ $(R>0)$ of the series.

It turns out that the converse proposition known as *Taylor's theorem* is also true: *if $f(z)$ is an analytic function in a domain D, then every point $z_0 \in D$ possesses a neighbourhood within which the function $f(z)$ can be represented as the sum of a power series*:

$$f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k \tag{58}$$

The radius of convergence R of this series is not smaller than the distance δ_0 from the point z_0 to the boundary S of the domain D.

Indeed, by Cauchy's integral formula (47), for any point z belonging to a circle $|z-z_0|<\delta<\delta_0$ we have

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta-z_0}\,\frac{f(\zeta)\,d\zeta}{1-\dfrac{z-z_0}{\zeta-z_0}}=$$

$$=\frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k \frac{f(\zeta)\,d\zeta}{\zeta-z_0} \tag{59}$$

where γ denotes the contour $|\zeta-z_0|=\delta$ and $\left|\dfrac{z-z_0}{\zeta-z_0}\right|=$ $=q<1$, $\zeta\in\gamma$. Since the series

$$\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k=\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}}$$

is uniformly convergent with respect to ζ on γ, the series on the right-hand side of (59) can be integrated term-by-term, which makes it possible to rewrite (59) in form (58) with

$$a_k=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{k+1}} \quad (k=0,\ 1,\ \ldots) \tag{60}$$

Now, taking into account (50), we arrive at the formula

$$a_k = \frac{f^{(k)}(z_0)}{k!} \quad (k=0, 1, \ldots) \tag{61}$$

Since series (58) is convergent in every circle $|z - z_0| < \delta$ where δ is an arbitrary number belonging to the interval $0 < \delta < \delta_0$, the radius of convergence of series (58) is not smaller than δ_0.

Power series (57) whose coefficients a_k are expressed in terms of the function $f(z)$ by formulas (60) and (61) is called *Taylor's series* (*Taylor's expansion*) *of the function* $f(z)$ *at the point* z_0.

From formula (61) follows *the uniqueness of Taylor's expansion of* $f(z)$ *in the neighbourhood of the given point* $z_0 \in D$.

4°. Uniqueness Theorem for Analytic Functions. Liouville's Theorem. Taylor's theorem implies the following **uniqueness theorem for analytic functions**: *if* $f(z)$ *is an analytic function in a domain D equal to zero on an infinite set of points E belonging to D and the set E has a limit point* z_0 *lying in D, then* $f(z) = 0$ *everywhere in D.*

Indeed, let d be a circle $|z - z_0| < \delta$ lying in D. By Taylor's theorem, for d we can write expansion (58). Let us denote by z_k $(k = 1, 2, \ldots)$ a sequence of points belonging to $E \cap d$ which converges to z_0. By the condition of the theorem, we have $f(z_k) = \sum\limits_{n=0}^{\infty} a_n (z_k - z_0)^n = 0$, whence, on passing to the limit for $z_k \to z_0$, $z_k \neq z_0$, we obtain $a_0 = 0$. In just the same way from the equality $\frac{f(z_k)}{z_k - z_0} = \sum\limits_{n=1}^{\infty} a_n (z_k - z_0)^{n-1} = 0$ we obtain $a_1 = 0$, and so on. Consequently, $a_k = 0$ for all $k = 0, 1, \ldots$, and hence $f(z) = 0$ in the circle d. Now let z^* be an arbitrary point of the domain D. Let us connect the points z_0 and z^* by a continuous curve l lying within D, and let δ_0 be the distance between l and the boundary S of the domain D. Let us make the centre of the circle $|z - \zeta| < \delta < \delta_0$ move from the point z_0 to the point z^*; using the fact that for every position of the centre ζ on l the equality $f(\zeta) = 0$ holds, we conclude that $f(z^*) = 0$, that is $f(z) = 0$ throughout D.

A point $z_0 \in D$ at which $f(z_0) = 0$ is called a *zero* (or a *root*) of the function $f(z)$. If z_0 is a zero of an analytic function $f(z)$ in a domain D, then, by virtue of (58), we have

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k, \quad n \geqslant 1, \quad a_n \neq 0$$

If $n = 1$ the zero z_0 is said to be *simple*, and if $n > 1$ we call z_0 a *multiple root (zero)* or an *n-fold root* or a *root of order n* or of *multiplicity n*. From (61) we see that *for z_0 to be an n-fold zero (root) of $f(z)$ it is necessary and sufficient that the conditions*

$$f^{(k)}(z_0) = 0; \quad k = 0, \ldots, n-1; \quad f^{(n)}(z_0) \neq 0$$

should hold.

Another consequence of Taylor's theorem is the following proposition known as ***Liouville's theorem***: *if $f(z)$ is an analytic function bounded throughout the whole complex plane of the variable z, then $f(z)$ is identically equal to a constant.*

Indeed, by virtue of (60), we have

$$|a_k| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(\delta e^{i\theta}) \frac{i\delta e^{i\theta}\, d\theta}{\delta^{k+1} e^{i(k+1)\theta}} \right| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f| \frac{d\theta}{\delta^k} < \frac{M}{\delta^k}$$

$$k = 0, 1, \ldots$$

where M is the supremum of $|f(z)|$ on the z-plane and δ is an arbitrary positive number. On passing to the limit in the last inequality for $\delta \to \infty$, we obtain $a_k = 0$ ($k = 1, 2, \ldots$), which means that $f(z) = a_0 = \text{const}$.

5°. **Laurent Series.** Let us consider the series

$$\sum_{k=-1}^{-\infty} a_k (z - z_0)^k, \quad z_0 \neq \infty \tag{62}$$

involving the negative powers of $(z - z_0)$: $(z - z_0)^k$ ($k = -1, -2, \ldots$). Each term of series (62) is an analytic function of the variable z for $0 < |z - z_0| < \infty$. The change of the variable $z - z_0 = 1/\zeta$ brings series (62) to the form of a power series:

$$\sum_{k=1}^{\infty} \alpha_{-k}\zeta^k \tag{63}$$

On putting $\zeta = 0$ for $z = \infty$, we readily see that if $|\zeta| < r_1$ is the circle of convergence of series (63), then series (62) is convergent outside the closed circle $|z - z_0| \leqslant r = 1/r_1$. Since for any $\rho > r$ series (62) is uniformly convergent outside the circle $|z - z_0| < \rho$, we conclude that, by virtue of Weierstrass' first theorem, the sum $S_1(z) = S_*\left(\frac{1}{z - z_0}\right)$ of this series (where $S_*(\zeta)$ is the sum of series (63)) is an analytic function at all points z satisfying the condition $|z - z_0| > r$.

If a series of form (62) is convergent to its sum $S_1(z)$ for $|z - z_0| > r$, and a power series $S_2(z) = \sum_{k=0}^{\infty} \alpha_k \times (z - z_0)^k$ is convergent in a circle $|z - z_0| < R$ where $R > r$, then the series $\sum_{k=-\infty}^{\infty} \alpha_k (z - z_0)^k$ is convergent in the annulus K: $r < |z - z_0| < R$, and its sum $S(z) = S_1(z) + S_2(z)$ is an analytic function in K.

The converse proposition known as the ***Laurent theorem*** also takes place: *if $f(z)$ is an analytic function in an annulus K: $r < |z - z_0| < R$, then at every point $z \in K$ this function can be represented in the form of the sum of a series*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{64}$$

where

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{k+1}} \quad (k = 0, \pm 1, \ldots) \tag{65}$$

and γ is an arbitrary circle $|\zeta - z_0| = \delta$ with a radius $\delta > 0$, $r < \delta < R$.

To prove this theorem, let us consider the annulus

$$K_1: r < r_1 < |z - z_0| < R_1 < R$$

(see Fig. 11). For any arbitrary point $z \in K_1$ we can write, using formula (48), the equality

$$f(z)=\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=R_1}\frac{f(\zeta)\,d\zeta}{\zeta-z}-\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=r_1}\frac{f(\zeta)\,d\zeta}{\zeta-z}=$$

$$=\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=R_1}\frac{1}{\zeta-z_0}\,\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}}\,f(\zeta)\,d\zeta+$$

$$+\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=r_1}\frac{1}{z-z_0}\,\frac{1}{1-\dfrac{\zeta-z_0}{z-z_0}}\,f(\zeta)\,d\zeta \quad (66)$$

Since the series

$$\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k=\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}} \quad (67)$$

and

$$\sum_{k=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^k=\frac{1}{1-\dfrac{\zeta-z_0}{z-z_0}} \quad (68)$$

are uniformly convergent when the point ζ lies on the circles $|\zeta-z_0|=R_1$ and $|\zeta-z_0|=r_1$ respectively, we can substitute (67) and (68) into the right-hand side of (66); this results in an equality of form (64) in which

$$a_k=\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=R_1}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{k+1}} \quad \text{for} \quad k=0,\ 1,\ \ldots \quad (69)$$

and

$$a_k=\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=r_1}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{k+1}} \quad \text{for} \quad k=-1,\ -2,\ \ldots \quad (70)$$

Since $f(\zeta)(\zeta-z_0)^{-k-1}$ is an analytic function of ζ in the annulus K, Cauchy's theorem implies that the integrals on the right-hand sides of (69) and (70) can be regarded as being taken over the contour $|\zeta-z_0|=\delta$, $r<\delta<R$.

The expression on the right-hand side of (64) is called the *Laurent series* (or the *Laurent expansion*) of the function

$f(z)$; the series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = f_1(z - z_0) \tag{71}$$

and

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = f_2\left(\frac{1}{z - z_0}\right) \tag{72}$$

are called the *regular part* and the *principal* (*singular*) *part* of series (64) respectively. A Taylor series is obviously a special case of a Laurent series.

We can easily prove *the uniqueness of Laurent expansion*

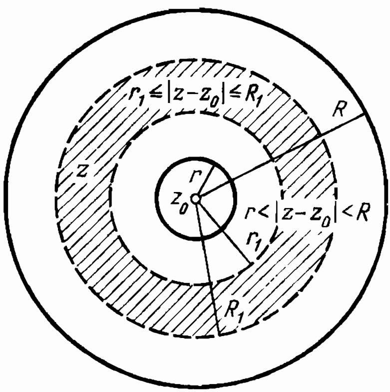

Fig. 11

(*series*) (64) *in the given annulus.* Indeed, if there are two such expansions, that is

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=-\infty}^{\infty} b_k (z - z_0)^k \tag{73}$$

we can multiply both sides of equality (73) by $(z - z_0)^{-n-1}$ and integrate the result over the circle γ, whence, by virtue of equalities (43) which can be written in the form

$$\int_{\gamma} \frac{dz}{(z - z_0)^k} = \begin{cases} 0 & \text{for } k \neq 1 \\ 2\pi i & \text{for } k = 1 \end{cases} \tag{74}$$

we obtain $a_n = b_n$ $(n = 0, \pm 1, \ldots)$. This proves the uniqueness of expansion (64).

6°. Singular Points and Residues of an Analytic Function. If a function $f(z)$ is analytic in a neighbourhood $|z - z_0| < \delta$ of a point z_0 in the complex plane of the variable z except the point z_0 itself (at that point the function may not be defined), z_0 is called an *isolated singular point* (or, simply, an *isolated singularity*) of the analytic function $f(z)$.

By virtue of Laurent's theorem, the function $f(z)$ can be expanded into Laurent series (64) in an annulus $0 < r < |z - z_0| < \delta$. Next, making r tend to zero, we see that Laurent series (64) obtained for $f(z)$ is convergent for all z satisfying the condition $0 < |z - z_0| < \delta$.

Depending on whether the collection of the nonzero coefficients among a_k $(k = -1, -2, \ldots)$ in Laurent expansion (64) is *void*, *finite* or *infinite*, the isolated singular point z_0 is called a *removable singular point*, a *pole* or an *essential singular point* respectively. The point z_0 is called a *pole of order* $n > 0$ if $a_{-n} \neq 0$ and $a_{-k} = 0$ for all $k > n$, if $n = 1$ the pole is said to be *simple*.

Laurent expansion (64) shows that if z_0 is a removable singular point of $f(z)$ then $\lim\limits_{z \to z_0} f(z) = a_0$, and if z_0 is a pole then $\lim\limits_{z \to z_0} f(z) = \infty$.

The definitions of a zero and of a pole imply that *if a point $z_0 \in D$ is a zero of multiplicity n (a pole of order n) of a function $f(z)$ analytic in the domain D, then this point is a pole of order n (a zero of multiplicity n) of the function* $1/f(z)$.

Indeed, there is a neighbourhood of the point z_0 in which the function $f(z)$ can be expanded into a series of the form

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k, \quad a_m \neq 0$$

where $m = n$ or $m = -n$ depending on whether the point z_0 is a zero or a pole of $f(z)$. Therefore

$$\frac{1}{f(z)} = (z - z_0)^{-m} \frac{1}{\varphi(z)} \quad \text{where} \quad \varphi(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$$

Since $\frac{1}{\varphi(z_0)} = \frac{1}{a_m} \neq 0$, the function $1/\varphi(z)$ is analytic in the vicinity of the point z_0, and therefore it can be represented in the form of the sum of a power series:

$$\frac{1}{\varphi(z)} = \sum_{k=0}^{\infty} b_k (z - z_0)^k, \quad b_0 = \frac{1}{a_m} \neq 0$$

Consequently,

$$\frac{1}{f(z)} = \sum_{k=0}^{\infty} b_k (z - z_0)^{k-m}$$

It follows that the point z_0 is a pole of order n or a zero of multiplicity n for the function $1/f(z)$ depending on whether $m = n$ or $m = -n$.

It can similarly be shown that *if z_0 is a removable singular point of an analytic function $f(z)$, then z_0 is a removable singular point of the function $1/f(z)$ in case* $\lim\limits_{z \to z_0} f(z) \neq 0$ *and a pole of $1/f(z)$ in case* $\lim\limits_{z \to z_0} f(z) = 0$.

If z_0 is an essential singular point of $f(z)$ and if $f(z) \neq 0$ in a neighbourhood of that point, then for the function $1/f(z)$ the point z_0 is an isolated singularity. Moreover, since z_0 can be neither a removable singular point nor a pole of $1/f(z)$ (because, if otherwise, z_0 would be a removable singular point or a zero of $f(z)$), the point z_0 is an essential singular point of $1/f(z)$.

The behaviour of $f(z)$ in the vicinity of its essential singularity is characterized by the following ***Sokhotsky-Weierstrass theorem***: *if z_0 is an essential singular point of the function $f(z)$ then for any complex number α there exists a sequence of points z_k ($k = 1, 2, \ldots$)* convergent to z_0 such that $\lim\limits_{z_k \to z_0} f(z_k) = \alpha$.

We shall begin with the case when $\alpha = \infty$. Let us represent the function $f(z)$ in the vicinity of the point z_0 in the form

$$f(z) = f_1(z - z_0) + f_2\left(\frac{1}{z - z_0}\right)$$

where f_1 and f_2 are expressed by formulas (71) and (72) respectively. Since the series on the left-hand side of (72) is convergent for $|z-z_0|>0$, the function $f_2(\zeta)=$ $=f_2\left(\frac{1}{z-z_0}\right)$, $\zeta=\frac{1}{z-z_0}$ (which is the sum of the series $\sum_{k=1}^{\infty} a_{-k}\zeta^k$ convergent for all the points of the complex $\zeta=$ plane) cannot be bounded. Indeed, if otherwise, Liouville's theorem would imply that $f_2(\zeta)$ is identically equal to a constant, that is there would be no principal part in the Laurent expansion, which is impossible because z_0 is an essential singular point of $f(z)$. Thus, the function $f_2(\zeta)=f_2\left(\frac{1}{z-z_0}\right)$ cannot be bounded in the vicinity of the point z_0 (which corresponds to $\zeta=\infty$), and therefore there exists a sequence ζ_k $(k=1, 2, \ldots)$ *divergent to* ∞ *such that* $\lim\limits_{\zeta_k\to\infty} f_2(\zeta_k)=\infty$. Consequently, the sequence $z_k=$ $=z_0+\frac{1}{\zeta_k}$ $(k=1, 2, \ldots)$ which converges to z_0 is such that $\lim\limits_{z_k\to z_0} f_2\left(\frac{1}{z_k-z_0}\right)=\infty$. Since $\lim\limits_{z_k\to z_0} f_1(z_k-z_0)=$ $=a_0$, we conclude that $\lim\limits_{z_k\to z_0} f(z_k)=\infty$.

Now we shall consider the case when α is a finite number. Let us take the function $f(z)-\alpha$ for which z_0 is obviously an essential singular point. If in every neighbourhood $|z-z_0|<1/k$ of the point z_0 there is a point z_k at which $f(z_k)=\alpha$ we shall have $\lim\limits_{z_k\to z_0} f(z_k)=\alpha$. In case the point z_0 has a neighbourhood in which $f(z)\neq\alpha$, then, as was already mentioned, z_0 will be an essential singular point for the function $1/[f(z)-\alpha]$ as well. Consequently, in the latter case there exists a sequence of points z_k $(k=1, 2, \ldots)$ convergent to z_0 such that

$$\lim_{z_k\to z_0}\frac{1}{f(z_k)-\alpha}=\infty$$

and consequently, $\lim\limits_{z_k\to z_0} f(z_k)=\alpha$.

We say that $z=\infty$ is an *isolated singular point at infinity* of an analytic function $f(z)$ if $\zeta=0$ is an isolated singular point for the function $\varphi(\zeta)=f(1/\zeta)$. Since the Laurent expansions of the functions $\varphi(\zeta)$ and $f(z)$ $(z=1/\zeta)$ in the neighbourhoods of the points $\zeta = 0$ and $z = \infty$ are connected by the relationship

$$\varphi(\zeta) = \sum_{k=-\infty}^{\infty} a_k \zeta^k = f(z) = \sum_{k=-\infty}^{\infty} a_k z^{-k}$$

it is possible to classify isolated singular points at infinity depending on the character of the collection of the nonzero coefficients among a_k ($k = -1, -2, \ldots$) in the Laurent expansion $f(z) = \sum_{k=-\infty}^{\infty} a_k z^{-k}$. Namely, depending on whether the set of these coefficients is *void*, *finite* or *infinite*, the isolated singular point at infinity $z = \infty$ is called a *removable singular point*, *a pole* or an *essential singular point.*

A function $f(z)$ analytic throughout the whole complex plane of the variable z is called an *entire* function. Depending on whether the point at infinity $z = \infty$ is a removable singular point, a pole or an essential singular point of an entire function $f(z)$, this function is identically equal to a *constant*, is a *polynomial* or, as we say, is an *entire transcendental function*. A function $f(z)$ having only poles in the extended complex plane of the variable z is called a *rational function.* The ratio $f(z)/\varphi(z)$ of two entire functions $f(z)$ and $\varphi(z)$ is called a *meromorphic function*. An example of a meromorphic function is the function $\sin z/\cos z = \tan z$.

Let a function $f(z)$ be analytic in a domain D everywhere except an isolated singular point $z_0 \in D$, and let γ be a piecewise smooth closed Jordan curve which lies inside D together with the domain D_γ bounded by it, the point z_0 belonging to D_γ. By Cauchy's theorem, the value of the integral

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

taken over γ in the positive direction (for which the finite domain D_γ always remains on the left) is one and the same for all γ; this value is called the *residue of the function $f(z)$ at the singular point z_0*. The residue is denoted as

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \operatorname*{Res}_{z=z_0} f(z) \tag{75}$$

When the residue is computed we can obviously take as γ a circle $|z - z_0| = \delta$ of a sufficiently small radius (see Fig. 12).

On substituting Laurent expansion (64) into the left-hand side of (75) and using equality (74), we obtain

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1}$$

In the case when z_0 is a pole of order n the residue a_{-1} can be found with the aid of the obvious formula

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)] \tag{76}$$

Let a function $f(z)$ be continuous in $D \cup S$ and analytic in D everywhere except isolated singular points $z_k \in D$

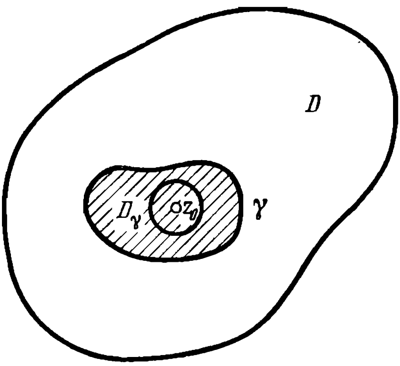

Fig. 12

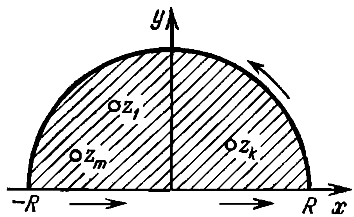

Fig. 13

($k = 1, \ldots, m$), and let the boundary S of the (finite) domain D be a piecewise smooth closed Jordan curve. Then formula (48) implies

$$\frac{1}{2\pi i} \int_S f(z)\, dz = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{|z - z_k| = \delta} f(z)\, dz = \sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z) \tag{77}$$

Formula (77) makes it possible to easily compute some definite integrals.

For instance, if it is known that a function $f(z)$ is continuous for $\operatorname{Im} z \geqslant 0$ and is analytic for $\operatorname{Im} z > 0$ everywhere except a finite number of isolated singular points $z_k > 0$, $\operatorname{Im} z_k > 0$ $(k = 1, \ldots, m)$, and if

$$|f(z)| < \frac{M}{|z|^2}, \quad M = \text{const} > 0 \tag{78}$$

for sufficiently large $|z|$, we can apply formula (77) to the domain D of the form of a semi-circle $|z| < R$, $\operatorname{Im} z > 0$ containing all the points z_k $(k = 1, \ldots, m)$; then (see Fig. 13) we obtain

$$\int_{-R}^{R} f(x)\,dx + \int_{0}^{\pi} f(Re^{i\theta})\, i\,Re^{i\theta}\,d\theta = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z) \tag{79}$$

When $R \to \infty$ the second integral on the left-hand side of (79) tends to zero (this follows from (78)). Therefore we derive from (79) the formula

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{m} \operatorname*{Res}_{z=z_k} f(z)$$

7°. Schwarz's Formula. Solution of Dirichlet Problem. Let us consider the following problem: it is required to find a function $f(z)$ analytic in the circle $|z| < 1$ whose real part $u(x, y)$ is continuous for $|z| \leqslant 1$ and assumes the limiting values $u^+(\zeta)$, described by a (known) continuous function $\varphi(\zeta)$, as the variable point z tends to the circumference $|\zeta| = 1$ of the circle $|z| < 1$ from its interior:

$$u^+(\zeta) = \varphi(\zeta), \quad |\zeta| = 1 \tag{80}$$

For the points belonging to the circumference $|\zeta| = R$ of the circle $|z| < R$ of radius $R < 1$ we have $f(\zeta) + \overline{f(\zeta)} = 2u(\zeta)$. On multiplying both members of this equality by $1/2\pi i(\zeta - z)$ $(|z| < R)$ and integrating over the contour $|\zeta| = R$, we obtain, by virtue of Cauchy's integral formula (47), the relation

$$f(z) + \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\overline{f(\zeta)}\,d\zeta}{\zeta - z} = \frac{1}{\pi i} \int_{|\zeta|=R} \frac{u(\zeta)\,d\zeta}{\zeta - z}, \quad |z| < R \tag{81}$$

Let us consider Taylor's expansion

$$f(z)=\sum_{k=0}^{\infty} a_k z^k$$

in the circle $|z|<1$; on denoting $\zeta=Re^{i\varphi}$ for $|\zeta|=R$ $\left(\text{then } \bar{\zeta}=Re^{-i\varphi}=\frac{R^2}{\zeta}\right)$, we can write $\overline{f(\zeta)}=\sum_{k=0}^{\infty}\bar{a}_k\bar{\zeta}^k=$ $=\sum_{k=0}^{\infty}\bar{a}_k\frac{R^{2k}}{\zeta^k}$. Therefore the second summand on the left-hand side of (81) can be written in the form

$$\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\overline{f(\zeta)}\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\sum_{k=0}^{\infty}\bar{a}_k R^{2k}\int_{|\zeta|=R}\frac{d\zeta}{\zeta^k(\zeta-z)},\quad |z|<R \tag{82}$$

From formula (74) we obtain

$$\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{d\zeta}{\zeta-z}=1,\quad |z|<R \tag{83}$$

To compute the integral on the right-hand side of (82) we apply formula (77) with $k>0$:

$$\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{d\zeta}{\zeta^k(\zeta-z)}=\operatorname*{Res}_{\zeta=0}\frac{1}{\zeta^k(\zeta-z)}+\operatorname*{Res}_{\zeta=z}\frac{1}{\zeta^k(\zeta-z)} \tag{84}$$

The residues on the right-hand side of (84) are found using formula (76):

$$\operatorname*{Res}_{\zeta=0}\frac{1}{\zeta^k(\zeta-z)}=\frac{1}{(k-1)!}\lim_{\zeta\to 0}\frac{d^{k-1}}{d\zeta^{k-1}}\left(\frac{1}{\zeta-z}\right)=-\frac{1}{z^k}$$

and

$$\operatorname*{Res}_{\zeta=z}\frac{1}{\zeta^k(\zeta-z)}=\lim_{\zeta\to z}\frac{1}{\zeta^k}=\frac{1}{z^k}$$

Consequently

$$\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{d\zeta}{\zeta^k(\zeta-z)}=-\frac{1}{z^k}+\frac{1}{z^k}=0 \tag{85}$$

$$|z|<R;\quad k=1,\ 2,\ \ldots$$

Using formulas (83) and (85) we bring (81) to the form

$$f(z)=\frac{1}{\pi i}\int\limits_{|\zeta|=R}\frac{u(\zeta)\,d\zeta}{\zeta-z}-\overline{a}_0 \tag{86}$$

where $\overline{a}_0=\overline{f(0)}=u(0,\ 0)-iv(0,\ 0)$. Further, taking into account condition (80), we pass to the limit for $R\to 1$ in (86) and thus find

$$f(z)=\frac{1}{\pi i}\int\limits_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta-z}-u(0,\ 0)+iv(0,\ 0) \tag{87}$$

For $z=0$ we obtain from (87) the equality

$$f(0)+\overline{f}(0)=2u(0,\ 0)=\frac{1}{\pi i}\int\limits_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta}$$

whence

$$u(0,\ 0)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\varphi(\zeta)\,d\zeta}{\zeta} \tag{88}$$

On substituting the value of $u(0,\ 0)$ expressed by (88) into (87) we arrive at *Schwarz's formula*:

$$f(z)=\frac{1}{2\pi}\int\limits_{|\zeta|=1}\frac{\zeta+z}{\zeta-z}\varphi(\zeta)\,d\theta+iC \tag{89}$$

$$\zeta=e^{i\theta},\quad C=v(0,\ 0)$$

This formula makes it possible to reconstruct a function $f(z)$ analytic in the circle $|z|<1$ from the boundary values of its real part on the circumference $|\zeta|=1$ of that circle to within an arbitrary pure imaginary constant.

Since for the boundary $|\zeta|=1$ of the circle $|z|<1$ we have

$$\frac{\zeta+z}{\zeta-z}=\frac{1-|z|^2+2i\,\mathrm{Im}\,\overline{\zeta}z}{|\zeta-z|^2}$$

Schwarz's formula (89) implies Poisson's formula

$$u(z)=u(x,\ y)=\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\varphi(e^{i\theta})\,d\theta \tag{90}$$

which expresses the solution of the Dirichlet problem for harmonic functions in the circle $|z| < 1$ (cf. formula (17) in Sec. 2°, § 2 of Chapter 1).

Using Riemann's theorem, the theorem on the correspondence of boundaries and formula (90) *we can prove the existence of the solution of the Dirichlet problem stated in the following general form: for a domain D in the complex plane of the variable $\zeta = \xi + i\eta$ bounded by a closed Jordan curve S it is required to find the harmonic function $u_*(\zeta) = u_*(\xi, \eta)$ which is continuous in $D \cup S$ and assumes on S given values described by a continuous function $g(\zeta)$.*

Indeed, the sought-for harmonic function $u_*(\xi, \eta)$ must be the real part of a function $F(\zeta)$ analytic in the domain D. Further, there is a function $z = f(\zeta)$ specifying a conformal mapping of the domain D onto the circle $|z| < 1$; the function $u(z) = u(x, y) = \text{Re}\, F[f^{-1}(z)]$ is harmonic in that circle and is continuous in the closed circle $|z| \leqslant 1$, on the boundary $|z| = 1$ of that circle it assumes the values described by the continuous function $\varphi(z) = g[f^{-1}(z)]$. The function $u(x, y) = u(z)$ is found with the aid of formula (90); finally, the sought-for harmonic function is expressed in terms of $u(x, y)$ in the form $u_*(\zeta) = u[f(\zeta)]$.

§ 4. Analytic Continuation

1°. The Notion of Analytic Continuation. Let D_1 and D_2 be two domains in the plane of the complex variable z, and let their intersection $d = D_1 \cap D_2$ be also a domain in the z-plane. Let us consider a function $f_1(z)$ analytic in the domain D_1. If there exists an analytic function $f_2(z)$ in D_2 coinciding with $f_1(z)$ in d, we say that *the function $f_2(z)$ is the analytic continuation of the function $f_1(z)$ from the domain D_1 to the domain D_2 across the common part d of these domains.* The uniqueness theorem for analytic functions obviously implies that if the analytic continuation exists then it must be *unique* (Fig. 14).

2°. The Continuity Principle. Let us suppose that two simply connected domains D_1 and D_2 are such that their boundaries have a common part which is a smooth Jordan curve γ, the intersection $D_1 \cap D_2$ being void.

By the *continuity principle for analytic functions* is meant the following proposition: *if $f_1(z)$ and $f_2(z)$ are analytic functions, in the domains D_1 and D_2 respectively, which are*

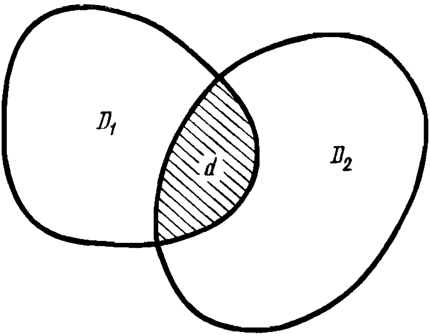

Fig. 14

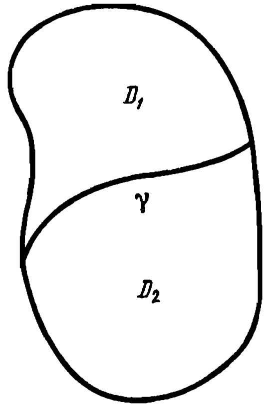

Fig. 15

continuous including the arc γ, and if it is additionally known that

$$f_1(z) = f_2(z) \quad \text{for} \quad z \in \gamma$$

then the function

$$f(z) = \begin{cases} f_1(z) & \text{for} \quad z \in D_1 \\ f_2(z) & \text{for} \quad z \in D_2 \\ f_1(z) = f_2(z) & \text{for} \quad z \in \gamma \end{cases}$$

is analytic in the domain $D = D_1 \cup D_2 \cup \gamma$ (in the case when the curve γ is non-closed we suppose that its end points are excluded from it; see Fig. 15).

The continuity principle will follow from Morera's theorem if we manage to show that the integral of $f(z)$ over any closed piecewise smooth Jordan curve S lying in D is equal to zero.

In the case when S lies in $D_1 \cup \gamma$ or in $D_2 \cup \gamma$ the equality

$$\int_S f(z)\,dz = 0$$

follows from Cauchy's theorem. Now let us consider the case when S is the boundary of a domain D_S whose intersections with both D_1 and D_2 are not void. According to Cauchy's theorem, the integrals of $f(z)$ taken over the contours $D_S \cap D_1$ and $D_S \cap D_2$ are equal to zero; in the sum of these integrals the parts of the arc γ contained in the contours $D_S \cap D_1$ and $D_S \cap D_2$ are described by the point z (z is the variable of integration) twice in opposite directions, and therefore in the case under consideration the integral we are interested in is also equal to zero.

3°. The Riemann-Schwarz Symmetry Principle. Let a part γ of the boundary of a simply connected domain D lying in the upper or in the lower half-plane be a segment of the real axis $\operatorname{Im} z = 0$, and let $f(z) = u(x, y) + iv(x, y)$ be

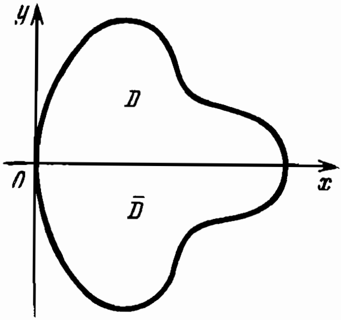

Fig. 16

an analytic function in D continuous including the line segment γ; further, let the imaginary part $v(x, 0)$ of this function be equal to zero on γ: $v(x, 0) = 0$ for $x \in \gamma$. Riemann and Schwarz proved the following proposition known as the ***Riemann-Schwarz symmetry principle*** (also called the ***reflection principle***): *let $\overline{D}$ be the domain symmetric to D about γ; then under the conditions stated above, the function $f(z)$ can be continued analytically from D to $\overline{D}$, and if $z \in \overline{D}$ then $f(z) = \overline{f}(\bar{z})$* (see Fig. 16).

To prove this principle we shall use the fact that in a neighbourhood of any point $z_0 \in D$ we have $f(z) =$

$= \sum_{k=0}^{\infty} a_k (z - z_0)^k$. It follows that the series $\sum_{k=0}^{\infty} \bar{a}_k (\bar{z} - \bar{z}_0)^k = \overline{\overline{f} (\bar{z})}$ is also convergent. By $f(\bar{z}) = \overline{f(\bar{z})}$ we shall mean the analytic function in the domain $\bar{D}$ equal to the sum of the power series $\sum_{k=0}^{\infty} \bar{a}_k (z - \bar{z}_0)^k$ $(z_0 \in D)$. Since $\operatorname{Im} f(z) = 0$ for $\operatorname{Im} z = 0$, we have $\bar{f}(x) = f(x)$ when $x \in \gamma$.

The function

$$F(z) = \begin{cases} f(z) & \text{for} \quad z \in D \\ f(x) = \bar{f}(x) & \text{for} \quad x \in \gamma \\ \bar{f}(z) & \text{for} \quad z \in \bar{D} \end{cases}$$

is analytic in the domain $D \cup \bar{D} \cup \gamma$, which follows from the continuity principle; hence, the Riemann-Schwarz symmetry principle has been proved.

Now let us consider a domain D a part γ_0 of whose boundary is an arc of a circle C; let D_* be a domain lying outside D, adjoining γ_0 and symmetric to D with respect to C. As is known, there exists a linear-fractional function specifying a conformal mapping under which the image of γ_0 is a segment γ of the real axis. Therefore the Riemann-Schwarz symmetry principle can also be stated thus: *if a function $f(z)$ is analytic in the domain D and continuous including the arc γ_0, and if* $\operatorname{Im} f(z) = 0$ *on γ_0, then the function $f(z)$ can be continued analytically from the domain D to the domain D_* across γ_0, and for $z \in D_*$ the equality*

$$f(z) = \overline{f(z_*)}$$

holds where z_* is the point symmetric to z about C.

§ 5. Formulas for Limiting Values of Cauchy-Type Integral and Their Applications

1°. Cauchy's Principal Value of a Singular Integral. Let S be a closed piecewise smooth Jordan curve, and let $f(t)$ be a continuous function defined on S. In Sec. 4°, § 2 of the present chapter we showed that Cauchy-type integral (49) of the function $f(t)$ along S is an analytic function at every point z not lying on S. For $z \in S$ the Cauchy-type integral obviously does not exist in the sense of the ordinary definition; however, under some additional assumptions concerning the function $f(t)$ and the curve S, this definition can be generalized in a proper manner.

Below we shall suppose that the curvature of the curve S is continuous. Let $t_0 \in S$ and let γ be the circumference of the circle $|t - t_0| < \varepsilon$ of a sufficiently small radius $\varepsilon > 0$; by S_ε we shall denote the part of S lying outside the closed circle $|t - t_0| \leqslant \varepsilon$.

It is obvious that the integral

$$I_\varepsilon = \int\limits_{S_\varepsilon} \frac{f(t)\,dt}{t - t_0}$$

exists in the sense of the ordinary definition.

If the limit

$$\lim_{\varepsilon \to 0} I_\varepsilon(t_0) = I(t_0)$$

exists, it is called *Cauchy's principal value of the singular integral*

$$\int\limits_{S} \frac{f(t)\,dt}{t - t_0}$$

and is denoted

$$I(t_0) = \text{v.p.} \int\limits_{S} \frac{f(t)\,dt}{t - t_0} \tag{91}$$

where v.p. is the abbreviation of the French *valeur principal* principal value.

Let us show that *if $f(t)$ satisfies Hölder's condition, that is if there exist constants $A > 0$ and $0 < h \leqslant 1$ such that*

$$|f(t_1) - f(t_2)| \leqslant A\,|t_1 - t_2|^h \tag{92}$$

for any $t_1, t_2 \in S$, then integral (91) *exists in the sense of Cauchy's principal value.*

Indeed, on the basis of Cauchy's integral formula (46), we can rewrite the expression of I_ε in the form

$$I_\varepsilon(t_0) = \int\limits_{S_\varepsilon} \frac{f(t) - f(t_0)}{t - t_0}\,dt + 2\pi i f(t_0) - f(t_0) \int\limits_{\gamma_1} \frac{dt}{t - t_0}$$

where γ_1 is the part of the contour γ lying outside the finite domain D with boundary S.

By condition (92), we conclude that the improper integral

$$\int_S \frac{f(t)-f(t_0)}{t-t_0}\,dt = \lim_{\varepsilon\to 0}\int_{S_\varepsilon}\frac{f(t)-f(t_0)}{t-t_0}\,dt$$

is uniformly convergent and is a continuous function of t_0.

On the other hand, we have

$$\lim_{\varepsilon\to 0}\int_{\gamma_1}\frac{dt}{t-t_0} = \lim_{\varepsilon\to 0} i\int_{\gamma_1} d\varphi = \pi i,\quad t-t_0 = \varepsilon e^{i\varphi}$$

Consequently, on passing to the limit for $\varepsilon \to 0$ in the expression of $I_\varepsilon(t_0)$ obtained above, we arrive at the relation

$$\lim_{\varepsilon\to 0} I_\varepsilon(t_0) = \int_S \frac{f(t)\,dt}{t-t_0} = \pi i f(t_0) + \int_S \frac{f(t)-f(t_0)}{t-t_0}\,dt \qquad (93)$$

In the case when the curve S is non-closed, under the assumption that $t_0 \in S$ is not an end point of S, Cauchy's principal value of the singular integral

$$\int_S \frac{f(t)\,dt}{t-t_0}$$

can also be defined as the limit of the expression

$$I_\varepsilon(t_0) = \int_{S_\varepsilon}\frac{f(t)-f(t_0)}{t-t_0}\,dt + f(t_0)\int_{S_\varepsilon}\frac{dt}{t-t_0}$$

for $\varepsilon \to 0$, and this limit is sure to exist when the function $f(t)$ satisfies Hölder's condition.

2°. Tangential Derivative of a Single-Layer Potential. In Chapter 1, § 4, Sec. 3°, when studying single-layer potential (59), we used representation (62) in which the function $v(x)$ satisfied conditions (60). Since the integral terms on the right-hand side of (62) are continuously differentiable functions when the point x passes from D^+ to D^- through an arbitrary point $x^0 \in S$, formulas (52) and (60) of Chapter 1 imply that *the tangential derivative of a single-layer potential exists and is continuous when the point x passes from D^+ to D^-.*

Let us write the expression of a single-layer potential in the form

$$u(z) = -\frac{1}{2\pi}\int_S \ln|t-z|\,\mu(t)\,ds, \quad z = x+iy$$

to show that

$$\frac{du(t_0)}{ds_0} = \frac{t_0'}{2\pi}\int_S \frac{\mu(t)\,\overline{t}'\,dt}{t-t_0} + \frac{i}{2\pi}\int_S \frac{d}{ds_0}\theta(t, t_0)\,ds \qquad (94)$$

for $z = t_0 \in S$ where s and s_0 are the curvilinear coordinates (reckoned as the arc lengths along S) of the points t and t_0, $t_0' = \frac{dt_0}{ds_0}$, $\overline{t}' = \frac{1}{\frac{dt}{ds}}$ and $\theta(t, t_0) = \arg(t-t_0)$. The integral in the first summand on the right-hand side of (94) is understood in the sense of the definition of Cauchy's principal value while the integral in the second summand is understood in the ordinary sense because the derivative $\frac{d}{ds_0}\theta(t, t_0)$ is a continuous function (see Chapter 1, § 4, Sec. 1°).

Let us denote by t_1 and t_2 two points on S whose curvilinear coordinates along S are $s_0-\varepsilon$ and $s_0+\varepsilon$ respectively where $\varepsilon > 0$ is a sufficiently small number; the part of the curve S lying outside the arc $\widehat{t_1t_0t_2}$ will be denoted S_ε.

It is evident that for

$$u_\varepsilon(t_0) = -\frac{1}{2\pi}\int_{S_\varepsilon} \ln|t-t_0|\,\mu(t)\,ds \qquad (95)$$

we have $\lim\limits_{\varepsilon\to 0} u_\varepsilon(t_0) = u(t_0)$ where the passage to the limit is uniform with respect to t_0. Differentiating both members of (95) with respect to s_0, we obtain

$$\frac{du_\varepsilon(t_0)}{ds_0} = -\frac{1}{2\pi}\int_{S_\varepsilon} \frac{d}{ds_0}\ln|t-t_0|\,\mu(t)\,ds +$$
$$+\frac{1}{2\pi}\mu(t_2)\ln|t_2-t_0| - \frac{1}{2\pi}\mu(t_1)\ln|t_1-t_0| \qquad (96)$$

Since

$$\mu(t_2)\ln|t_2-t_0| - \mu(t_1)\ln|t_1-t_0| =$$
$$= [\mu(t_2)-\mu(t_1)]\ln|t_2-t_0| - \mu(t_1)\ln\frac{|t_1-t_0|}{|t_2-t_0|}$$

and

$$\lim_{\varepsilon \to 0} \frac{|t_1 - t_0|}{|t_2 - t_0|} = 1$$

and since the function μ possesses a continuous derivative of the second order, we have

$$\lim_{\varepsilon \to 0} [\mu(t_2) \ln |t_2 - t_0| - \mu(t_1) \ln |t_1 - t_0|] = 0$$

On the other hand, the equality

$$|t - t_0| = (t - t_0) e^{-i\theta(t, t_0)}$$

implies that

$$\frac{d}{ds_0} \ln |t - t_0| = -\frac{t_0'}{t - t_0} - i \frac{d}{ds_0} \theta(t, t_0)$$

for $t \in S_\varepsilon$, and hence

$$-\frac{1}{2\pi} \int_{S_\varepsilon} \frac{d}{ds_0} \ln |t - t_0| \mu(t)\, ds =$$

$$= \frac{t_0'}{2\pi} \int_{S_\varepsilon} \frac{\bar{t}' \mu(t)\, dt}{t - t_0} + \frac{i}{2\pi} \int_{S_\varepsilon} \frac{d}{ds_0} \theta(t, t_0)\, ds$$

Taking into account the fact that the function $\bar{t}'\mu(t)$ is sure to satisfy Hölder's condition (92) we obtain, by virtue of the definition of Cauchy's principal value of a singular integral, the relation

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{\bar{t}' \mu(t)\, dt}{t - t_0} = \int_S \frac{\bar{t}' \mu(t)\, dt}{t - t_0}$$

Therefore, passing to the limit in equality (96) for $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} \frac{du_\varepsilon(t_0)}{ds_0} = \frac{t_0'}{2\pi} \int_S \frac{\bar{t}' \mu(t)\, dt}{t - t_0} + \frac{i}{2\pi} \int_S \frac{d}{ds_0} \theta(t, t_0)\, ds$$

and the passage to the limit in the last relation is *uniform with respect to* t_0, whence, on the basis of the well-known theorem proved in mathematical analysis, we conclude that *the tangential derivative of a single-layer potential can be represented in integral form* (94).

3°. Limiting Values of Cauchy-Type Integral. We shall suppose that the Cauchy-type integral

$$F(z) = \frac{1}{2\pi i} \int_S \frac{f(t)\, dt}{t - z} \tag{97}$$

is taken over a contour S which is a closed Jordan curve possessing continuous curvature, and that the function $f(t)$ is one-valued and twice continuously differentiable.

Under these assumptions expression (97) can be written in the form

$$F(z) = u(z) + v(z) \tag{98}$$

where

$$u(z) = \frac{1}{2\pi} \int_S f(t) \frac{\partial}{\partial \nu_t} \ln |t - z|\, ds, \quad t = t(s) \tag{99}$$

is the double-layer potential with the "dipole moment per unit length" $f(t)$ and

$$v(z) = \frac{1}{2\pi i} \int_S f(t) \frac{\partial}{\partial s} \ln |t - z|\, ds = -\frac{1}{2\pi i} \int_S \ln |t - z|\, f'_s\, ds \tag{100}$$

is the single-layer potential with density $\frac{1}{i} f'(s)$.

According to what was shown in Secs 2° and 3°, § 4 of Chapter 1, the function $v(z)$ is continuous throughout the whole complex plane of the variable z, and the limiting values of $u(z)$ for z tending to $t_0 \in S$ from D^+ and D^- are expressed by the equalities

$$u^+(t_0) = \frac{1}{2\pi} \int_S f(t) \frac{\partial}{\partial \nu_t} \ln |t - t_0|\, ds + \frac{1}{2} f(t_0) \tag{101}$$

and

$$u^-(t_0) = \frac{1}{2\pi} \int_S f(t) \frac{\partial}{\partial \nu_t} \ln |t - t_0|\, ds - \frac{1}{2} f(t_0) \tag{102}$$

respectively (see formulas (54) and (55) in Chapter 1). Besides, from (100) we obtain

$$v(t_0) = -\frac{1}{2\pi i}\int_S \ln|t-t_0|\, f'(s)\, ds =$$

$$= -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \ln|t-t_0|\, f'(s)\, ds =$$

$$= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left\{\int_{S_\varepsilon} \frac{\partial}{\partial s}\ln|t-t_0|\, f(t)\, ds + \right.$$

$$\left. + f(t_2)\ln|t_2-t_0| - f(t_1)\ln|t_1-t_0|\right\}$$

where S_ε is the part of S considered in the foregoing section.

Since

$$\frac{\partial}{\partial s}\ln|t-t_0| = \frac{\partial}{\partial s}\ln(t-t_0) - i\frac{\partial}{\partial s}\theta(t, t_0) =$$

$$= \frac{t'}{t-t_0} - i\frac{\partial}{\partial s}\theta(t, t_0)$$

and

$$\lim_{\varepsilon\to 0}[f(t_2)\ln|t_2-t_0| - f(t_1)\ln|t_1-t_0|] = 0$$

we obtain for $v(t_0)$ the expression

$$v(t_0) = \frac{1}{2\pi i}\int_S \frac{f(t)\, dt}{t-t_0} - \frac{1}{2\pi}\int_S \frac{\partial}{\partial s}\theta(t, t_0)\, f(t)\, ds =$$

$$= \frac{1}{2\pi i}\int_S \frac{\partial}{\partial s}\ln|t-t_0|\, f(t)\, ds \quad (103)$$

From (101), (102) and (103) we conclude that *there exist limiting values $F^+(t_0)$ and $F^-(t_0)$ of Cauchy-type integral (97) when z tends to $t_0 \in S$ from D^+ and D^- respectively which are expressed by the formulas*

$$F^+(t_0) = \frac{1}{2} f(t_0) + \frac{1}{2\pi}\int_S f(t)\frac{\partial}{\partial \nu_t}\ln|t-t_0|\, ds +$$

$$+ \frac{1}{2\pi i}\int_S \frac{\partial}{\partial s}\ln|t-t_0|\, f(t)\, ds$$

and

$$F^-(t_0) = -\frac{1}{2} f(t_0) + \frac{1}{2\pi} \int_S f(t) \frac{\partial}{\partial \nu_t} \ln|t - t_0|\, ds + $$
$$+ \frac{1}{2\pi i} \int_S \frac{\partial}{\partial s} \ln|t - t_0|\, f(t)\, ds$$

From these formulas, using the obvious equality

$$\frac{\partial}{\partial s} \ln|t - t_0| + i \frac{\partial}{\partial \nu_t} \ln|t - t_0| = \frac{t'}{t - t_0}$$

we obtain

$$F^+(t_0) = \frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int_S \frac{f(t)\, dt}{t - t_0} \tag{104}$$

and

$$F^-(t_0) = -\frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int_S \frac{f(t)\, dt}{t - t_0} \tag{105}$$

From (104) and (105) immediately follow the equalities

$$F^+(t_0) - F^-(t_0) = f(t_0) \tag{106}$$

and

$$F^+(t_0) + F^-(t_0) = \frac{1}{\pi i} \int_S \frac{f(t)\, dt}{t - t_0} \tag{107}$$

known as the *Sokhotsky-Plemelj formulas.*

4°. The Notion of a Piecewise Analytic Function. The conclusions drawn in Secs 1°, 2° and 3° of § 5 also remain valid for some more general conditions on the curve S and on the functions $f(t)$ and $\mu(t)$ defined for $t \in S$. In particular, formulas (104), (105), (106) and (107) remain valid when the function $f(t)$ satisfies Hölder's condition and S is a *Lyapunov curve*, *that is a curve for which the function* $\theta(t)$ *describing the dependence on* t *of the angle between the tangent line to* S *at the point* t *and some constant direction* (for instance, the direction of the real axis in the complex plane of the variable z) *also satisfies Hölder's condition.*

From representation (93) of the singular integral understood in the sense of Cauchy's principal value and from formulas (104) and (105) it obviously follows that $F^+(t_0)$ *and* $F^-(t_0)$ *are continuous functions.* Moreover, *if* $f(t)$ *satisfies Hölder's condition of order* h, $0 < h < 1$, *then so are the limiting values* $F^+(t_0)$ *and* $F^-(t_0)$. Below we shall make use of this assertion (but its proof will not be presented).

A function $\Phi(z)$ defined in a domain D is said to be *everywhere continuously extendible to the boundary S* of D if the limit

$$\lim_{z \to t} \Phi(z), \quad z \in D$$

exists for every $t \in S$.

The argument given above implies that, *under the conditions imposed on S and f, the function represented by the Cauchy-type integral is continuously extendible to S both from* D^+ *and from* D^-.

A function $\Phi(z)$ which is analytic both in D^+ and in D^- and which can be extended continuously everywhere to the boundary S of these domains will be referred to as a *piecewise analytic function* on the plane of the complex variable z. The conclusions drawn in the foregoing section imply that *if the function* $f(t)$ *in Cauchy-type integral* (97) *satisfies Hölder's condition, then the function* $F(z)$ *represented by formula* (97) *is piecewise analytic.*

The difference $\Phi^+(t) - \Phi^-(t) = g(t)$ will be referred to as the *jump* of the piecewise analytic function $\Phi(z)$.

In the case when $\Phi^+(t)$ and $\Phi^-(t)$ are continuous functions and $g(t) = 0$ everywhere on S, the continuity principle proved in Sec. 2°, § 4 implies that $\Phi(z)$ *is an analytic function throughout the whole complex plane of the variable z, and hence, by virtue of Sec.* 6°, § 3 *of the present chapter, the function* $\Phi(z)$ *is either identically equal to a constant (in particular, to zero) or is a polynomial or an entire transcendental function.*

5°. Application to Boundary-Value Problems. In applications the following *boundary-value problem* is frequently encountered: *it is required to find a piecewise analytic function* $\Phi(z)$ *satisfying the boundary condition*

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in S \tag{108}$$

where $g(t)$ *is a given function satisfying Hölder's condition.*

In the case when S is a closed Lyapunov curve, formula (106) implies that one of the solutions of problem (108) is expressed by the Cauchy-type integral

$$\Phi_1(z) = \frac{1}{2\pi i}\int_S \frac{g(t)\,dt}{t-z}$$

Let $\Phi(z)$ denote the general solution of this problem; then, by (108), for the difference $\Phi(z) - \Phi_1(z) = \Omega(z)$ the relation

$$\Omega^+(t) - \Omega^-(t) = 0, \quad t \in S$$

holds, and consequently $\Omega(z) = P(z)$, that is

$$\Phi(z) = \frac{1}{2\pi i}\int_S \frac{g(t)\,dt}{t-z} + P(z) \tag{109}$$

where $P(z)$ is an arbitrary entire function.

If we require additionally that the solution $\Phi(z)$ of problem (108) should have a pole of order n, $n \geqslant 1$ at infinity, or should be bounded, then in formula (109) we must write, instead of an arbitrary entire function $P(z)$, an arbitrary polynomial of the nth degree or an arbitrary constant C respectively.

According to (109), in the neighbourhood of the point at infinity we have

$$\Phi(z) = -\frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int_S g(t)\,t^k\,dt + P_n(z)$$

and therefore *the solution of problem* (108) *having a zero of multiplicity n at infinity always exists for $n = 1$, and for $n > 1$ it exists only when the conditions*

$$\int_S g(t)\,t^k\,dt = 0 \quad (k = 0, \ldots, n-1)$$

hold; in both cases the solution is unique and is expressed by formula (109) *in which $P(z) = 0$.*

The solution of the problem of determining a piecewise analytic function $\Psi(z)$ satisfying the boundary condition

$$\Psi^+(t) + \Psi^-(t) = g(t)$$

is obviously given by the formula

$$\Psi(z)=\begin{cases}\Phi(z) & \text{for} \quad z\in D^{+}\\ -\Phi(z) & \text{for} \quad z\in D^{-}\end{cases}$$

where $\Phi(z)$ is the solution of problem (108).

In the case when S is a straight line, say the real axis, and D^{+} and D^{-} denote the upper and the lower half-plane respectively, the solution of problem (108) *bounded throughout the whole complex plane z is given by the formula*

$$\Phi(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)\,dt}{t-z}+C \tag{110}$$

where C is an arbitrary constant.

To prove the assertion we have stated let us consider the function

$$w=\frac{z-i}{z+i}$$

specifying a conformal mapping of the upper half-plane D^{+} onto the circle $|w|<1$ (see Sec. 5°, § 1 of the present chapter).

If $\Phi(z)$ denotes the solution of problem (108) in the case under consideration, then the function

$$F(w)=\Phi\left(i\,\frac{1+w}{1-w}\right) \tag{111}$$

is the solution of the boundary-value problem

$$F^{+}(\tau)-F^{-}(\tau)=g_1(\tau), \quad |\tau|=1 \tag{112}$$

where

$$g_1(\tau)=g\left(i\,\frac{1+\tau}{1-\tau}\right)$$

By virtue of (109), the bounded solution of problem (112) has the form

$$F(w)=\frac{1}{2\pi i}\int_{|\tau|=1}\frac{g_1(\tau)\,d\tau}{\tau-w}+C_1 \tag{113}$$

where C_1 is an arbitrary constant. By virtue of (111), we obtain from (113) the expression

$$\Phi(z) = F\left(\frac{z-i}{z+i}\right) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{z+i}{t+i}\,\frac{g(t)\,dt}{t-z} + C_1 =$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(t)\,dt}{t-z} + C$$

where

$$C = C_1 - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(t)\,dt}{t+i}$$

The integral on the right-hand side of formula (110) is understood as the limit of the expression

$$\int_{-N'}^{N''} \frac{g(t)\,dt}{t-z} \tag{114}$$

when the positive constants N' and N'' independently tend to infinity. It is evident that in this case we must require that the function $g(t)$ should have the form

$$g(t) = O(1/|t|^h), \quad h > 0$$

for sufficiently large $|t|$ where $O(1/|t|^h)$ denotes an infinitesimal of the same order as $1/|t|^h$ for $t \to \infty$. In the case when

$$g(t) = \text{const} + O(1/|t|^h), \quad h > 0, \quad \text{const} \neq 0$$

for sufficiently large $|t|$ we should put $N' = N''$ in expression (114), that is the integral on the right-hand side of (110) should be understood in the sense of Cauchy's principal value.

Using solution (110) of problem (108) we can easily derive *Schwarz's formula*

$$F(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-z} + iC \tag{115}$$

which determines, to within a pure imaginary arbitrary additive constant iC, a bounded and analytic function $F(z) = u(x, y) +$

$+ iv(x, y)$ *in the upper half-plane* D^+ *which is continuous including the real axis* $y = 0$ *and satisfies the boundary condition*

$$\operatorname{Re} F(t) = f(t), \quad -\infty < t < \infty \tag{116}$$

where $f(t) = O(1/|t|^h)$ $(h > 0)$ *for sufficiently large values* of $|t|$.

Indeed, let us write condition (116) in the form

$$F^+(t) + \overline{F^+(t)} = 2f(t), \quad -\infty < t < \infty \tag{117}$$

and let us construct the function

$$G(z) = \begin{cases} F(z) & \text{for} \quad z \in D^+ \\ -\overline{F(\bar{z})} = -\bar{F}(z) & \text{for} \quad z \in D^- \end{cases}$$

The function $G(z)$ can be extended continuously to the whole real axis both from D^+ and from D^-. Since $\bar{F}(z) = u(x, -y) - iv(x, -y)$ for $z \in D^-$, condition (117) is equivalent to the condition

$$G^+(t) - G^-(t) = 2f(t), \quad -\infty < t < \infty$$

Therefore, by (110), we have

$$G(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t - z} + \text{const}$$

The function $F(z) = G(z)$ is obviously the solution of problem (116) when const $= iC$ where C is an arbitrary real constant.

It should be noted that *formula* (115) *also gives the solution of problem* (116) *in the more general case when the function* $f(t)$ *has a finite number of singularities on the real axis but is integrable over that axis.*

Using this remark we shall show that Schwarz's formula (115) makes it possible to express in quadratures the solution of the following boundary-value problem: *it is required to find a function* $\Phi(z)$ *which is analytic in the upper half-plane* D^+ *and continuous including the real axis except two points* $z = -a$ *and* $z = a$ *at which it may turn into infinity but is integrable over the real axis, has a simple zero at infinity* *and satisfies the boundary conditions*

$$\operatorname{Re} \Phi(t) = f(t), \quad -a < t < a, \qquad a > 0 \tag{118}$$

and

$$\operatorname{Im} \Phi(t) = 0, \quad -\infty < t < -a, \quad a < t < \infty \tag{119}$$

where $f(t)$ *is a given real function in the interval* $-a < z < a$ *satisfying Hölder's condition.*

Indeed, let us choose a one-valued branch of the function $\sqrt{a^2 - z^2}$ which assumes real values for $-a < z < a$ and let us consider the new function

$$F(z) = \sqrt{a^2 - z^2}\, \Phi(z)$$

where $\Phi(z)$ is the sought-for solution of problem (118), (119).

The function $F(z)$ is analytic and bounded in the upper half-plane D^+ and satisfies the boundary conditions

$$\operatorname{Re} F(t) = \begin{cases} \sqrt{a^2 - t^2}\, f(t) & \text{for} \quad -a < t < a \\ 0 & \text{for} \quad -\infty < t < -a, \\ & \qquad a < t < \infty \end{cases}$$

This function is given by formula (115):

$$F(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\sqrt{a^2 - t^2}}{t - z} f(t)\, dt + iC$$

whence we find

$$\Phi(z) = \frac{1}{\pi i} \int_{-a}^{a} \sqrt{\frac{a^2 - t^2}{a^2 - z^2}} \frac{f(t)\, dt}{t - z} + \frac{iC}{\sqrt{a^2 - z^2}} \tag{120}$$

The constant C can be chosen so that the function $\Phi(z)$ is bounded at one of the end points of the interval $(-a, a)$. For instance, if we take

$$C = \frac{1}{\pi} \int_{-a}^{a} \sqrt{a^2 - t^2} \frac{f(t)\, dt}{t - a}$$

then formula (120) expresses the uniquely determined solution

$$\Phi(z) = \frac{1}{\pi i} \int_{-a}^{a} \sqrt{\frac{(a+t)(a-z)}{(a-t)(a+z)}} \frac{f(t)\,dt}{t-z} \qquad (121)$$

of problem (118), (119) which is bounded at the point $z = a$ where by $\sqrt{\frac{a-z}{a+z}}$ is meant the branch of the function turning into i for $z \to \infty$.

§ 6. Functions of Several Variables

1°. Notation and Basic Notions. An ordered n-tuple $z = (z_1, \ldots, z_n)$ of the values $z_1, \ldots, z_n$ of the complex variables $z_k = x_k + iy_k$ $(k = 1, \ldots, n)$ will be referred to as a *point of the n-dimensional complex vector space* C^n. The space C^n can be interpreted as the $2n$-dimensional Euclidean space of the real variables $x_1, \ldots, x_n;\ y_1, \ldots, y_n$.

A set of points $z \in C^n$ satisfying the conditions

$$| z_k - z_k^0 | < r_k \quad (k = 1, \ldots, n)$$

where r_k are positive numbers is called an *open polycylinder of radius* $r = (r_1, \ldots, r_n)$ *with centre at the point* z^0 and is denoted $C(r, z^0)$; a set of points $z \in C^n$ satisfying the conditions

$$| z_k - z_k^0 | \leqslant r_k \quad (k = 1, \ldots, n)$$

will be called a *closed polycylinder* and will be denoted $\overline{C(r, z^0)}$. The points $z \in C^n$ for which the equalities $| z_k - z_k^0 | = r_k$ $(k = 1, \ldots, n)$ are fulfilled form the *skeleton* of the polycylinder $C(r, z^0)$.

The concept of a polycylinder makes it possible to introduce the notions of a neighbourhood of a given point, of an interior point, of a limit point and of an isolated point for a set E of points belonging to the space C^n, and also the notions of an open set, of a closed set and of a bounded set in C^n.

Let E and E_1 be some sets lying in C^n and in the complex plane of the variable w respectively. In the case when there is a law according to which to every value $z \in E$ there corresponds a definite value $w \in E_1$, we say that w is a *one-valued function of the variable* z or, equivalently, a *one-valued function of several complex variables* $z_1, \ldots, z_n$; in this case we write

$$w = f(z) = f(z_1, \ldots, z_n)$$

A function $f(z)$ of several variables defined on a set E is said to be *continuous in the variables* $z_1, \ldots, z_n$ at a limit point $z^0 \in E$ of that set if, given an arbitrary number $\varepsilon > 0$, there is a system of positive numbers $\delta = (\delta_1, \ldots, \delta_n)$ such that for any two points $z' \in E \cap C(\delta, z^0)$ and $z'' \in E \cap C(\delta, z^0)$ the inequality $|f(z') - f(z'')| < \varepsilon$ holds.

The definitions of uniform continuity of a function $f(z)$ defined on a set E and of convergence and uniform convergence of a sequence of functions $f_n(z)$ $(n = 1, \ldots)$, $z \in E$ are stated in just the same way as in the case of a function of one complex variable.

A finite sum of the form

$$\sum a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n} = P_m(z)$$

where $a_{k_1 \ldots k_n}$ are given complex numbers with indices $k_1, \ldots, k_n$ assuming nonnegative integral values such that $\sum_{j=1}^{n} k_j = m$, is called a *homogeneous polynomial of degree m in the variables* $z_1, \ldots, z_n$. It is evident that *$P_m(z)$ is a continuous function* for all finite values of z.

2°. The Notion of an Analytic Function of Several Variables. Let $w = f(z) = u(x, y) + iv(x, y)$ $(x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n))$ be a function defined in a domain D of the space C^n whose real and imaginary parts considered as functions of the real variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ are continuous together with their partial derivatives of the first order in their domain of definition.

Let the variables z_k receive some increments Δz_k $(k = 1, \ldots, n)$. Then the corresponding increment Δw of the function $w = f(z)$ can be written in the form

$$\Delta w = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial z_k} \Delta z_k + \frac{\partial f}{\partial \bar{z}_k} \Delta \bar{z}_k \right) + o(|\Delta z|) \qquad (122)$$

where

$$\frac{\partial}{\partial z_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}-i\frac{\partial}{\partial y_k}\right),\quad \frac{\partial}{\partial \bar{z}_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}+i\frac{\partial}{\partial y_k}\right)$$

$$\Delta z_k=\Delta x_k+i\,\Delta y_k,\quad \Delta\bar{z}_k=\Delta x_k-i\,\Delta y_k$$

and $o(|\Delta z|)$ is an infinitesimal of a higher order than $|\Delta z|=\sum\limits_{k=1}^{n}|\Delta z_k|$.

If the part

$$\sum_{k=1}^{n}\left(\frac{\partial f}{\partial z_k}\Delta z_k+\frac{\partial f}{\partial \bar{z}_k}\Delta\bar{z}_k\right)$$

of increment (122) of the function $f(z)$ is a linear form dependent solely on Δz_k $(k=1,\ldots,n)$ at each point $z\in D$, that is if for every point $z\in D$ the equalities

$$\frac{\partial f}{\partial \bar{z}_k}=0\quad (k=1,\ldots,n) \tag{123}$$

hold, the function $f(z)$ is said to be *analytic in the domain D*.

Relations (123) are the complex representation of the following systems of equalities involving real quantities:

$$\frac{\partial u}{\partial x_k}-\frac{\partial v}{\partial y_k}=0,\quad \frac{\partial u}{\partial y_k}+\frac{\partial v}{\partial x_k}=0\quad (k=1,\ldots,n) \tag{CR}$$

Equalities (CR) are called the *Cauchy-Riemann system of partial differential equations for several independent variables* corresponding to an analytic function $f(z)$ of several variables.

The expression

$$dw=df=\sum_{k=1}^{n}\frac{\partial f}{\partial z_k}dz_k \tag{124}$$

is called the *total differential* of the analytic function $f(z)$.

The above definition of an analytic function $f(z)=f(z_1,\ldots,z_n)$ implies that $f(z)$ (considered as a function of several variables $z_1,\ldots,z_n$) is continuous in the variables $z_1,\ldots,z_n$ in the sense of the definition stated in Sec. 1°, § 6 of the present chapter; it also follows that $f(z_1,\ldots,z_n)$

is analytic with respect to each of the variables z_k separately in the sense of the definition of an analytic function of one independent variable (see Sec. 1°, § 1 of the present chapter). It turns out that the converse proposition is also true: *if the function $f(z)$ is analytic with respect to each of the variables z_k ($k = 1, \ldots, n$) separately in the domain D then it is analytic in $z \in D$ in the sense of the definition of an analytic function of several complex variables.* This proposition is known as ***Hartoges' theorem***; its proof will not be presented here.

The coefficient $\partial f/\partial z_k$ in dz_k on the right-hand side of formula (124) is called the *partial derivative* of the analytic function $f(z)$ with respect to the variable z_k, and it can be computed using the formula

$$\frac{\partial f}{\partial z_k} = \lim_{\Delta z_k \to 0} \frac{f(z_1, \ldots, z_k + \Delta z_k, \ldots, z_n) - f(z_1, \ldots, z_k, \ldots, z_n)}{\Delta z_k} =$$

$$= \frac{\partial u}{\partial x_k} + i\frac{\partial v}{\partial x_k} = -i\frac{\partial u}{\partial y_k} + \frac{\partial v}{\partial y_k}$$

It can readily be checked directly that every polynomial in the variables $z_1, \ldots, z_n$ is an analytic function.

3°. Multiple Power Series. A functional series in several variables of the form

$$\sum \alpha_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n} \tag{125}$$

where $\alpha_{k_1 \ldots k_n}$ are given numbers and the summation is carried out over all the values of the indices k_j ($j = 1, \ldots, n$) ranging from zero to infinity is referred to as a *multiple power series* or a *power series in several variables* $z_1, \ldots, z_n$.

From the course of mathematical analysis we know that *if a power series*

$$\sum_{k=0}^{\infty} \alpha_k z^k$$

in one complex variable z is convergent at a point $z_0 \neq 0$ of the complex z-plane then this series is absolutely convergent in the circle $|z| < |z_0|$ (this is ***Abel's theorem***). It turns out that for general series of form (125) this proposition is not true. However, *if the coefficients $\alpha_{k_1 \ldots k_n}$ of power series*

(125) *satisfy the additional requirements*

$$|a_{k_1\ldots k_n}|\leqslant\frac{g}{r_1^{k_1}\ldots r_n^{k_n}},\quad r_j>0 \tag{126}$$

for all values of the indices where g is a positive number independent of $k_1, \ldots, k_n$ then this series is absolutely convergent in the open polycylinder $C(r, 0)$, $r=(r_1, \ldots, r_n)$, and the convergence of the series is uniform in every closed bounded subset of points belonging to the polycylinder $C(r, 0)$.

Let us consider the series

$$\sum_{k_1\geqslant 0,\ \ldots,\ k_n\geqslant 0} g\left|\frac{z_1}{r_1}\right|^{k_1}\ldots\left|\frac{z_n}{r_n}\right|^{k_n}$$

for $z\in C(r, 0)$; its terms form a multiple geometric progression whose sum is equal to the expression

$$g\left(1-\frac{|z_1|}{r_1}\right)^{-1}\ldots\left(1-\frac{|z_n|}{r_n}\right)^{-1}$$

If conditions (126) are fulfilled then each of the terms of series (125) satisfies the inequality

$$|a_{k_1\ldots k_n}z_1^{k_1}\ldots z_n^{k_n}|\leqslant g\left|\frac{z_1}{r_1}\right|^{k_1}\ldots\left|\frac{z_n}{r_n}\right|^{k_n} \tag{127}$$

From (127) follows the proposition stated above.

Since series (125) satisfying conditions (126) is absolutely convergent in the polycylinder $C(r, 0)$, we can group its terms so that the series takes the form

$$\sum_{m=0}^{\infty} P_m(z) \tag{128}$$

where $P_m(z)$ are homogeneous polynomials of degree m in the variables $z_1, \ldots, z_n$.

Since series (128) is uniformly convergent on each closed subset of the polycylinder $C(r, 0)$ and the polynomials P_m are analytic functions with respect to each of the variables z_k $(k=1, \ldots, n)$ we conclude, on the basis of Weierstrass' first theorem (see Sec. 2°, § 3 of the present chapter), that the sum $s(z)$ of the series is analytic with respect to each of these variables; if, for instance, we differentiate series

(128) once with respect to the variable z_k, the resultant (differentiated) series possesses the property that its every term does not exceed in its modulus the corresponding term of the geometric progression whose sum is the expression

$$\frac{g}{r_k}\left(1-\frac{|z_1|}{r_1}\right)^{-1}\cdots\left(1-\frac{|z_k|}{r_k}\right)^{-2}\cdots\left(1-\frac{|z_n|}{r_n}\right)^{-1}$$

Consequently, by Hartogs' theorem, *the sum $s(z)$ of power series* (125) *is an analytic function in the polycylinder $C(r, 0)$.*

All that was established above remains true for a power series of the form

$$\sum \alpha_{k_1\ldots k_n}(z_1-z_1^0)^{k_1}\ldots(z_n-z_n^0)^{k_n}$$

where $z^0 = (z_1^0, \ldots, z_n^0)$ is a finite point of the space C^n.

4°. Cauchy's Integral Formula and Taylor's Theorem. Let $f(z)$ be an analytic function in a domain $D \subset C^n$ and let z^0 be a point belonging to D. For sufficiently small $r_1, \ldots, r_n$ the polycylinder $\overline{C(r, z^0)}$ lies inside D. Let us choose, for $k \neq j$, some fixed values of the variables z_k belonging to the circles $|z_k - z_k^0| < r_k$; then $f(z)$ considered as an analytic function of the variable z_j in the circle $|z_j - z_j^0| < r_j$ can be represented in the form

$$f(z)=\frac{1}{2\pi i}\int\limits_{|t_j-z_j^0|=r_j}\frac{f(z_1, \ldots, t_j, \ldots, z_n)}{t_j-z_j}\,dt_j$$

with the aid of Cauchy's formula (47).

On repeating this argument for all $j = 1, \ldots, n$, we conclude that for all $z \in C(r, z^0)$ the following *Cauchy integral formula* takes place:

$$f(z)=\frac{1}{(2\pi i)^n}\int\limits_{|t_1-z_1^0|=r_1}dt_1\ldots\int\limits_{|t_n-z_n^0|=r_n}\frac{f(t)\,dt_n}{(t_1-z_1)\ldots(t_n-z_n)} \tag{129}$$

The integration on the right-hand side of formula (129) can be carried out in any order because the integrand expression is continuous as a function of the point $t = (t_1, \ldots, t_n)$. Arguing like in Sec. 4°, § 3, we conclude from formula (129) that *an analytic function* $f(z)$ *possesses derivatives of all orders with respect to the variables* $z_1, \ldots, z_n$; *by Hartogs' theorem, these derivatives are themselves analytic functions in the polycylinder* $C(r, z^0)$. Since the point $z_0 \in D$ has been taken quite arbitrarily we have thus proved the existence and the analyticity of all derivatives of an analytic function $f(z)$.

For $z \in C(r, z^0)$ we have

$$\frac{1}{(t_1 - z_1) \ldots (t_n - z_n)} =$$

$$= \frac{1}{(t_1 - z_1^0) \ldots (t_n - z_n^0)} \frac{1}{\left(1 - \frac{z_1 - z_1^0}{t_1 - z_1^0}\right) \ldots \left(1 - \frac{z_n - z_n^0}{t_n - z_n^0}\right)} =$$

$$= \frac{1}{(t_1 - z_1^0) \ldots (t_n - z_n^0)} \sum \left(\frac{z_1 - z_1^0}{t_1 - z_1^0}\right)^{k_1} \ldots \left(\frac{z_n - z_n^0}{t_n - z_n^0}\right)^{k_n}$$

and the series on the right-hand side is uniformly convergent with respect to the point t on the skeleton of the polycylinder $C(r, z^0)$; therefore from formula (129) we obtain the equality

$$f(z) = \sum \beta_{k_1 \ldots k_n} (z_1 - z_1^0)^{k_1} \ldots (z_n - z_n^0)^{k_n} \tag{130}$$

where

$$\beta_{k_1 \ldots k_n} = \frac{1}{(2\pi i)^n} \int_{|t_1 - z_1^0| = r_1} dt_1 \ldots$$

$$\ldots \int_{|t_n - z_n^0| = r_n} \frac{f(t)\, dt_n}{(t_1 - z_1^0)^{k_1+1} \ldots (t_n - z_n^0)^{k_n+1}} \tag{131}$$

From formula (131) we derive

$$|\beta_{k_1 \ldots k_n}| \leqslant \frac{M}{r_1^{k_1} \ldots r_n^{k_n}}$$

where $M = \max |f(z)|$ for $z \in C(r, z^0)$, whence it follows that the power series on the right-hand side of (130) is absolutely and uniformly convergent in any polycylinder $\overline{C(\rho, z^0)}$ where $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_k < r_k$ $(k = 1, \ldots, n)$.

We have thus proved the following ***Taylor theorem***: *a function* $f(z)$ *of several variables analytic in a domain* D *can be represented in a neighbourhood of every point* $z^0 \in D$ *in the form of the sum of absolutely convergent power series* (130) *whose coefficients are computed according to formula* (131).

By virtue of formula (129), we have

$$\frac{\partial^l f(z)}{\partial z_k^l} = \frac{l!}{(2\pi i)^n} \int\limits_{|t_1 - z_1^0| = r_1} dt_1 \ldots \int\limits_{|t_k - z_k^0| = r_k} dt_k \ldots \int\limits_{|t_n - z_n^0| = r_n} \frac{f(t)\, dt_n}{(t_1 - z_1) \ldots (t_k - z_k)^{l+1} \ldots (t_n - z_n)}$$

and therefore for the coefficients $\beta_{k_1 \ldots k_n}$ given by formula (131) we also have the expression

$$\beta_{k_1 \ldots k_n} = \frac{1}{k_1! \ldots k_n!} \left(\frac{\partial^{k_1 + \ldots + k_n} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}} \right)_{z = z^0} \tag{132}$$

The properties of power series we have established and Taylor's theorem allow us to conclude that the definition of an analytic function given in Sec. 2°, § 6 of the present chapter is equivalent to the following definition: *a one-valued function $F(z)$ defined in a domain D is called analytic if for each point $z^0 \in D$ there is a neighbourhood in which $F(z)$ can be represented in the form of an (absolutely) convergent power series*

$$F(z) = \sum \gamma_{k_1 \ldots k_n} (z_1 - z_1^0)^{k_1} \ldots (z_n - z_n^0)^{k_n} \tag{133}$$

Using Cauchy's integral formula, we can show directly that the coefficients $\gamma_{k_1 \ldots k_n}$ are uniquely determined by formulas (131) and (132) in which the function f should be replaced by F. In the case of real variables $z_1, \ldots, z_n$ this definition of an analytic function coincides with the well-known definition of an analytic function of several real variables stated in the course of mathematical analysis.

5°. Analytic Functions of Real Variables. A function $f(x)$ defined in a domain D of the Euclidean space E_n is said to be *analytic* in D if for each point $x^0 \in D$ there exists a parallelepiped $|x_k - x_k^0| < \delta_k$ $(k = 1, \ldots, n)$ within which $f(x)$ can be represented in the form of the sum of an absolutely convergent power series

$$f(x) = \sum_{k_1 \geqslant 0, \ldots, k_n \geqslant 0} a_{k_1 \ldots k_n} (x_1 - x_1^0)^{k_1} \ldots (x_n - x_n^0)^{k_n} \tag{134}$$

It is evident that the coefficients of this series are expressed in terms of $f(x)$ by the formulas

$$a_{k_1 \ldots k_n} = \frac{1}{k_1! \ldots k_n!} \left(\frac{\partial^{k_1 + \ldots + k_n} f}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \right)_{x = x^0}$$

The class of analytic functions is rather wide. In particular, it includes the harmonic functions. When the analyticity of harmonic functions is proved, Poisson's formula (see formula (20) in Chapter 1) is used. Here we shall limit ourselves to the consideration of harmonic functions for the case $n = 2$.

Thus, let $x_1 = x$, $x_2 = y$ and $x_1^0 = x_0$, $x_2^0 = y_0$, and let D be a domain in the complex plane of the variable $z = x + iy$ in which a harmonic function $u(x, y)$ is defined. Let us consider a circle $|z - z_0| < R$ where $z_0 = x_0 + iy_0$ is an arbitrary fixed point belonging to D and R is a positive number smaller than the distance from the point z_0 to the boundary of the domain D. In this circle the function $u(x, y)$ can be represented with the aid of Schwarz's formula (see formula (86) in the present chapter):

$$u(x, y) = \operatorname{Re} \left(\frac{1}{\pi i} \int_{|t - z_0| = R} \frac{u(t)\, dt}{t - z} + C \right) \tag{135}$$

Since for $|z - z_0| < |t - z_0|$ we have

$$\frac{1}{t - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(t - z_0)^{k+1}}$$

formula (135) implies

$$u(x, y) = \operatorname{Re} \sum_{k=0}^{\infty} \beta_k (z - z_0)^k \tag{136}$$

where

$$\beta_0 = \frac{1}{\pi i} \int_{|t - z_0| = R} \frac{u(t)\, dt}{t - z_0} + C$$

and

$$\beta_k = \frac{1}{\pi i} \int_{|t - z_0| = R} \frac{u(t)\, dt}{(t - z_0)^{k+1}} \quad (k = 1, 2, \ldots)$$

On regrouping the terms on the right-hand side of (136) in the appropriate manner (this operation is legitimate since the power series is absolutely convergent), we arrive at a series in powers of $(x - x_0)$ and $(y - y_0)$:

$$u(x, y) = \sum_{k,\, j=0}^{\infty} \gamma_{kj} (x - x_0)^k (y - y_0)^j \tag{137}$$

The coefficients of the last series are computed with the aid of the formula

$$\gamma_{kj} = \frac{1}{k!\,j!} \left(\frac{\partial^{k+j} u}{\partial x^k \, \partial y^j} \right)_{z=z_0}$$

The series on the right-hand side of (136) being convergent for $|z - z_0| < R$, power series (137) converges absolutely in the parallelepiped $|x - x_0| < r_1$, $|y - y_0| < r_2$ where $r_1^2 + r_2^2 < R^2$, whence follows the analyticity of the function $u(x, y)$.

Since series (137) is absolutely convergent in the polycylinder $C(r, z_0)$ lying in the complex space C^2 of the variables $z_1 = x + ix'$, $z_2 = y + iy'$ for $r_1^2 + r_2^2 < R^2$, its sum

$$u(z_1, z_2) = \sum_{k,\, j=0}^{\infty} \gamma_{kj} (z_1 - x_0)^k (z_2 - y^0)^j$$

which is an analytic function in $C(r, z_0)$ (see Sec. 3°, § 6 of the present chapter) can naturally be called the *analytic continuation* of the harmonic function $u(x, y)$ from the parallelepiped $|x - x_0| < r_1$, $|y - y_0| < r_2$ to the polycylinder $C(r, z_0)$.

6°. Conformal Mappings in Euclidean Spaces. Let us consider a system of real functions $y_i = y_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ defined in a domain D of the Euclidean space E_n of the points $x = (x_1, \ldots, x_n)$. We shall suppose that these functions are continuous together with their first-order partial derivatives and that they specify a one-to-one mapping of the domain D onto a domain $D_1 \subset E_n$.

Using vector notation we can write the function specifying this mapping in the form

$$y = y(x) \tag{138}$$

The squares $|dx|^2 = dx\,dx$, $|dy|^2 = dy\,dy$ of the distances between the points x, $x + dx$ and y, $y + dy$ (the squares of the elements of length) will be denoted as ds^2 and $d\sigma^2$ respectively.

Function (138) is said to specify the *Gauss conformal mapping* if there exists a scalar function $\lambda(x)$ such that

$$d\sigma^2 = \lambda\, ds^2 \tag{139}$$

In other words, the conformality of the mapping specified by (138) means that under the mapping there is uniform magnification (of elements of length)* for all directions issued from the point x.

Condition (139) is equivalent to the equalities

$$\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_i} = \lambda(x), \quad \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_k} = 0 \tag{140}$$
$$i \neq k; \quad (i,\ k = 1, \ldots, n)$$

Under the mapping specified by (138) two infinitesimal vectors dx and δx starting from a point x go into two vectors dy and δy starting from the point $y = y(x)$. Since

$$dy_i = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} dx_k \quad \text{and} \quad \delta y_i = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \delta x_k$$

we obtain, by virtue of (139), the relation

$$\cos \widehat{dy\,\delta y} = \frac{dy\ \delta y}{|dy|\,|\delta y|} = \frac{dx\ \delta x}{|dx|\,|\delta x|} = \cos \widehat{dx\,\delta x}$$

Consequently, a characteristic feature of a conformal mapping is the *angle-preserving property.*

The following simple transformations are examples of conformal mappings in space: a *parallel translation* $y = x + h$, a *transformation of similitude* $y = \mu x$ and an *orthogonal transformation* $y = Cx$ where $h = (h_1, \ldots, h_n)$ is a constant vector, μ is a constant scalar and C is a constant orthogonal matrix. In the first and the third of these examples we have $\lambda = 1$, and in the second example $\lambda = \mu^2$.

The mapping specified by the function

$$y = \frac{x}{|x|^2} \tag{141}$$

* See footnote on page 107.— *Tr.*

is defined for all finite values of x different from $x = 0$; this mapping is called the *inversion* or the *reflection of the space* E_n *in the unit sphere* $|x| = 1$ or the *symmetry transformation of* E_n *with respect to the unit sphere.*

On multiplying scalarly both members of equality (141) by x, we obtain

$$xy = 1 \tag{142}$$

From (141) and (142) we conclude that $|x|\,|y| = 1$, that is under the inversion two points x and y corresponding to each other belong to one ray starting from the point $x = 0$, the product of the distances from these points to the point $x = 0$ being equal to unity. Since $1/|x|^2 = |y|^2$, we directly derive the formula expressing the one-valued mapping inverse to (141) for $x \neq 0$, $y \neq 0$:

$$x = \frac{y}{|y|^2}$$

For $x \neq 0$ the differentiation of equality (141) results in

$$dy = \frac{|x|^2\,dx - 2\,(x\,dx)\,x}{|x|^4}$$

and therefore $|dy|^2$ is expressed by the formula

$$|dy|^2 = \frac{|dx|^2}{|x|^4}$$

This means that the inversion specified by formula (141) is a conformal mapping for $x \neq 0$, and $\lambda = 1/|x|^4$.

In the case $n = 2$ system (140) is equivalent to one of the following two linear systems of partial differential equations:

$$\frac{\partial y_1}{\partial x_1} - \frac{\partial y_2}{\partial x_2} = 0, \quad \frac{\partial y_1}{\partial x_2} + \frac{\partial y_2}{\partial x_1} = 0$$

and

$$\frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2} = 0, \quad \frac{\partial y_1}{\partial x_2} - \frac{\partial y_2}{\partial x_1} = 0$$

Consequently, in this case the theory of conformal mappings reduces completely to the theory of one-sheeted analytic functions of one complex variable $z = x_1 + ix_2$ or $\bar{z} = x_1 - ix_2$ respectively.

In the case $n > 2$ system of equations (140) is non-linear with respect to $y_1, \ldots, y_n$, and the number of the equations in the system exceeds that of the sought-for functions.

The extent to which system (140) is overdetermined is characterized by the following *Liouville theorem*: *for $n > 2$ the conformal mappings in the Euclidean space E_n are exhausted by a finite number of superpositions of the following four types of mappings: the parallel translation, the transformation of similitude, the orthogonal transformation and the inversion* (also called, as was already mentioned, the *symmetry transformation with respect to unit sphere* or the *reflection in unit sphere*). Here we shall not present the proof of this theorem.

CHAPTER 3

HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

§ 1. Wave Equation

1°. Wave Equation with Three Spatial Variables. Kirchhoff's Formula. Below we shall suppose that in the space E_{n+1} of points (x, t) the symbol x denotes the collection of the spatial variables $x_1, \ldots, x_n$ and t denotes time.

It was proved in Sec. 2°, § 3 of Introduction that *if a function* $\mu(x_1, x_2, x_3)$ *defined in the space* E_3 *of the variables* x_1, x_2, x_3 *possesses continuous partial derivatives of the second order, then the function*

$$u(x_1, x_2, x_3, t) = tM(\mu)$$

where

$$M(\mu) = \int_{|\xi|=1} \mu(x_1 + t\xi_1, x_2 + t\xi_2, x_3 + t\xi_3)\, d\sigma_\xi \tag{1}$$

is a regular solution of the wave equation with three spatial variables:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{2}$$

Since the element of area ds_y of the sphere $|y - x|^2 = t^2$ is equal to $t^2\, d\sigma_\xi$ where $d\sigma_\xi$ is the element of area of the unit sphere $|\xi|^2 = 1$, the expression

$$\frac{1}{4\pi} M(\mu) = \frac{1}{4\pi t^2} \int_{|y-x|^2=t^2} \mu(y_1, y_2, y_3)\, ds_y \tag{3}$$

is the integral mean of the function $\mu(x_1, x_2, x_3)$ over the sphere $|y - x|^2 = t^2$.

It is evident that as well as $tM(\mu)$ the function $\partial/\partial t \times \times [tM(\mu)]$ is also a regular solution of equation (2) provided that the function $\mu(x_1, x_2, x_3)$ possesses continuous partial derivatives of the third order in the space E_3 where it is defined.

It can easily be shown that *the function*

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi} tM(\psi) + \frac{1}{4\pi} \frac{\partial}{\partial t} [tM(\varphi)] \tag{4}$$

is the regular solution of the Cauchy problem for wave equation (2) *with initial conditions*

$$u(x_1, x_2, x_3, 0) = \varphi(x_1, x_2, x_3) \tag{5}$$

and

$$\left. \frac{\partial u(x_1, x_2, x_3, t)}{\partial t} \right|_{t=0} = \psi(x_1, x_2, x_3) \tag{6}$$

where $\varphi(x_1, x_2, x_3)$ *and* $\psi(x_1, x_2, x_3)$ *are real functions defined in the space* E_3 *of the variables* x_1, x_2, x_3 *and possessing continuous partial derivatives of the third and of the second order respectively.*

Indeed, as was already mentioned above, each summand on the right-hand side of (4) is a regular solution of equation (2) for all points (x_1, x_2, x_3, t) belonging to the space E_4 of the variables x_1, x_2, x_3, t. From (1) and (4) it follows that for $t = 0$ we have

$$u(x_1, x_2, x_3, 0) = \frac{1}{4\pi} \int\limits_{|\xi|=1} \varphi(x_1, x_2, x_3)\, d\sigma_\xi = \varphi(x_1, x_2, x_3)$$

Further, since

$$\frac{\partial u(x_1, x_2, x_3, t)}{\partial t} = \frac{1}{4\pi} \frac{\partial}{\partial t} [tM(\psi)] + \frac{1}{4\pi} t \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) M(\varphi)$$

we have

$$\left. \frac{\partial}{\partial t} u(x_1, x_2, x_3, t) \right|_{t=0} = \frac{1}{4\pi} \int\limits_{|\xi|=1} \psi(x_1, x_2, x_3)\, d\sigma_\xi = \psi(x_1, x_2, x_3)$$

Equality (4) expressing the solution of Cauchy's problem (5), (6) for wave equation (2) in the case of three spatial variables x_1, x_2, x_3 is known as *Kirchhoff's formula.*

The physical phenomenon described by a solution $u(x, t)$ of the wave equation is spoken of as *propagation of a wave*, and the solution $u(x, t)$ is often referred to as a *wave.*

Formulas (33) and (34) established in Introduction imply that

$$\frac{\partial}{\partial t} tM(\varphi) = M(\varphi) + \frac{1}{t} \int\limits_S \frac{\partial \varphi}{\partial \nu} ds_y$$

where ν is the outer normal to S at the point y, and therefore Kirchhoff's formula implies that *in the case of three spatial variables the wave corresponding to Cauchy's problem* (5), (6) *is completely determined at the point* (x_1, x_2, x_3, t) *of the space* E_4 *by the values of* φ, $\frac{\partial \varphi}{\partial \nu}$ *and* ψ *assumed on the sphere* $(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 = t^2$ *of radius* $|t|$ *with centre at the point* (x_1, x_2, x_3). In the theory of propagation of waves, and, in particular, in the theory of sound, this fact is known as *Huygens' principle.*

2°. Wave Equation with Two Spatial Variables. Poisson's Formula. Let us consider the wave equation with two spatial variables:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{7}$$

The solution $u(x_1, x_2, t)$ of the Cauchy problem for this equation with the initial data

$$u(x_1, x_2, 0) = \varphi(x_1, x_2) \tag{8}$$

and

$$\frac{\partial}{\partial t} u(x_1, x_2, t)\Big|_{t=0} = \psi(x_1, x_2) \tag{9}$$

can be derived from Kirchhoff's formula (4) provided that the functions φ and ψ possess continuous partial derivatives of the third and of the second order respectively.

To obtain the solution we use the fact that when the functions φ and ψ on the right-hand side of formula (4) depend solely on the two variables x_1 and x_2 this formula gives the function

$$u(x_1, x_2, t) = \frac{1}{4\pi t} \int\limits_{|y|^2=t^2} \psi(x_1+y_1, x_2+y_2)\, ds_y + \\ + \frac{1}{4\pi} \frac{\partial}{\partial t} \left[\frac{1}{t} \int\limits_{|y|^2=t^2} \varphi(x_1+y_1, x_2+y_2)\, ds_y \right] \quad (10)$$

which is independent of x_3 and satisfies both equation (7) and initial conditions (8), (9).

As is known, the projection $dy_1\, dy_2$ of the element of area ds_y of the sphere $|y|^2 = t^2$ on the circle $y_1^2 + y_2^2 \leqslant t^2$ is expressed in terms of ds_y by the formula $dy_1\, dy_2 = ds_y \times \times \cos(\widehat{i_3, \nu}) = \frac{y_3}{|t|} ds_y$ where i_3 is the unit vector along the x_3-axis and ν is the normal to the sphere $|y|^2 = t^2$ at the point (y_1, y_2, y_3). To compute the integrals on the right-hand side of formula (4) we must project on the circle $y_1^2 + y_2^2 \leqslant t^2$ both the upper hemisphere ($y_3 > 0$) and the lower hemisphere ($y_3 < 0$) of the sphere $|y|^2 = t^2$; therefore formula (10) can be written in the form

$$u(x_1, x_2, t) = \frac{1}{2\pi} \int\limits_d \frac{\psi(y_1, y_2)\, dy_1\, dy_2}{\sqrt{t^2-(y_1-x_1)^2-(y_2-x_2)^2}} + \\ + \frac{1}{2\pi} \frac{\partial}{\partial t} \int\limits_d \frac{\varphi(y_1, y_2)\, dy_1\, dy_2}{\sqrt{t^2-(y_1-x_1)^2-(y_2-x_2)^2}} \quad (11)$$

where d is the circle $(y_1 - x_1)^2 + (y_2 - x_2)^2 \leqslant t^2$.

Equality (11) is called *Poisson's formula*. This formula shows that in order to determine the wave $u(x_1, x_2, t)$ at a point (x_1, x_2, t) we must know not only the values of the solution and of its time derivative described by the functions $\varphi(x_1, x_2)$ and $\psi(x_1, x_2)$ on the circumference $(y_1 - x_1)^2 + (y_2 - x_2)^2 = t^2$ of the circle d but also the values of the functions $\varphi(x_1, x_2)$ and $\psi(x_1, x_2)$ at all the points inside the circle d. This means that *in the case of the two spatial variables x_1 and x_2 Huygens' principle does not apply to wave processes.*

3°. Equation of Oscillation of a String. D'Alembert's Formula. Let us consider the case when the initial data φ and ψ depend solely on one spatial variable $x = x_1$. For this case formula (11) yields

$$u(x, t) = \frac{1}{2\pi} \int_{-t}^{t} \psi(x+\eta_1)\, d\eta_1 \int_{-\sqrt{t^2-\eta_1^2}}^{\sqrt{t^2-\eta_1^2}} \frac{d\eta_2}{\sqrt{t^2-\eta_1^2-\eta_2^2}} +$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial t} \int_{-t}^{t} \varphi(x+\eta_1)\, d\eta_1 \int_{-\sqrt{t^2-\eta_1^2}}^{\sqrt{t^2-\eta_1^2}} \frac{d\eta_2}{\sqrt{t^2-\eta_1^2-\eta_2^2}} =$$

$$= \frac{1}{2\pi} \int_{-t}^{t} \psi(x+\eta_1) \arcsin \frac{\eta_2}{\sqrt{t^2-\eta_1^2}} \Bigg|_{-\sqrt{t^2-\eta_1^2}}^{\sqrt{t^2-\eta_1^2}} d\eta_1 +$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial t} \int_{-t}^{t} \varphi(x+\eta_1) \arcsin \frac{\eta_2}{\sqrt{t^2-\eta_1^2}} \Bigg|_{-\sqrt{t^2-\eta_1^2}}^{\sqrt{t^2-\eta_1^2}} d\eta_1 =$$

$$= \frac{1}{2} \int_{-t}^{t} \psi(x+\eta)\, d\eta + \frac{1}{2}\frac{\partial}{\partial t} \int_{-t}^{t} \varphi(x+\eta)\, d\eta =$$

$$= \frac{1}{2}\varphi(x+t) + \frac{1}{2}\varphi(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\tau)\, d\tau$$

The formula

$$u(x, t) = \frac{1}{2}\varphi(x+t) + \frac{1}{2}\varphi(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\tau)\, d\tau \quad (12)$$

expresses the solution of the Cauchy problem for the *equation of oscillation of a string*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad (13)$$

with the initial data

$$u(x, 0) = \varphi(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x, t)\Big|_{t=0} = \psi(x)$$

Relation (12) is called *D'Alembert's formula.*

4°. The Notion of the Domains of Dependence, Influence and Propagation. In Secs 1°-3°, § 1 of the present chapter we considered the Cauchy problem in which the initial data were prescribed throughout the whole space E_n of the variables $x = (x_1, \ldots, x_n)$.

The set of points belonging to the space E_n which possesses the property that the value of the solution $u(x, t)$ of the wave equation at the point (x, t) of the space E_{n+1} is completely determined by the values of the functions $\varphi(x)$

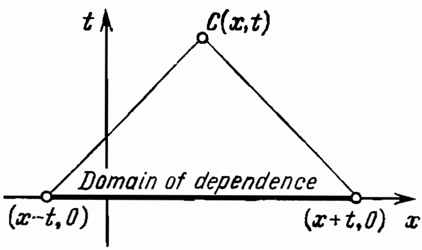

Fig. 17

and $\psi(x)$ assumed on that set is called the *domain of dependence* corresponding to the point (x, t). Those points for which the corresponding values of $\varphi(x)$ and $\psi(x)$ do not affect the value of $u(x, t)$ at the point (x, t) are not of course included in the domain of dependence (Fig. 17).

As was already mentioned, in the cases $n = 2$ and $n = 1$ the domains of dependence corresponding to the point (x, t) are the circle $|y - x|^2 \leqslant t^2$ and the line segment $|y - x|^2 \leqslant t^2$ in the space E_n respectively, and in the case $n = 3$ the domain of dependence is determined according to Huygens' principle.

Now let us suppose that the initial data are prescribed not in the whole space E_n but on some domain G lying in E_n, that is

$$u(x, 0) = \varphi(x), \quad \frac{\partial u(x, t)}{\partial t} = \psi(x) \quad \text{for} \quad t = 0, \quad x \in G \tag{14}$$

As is seen from formulas (4), (11) and (12), the values of $\varphi(x)$ and $\psi(x)$ prescribed on G affect the values of $u(x, t)$ assumed at all those points (x, t) of the space E_{n+1} which possess the property that the intersection of the two sets G and $\{|y-x|^2 \leqslant t^2\}$ is not void. The set of all such points is usually referred to as the *domain of influence* (see Fig. 18).

The set of points $(x, t) \in E_{n+1}$ for which the corresponding values of $u(x, t)$ are completely determined by the given

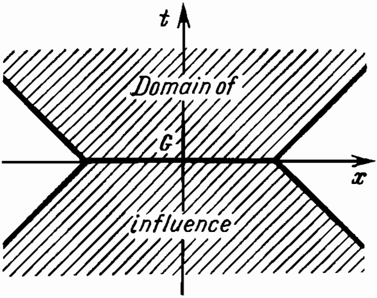

Fig. 18

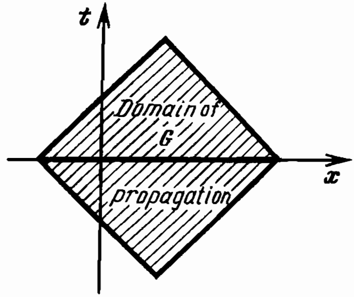

Fig. 19

values of $\varphi(x)$ and $\psi(x)$ on G is called the *domain of propagation of the wave $u(x, t)$ with the initial data prescribed on G* (Fig. 19).

Formulas (4), (11) and (12) show that for initial data (14) the domain of propagation of the wave $u(x, t)$ consists of those and only those points (x, t) of the space E_{n+1} which possess the following property: for $n = 3$ the sphere $|y-x|^2 = t^2$, which is the intersection of the characteristic cone $|y-x|^2 = (\tau - t)^2$ with vertex at the point (x, t) and the hyperplane $\tau = 0$, belongs to G, for $n = 2$ not only the contour $|y-x|^2 = t^2$ (which is the intersection of the characteristic cone $|y-x|^2 = (\tau - t)^2$ with vertex at the point (x, t) and the plane $\tau = 0$), but also the whole circle $|y-x|^2 \leqslant t^2$ belongs to G and, finally, for $n = 1$ not only the points $x - t$ and $x + t$ at which the characteristic straight lines $y - x = \tau - t$ and $y - x = t - \tau$ (these lines form the degenerate characteristic cone $(y-x)^2 = (\tau - t)^2$) passing through the point (x, t) intersect the straight line $\tau = 0$, but also the whole line segment between these points belongs to G.

§ 2. Non-Homogeneous Wave Equation

1°. The Case of Three Spatial Variables. Retarded Potential. Let the initial data be prescribed not on the plane $t = 0$ but on the plane $t = \tau_1$ where τ_1 is a parameter. We shall denote by $v(x_1, x_2, x_3, t, \tau_1)$ the solution of wave equation (2) satisfying the initial conditions

$$\begin{gathered} v(x_1, x_2, x_3, \tau_1, \tau_1) = 0 \\ \frac{\partial}{\partial t} v(x_1, x_2, x_3, t, \tau_1)\Big|_{t=\tau_1} = g(x_1, x_2, x_3, \tau_1) \end{gathered} \tag{15}$$

where $g(x_1, x_2, x_3, \tau_1)$ is a given real function possessing continuous partial derivatives of the second order.

On replacing t by $t - \tau_1$, we obtain from Kirchhoff's formula (4) the following expression for v:

$$v(x_1, x_2, x_3, t, \tau_1) = \frac{1}{4\pi(t-\tau_1)} \int\limits_{|y-x|=|t-\tau_1|} g(y_1, y_2, y_3, \tau_1)\, ds_y$$

Let us show that *the function*

$$u(x_1, x_2, x_3, t) = \int\limits_0^t v(x_1, x_2\, x_3, t, \tau_1)\, d\tau_1 \tag{16}$$

is the solution of the Cauchy problem with the data

$$u(x_1, x_2, x_3, 0) = 0, \quad \frac{\partial}{\partial t} u(x_1, x_2, x_3, t)\Big|_{t=0} = 0 \tag{17}$$

for the non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial t^2} = -g(x_1, x_2, x_3, t) \tag{18}$$

Indeed, by virtue of (15), it is readily seen that the function $u(x_1, x_2, x_3, t)$ satisfies initial conditions (17).

Further, from (15) and (16) we derive

$$\frac{\partial^2 u}{\partial t^2} = g(x_1, x_2, x_3, t) + \int\limits_0^t \frac{\partial^2}{\partial t^2} v(x_1, x_2, x_3, t, \tau)\, d\tau \tag{19}$$

Finally, from (16) and (19) it follows that

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}-\frac{\partial^2 u}{\partial t^2}=-g(x_1, x_2, x_3, t)+$$

$$+\int_0^t\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-\frac{\partial^2}{\partial t^2}\right)v(x_1, x_2, x_3, t, \tau)\,d\tau=$$

$$=-g(x_1, x_2, x_3, t)$$

which proves the assertion stated above.

The change of the variable $t-\tau_1=\tau$ brings formula (19) to the form

$$u(x, t)=\frac{1}{4\pi}\int_0^t \tau\,d\tau\int_{r^2=\tau^2} g(y_1, y_2, y_3, t-\tau)\,\frac{ds_y}{\tau^2}=$$

$$=\frac{1}{4\pi}\int_{r^2\leqslant t^2}\frac{g(y_1, y_2, y_3, t-r)}{r}\,d\tau_y \qquad (20)$$

where $r=|y-x|$.

The function $u(x, t)$ defined by formula (20) coincides with the potential function of a volume distribution of mass over the sphere $r^2\leqslant t^2$ with density $g(y_1, y_2, y_3, t-r)$ and is the solution of problem (17), (18). The function g in formula (20) involves the values of time $t-r$ preceding the instant t at which the wave is observed; that is why expression (20) is called the *retarded* (or the *delayed*) *potential.*

2°. The Case of Two, or One, Spatial Variables. The above procedure of constructing the solution of the Cauchy problem for equation (18) can also be applied to the case of two, or one, spatial variables.

Since, by virtue of (11), the function

$$v(x_1, x_2, t, \tau)=\frac{1}{2\pi}\int_d\frac{g(y_1, y_2, \tau)\,dy_1\,dy_2}{\sqrt{(t-\tau)^2-(y_1-x_1)^2-(y_2-x_2)^2}}$$

is the solution of equation (7) satisfying the conditions

$$v(x_1, x_2\,\tau, \tau)=0 \quad\text{and}\quad \frac{\partial}{\partial t}v(x_1, x_2, t, \tau)\Big|_{t=\tau}=g(x_1, x_2, \tau)$$

we see that the expression

$$u(x_1, x_2, t) = \frac{1}{2\pi}\int_0^t d\tau \int_d \frac{g(y_1, y_2, \tau)\, dy_1\, dy_2}{\sqrt{(t-\tau)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \tag{21}$$

where d is the circle $(x_1 - y_1)^2 + (x_2 - y_2)^2 \leqslant (t - \tau)^2$, is the solution of the Cauchy problem for the non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} = -g(x_1, x_2, t) \tag{22}$$

with the initial conditions

$$u(x_1, x_2, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} u(x_1, x_2, t)|_{t=0} = 0$$

It can similarly be shown that the function

$$v(x, t, \tau) = \frac{1}{2}\int_{x-t+\tau}^{x+t-\tau} g(\tau_1, \tau)\, d\tau_1$$

is the solution of equation of oscillation of a string (13) satisfying the initial conditions

$$v(x, \tau, \tau) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} v(x, t, \tau)|_{t=\tau} = g(x, \tau)$$

and that the function

$$u(x, t) = \int_0^t v(x, t, \tau)\, d\tau = \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} g(\tau_1, \tau)\, d\tau_1 \tag{23}$$

is the solution of the non-homogeneous equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = -g(x, t) \tag{24}$$

satisfying the initial conditions

$$u(x, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} u(x, t)|_{t=0} = 0$$

Here we suppose that the functions $g(x_1, x_2, t)$ and $g(x, t)$ in equations (22) and (24) (and, consequently, in formulas (21) and (23) as well) possess continuous partial derivatives of the second and of the first order respectively.

§ 3. Well-Posed Problems for Hyperbolic Partial Differential Equations

1° Uniqueness of the Solution of the Cauchy Problem. We shall show that *the Cauchy problem stated above for the wave equation (both homogeneous and non-homogeneous) cannot have more than one solution.* For the sake of simplicity, we shall limit ourselves to the investigation of the case of one spatial variable $x_1 = x$.

Let us suppose that $u_1(x, t)$ and $u_2(x, t)$ are two solutions of the Cauchy problem for equation (24); then their difference $u_1(x, t) - u_2(x, t) = u(x, t)$ is the solution of equation of oscillation of a string (13) satisfying the initial conditions

$$u(x, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} u(x, t)|_{t=0} = 0 \tag{25}$$

Hence, we must prove that homogeneous equation (13) cannot possess a nonzero solution satisfying homogeneous initial conditions (25). On integrating the obvious identity

$$-2\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right) =$$

$$= -2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 = 0$$

over the triangular domain Δ with vertices at the points $A(x-t, 0)$, $B(x+t, 0)$ and $C(x, t)$, and using formula (GO), we obtain

$$\int_\Delta \left[-2\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\frac{\partial u}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial \xi}\right)^2 + \frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial \tau}\right)^2\right] d\xi\, d\tau =$$

$$= \int_{AB+BC+CA} -2\frac{\partial u}{\partial \xi}\frac{\partial u}{\partial \tau}\, d\tau - \left(\frac{\partial u}{\partial \xi}\right)^2 d\xi - \left(\frac{\partial u}{\partial \tau}\right)^2 d\xi = 0 \tag{26}$$

By (25), the equalities $\partial u/\partial \xi = 0$ and $\partial u/\partial \tau = 0$ hold along AB. Besides, since the line segments BC and CA are described by the equations $\xi = -\tau + x + t$ and $\xi = \tau + x - t$, we have for these line segments $d\xi = -d\tau$ and $d\xi = d\tau$ respectively. Therefore equality (26) can be rewritten in the form

$$\int_{BC} \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \tau}\right)^2 d\tau - \int_{CA} \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau}\right)^2 d\tau = 0$$

that is

$$\int_0^t \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \tau}\right)^2 d\tau + \int_0^t \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau}\right)^2 d\tau = 0$$

whence it follows that $\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \tau} = 0$ on BC and $\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau} = 0$ on AC. Consequently, at the vertex $C(x, t)$ of the triangle Δ the equalities $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0$ and $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$ hold, that is $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial t} = 0$.

Since the point $C(x, t)$ has been chosen quite arbitrarily, the equalities $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial t} = 0$ hold throughout the whole plane of the variables x, t. This means that $u(x, t) = \text{const}$. Finally, from (25) it follows that $u(x, 0) = 0$, whence we conclude that $u(x, t) = 0$ everywhere.

Now let us suppose that $u_1(x, t)$ is the solution of non-homogeneous equation (24) satisfying the non-homogeneous initial conditions

$$u_1(x, 0) = \varphi(x) \quad \text{and} \quad \frac{\partial}{\partial t} u_1(x, t)\big|_{t=0} = \psi(x) \qquad (27)$$

We shall denote by $u_2(x, t)$ the solution of homogeneous equation (13) satisfying non-homogeneous initial conditions (27) (as is known, this solution is expressed by D'Alembert's formula); then the difference $u_1(x, t) - u_2(x, t) = u(x, t)$ is obviously the solution of non-homogeneous equation (24) satisfying homogeneous initial conditions (25). Such a reduction is often used in practical problems.

2°. Correctness of the Cauchy Problem for Wave Equation. We shall show that the Cauchy problem stated for the wave equation is well-posed (correctly set). In other words, *to small variations of the initial data φ and ψ and of the right-hand member g of the wave equation there corresponds a small variation of the solution of the Cauchy problem.* This follows from Kirchhoff's, Poisson's and D'Alembert's formulas and also from formulas (16), (21) and (23).

For the sake of simplicity, we shall limit ourselves to the proof of this assertion for the case of homogeneous Cauchy problem (25) for non-homogeneous equation (24). Without loss of generality we may assume that $t > 0$.

If the difference $g_1(x, t) - g_2(x, t) = g(x, t)$ between the right-hand members of the non-homogeneous equations

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial t^2} = -g_1(x, t) \quad \text{and} \quad \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial t^2} = -g_2(x, t)$$

is sufficiently small, that is $|g(x, t)| < \varepsilon$, then, for the difference $u_1(x, t) - u_2(x, t) = u(x, t)$ between the solutions $u_1(x, t)$ and $u_2(x, t)$ of these equations satisfying homogeneous initial conditions (25), we obtain, by virtue of formula (23), the inequality

$$|u(x, t)| = \frac{1}{2}\left|\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} g(\tau_1, \tau)\, d\tau_1\right| < \varepsilon \int_0^t (t-\tau)\, d\tau = \varepsilon \frac{t^2}{2}$$

Consequently, *to a small variation of the right-hand member of non-homogeneous equation* (24) *in the domain of definition of that member there corresponds a small variation of the solution of Cauchy problem* (24), (25) *provided that this domain is bounded with respect to the variable t.* Taking into account the uniqueness property of the solution, we conclude that *the Cauchy problem for the wave equation is well-posed.*

3°. General Statement of the Cauchy Problem. Up till now we investigated the case when initial data (27) were prescribed on the hyperplane $t = 0$ in the space E_{n+1} of the variables $x_1, \ldots, x_n, t$. In this section we shall consider a more general case when the data are prescribed on a manifold L different from $t = 0$; we shall also investigate the form of the initial data themselves for which the resultant problem is well-posed. For the sake of simplicity, we shall limit ourselves to the example of equation (13).

Let us denote by D a domain in the plane of the variables x, t bounded by a piecewise smooth Jordan curve S. By $u(x, t)$ we shall denote a regular solution of equation (13) in the domain D possessing continuous partial derivatives in $D \cup S$.

On integrating the identity

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t_1}\right)=0 \tag{28}$$

over the domain D and using formula (GO), we obtain

$$\int_D\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t_1}\right)\right]dx_1\,dt_1=$$

$$=\int_S \frac{\partial u}{\partial x_1}\,dt_1+\frac{\partial u}{\partial t_1}\,dx_1=0 \tag{29}$$

Let L be a non-closed Jordan curve with continuous curvature satisfying the following two requirements: (a) every

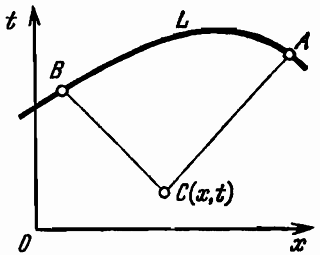

Fig. 20

straight line belonging to one of the two families $x+t=$ $=\text{const}$ and $x-t=\text{const}$ of the characteristics of equation (13) intersects the curve L at not more than one point; (b) the direction of the tangent line to the curve L does not coincide with the characteristic direction corresponding to equation (13) at any point belonging to L.

Let us suppose that the characteristics $x_1-x=t_1-t$ and $x_1-x=t-t_1$ issued from a point $C(x, t)$ intersect the curve L at two points A and B (see Fig. 20). On applying formula (29) to the domain bounded by the arc AB of the curve L and the segments of the characteristics CA and CB, we obtain

$$\int_{AB+BC+CA}\frac{\partial u}{\partial x_1}\,dt_1+\frac{\partial u}{\partial t_1}\,dx_1=0 \tag{30}$$

Since for CA and BC we have $dx_1 = dt_1$ and $dx_1 = -dt_1$ respectively, formula (30) can be written in the form

$$\int_{AB} \frac{\partial u}{\partial x_1} dt_1 + \frac{\partial u}{\partial t_1} dx_1 - 2u(C) + u(A) + u(B) = 0$$

whence we find

$$u(C) = \frac{1}{2} u(A) + \frac{1}{2} u(B) + \frac{1}{2} \int_{AB} \frac{\partial u}{\partial x_1} dt_1 + \frac{\partial u}{\partial t_1} dx_1 \quad (31)$$

If the solution $u(x, t)$ of equation (13) satisfies the conditions

$$u\,|_L = \varphi, \quad \left.\frac{\partial u}{\partial l}\right|_L = \psi \quad (32)$$

where φ and ψ are given real functions which are continuously differentiable twice and once respectively and l is a vector

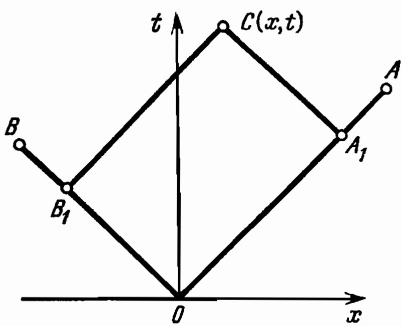

Fig. 21

defined on L such that it varies sufficiently smoothly and does not coincide with the tangent to the curve L at any point, then, on determining $\partial u/\partial x_1$ and $\partial u/\partial t_1$ from the equalities

$$\frac{\partial u}{\partial x_1} \frac{dx_1}{ds} + \frac{\partial u}{\partial t_1} \frac{dt_1}{ds} = \frac{d\varphi}{ds}, \quad \frac{\partial u}{\partial x_1} \frac{dx_1}{dl} + \frac{\partial u}{\partial t_1} \frac{dt_1}{dl} = \psi$$

where s is the arc length of L, and substituting the given values of u, $\partial u/\partial x_1$ and $\partial u/\partial t_1$ into the right-hand side of (31), we obtain the regular solution of equation (13) satisfying conditions (32).

The problem of determining the regular solution of equation (13) satisfying conditions (32) is also called the *Cauchy problem*. The argument presented above shows that *the Cauchy problem stated in the indicated way possesses a single stable solution.*

4°. **Goursat Problem.** Now let L consist of two line segments OA and OB lying on the characteristics $x_1 - t_1 = 0$ and $x_1 + t_1 = 0$ respectively. The two characteristics $x_1 - x = t - t_1$ and $x_1 - x = t_1 - t$ issued from the point $C(x, t)$ intersect OA and OB at the points $A_1\left(\frac{x+t}{2}, \frac{x+t}{2}\right)$ and $B_1\left(\frac{x-t}{2}, -\frac{x-t}{2}\right)$ respectively (see Fig. 21).

The application of formula (29) to the characteristic rectangle OA_1CB_1 results in

$$\int\limits_{OA_1+A_1C+CB_1+B_1O} \frac{\partial u}{\partial x_1}\, dt_1 + \frac{\partial u}{\partial t_1}\, dx_1 = 0$$

that is

$$\int\limits_{OA_1} \frac{\partial u}{\partial x_1}\, dx_1 + \frac{\partial u}{\partial t_1}\, dt_1 - \int\limits_{A_1C} \frac{\partial u}{\partial x_1}\, dx_1 + \frac{\partial u}{\partial t_1} dt_1 +$$

$$+ \int\limits_{CB_1} \frac{\partial u}{\partial x_1}\, dx_1 + \frac{\partial u}{\partial t_1}\, dt_1 - \int\limits_{B_1O} \frac{\partial u}{\partial x_1}\, dx_1 + \frac{\partial u}{\partial t_1}\, dt_1 =$$

$$= 2u(A_1) - 2u(O) - 2u(C) + 2u(B_1) = 0$$

whence we obtain

$$u(C) = u(A_1) + u(B_1) - u(0) \qquad (33)$$

If it is known that

$$u|_{OA} = \varphi(x), \quad u|_{OB} = \psi(x), \quad \varphi(0) = \psi(0) \qquad (34)$$

then we obtain from (33) the expression

$$u(x, t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0) \qquad (35)$$

The problem of determining the regular solution of equation (13) satisfying conditions (34) is referred to as the

Goursat problem. The single stable solution of this problem is given by formula (35). In the Goursat problem the data (the "Goursat data" or the "Goursat conditions") are prescribed on characteristic curves (straight lines) of equation (13).

5°. **Some Improperly Posed Problems.** Since formula (35) determines uniquely the solution of Goursat problem (34) in the characteristic rectangle OAO_1B with the data prescribed on the two adjoining sides OA and OB of the rectangle we cannot additionally prescribe arbitrary values of $u(x, t)$ on the sides O_1A and O_1B. It follows that the *Dirichlet problem* (in which the data are prescribed on a closed contour) for a hyperbolic partial differential equation is improperly posed (not well-posed).

With the aid of a simple example it can also be shown that *the Cauchy problem for the Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{36}$$

is improperly posed.

Indeed, let us consider the following problem: it is required to find the regular solution $u(x, y)$ of equation (36) satisfying the initial conditions

$$u(x, 0) = \tau(x) = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = \nu(x) = \frac{\sin nx}{n}$$

Since $\Delta^k \nu(x) = \frac{d^{2k}}{dx^{2k}} \nu(x) = (-1)^k n^{2k+1} \frac{\sin nx}{n^2}$, we find, using formula (26) established in Introduction, the expression

$$u(x, y) = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} (-1)^k n^{2k+1} \frac{\sin nx}{n^2} =$$

$$= \frac{\operatorname{sh} ny \sin nx}{n^2} \tag{37}$$

Taking sufficiently large n we can make the function $\nu(x)$ become arbitrarily small, while the corresponding solution of form (37) of the Cauchy problem for equation (36) is unbounded for $n \to \infty$. Consequently, the solution we have obtained is unstable, and hence the problem under consideration is not well-posed. This example was suggested by J. S. Hadamard.

The Cauchy problem and the Goursat problem are also well-posed for more general hyperbolic partial differential equations of the second order.

§ 4. General Linear Hyperbolic Partial Differential Equation of the Second Order in Two Independent Variables

1°. Riemann's Function. In Sec. 2°, § 2 of Introduction we proved that under some general assumptions concerning the coefficients of a linear hyperbolic partial differential equation of the second order there exists a non-singular transformation of the independent variables which brings the equation to the normal form

$$\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial^2 u_1}{\partial y^2}+A(x, y)\frac{\partial u_1}{\partial x}+B(x, y)\frac{\partial u_1}{\partial y}+ \\ +C(x, y)u_1=F_1(x, y) \quad (38)$$

Equation (38) can also be written in terms of the characteristic variables $\xi = x+y$ and $\eta = x-y$:

$$Lu=\frac{\partial^2 u}{\partial\xi\,\partial\eta}+a\frac{\partial u}{\partial\xi}+b\frac{\partial u}{\partial\eta}+cu=F \quad (39)$$

where

$$4a=A+B,\quad 4b=A-B,\quad 4c=C,\quad 4F=F_1$$

$$u(\xi, \eta)=u_1\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$$

It is evident that the characteristic curves of equation (39) are the straight lines ξ = const and η = const.

Under the assumption that the coefficients a and b of equation (39) are differentiable we can define the *adjoint operator of* L denoted as L^*:

$$L^*v=\frac{\partial^2 v}{\partial\xi\,\partial\eta}-\frac{\partial}{\partial\xi}(av)-\frac{\partial}{\partial\eta}(bv)+cv$$

The solution $v(\xi, \eta)$ of the *adjoint equation*

$$L^*v=\frac{\partial^2 v}{\partial\xi\,\partial\eta}-\frac{\partial}{\partial\xi}(av)-\frac{\partial}{\partial\eta}(bv)+cv=0 \quad (40)$$

which satisfies the conditions

$$
\begin{aligned}
v(\xi_1, \eta) &= \exp \int_{\eta_1}^{\eta} a(\xi_1, \eta_2)\, d\eta_2 \\
v(\xi, \eta_1) &= \exp \int_{\xi_1}^{\xi} b(\xi_2, \eta_1)\, d\xi_2
\end{aligned}
\tag{41}
$$

on the characteristics $\xi = \xi_1$ and $\eta = \eta_1$ where (ξ_1, η_1) is an arbitrary fixed point of the domain D of definition of equation (39), is called *Riemann's function.*

Let us show that *when the additional requirement that the functions $\partial a/\partial\xi$, $\partial b/\partial\eta$ and c should be continuous is fulfilled Riemann's function exists.*

Indeed, the integration of equation (40) results in

$$
\begin{aligned}
& v(\xi, \eta) - v(\xi, \eta_1) - v(\xi_1, \eta) + v(\xi_1, \eta_1) - \\
& \quad - \int_{\xi_1}^{\xi} b(\xi_2, \eta) v(\xi_2, \eta)\, d\xi_2 - \int_{\eta_1}^{\eta} a(\xi, \eta_2) v(\xi, \eta_2)\, d\eta_2 + \\
& + \int_{\xi_1}^{\xi} d\xi_2 \int_{\eta_1}^{\eta} c(\xi_2, \eta_2) v(\xi_2, \eta_2)\, d\eta_2 + \int_{\xi_1}^{\xi} b(\xi_2, \eta_1) v(\xi_2, \eta_1)\, d\xi_2 + \\
& \qquad + \int_{\eta_1}^{\eta} a(\xi_1, \eta_2) v(\xi_1, \eta_2)\, d\eta_2 = 0
\end{aligned}
\tag{42}
$$

Since, by virtue of (41), we have

$$
v(\xi, \eta) - \int_{\xi_1}^{\xi} b(\xi_2, \eta_1) v(\xi_2, \eta_1)\, d\xi_2 = 1
$$

$$
v(\xi_1, \eta) - \int_{\eta_1}^{\eta} a(\xi_1, \eta_2) v(\xi_1, \eta_2)\, d\eta_2 = 1
$$

and

$$
v(\xi_1, \eta_1) = 1
$$

equality (42) can be written in the form of Volterra's linear integral equation of the second kind with respect to $v(\xi, \eta)$:

$$v(\xi, \eta) - \int_{\xi_1}^{\xi} b(\xi_2, \eta) v(\xi_2, \eta)\, d\xi_2 - \int_{\eta_1}^{\eta} a(\xi, \eta_2) v(\xi, \eta_2)\, d\eta_2 + \\ + \int_{\xi_1}^{\xi} d\xi_2 \int_{\eta_1}^{\eta} c(\xi_2, \eta_2) v(\xi_2, \eta_2)\, d\eta_2 = 1 \qquad (43)$$

In Sec. 4°, § 2 of Chapter 5 we shall prove that equation (43) possesses a uniquely determined solution, and therefore we can consider the existence of Riemann's function to be proved.

Since Riemann's function depends not only on the variables ξ and η but also on ξ_1 and η_1 we shall denote it

$$v = R(\xi, \eta; \xi_1, \eta_1)$$

From (41) we obtain

$$\begin{gathered} \frac{\partial R(\xi_1, \eta; \xi_1, \eta_1)}{\partial \eta} - a(\xi_1, \eta) R(\xi_1, \eta; \xi_1, \eta_1) = 0 \\ \frac{\partial R(\xi, \eta_1; \xi_1, \eta_1)}{\partial \xi} - b(\xi, \eta_1) R(\xi, \eta_1; \xi_1, \eta_1) = 0 \qquad (44) \\ R(\xi_1, \eta_1; \xi_1, \eta_1) = 1 \end{gathered}$$

and

$$\begin{gathered} \frac{\partial R(\xi, \eta; \xi, \eta_1)}{\partial \eta_1} + a(\xi, \eta_1) R(\xi, \eta; \xi, \eta_1) = 0 \\ \frac{\partial R(\xi, \eta; \xi_1, \eta)}{\partial \xi_1} + b(\xi_1, \eta) R(\xi, \eta; \xi_1, \eta) = 0 \qquad (45) \\ R(\xi, \eta; \xi, \eta) = 1 \end{gathered}$$

If $u(\xi_1, \eta_1)$ is a sufficiently smooth function defined in the domain D, then the obvious identity

$$\frac{\partial^2}{\partial \xi_1 \partial \eta_1} [u(\xi_1, \eta_1) R(\xi_1, \eta_1; \xi, \eta) - \\ - R(\xi_1, \eta_1; \xi, \eta) Lu(\xi_1, \eta_1)] = \\ = \frac{\partial}{\partial \xi_1}\left[u\left(\frac{\partial R}{\partial \eta_1} - aR\right)\right] + \frac{\partial}{\partial \eta_1}\left[u\left(\frac{\partial R}{\partial \xi_1} - bR\right)\right] \qquad (46)$$

holds.

On integrating (46) with respect to ξ_1 and with respect to η_1 over the intervals $\xi_0 \leqslant \xi_1 \leqslant \xi$ and $\eta_0 \leqslant \eta_1 \leqslant \eta$ where (ξ_0, η_0) is an arbitrary point belonging to the domain D, we obtain, by virtue of (44), the equality

$$u(\xi, \eta) = u(\xi_0, \eta_0) R(\xi_0, \eta_0; \xi, \eta) +$$

$$+ \int_{\xi_0}^{\xi} R(\xi_1, \eta_0; \xi, \eta) \left[\frac{\partial u(\xi_1, \eta_0)}{\partial \xi_1} + b(\xi_1, \eta_0) u(\xi_1, \eta_0) \right] d\xi_1 +$$

$$+ \int_{\eta_0}^{\eta} R(\xi_0, \eta_1; \xi, \eta) \left[\frac{\partial u(\xi_0, \eta_1)}{\partial \eta_1} + a(\xi_0, \eta_1) u(\xi_0, \eta_1) \right] d\eta_1 +$$

$$+ \int_{\xi_0}^{\xi} d\xi_1 \int_{\eta_0}^{\eta} R(\xi_1, \eta_1; \xi, \eta) Lu(\xi_1, \eta_1) d\eta_1 \quad (47)$$

Let us put $u(\xi, \eta) = R(\xi_0, \eta_0; \xi, \eta)$ in (47), according to (45), this yields

$$\int_{\xi_0}^{\xi} d\xi_1 \int_{\eta_0}^{\eta} R(\xi_1, \eta_1; \xi, \eta) LR(\xi_0, \eta_0, \xi_1, \eta_1) d\eta_1 = 0 \quad (48)$$

Identity (48) implies that *Riemann's function* $R(\xi, \eta; \xi_1, \eta_1)$ *is a solution of the homogeneous equation*

$$LR(\xi, \eta; \xi_1, \eta_1) = 0 \quad (49)$$

with respect to the last two variables ξ_1 *and* η_1.

On the basis of (45) and (49) we can directly check the following property: *for a continuous right-hand member* $F(\xi, \eta)$ *of equation* (39) *one of its particular solutions is the function*

$$u_0(\xi, \eta) = \int_{\xi_0}^{\xi} d\xi_1 \int_{\eta_0}^{\eta} R(\xi_1, \eta_1; \xi, \eta) F(\xi_1, \eta_1) d\eta_1$$

2°. Goursat Problem. Let us take as $u(\xi, \eta)$ in identity (47) a solution of equation (39); then the integration by parts results in

$$u(\xi, \eta) = R(\xi, \eta_0; \xi, \eta)\, u(\xi, \eta_0) + R(\xi_0, \eta; \xi, \eta)\, u(\xi_0, \eta) - R(\xi_0, \eta_0; \xi, \eta)\, u(\xi_0, \eta_0) + \int_{\xi_0}^{\xi} \left[b(t, \eta_0) R(t, \eta_0; \xi, \eta) - \frac{\partial}{\partial t} R(t, \eta_0; \xi, \eta) \right] \times u(t, \eta_0)\, dt + \int_{\eta_0}^{\eta} \left[a(\xi_0, \tau) R(\xi_0, \tau; \xi, \eta) - \frac{\partial}{\partial \tau} R(\xi_0, \tau; \xi, \eta) \right] u(\xi_0, \tau)\, d\tau + \int_{\xi_0}^{\xi} dt \int_{\eta_0}^{\eta} R(t, \tau; \xi, \eta) F(t, \tau)\, d\tau \tag{50}$$

The properties of Riemann's function enumerated above obviously imply that *if* $u(\xi, \eta_0)$ *and* $u(\xi_0, \eta)$ *in formula* (50) *are replaced by arbitrary continuously differentiable functions and* $u(\xi_0, \eta_0)$ *is replaced by an arbitrary constant, we obtain a regular solution* $u(\xi, \eta)$ *of equation* (39).

Consequently, *the Goursat problem for equation* (39) *with the conditions*

$$u(\xi, \eta_0) = \varphi(\xi), \quad u(\xi_0, \eta) = \psi(\eta)$$

where $\varphi(\xi)$ *and* $\psi(\eta)$ *are given continuously differentiable functions satisfying the equality* $\varphi(\xi_0) = \psi(\eta_0)$, *has a uniquely determined stable solution* $u(\xi, \eta)$ *which is expressed by the formula*

$$u(\xi, \eta) = R(\xi, \eta_0; \xi, \eta)\, \varphi(\xi) + R(\xi_0, \eta; \xi, \eta)\, \psi(\eta) - R(\xi_0, \eta_0; \xi, \eta)\, \varphi(\xi_0) + \int_{\xi_0}^{\xi} \left[b(t, \eta_0) R(t, \eta_0; \xi, \eta) - \frac{\partial}{\partial t} R(t, \eta_0; \xi, \eta) \right] \varphi(t)\, dt + \int_{\eta_0}^{\eta} \left[a(\xi_0, \tau) R(\xi_0, \tau; \xi, \eta) - \frac{\partial}{\partial \tau} R(\xi_0, \tau; \xi, \eta) \right] \psi(\tau)\, d\tau + \int_{\xi_0}^{\xi} dt \int_{\eta_0}^{\eta} R(t, \tau; \xi, \eta)\, d\tau$$

3°. Cauchy Problem. Let us denote by σ a non-closed Jordan curve with continuous curvature lying in the domain D and possessing the property that it is not tangent to characteristic curves of equation (39) at any of its points.

Let us suppose that the characteristics $\xi_1 = \xi$ and $\eta_1 = \eta$ issued from a point P (ξ, η) intersect the arc σ at two points Q' and Q respectively, and let us denote by G the finite domain in the plane of the variables ξ, η bounded by the part QQ' of the arc σ and by the segments PQ and PQ' of the characteristics.

For arbitrary twice continuously differentiable functions $u(\xi_1, \eta_1)$ and $v(\xi_1, \eta_1)$ defined in the domain G there holds the identity

$$2(vLu - uL^*v) = \frac{\partial}{\partial \eta_1}\left(\frac{\partial u}{\partial \xi_1} v - \frac{\partial v}{\partial \xi_1} u + 2buv\right) + \\ + \frac{\partial}{\partial \xi_1}\left(\frac{\partial u}{\partial \eta_1} v - \frac{\partial v}{\partial \eta_1} u + 2auv\right)$$

On integrating this identity over the domain G, we obtain, using formula (GO), the relation

$$2\int_G (vLu - uL^*v)\, d\xi\, d\eta = \int_S \left(\frac{\partial u}{\partial \eta_1} v - \frac{\partial v}{\partial \eta_1} u + 2auv\right) d\eta_1 - \\ - \left(\frac{\partial u}{\partial \xi_1} v - \frac{\partial v}{\partial \xi_1} u + 2buv\right) d\xi_1 \quad (51)$$

where S is the boundary of the domain G.

Let $u(\xi_1, \eta_1) = u(P')$ in formula (51) be a solution of equation (39) and let $v(\xi_1, \eta_1) = v(P') = R(\xi_1, \eta_1; \xi, \eta) = R(P', P)$ where $P = P(\xi, \eta)$. Then identity (51) yields

$$u(P) = \frac{1}{2} u(Q) R(Q, P) + \frac{1}{2} u(Q') R(Q', P) + \\ + \int_G F(P') R(P', P)\, d\xi_1\, d\eta_1 - \\ - \frac{1}{2}\int_{QQ'} \left[\frac{\partial u(P')}{\partial N} R(P', P) - u(P') \frac{\partial R(P', P)}{\partial N}\right] d\sigma_{P'} - \\ - \int_{QQ'} \left[a(P') \frac{\partial \xi_1}{\partial N} + b(P') \frac{\partial \eta_1}{\partial N}\right] R(P', P)\, u(P')\, d\sigma_{P'} \quad (52)$$

where

$$\frac{\partial}{\partial N}=\frac{\partial \xi_1}{\partial \nu}\frac{\partial}{\partial \eta_1}+\frac{\partial \eta_1}{\partial \nu}\frac{\partial}{\partial \xi_1}$$

and ν is the outer normal to the curve σ at the point P'.

Conversely, if u and $\partial u/\partial N$ on the right-hand side of formula (52) are arbitrary sufficiently smooth functions defined on σ, then the function $u(P)$ determined by this formula is a solution of equation (39).

If the sought-for solution $u(P)$ of equation (39) and its derivative $\partial u(P)/\partial l$, where l is a vector defined on σ such that it is not tangent to σ at any point, are known as functions defined on σ, that is

$$u(P)=\Phi(P), \quad \frac{\partial u(P)}{\partial l}=\Psi(P), \quad P\in\sigma \tag{53}$$

where Φ and Ψ are given twice continuously differentiable and once continuously differentiable functions respectively, the derivative $\partial u/\partial N$ can always be determined uniquely.

Consequently, *formula* (52) *gives the solution of Cauchy problem* (39), (53). The process of the derivation of formula (52) obviously shows that *the solution of this problem is unique and stable.*

Besides the problems considered in the present chapter, an important role is also played in applications by the so-called *mixed problems* for hyperbolic partial differential equations, but here we shall not dwell on them.

CHAPTER 4

PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

§ 1. Heat Conduction Equation. First Boundary-Value Problem

1°. Extremum Principle. The simplest example of a parabolic partial differential equation is the *heat conduction equation*:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \tag{1}$$

Since the differential equation describing the characteristic curves corresponding to equation (1) has the form $dt^2 = 0$, equation (1) possesses only one family of characteristic curves $t =$ const which are straight lines parallel to the x-axis.

Let us consider a domain D in the plane of the variables x, t which is bounded by segments OA and BN of the straight lines $t = 0$ and $t = T$ respectively where T is a positive number and by two curves OB and AN each of which intersects any straight line $t =$ const at one point; we shall also suppose that if the equations of these curves are given in the forms $x = \alpha(t)$ and $x = \beta(t)$ respectively, then $\alpha(t) < \beta(t)$ for $0 \leqslant t \leqslant T$.

Let us denote by S the part of the boundary of the domain D consisting of OA, OB and AN; here we suppose that $B \in S$ and $N \in S$ (see Fig. 22).

A function $u(x, t)$ possessing continuous partial derivatives $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ on the set $D \cup BN$ and satisfying equation (1) in the domain D will be referred to as a *regular solution* of that equation.

The extremum principle: *a regular solution* $u(x, t)$ *of equation* (1) *which is continuous* in $D \cup S \cup BN$ ***attains its extremum*** on S.

Here we limit ourselves to proving this principle for the case of maximum.

We shall denote by M the maximum of $u(x, t)$ on the closed set $D \cup S \cup BN$. Let us suppose that the function

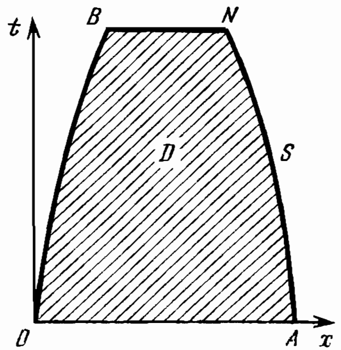

Fig. 22

$u(x, t)$ attains its maximum M not on S but at a point $(x_0, t_0) \in D \cup BN$. It is easy to show that this assumption leads to a contradiction.

Indeed, let us construct the function

$$v(x, t) = u(x, t) + a(T - t) \tag{2}$$

where a is a positive constant. Since $0 \leqslant t \leqslant T$, formula (2) implies that

$$u(x, t) \leqslant v(x, t) \leqslant u(x, t) + aT \tag{3}$$

everywhere in $D \cup S \cup BN$.

Let M_u^S and M_v^S be the maxima of $u(x, t)$ and $v(x, t)$ on S respectively. By the hypothesis, $M_u^S < M$. Let us choose the number a so that the inequality

$$a < \frac{M - M_u^S}{T} \tag{4}$$

holds.

From (3) and (4) we obtain

$$M_v^S \leqslant M_u^S + aT < M_u^S + \frac{M - M_u^S}{T} T = M = \\ = u(x_0, t_0) \leqslant v(x_0, t_0)$$

It follows that the function $v(x, t)$ cannot attain its maximum on S. Consequently, the maximum of this function on $D \cup S \cup BN$ is attained at a point $(x_1, t_1) \in D \cup BN$.

Let us first suppose that $(x_1, t_1) \in D$. Since (x_1, t_1) is a point of maximum of the function $v(x, t)$ on $D \cup S \cup BN$, we have $\partial v/\partial t = 0$ and $\partial^2 v/\partial x^2 \leqslant 0$ at that point, that is

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \geqslant 0 \tag{5}$$

Now let $(x_1, t_1) \in BN$. Since the function $v(x, t)$ attains its maximum on the set $D \cup S \cup BN$ at the point (x_1, t_1), there must be $\partial v/\partial t \geqslant 0$ at that point. Further, since (x_1, T) is the point of maximum of $v(x, T)$ considered as function of x, we must have $\frac{\partial^2 v(x_1, T)}{\partial x_1^2} \leqslant 0$. Consequently, inequality (5) is fulfilled at the point (x_1, T) as well.

On substituting the values of $\partial v/\partial t$ and $\partial^2 v/\partial x^2$ found from equality (2) into the left-hand side of (5), we derive

$$\frac{\partial u}{\partial t} - a - \frac{\partial^2 u}{\partial x^2} \geqslant 0 \quad \text{for} \quad x = x_1, \quad t = t_1$$

whence, since $u(x, t)$ is a solution of equation (1), we obtain $-a \geqslant 0$, which is impossible because $a > 0$.

The contradiction we have arrived at proves the assertion of the extremum principle for the case of maximum. The case of minimum is considered quite analogously.

2°. First Boundary-Value Problem for Heat Conduction Equation. The extremum principle proved in the foregoing section makes it possible to establish *the uniqueness and the stability of the solution of the following problem called the first boundary-value problem for the heat conduction equation: it is required to find the regular solution $u(x, t)$ of equation (1) in the domain D which is continuous in $D \cup S \cup BN$ and satisfies the conditions*

$$\begin{gathered} u\,|_{OB} = \psi_1(t), \quad u\,|_{AN} = \psi_2(t), \quad u\,|_{OA} = \varphi(x) \\ \psi_1(0) = \varphi(0), \quad \psi_2(A) = \varphi(A) \end{gathered} \tag{6}$$

where ψ_1, ψ_2 and φ are given real continuous functions.

Indeed, let us suppose that $u_1(x, t)$ and $u_2(x, t)$ are two regular solutions of equation (1) satisfying boundary conditions (6); then the function $u(x, t) = u_1(x, t) - u_2(x, t)$

is a regular solution of equation (1) which turns into zero on S. Consequently, according to the extremum principle, we have $u(x, t) = 0$ in $D \cup S \cup BN$, whence it follows that the solution of the first boundary-value problem (1), (6) is unique.

Now let us suppose that the absolute value of the difference between the boundary values on S of two regular solutions $u_1(x, t)$ and $u_2(x, t)$ of equation (1) is less than ε ($\varepsilon > 0$). Then, by virtue of the extremum principle, we must have $|u_1(x, t) - u_2(x, t)| < \varepsilon$ everywhere in $D \cup S \cup BN$, which means that the solution of the first boundary-value problem depends continuously on the boundary data prescribed on S, and hence we have proved the stability of the solution of this problem.

Now we shall prove the *existence of the solution* of the first boundary-value problem for equation (1) under the assumption that OB and AN are the segments of straight lines connecting the points $O(0, 0)$, $B(0, T)$ and $A(l, 0)$, $N(l, T)$ respectively and that

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \leqslant t < T \tag{7}$$

and

$$u(x, 0) = \varphi(x), \quad 0 \leqslant x \leqslant l \tag{8}$$

where $\varphi(x)$ is a continuously differentiable function defined in the interval $0 \leqslant x \leqslant l$ which turns into zero for $x = 0$ and for $x = l$.

As is known from the course of mathematical analysis, in the interval $0 \leqslant x \leqslant l$ the function $\varphi(x)$ can be expanded into absolutely and uniformly convergent Fourier's series

$$\varphi(x) = \sum_{k=1}^{\infty} a_k \sin \frac{\pi k}{l} x \tag{9}$$

where

$$a_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{\pi k}{l} x \, dx \quad (k = 1, 2, \ldots)$$

Using formula (40) established in Introduction in which we put $n = 2$, $x_1 = x$, $x_2 = t$, $\tau_k(x) = \sin \frac{\pi k}{l} x$ we obtain the regular solution

$$u_k(x, t) = e^{-\pi^2 k^2 t/l^2} \sin \frac{\pi k}{l} x \tag{10}$$

of equation (1) satisfying the boundary conditions $u_k(0, t) = u(l, t) = 0$, $u_k(x, 0) = \sin \frac{\pi k}{l} x$.

It is evident that the function $u(x, t)$ represented as the sum of the series

$$u(x, t) = \sum_{k=1}^{\infty} a_k e^{-\frac{\pi^2 k^2 t}{l^2}} \sin \frac{\pi k}{l} x \tag{11}$$

is the sought-for solution of boundary-value problem (1), (7), (8). For $t > 0$, the absolute and uniform convergence of series (11), in the neighbourhood of the point (x, t), and of the series obtained from (11) by means of the differentiation with respect to x and with respect to t any number of times, follows from the fact that

$$\lim_{k\to\infty} \left(\frac{\pi k}{l}\right)^m e^{-\frac{\pi^2 k^2}{l^2} t} = 0 \quad (m = 0, 1, \ldots)$$

When initial data (8) are prescribed on a segment of the straight line $t = t_0$ and when $t_0 \leqslant t \leqslant T$ in conditions (7) the solution of the first boundary-value problem in the rectangle $0 < x < l$, $t_0 < t < T$ is also expressed by formula (11) in which t should be replaced by $t - t_0$.

It should be noted that the series on the right-hand side of formula (11) may not make sense at all for $t < t_0$. That is why the first boundary-value problem for equation (1) is not stated when $t < t_0$ where $t = t_0$ is the set on which the boundary data are prescribed.

All that was established above obviously remains valid in a more general case when the number of spatial variables exceeds unity; the only distinction is that in this case we should take multiple series instead of one-fold series (9) and (11).

§ 2. Cauchy-Dirichlet Problem

1°. Statement of Cauchy-Dirichlet Problem and the Proof of the Existence of Its Solution. Let D be the infinite strip

$-\infty < x < \infty$, $0 \leqslant t \leqslant T$ where T is a fixed positive number, the case $T = \infty$ not being excluded (see Fig. 23).

A bounded and continuous function $u(x, t)$ defined in the strip D, possessing continuous partial derivatives $\partial^2 u/\partial x^2$

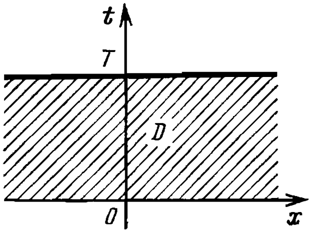

Fig. 23

and $\partial u/\partial t$ inside D and satisfying equation (1) will be referred to as a *regular solution* of that equation.

The *Cauchy-Dirichlet problem* is stated thus: *it is required to find the regular solution* $u(x, t)$ *of equation* (1) *in the strip* D *satisfying the condition*

$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty \tag{12}$$

where $\varphi(x)$ $(-\infty < x < \infty)$ *is a given real bounded continuous function.*

As was already mentioned in Sec. 3°, § 3 of Introduction, the function

$$E(x, \xi, t, 0) = \frac{1}{\sqrt{t}} e^{-\frac{(\xi - x)^2}{4t}}, \quad t > 0 \tag{13}$$

satisfies equation (1) at all the points (x, t) of the half-plane $t > 0$.

Let us prove that *the function* $u(x, t)$ *determined by the formula*

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(\xi - x)^2}{4t}} d\xi \tag{14}$$

is the solution of the Cauchy-Dirichlet problem.

From the course of mathematical analysis it is known that the integral on the right-hand side of (14) is uniformly convergent in a neighbourhood of any interior point (x, t) of the strip D.

The change of the variable of integration $\xi = x + 2\eta\sqrt{t}$ brings formula (14) to the form

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2\eta\sqrt{t})\, e^{-\eta^2}\, d\eta \tag{15}$$

Since $\sup\limits_{-\infty<x<\infty} |\varphi(x)| < M$ where M is a positive number and since the integral on the right-hand side of (15) is absolutely convergent, we have

$$|u(x, t)| < \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta$$

whence, since

$$\int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta = \sqrt{\pi} \tag{16}$$

it follows that

$$|u(x, t)| \leqslant M$$

The integrals resulting from the differentiation any number of times of the integral on the right-hand side (14) with respect to x and t under the integral sign are uniformly convergent in a neighbourhood of every point (x, t) $(t > 0)$, and the function $E(x, \xi, t, 0)$ satisfies equation (1) for $t > 0$; this leads to the conclusion that the function $u(x, t)$ determined by formula (14) satisfies equation (1) in the strip D.

Finally, on passing to the limit for $t \to 0$ (this operation is legitimate because the integral converges uniformly in a neighbourhood of every point $(x, 0)$ for $t \geqslant 0$), we obtain from (15), by virtue of (16), the limiting relation

$$\lim_{t\to 0} u(x, t) = \varphi(x)$$

2°. Uniqueness and Stability of the Solution of Cauchy-Dirichlet Problem. The uniqueness and the stability of the solution of the Cauchy-Dirichlet problem are immediate consequences of the following proposition (*the extremum* *principle for a strip*): *a regular solution* $u(x, t)$ *of equation* (1) *in the strip* D *satisfies the inequalities*

$$m \leqslant u(x, t) \leqslant M \tag{17}$$

where

$$m = \inf u(x, 0), \quad M = \sup u(x, 0), \quad -\infty < x < \infty$$

To prove the first inequality in (17) let us consider the function $v(x, t) = x^2 + 2t$ which is a solution of equation (1).

We shall denote by n the infimum of $u(x, t)$ for $(x, t) \in D$ and consider the function

$$w(x, t) = u(x, t) - m + \varepsilon \frac{v(x, t)}{v(x_0, t_0)} \tag{18}$$

where ε is an arbitrary positive number and (x_0, t_0) is an arbitrary fixed point inside the strip D.

The function $w(x, t)$ expressed by formula (18) satisfies equation (1); for $t = 0$ we have

$$w(x, 0) = u(x, 0) - m + \varepsilon \frac{x^2}{x_0^2 + 2t_0} \geqslant 0 \tag{19}$$

and for $|x| = |x_0| + \sqrt{\frac{(m-n)\,v(x_0, t_0)}{\varepsilon}}$ we have

$$w(x, t) \geqslant u(x, t) - n \geqslant 0 \tag{20}$$

Inequalities (19) and (20) together with the extremum principle proved in the foregoing section (which should be applied to the rectangle

$$0 \leqslant t \leqslant T, \quad -|x_0| - \sqrt{\frac{(m-n)\,v(x_0, t_0)}{\varepsilon}} \leqslant x \leqslant |x_0| + \sqrt{\frac{(m-n)\,v(x_0, t_0)}{\varepsilon}}$$

containing the point (x_0, t_0)) imply that

$$w(x_0, t_0) = u(x_0, t_0) - m + \varepsilon \geqslant 0$$

that is $u(x_0, t_0) \geqslant m - \varepsilon$. Since ε is quite arbitrary, it follows that $u(x_0, t_0) \geqslant m$. Thus, $u(x, t) \geqslant m$ everywhere in D.

Replacing $u(x, t)$ by $-u(x, t)$ and repeating the above argument, we readily prove the second inequality in (17).

Problem (1), (12) is called the Cauchy-Dirichlet problem because, if the variable t is interpreted as time, relation (12) can be regarded as an initial condition. However, this relation can also be considered as a boundary condition set for the boundary $t = 0$ of the upper half-plane $t > 0$ of the variables x, t.

3°. **Non-Homogeneous Heat Conduction Equation.** In this section we shall consider the non-homogeneous equation $\partial^2 u/\partial x^2 - \partial u/\partial t = -g(x, t)$ where $g(x, t)$ is a given real bounded continuous function defined for $-\infty < x < \infty$, $0 \leqslant t < \infty$. Let the initial data be prescribed on the straight line $t = \tau$ (instead of $t = 0$) where τ is a fixed positive number. For the function

$$v(x, t, \tau) = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t-\tau}} \int_{-\infty}^{\infty} e^{-\frac{(\xi - x)^2}{4(t-\tau)}} g(\xi, \tau)\, d\xi, \quad t > \tau$$

we can readily show in just the same way as in the foregoing section, that

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = 0 \quad \text{for} \quad t > \tau \quad \text{and} \quad v(x, \tau, \tau) = g(x, \tau)$$

From these equalities we conclude that *the function*

$$u(x, t) = \int_0^t v(x, t, \tau)\, d\tau$$

is the solution of the non-homogeneous heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = -g(x, t)$$

satisfying the condition $u(x, 0) = 0$.

§ 3. On Smoothness of Solutions of Partial Differential Equations

1°. **The Case of Elliptic and Parabolic Partial Differential Equations.** As was already proved in Sec. 5°, § 6 of Chapter 2, a harmonic function $u(x, y)$ in a domain D is an analytic function of the variables x and y in that domain. Moreover, it can be proved that the solutions of linear elliptic partial differential equations with analytic coefficients are analytic functions in the domain where they are regular.

From Poisson's formula (see Sec. 2°, § 2, Chapter 1) it follows that the solution $u(x, y)$ of the Dirichlet problem for Laplace's equation in the circle $|z| < 1$ is an *analytic* function of the real variables x and y for $|z| < 1$ when the only requirement that the function g describing the boundary values should be continuous on the contour $|z| = 1$ is fulfilled (even when g is continuous on $|z| = 1$ but is not differentiable at any point of the contour).

In Sec. 2°, § 1 of the present chapter it was shown that the solution $u(x, t)$ of the first boundary-value problem (7), (8) for heat conduction equation (1) possesses partial derivatives of all orders with respect to the variables x and t in the domain D: $0 < x < l$, $0 < t < T$ provided that the first derivative of the function $u(x, 0) = \varphi(x)$ is continuous. Similarly, in Sec. 1°, § 2 of the present chapter we concluded from formula (14) that the boundedness and the continuity of the function $\varphi(x) = u(x, 0)$ $(-\infty < x < \infty)$ guarantee the existence of partial derivatives of *all orders* of the solution $u(x, t)$ of Cauchy-Dirichlet problem (12) for equation (1).

2°. The Case of Hyperbolic Partial Differential Equations. The assertions of the foregoing section are not valid for the Cauchy problem and the Goursat problem for the equation of oscillation of a string.

For instance, formula (35) of Chapter 3 implies that *the type of smoothness of the solution $u(x, t)$ of Goursat problem* (13), (34) *is the same as that of the functions* $\varphi(x)$ and $\psi(x)$ *describing the given data*, that is for the partial derivatives of the kth order of the sought-for solution $u(x, t)$ of this problem to exist we must require that the derivatives of the kth order of the functions $\varphi(x)$ and $\psi(x)$ should exist. The function $u(x, t)$ determined by this formula is called, irrespective of the type of smoothness of the functions $\varphi(x)$ and $\psi(x)$, the *generalized solution* of problem (13), (34) stated in Chapter 3. If the function $\varphi(x)$ (or $\psi(x)$) has a discontinuity for $x = \xi$, then the function $u(x, t)$ also has a discontinuity on the characteristic $x + t = 2\xi$ (or $x - t = 2\xi$), that is *the discontinuities of the functions* $\varphi(x)$ *and* $\psi(x)$ *describing the given data generate discontinuities of the wave* $u(x, t)$ *on the characteristics of the equation of oscillation of a string.*

CHAPTER 5

INTEGRAL EQUATIONS

§ 1. Iterative Method for Solving Integral Equations

1°. General Remarks. In this chapter we shall study the *Fredholm integral equations of the second kind*

$$\varphi(x) - \lambda \int\limits_{D(S)} K(x, y)\,\varphi(y)\,dy = f(x), \quad x \in D \text{ (or } x \in S) \qquad (*)$$

where the integral is taken over a bounded domain D of the Euclidean space E_n or over its smooth boundary S, the *kernel* $K(x, y)$ and the *right member* $f(x)$ are given real continuous functions of the points x and y, $\varphi(x)$ is the *unknown function* and λ is a *real parameter.*

The integral equation

$$\varphi^0(x) - \lambda \int\limits_{D(S)} K(x, y)\,\varphi^0(y)\,dy = 0 \qquad (1)$$

is called the *homogeneous integral equation* corresponding to the given Fredholm integral equation of the second kind of form (*). Further, the homogeneous integral equation

$$\psi(x) - \lambda \int\limits_{D(S)} K(y, x)\,\psi(y)\,dy = 0 \qquad (2)$$

is referred to as the *homogeneous integral equation adjoint to* (or *associated with*) integral equation (1).

Below we state and prove the basic propositions of the theory of integral equations for the special case when D is a finite interval (a, b) lying on the real axis, that is for the equation

$$\varphi(x) - \lambda \int_a^b K(x, y)\,\varphi(y)\,dy = f(x) \qquad (3)$$

We shall also consider the integral equation with a variable upper limit of integration:

$$\varphi(x) - \lambda \int_a^x K(x, y)\, \varphi(y)\, dy = f(x), \quad x > a \tag{4}$$

Equation (4) is called *Volterra's integral equation of the second kind.*

It can easily be seen that *the general solution* $\Phi(x)$ *of the Fredholm integral equation of the second kind, provided it exists, has the form*

$$\Phi(x) = \varphi^0(x) + \varphi(x) \tag{5}$$

where $\varphi^0(x)$ *is the general solution of homogeneous equation* (1) *corresponding to* (3) *which in the case under consideration has the form*

$$\varphi^0(x) - \lambda \int_a^b K(x, y)\, \varphi^0(y)\, dy = 0 \tag{6}$$

and $\varphi(x)$ *is a particular solution of non-homogeneous equation* (3).

Indeed, if $\Phi(x)$ and $\varphi(x)$ are the general solution and a particular solution respectively of non-homogeneous equation (3), then their difference $\varphi^0(x) = \Phi(x) - \varphi(x)$ satisfies equation (6), which proves equality (5).

2°. Solution of Fredholm Integral Equation of the Second Kind for Small Values of the Parameter Using Iterative Method. *In the case when the parameter* λ *satisfies the condition*

$$|\lambda| < \frac{1}{M} \tag{7}$$

where M *is a positive number such that*

$$\int_a^b |K(x, y)|\, dy \leqslant M, \quad a \leqslant x \leqslant b \tag{8}$$

the solution $\varphi(x)$ *of equation* (3) *exists and can be constructed using the iterative method (the method of successive approximations).*

The idea of this method is that the function $\varphi(x)$ is constructed as the limit of the sequence of functions

$$\varphi_0(x)=f(x), \quad \varphi_n(x)=f(x)+\lambda\int_a^b K(x, y)\varphi_{n-1}(y)\,dy \quad (9)$$
$$n=1, 2, \ldots$$

As is known from the course of mathematical analysis, the convergence of sequence (9) is equivalent to the convergence of the series

$$\varphi_0(x)+\sum_{n=1}^{\infty}[\varphi_n(x)-\varphi_{n-1}(x)] \quad (10)$$

From (8) we obtain the inequalities

$$|\varphi_0(x)|\leqslant m$$
$$|\varphi_n(x)-\varphi_{n-1}(x)|\leqslant m|\lambda|^n M^n \quad (n=1, 2, \ldots) \quad (11)$$

where $m=\max\limits_{a\leqslant x\leqslant b}|f(x)|$.

Thus, the absolute value of every term of series (10) does not exceed the corresponding term of the positive number series $\sum\limits_{n=0}^{\infty} m|\lambda|^n M^n$, the latter being convergent by virtue of inequality (7). Consequently, series (10) is uniformly and absolutely convergent; therefore sequence (9) of continuous functions is also uniformly and absolutely convergent, its limit being a continuous function $\varphi(x)$:

$$\varphi(x)=\lim_{n\to\infty}\varphi_n(x)=\varphi_0(x)+\sum_{n=1}^{\infty}[\varphi_n(x)-\varphi_{n-1}(x)]$$

Passing to the limit for $n\to\infty$ in the equality

$$\varphi_n(x)=f(x)+\lambda\int_a^b K(x, y)\varphi_{n-1}(y)\,dy$$

(all conditions guaranteeing that this operation is legitimate are fulfilled) we obtain

$$\varphi(x)=f(x)+\lambda\int_a^b K(x, y)\varphi(y)\,dy$$

which means that the function $\varphi(x)$ is a solution of integral equation (3).

It can easily be shown that *equation* (3) *has no solutions other than the function* $\varphi(x)$ we have constructed. Indeed, let us suppose that not only the function $\varphi(x)$ is a solution of equation (3) but another function $\psi(x)$ is also a solution. Then the difference $\theta(x)=\varphi(x)-\psi(x)$ of these solutions must satisfy homogeneous equation (6), that is

$$\theta(x)=\lambda\int_a^b K(x, y)\,\theta(y)\,dy$$

whence we find that

$$\theta_0\leqslant|\lambda|M\theta_0 \quad \text{where} \quad \theta_0=\max_{a\leqslant x\leqslant b}|\theta(x)|$$

The inequality we have obtained contradicts inequality (7) when

$$\theta_0\neq 0$$

Consequently,

$$\theta_0=0$$

and therefore $\theta(x)=0$ that is, $\psi(x)=\varphi(x)$.

3°. **Volterra Integral Equation of the Second Kind.** On repeating the above argument in the case of Volterra's integral equation of the second kind (4), we obtain

$$\varphi_0(x)=f(x), \quad \varphi_n(x)=f(x)+\lambda\int^x K(x, y)\,\varphi_{n-1}(y)\,dy \qquad (12)$$

and

$$|\varphi_n(x)-\varphi_{n-1}(x)|\leqslant m\frac{|\lambda|^n M_*^n (x-a)^n}{n!}, \quad n=1, 2, \ldots \qquad (13)$$

$$m=\max|f(x)|, \quad M_*=\max|K(x, y)|$$

Since the functional series

$$m\sum_{n=0}^{\infty}\frac{|\lambda|^n M_*^n (x-a)^n}{n!}=me^{|\lambda|M_*(x-a)}$$

with positive terms is uniformly convergent for any finite value of the parameter λ, *inequalities* (13) *imply that sequence* *of functions* (12) *is uniformly convergent, and therefore the function*

$$\varphi^*(x) = \lim_{n\to\infty} \varphi_n(x)$$

is a solution of integral equation (4).

Let us prove that *for any fixed value of the parameter* λ *equation* (4) *cannot possess more than one solution.*

To this end let us suppose that $\varphi(x)$ and $\psi(x)$ are two continuous solutions of equation (4). Then the difference $\theta(x) = \varphi(x) - \psi(x)$ of these solutions must satisfy the homogeneous equation

$$\theta(x) = \lambda \int_a^x K(x, y)\,\theta(y)\,dy \tag{14}$$

and consequently

$$|\theta(x)| \leqslant |\lambda|\, M_* m_* (x - a) \tag{15}$$

where $M_* = \max |K(x, y)|$ and $m_* = \max |\theta(x)|$. From (14), by virtue of (15), we derive the inequality

$$|\theta(x)| \leqslant |\lambda|^2 M_*^2 m_* \frac{(x-a)^2}{2}$$

On repeating this procedure n times we arrive at the inequality

$$|\theta(x)| \leqslant |\lambda|^n M_*^n m_* \frac{(x-a)^n}{n!} \tag{16}$$

which holds for any natural n. From inequality (16) we obtain, on passing to the limit for $n \to \infty$, the equality $\theta(x) = 0$, which means that $\psi(x) = \varphi(x)$.

Thus, we have come to the conclusion that *Volterra's integral equation* (4) *has a uniquely determined solution for any finite value of the parameter* λ *on condition that its kernel* $K(x, y)$ *and its right member* $f(x)$ *are continuous.* Here lies the essential difference between Volterra's integral equation of the second kind and Fredholm's integral equation of the second kind: later we shall show that Fredholm's equation may not possess solutions for some values of λ and that for some other values of λ it can have several solutions.

§ 2. Fredholm Theorems

1°. Fredholm Integral Equation of the Second Kind with Degenerate Kernel. The kernel $K(x, y)$ of integral equation (3) is said to be *degenerate* when it has the form

$$K(x, y) = \sum_{i=1}^{N} p_i(x) q_i(y) \tag{17}$$

where $p_i(x)$ and $q_i(y)$ $(i = 1, \ldots, N)$ are given real continuous functions defined in the intervals $a \leqslant x \leqslant b$ and $a \leqslant y \leqslant b$ respectively. Without loss of generality, we can assume that the systems of functions $\{p_i(x)\}$ and $\{q_i(y)\}$ are linearly independent.

Let us consider Fredholm's integral equation of the second kind

$$\varphi(x) - \lambda \sum_{i=1}^{N} \int_a^b p_i(x) q_i(y) \varphi(y)\, dy = f(x) \tag{18}$$

Integral equation (18) can be written in the form

$$\varphi(x) = f(x) + \lambda \sum_{i=1}^{N} c_i p_i(x) \tag{19}$$

where

$$c_i = \int_a^b q_i(y) \varphi(y)\, dy, \quad i = 1, \ldots, N \tag{20}$$

are some unknown constants.

Let us try to choose the constants c_i $(i = 1, \ldots, N)$ so that the function $\varphi(x)$ specified by formula (19) satisfies integral equation (18). To this end we substitute expression (19) of $\varphi(x)$ into the left-hand side of (18). After some simple calculations we obtain

$$\sum_{i=1}^{N} p_i(x) \left[c_i - \int_a^b q_i(y) f(y)\, dy - \lambda \sum_{j=1}^{N} \int_a^b c_j q_i(y) p_j(y)\, dy \right] = 0$$

whence, since the functions $p_i(x)$ $(i=1, \ldots, N)$ are linearly independent, it follows that

$$c_i - \int_a^b q_i(y) f(y)\, dy - \lambda \sum_{j=1}^{N} c_j \int_a^b q_i(y) p_j(y)\, dy = 0$$

Hence, we have

$$c_i - \lambda \sum_{j=1}^{N} \alpha_{ij} c_j = \gamma_i, \quad i=1, \ldots, N \tag{21}$$

where

$$\alpha_{ij} = \int_a^b q_i(y) p_j(y)\, dy \quad \text{and} \quad \gamma_i = \int_a^b f(y) q_i(y)\, dy \tag{22}$$

Thus, the problem of the determination of the solution $\varphi(x)$ of integral equation (18) has been reduced to the solution of the system of algebraic linear equations (21).

The homogeneous integral equation corresponding to (18) has the form

$$\varphi^0(x) - \lambda \int_a^b \sum_{i=1}^{N} p_i(x) q_i(y) \varphi^0(y)\, dy = 0 \tag{23}$$

and it can be in just the same way reduced to the homogeneous algebraic linear system

$$c_i^0 - \lambda \sum_{j=1}^{N} \alpha_{ij} c_j^0 = 0 \tag{24}$$

corresponding to (21).

As is known, in the theory of algebraic linear systems of form (21) the fundamental role is played by the matrix

$$M(\lambda) = \begin{Vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & \ldots & -\lambda\alpha_{1N} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & \ldots & -\lambda\alpha_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ -\lambda\alpha_{N1} & -\lambda\alpha_{N2} & \ldots & 1-\lambda\alpha_{NN} \end{Vmatrix}$$

In linear algebra it is proved that when the condition

$$\det M(\lambda) \neq 0 \tag{25}$$

is fulfilled, system (21) is always solvable (for any right members γ_i) and that its solution is unique.

The expression det $M(\lambda)$ is a polynomial of the Nth degree with respect to λ, and consequently condition (25) may be violated only for a finite number of values of λ: $\lambda_1, \ldots, \lambda_m$ $(m \leqslant N)$. The numbers $\lambda_1, \ldots, \lambda_m$ are the zeros of the polynomial det $M(\lambda)$; they are called the *characteristic values* (*numbers*) *of the kernel* $K(x, y)$.

Thus, for every *finite* value of λ different from λ_k $(k = 1, \ldots, m)$ system (21) possesses a single solution $c_1, \ldots, c_N$. On substituting the solution $c_1, \ldots, c_N$ found from system (21) into the right-hand side of formula (19), we obtain the solution $\varphi(x)$ of integral equation (18). We have thus proved the following theorem: *if λ is not a characteristic value of the kernel $K(x, y)$, integral equation* (18) *is solvable for any continuous right member $f(x)$, and its solution is unique* (this is *Fredholm's first theorem*).

According to formula (1), the adjoint integral equation corresponding to (23) has the form

$$\psi(x) - \lambda \sum_{i=1}^{N} \int_a^b p_i(y)\, q_i(x)\, \psi(y)\, dy = 0 \tag{26}$$

Equation (26) is equivalent to the homogeneous algebraic linear system

$$d_i - \lambda \sum_{j=1}^{N} \alpha_{ji} d_j = 0 \tag{27}$$

where

$$d_i = \int_a^b p_i(y)\, \psi(y)\, dy; \quad i = 1, \ldots, N$$

System (27) is the adjoint algebraic system corresponding to (24).

If $\lambda = \lambda_k$ $(k = 1, \ldots, m)$ and if the rank of the matrix $M(\lambda)$ is equal to r, then, as is known from linear algebra, homogeneous system (24) and its adjoint system (27) have $N - r$ linearly independent solutions each:

$$c_1^{0j}, \ldots, c_N^{0j} \quad (j = 1, \ldots, N - r)$$

and

$$d_1^j, \ldots, d_N^j \quad (j=1, \ldots, N-r)$$

On substituting the solutions of systems (24) and (27) thus found into the right-hand sides of the formulas

$$\varphi_l^0(x)=\lambda \sum_{i=1}^{N} c_i^{0l} p_i(x), \quad (l=1, \ldots, N-r)$$

and

$$\psi_l(x)=\lambda \sum_{i=1}^{N} d_i^l q_i(x), \quad (l=1, \ldots, N-r)$$

we obtain $N-r$ linearly independent solutions of homogeneous integral equation (23) and of equation (26) respectively. Hence, *homogeneous integral equation* (23) *corresponding to* (18) *and adjoint (homogeneous) integral equation* (26) *corresponding to* (23) *have exactly* $N-r$ *linearly independent solutions each* (this is *Fredholm's second theorem*).

The functions $\varphi_l^0(x)$ $(l=1, \ldots, N-r)$ are called the *eigenfunctions of the kernel* $K(x, y)$ *corresponding to the characteristic value* λ_k.

From linear algebra it is known that for $\lambda=\lambda_k$ $(k=1, \ldots, m)$ system (21) may not be solvable for some right members. For the system to be solvable it is necessary and sufficient that the numbers γ_i $(i=1, \ldots, N)$ should satisfy the conditions

$$\sum_{j=1}^{N} \gamma_j d_j^l=0, \quad (l=1, \ldots, N-r) \tag{28}$$

By (22), conditions (28) are equivalent to the system of equalities

$$\int_a^b f\psi_l(x)\,dx=\lambda \sum_{i=1}^{N} d_i^l \int_a^b q_i(x) f(x)\,dx=0, \tag{29}$$

$$l=1, \ldots, N-r$$

Thus, we have come to the conclusion that *for integral equation* (18) *to be solvable for* $\lambda=\lambda_k$ $(k=1, \ldots, m)$ *it is necessary and sufficient that its right member* $f(x)$ *should be* *orthogonal to all the solutions* $\psi_l(x)$ $(l=1, \ldots, N-r)$ *of adjoint homogeneous integral equation* (26) *corresponding to* (23) (this is *Fredholm's third theorem*).

2°. The Notions of Iterated and Resolvent Kernels. In Sec. 2°, § 1 of the present chapter we proved that when inequality (7) holds successive approximations (9) converge to the solution $\varphi(x)$ of integral equation (3) (we supposed that the functions $K(x, y)$ and $f(x)$ were continuous).

The functions

$$K_n(x, y)=\int_a^b K(x, y_1) K_{n-1}(y_1, y)\, dy_1 \quad (n=2, 3, \ldots)$$

are called *iterated kernels.*

Using iterated kernels we can rewrite successive approximations (9) in the form

$$\varphi_n(x)=f(x)+\lambda \int_a^b \sum_{j=1}^{n} \lambda^{j-1} K_j(x, y) f(y)\, dy \tag{30}$$

The repetition of the argument which was used in Sec. 2°, § 1 of the present chapter in the proof of the convergence of sequence (9) shows that if condition (7) is fulfilled, the series

$$\sum_{j=1}^{\infty} \lambda^{j-1} K_j(x, y)$$

is uniformly convergent for $a \leqslant x \leqslant b$, $a \leqslant y \leqslant b$. The sum $R(x, y; \lambda)$ of this series is called the *resolvent kernel corresponding to the kernel* $K(x, y)$ (or *to integral equation* (3)). From equality (30) it is obviously seen that the resolvent kernel makes it possible to rewrite the expression for the solution $\varphi(x)$ of equation (3) in the form

$$\varphi(x)=f(x)+\lambda \int_a^b R(x, y; \lambda) f(y)\, dy \tag{31}$$

It should be noted that the function $R(x, y; \lambda)$ is continuous with respect to the variables x, y in the square $a \leqslant x \leqslant b$, $a \leqslant y \leqslant b$ (and is analytic with respect to λ for all values of λ, both real and complex, belonging to the circle $|\lambda| < 1/M$). Therefore from formula (31) it follows directly that *if $f(x)$ is a continuous function then so is the solution $\varphi(x)$ of equation* (3).

3°. Fredholm Integral Equation of the Second Kind with an Arbitrary Continuous Kernel. Now let us proceed to study integral equation (3) for the general case when condition (7) must not necessarily be fulfilled.

From the course of mathematical analysis it is known that if $K(x, y)$ is a continuous function in the square $a \leqslant x \leqslant b$, $a \leqslant y \leqslant b$, then, given an arbitrary number $\varepsilon > 0$, there exist linearly independent systems $\{p_i(x)\}$, $a \leqslant x \leqslant b$, and $\{q_i(y)\}$, $a \leqslant y \leqslant b$ ($i = 1, \ldots, N$) of continuous functions such that

$$K(x, y) = \sum_{i=1}^{N} p_i(x) q_i(y) + K_\varepsilon(x, y) \tag{32}$$

where $K_\varepsilon(x, y)$ is a continuous function satisfying the condition

$$(b-a)\,|K_\varepsilon(x, y)| < \varepsilon, \quad a \leqslant x \leqslant b, \quad a \leqslant y \leqslant b \tag{33}$$

In particular, according to Weierstrass' theorem proved in mathematical analysis, as functions $p_i(x)$ and $q_i(y)$ can serve some polynomials.

Let us represent the kernel $K(x, y)$ of equation (3) using formula (32) and rewrite that equation in the form

$$\varphi(x) - \lambda \int_a^b K_\varepsilon(x, y)\,\varphi(y)\,dy = F(x) \tag{34}$$

where

$$F(x) = f(x) + \lambda \sum_{i=1}^{N} \int_a^b p_i(x)\,q_i(y)\,\varphi(y)\,dy \tag{35}$$

Let λ be an arbitrary finite fixed value; we can choose a number $\varepsilon > 0$ so small that the inequality

$$|\lambda| < \frac{1}{\varepsilon} \tag{36}$$

holds.

According to (33) and (36), condition (7) is fulfilled for integral equation (34), and therefore this equation can be uniquely resolved with respect to $\varphi(x)$. Let $R_\varepsilon(x, y; \lambda)$ be the resolvent kernel corresponding to $K_\varepsilon(x, y)$; then equation (34) can be written in the form

$$\varphi(x) = F(x) + \lambda \int_a^b R_\varepsilon(x, y; \lambda) F(y)\, dy \qquad (37)$$

After some simple calculations, the substitution of expression (35) of $F(x)$ into the right-hand side of (37) yields

$$\varphi(x) - \lambda \int_a^b \sum_{i=1}^{N} r_i(x) q_i(y) \varphi(y)\, dy = g(x) \qquad (38)$$

where

$$r_i(x) = p_i(x) + \lambda \int_a^b R_\varepsilon(x, y; \lambda) p_i(y)\, dy$$

and

$$g(x) = f(x) + \lambda \int_a^b R_\varepsilon(x, y; \lambda) f(y)\, dy$$

Thus, *for any finite fixed value of* λ *integral equation* (3) *is equivalent to Fredholm's integral equation of the second kind* (38) *with a degenerate kernel.*

Now, using the Fredholm theorems proved in the foregoing section for an integral equation with a degenerate kernel, we arrive at the so-called *Fredholm alternative*: *for every fixed value of* λ, *either homogeneous integral equation* (6) *corresponding to* (3) *has no solution different from zero* (*and then equation* (3) *always has a uniquely determined solution for any right member* $f(x)$) *or homogeneous equation* (6) *possesses solutions not identically equal to zero* (*and then both homogeneous equation* (6) *and its homogeneous adjoint equation possess an equal number of linearly independent solutions each*); *in this case equation* (3) *may not be solvable for some functions* $f(x)$: *for non-homogeneous equation* (3) *to be solvable in this case it is necessary and sufficient that its right member* $f(x)$ *should be orthogonal to all the solutions of the homogeneous adjoint equation corresponding to* (6), *that is*

$$\int_a^b f(x) \psi_l(x)\, dx = 0 \quad (l = 1, \ldots, \rho)$$

where $\psi_l(x)$ $(l = 1, \ldots, \rho)$ *are all linearly independent solutions of the homogeneous adjoint equation corresponding* to (6).

The function which is identically equal to zero obviously satisfies both homogeneous equation (6) and the adjoint equation corresponding to it. In what follows, when speaking of a *solution* of a homogeneous integral equation (or of the adjoint equation corresponding to it) we shall always mean a solution which is not identically equal to zero.

A number λ for which homogeneous equation (6) possesses solutions $\varphi_l(x)$ $(l = 1, \ldots, \rho)$ will be called, like in Sec. 1°, § 2 of the present chapter, a *characteristic value* (or a *characteristic number*) *of the integral operator with the kernel* $K(x, y)$, *and the functions* $\varphi_l(x)$ *will be called the eigenfunctions of that kernel* (or *of that operator*) *corresponding to* (or *associated with* or *belonging to*) *the characteristic value* λ.

It should be noted that if we write equation (3) in the form

$$\varphi(y) - \lambda \int_a^b K(y, t)\, \varphi(t)\, dt = f(y)$$

and then multiply its both members by $\lambda K(x, y)$ and integrate the result from a to b, this will yield

$$\varphi(x) - \lambda^2 \int_a^b K_2(x, t)\, \varphi(t)\, dt = f_2(x)$$

where

$$f_2(x) = f(x) + \lambda \int_a^b K(x, y)\, f(y)\, dy$$

Continuing this process, we obtain the relations

$$\varphi(x) - \lambda^m \int_a^b K_m(x, y)\, \varphi(y)\, dy = f_m(x)$$

where

$$f_m(x) = f_{m-1}(x) + \lambda \int_a^b K(x, y)\, f_{m-1}(y)\, dy, \quad f_1(x) = f(x)$$

Thus, we have arrived at the conclusion that *if* λ *is a characteristic value of the kernel* $K(x, y)$ *and* $\varphi(x)$ *is an eigenfunction associated with* λ, *then* λ^m *is a characteristic value of the iterated kernel* $K_m(x, y)$ *and* $\varphi(x)$ *is an eigenfunction of that kernel belonging to the characteristic value* λ^m. The converse proposition is also true but here we shall not dwell on its proof.

All the propositions concerning equation (3) which were stated and proved above can be immediately extended to the case of general equation (*) with a continuous kernel $K(x, y)$ and a continuous right member $f(x)$ $(x, y \in E_n)$.

Moreover, proceeding from the remark made above, we can conclude that these propositions also remain valid in the case of a kernel of the form

$$K(x, y) = \frac{K^*(x, y)}{|x-y|^\alpha} \qquad 0 < \alpha < n_0$$

where $K^*(x, y)$ is a function continuous with respect to the variable point $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and n_0 denotes the dimension of the domain D or of its boundary S. This can easily be shown if we take into account the fact that the kernel $K_m(x, y)$ of the integral equation

$$\varphi(x) - \lambda^m \int\limits_{D(S)} K_m(x, y)\, \varphi(y)\, dy = f_m(x)$$

obtained from the kernel $K(x, y)$ by means of m-fold iteration is a continuous function of the point (x, y) for a sufficiently large value of m.

It is also evident that if the function $f(x)$ is continuous everywhere except a finite number of points or of smooth manifolds whose dimensions are less than n_0, and if $f(x)$ is absolutely integrable over the domain D (or over its boundary S), the Fredholm alternative remains true. It is this class of equations to which belong the integral equations mentioned in Sec. 2°, § 5 of Chapter 1; to the latter integral equations were reduced the problems concerning the existence of the solutions (including the elementary solutions) of general elliptic partial differential equations of the second order.

For these integral equations the value of the integral

$$\int_D |K(x, y)|\, dy$$

can be made arbitrarily small *for a domain D of a sufficiently small diameter*, and therefore condition (7) of Sec. 2°, § 1 of the present chapter may be regarded as being fulfilled; consequently *for the domains of this kind the existence of the solutions of the indicated integral equations is guaranteed.*

4°. The Notion of Spectrum. The set of all characteristic values of an integral equation with a kernel $K(x, y)$ is called the *spectrum* of this kernel. The investigation of the spectrum plays an important role in the theory of integral equations.

As was shown earlier, the spectrum of the kernel of Volterra's equation is void (see Sec. 3°, § 1 of the present chapter), and in the case of a degenerate kernel of Fredholm's equation the spectrum consists of a finite number of characteristic values (see Sec. 1°, § 2 of the present chapter). Among the other kernels whose spectra are thoroughly investigated it is advisable to mention the so-called symmetric real kernels.

A (real) kernel $K(x, y)$ is said to be *symmetric* if the equality $K(x, y) = K(y, x)$ holds for all the values of x and y belonging to the domain of definition of the kernel.

It can easily be seen that *if* $\varphi_1(x)$ *and* $\varphi_2(x)$ *are two eigenfunctions* (of a symmetric kernel $K(x, y)$) *corresponding to two different characteristic values* λ_1 *and* λ_2 *respectively, then*

$$\int_a^b \varphi_1(x)\, \varphi_2(x)\, dx = 0$$

Indeed, the kernel $K(x, y)$ being symmetric, we have

$$(\lambda_1 - \lambda_2)\int_a^b \varphi_1(x)\, \varphi_2(x)\, dx = \lambda_1\lambda_2 \int_a^b \varphi_1(x)\, dx \int_a^b K(x, y)\, \varphi_2(y)\, dy -$$

$$- \lambda_1\lambda_2 \int_a^b \varphi_2(x)\, dx \int_a^b K(x, y)\, \varphi_1(y)\, dy =$$

$$= \lambda_1\lambda_2 \int_a^b \varphi_1(x)\, dx \int_a^b [K(x, y) - K(y, x)]\, \varphi_2(y)\, dy = 0$$

whence, since $\lambda_1 \neq \lambda_2$, it follows that the assertion we have stated is true.

In its turn, this assertion implies that *the characteristic values of an integral operator with a symmetric kernel cannot be complex.*

Indeed, let us suppose that a characteristic value λ and the corresponding eigenfunction $\varphi(x)$ are complex, that is

$$\lambda = \lambda_1 + i\lambda_2, \quad \varphi(x) = \varphi_1(x) + i\varphi_2(x)$$

Then $\bar{\lambda} = \lambda_1 - i\lambda_2$ and $\overline{\varphi(x)} = \varphi_1(x) - i\varphi_2(x)$ are also a characteristic value and an eigenfunction corresponding to it respectively.

If $\lambda_2 \neq 0$, then, as was already proved, we have

$$\int_a^b \varphi(x)\overline{\varphi(x)}\,dx = \int_a^b |\varphi|^2\,dx = 0$$

It follows that $\varphi(x)$ is identically equal to zero, which is impossible by virtue of the definition of an eigenfunction. Hence, $\lambda_2 = 0$, that is λ is in fact a real number, which is what we intended to prove.

Among the other important properties of a symmetric kernel we shall mention the following one: *the spectrum of an integral equation with a symmetric real kernel is not void.* Here we shall not prove this property because we do not need it for our further aims.

5°. Volterra Integral Equation of the Second Kind with Multiple Integral. The application of the argument given in Sec. 3°, § 1 of the present chapter to an integral equation of the form

$$\varphi(\xi, \eta) - \lambda \int_{\xi_1}^{\xi} dt \int_{\eta_1}^{\eta} K(\xi, \eta;\ t, \tau)\,\varphi(t, \tau)\,d\tau = f(\xi, \eta) \quad (39)$$

where $K(\xi, \eta;\ t, \tau)$ and $f(\xi, \eta)$ are given real continuous functions, leads to the conclusion that this equation possesses a single solution for any fixed value of the real parameter λ. That is why equation (39) is also called the Volterra integral equation of the second kind.

The one-to-one transformation

$$v(\xi, \eta) = w(\xi, \eta) + \int_{\xi_1}^{\xi} w(t, \eta)\, b(t, \eta) \exp\left(\int_{t}^{\xi} b(t_1, \eta)\, dt_1\right) dt + \int_{\eta_1}^{\eta} w(\xi, \tau)\, a(\xi, \tau) \exp\left(\int_{\tau}^{\eta} a(\xi, \tau_1)\, d\tau_1\right) d\tau$$

of the unknown function $v(\xi, \eta)$ in equation (43) of Chapter 3 to the new function $w(\xi, \eta)$ reduces this equation to an integral equation of form (39):

$$w(\xi, \eta) + \int_{\xi_1}^{\xi} dt \int_{\eta_1}^{\eta} K_0(\xi, \eta;\ t, \tau)\, w(t, \tau)\, d\tau = 1$$

where

$$K_0(\xi, \eta;\ t, \tau) = c(t, \tau) - b(t, \eta)\, a(t, \tau) \exp\left(\int_{\tau}^{\eta} a(t, \tau_1)\, d\tau_1\right) - a(\xi, \tau)\, b(t, \tau) \exp\left(\int_{t}^{\xi} b(t_1, \tau)\, dt_1\right) + b(t, \tau) \int_{t}^{\xi} c(t_1, \tau) \exp\left(\int_{t}^{t_1} b(t_2, \tau)\, dt_2\right) dt_1 + a(t, \tau) \int_{\tau}^{\eta} c(t, \tau_1) \exp\left(\int_{\tau}^{\tau_1} a(t, \tau_2)\, d\tau_2\right) d\tau_1$$

6°. Volterra Integral Equation of the First Kind. Let us consider the *Volterra integral equation of the first kind*

$$\int_{a}^{x} K(x, y)\, \varphi(y)\, dy = f(x)$$

whose kernel $K(x, y)$ and the right member $f(x)$ satisfy the following conditions: (1) the derivatives $K_x(x, y)$ and $f'(x)$ exist and are continuous functions, and (2) the expression $K(x, x)$ does not turn into zero for any value of x. The differentiation of this equation with respect to x brings it to the Volterra integral equation of the second kind

$$\varphi(x)+\int_a^x K^*(x, y)\,\varphi(y)\,dy=f^*(x)$$

where

$$K^*(x, y)=\frac{K_x(x, y)}{K(x, x)} \quad \text{and} \quad f^*(x)=\frac{f'(x)}{K(x, x)}$$

In the case when the above conditions do not hold the investigation of the Volterra integral equation of the first kind is rather intricate. However, for some special cases it is possible to elaborate methods with the aid of which the solutions of the equations can even be expressed in quadratures.

For instance, let us consider the integral equation

$$\int_0^x \frac{\varphi(y)\,dy}{(x-y)^{\alpha}}=f(x), \quad 0<\alpha<1, \quad x>0$$

whose right member $f(x)$ is a continuous function; it is known as *Abel's integral equation*. Let us rewrite the equation in the form

$$\int_0^t \frac{\varphi(y)\,dy}{(t-y)^{\alpha}}=f(t)$$

On multiplying the last equality by the kernel $\frac{1}{(x-t)^{1-\alpha}}$ and integrating with respect to t from zero to x, we obtain the identity

$$\int_0^x \frac{dt}{(x-t)^{1-\alpha}}\int_0^t \frac{\varphi(y)\,dy}{(t-y)^{\alpha}}=\int_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}$$

Now, taking into account the relations

$$\int_0^x \frac{dt}{(x-t)^{1-\alpha}}\int_0^t \frac{\varphi(y)\,dy}{(t-y)^{\alpha}}=\int_0^x \varphi(y)\,dy\int_y^x \frac{dt}{(x-t)^{1-\alpha}(t-y)^{\alpha}}$$

and

$$\int_y^x \frac{dt}{(x-t)^{1-\alpha}(t-y)^{\alpha}}=\frac{\pi}{\sin \pi\alpha}$$

we derive from the above identity the equality

$$\varphi(x)=\frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}$$

In case the function $f(x)$ is continuously differentiable we can rewrite the last expression in the form

$$\varphi(x)=\frac{\sin \pi\alpha}{\pi}\left[\frac{f(0)}{x^{1-\alpha}}+\int_0^x \frac{f'(t)\,dt}{(x-t)^{1-\alpha}}\right]$$

In particular, putting $\alpha=1/2$ we obtain the solution of integral equation (63) of Introduction which is encountered in the study of the *tautochrone problem*:

$$\varphi(x)=\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)\,dt}{\sqrt{x-t}}$$

§ 3. Applications of the Theory of Linear Integral Equations of the Second Kind

1°. Application of Fredholm Alternative to the Theory of Boundary-Value Problems for Harmonic Functions. In Sec. 2°, § 4 of Chapter 1 we proved that if the solution of the Dirichlet problem is constructed in the form of double-layer potential (47) then the function μ must satisfy the Fredholm integral equation of the second kind of form (57):

$$\mu(s)-\lambda\int_S K(s, t)\,\mu(t)\,dt=-2g(s), \quad \lambda=-1 \qquad (40)$$

If we manage to show that $\lambda=-1$ is not a characteristic value of the kernel $K(s, t)$, the Fredholm alternative will imply that integral equation (40) *is solvable* for any right member. This will imply the solvability of the Dirichlet problem with boundary condition (56) stated in Chapter 1.

The homogeneous equation corresponding to (40) has the form

$$\mu_0(s) - \lambda \int_S K(s, t)\, \mu_0(t)\, dt = 0 \tag{41}$$

For $\lambda = -1$ this equation has in fact no solutions different from zero. Indeed, let us suppose that μ_0 is a solution of equation (41). The double-layer potential $u_0(x, y)$ corresponding to μ_0 possesses the property

$$u_0^+(s) = 0, \quad x^0(s) \in S$$

which follows from formulas (54) in Chapter 1 and formula (41) written above.

When the variable point (x, y) tends to the boundary S from the interior of the domain D^+ the limit of the function $u_0(x, y)$ is equal to zero, and therefore, by the uniqueness property of a harmonic function, the equality $u_0(x, y) = 0$ must be fulfilled for all $(x, y) \in D^+$. Consequently, for the value of the normal derivative $\partial u_0/\partial \nu$ on S we have

$$\left(\frac{\partial u_0}{\partial \nu}\right)^+ = 0 \tag{42}$$

Now, by virtue of the property of the normal derivative of a double-layer potential expressed by the equality

$$\left(\frac{\partial u_0}{\partial \nu}\right)^+ = \left(\frac{\partial u_0}{\partial \nu}\right)^-$$

(see Sec. 2°, § 4 in Chapter 1) and by virtue of equality (42), we conclude that

$$\left(\frac{\partial u_0}{\partial \nu}\right)^- = 0 \tag{43}$$

Like in Sec. 3°, § 4 of Chapter 1, it can easily be shown that the function $u_0(x, y)$, which is harmonic in D^-, represents a double-layer potential and satisfies condition (43), is identically equal to a constant. Finally, since $u_0(x, y)$ turns into zero at infinity there must be $u_0(x, y) = 0$ everywhere in D^-.

Now, using again formulas (54) and (55) in Chapter 1. we can assert that

$$-\mu_0(s) = u_0^+(s) - u_0^-(s) = 0$$

This proves that $\lambda = -1$ *is not a characteristic value of the kernel* $K(s, t)$.

It can be proved that the kernel $K^*(s, t)$ of Fredholm's integral equation (69) in Chapter 1 to which Neumann problem (67) was reduced has the characteristic value $\lambda = -1$; however, in this case condition (68) is fulfilled, and therefore the Fredholm alternative guarantees the solvability of this equation.

But here, in order to prove that condition (68) is sufficient for Neumann problem (67) to be solvable, we shall use another technique.

Let $u(x, y)$ be the sought-for solution of the Neumann problem in the domain D^+, and let $v(x, y)$ be the harmonic conjugate function to $u(x, y)$.

Since the derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are supposed to be continuous functions in $D^+ \cup S$, condition (CR) and (67) imply the following expression for $\partial v/\partial s$:

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds} = -\frac{\partial u}{\partial y}\frac{dx}{ds} + \frac{\partial u}{\partial x}\frac{dy}{ds} =$$

$$= \frac{\partial u}{\partial y}\frac{dy}{d\nu} + \frac{\partial u}{\partial x}\frac{dx}{d\nu} = \frac{\partial u}{\partial \nu} = g(s)$$

It follows that

$$v(s) = \int_0^s g(t)\,dt + C, \quad 0 \leqslant s \leqslant l$$

where l is the length of contour S bounding the domain D^+.

Since

$$v(0) = C \quad \text{and} \quad v(l) = \int_0^l g(t)\,dt + C$$

we see that for the function $v(s)$ to be continuous at the points $s = 0$ and $s = l$, that is for the equality $v(0) = v(l)$

to be fulfilled, the function $g(t)$ should satisfy the condition

$$\int_0^l g(t)\,dt = 0$$

which exactly coincides with condition (68) indicated in Chapter 1.

The existence of the function $v(x, y)$ harmonic in the domain D^+ and satisfying the boundary condition

$$v|_S = \int_0^s g(t)\,dt + C$$

was proved above (as has just been proved, the Dirichlet problem is solvable).

The harmonic function $u(x, y)$ (the sought-for solution of the Neumann problem) can be constructed using $v(x, y)$ with the aid of the method indicated in Sec. 5° § 2 of Chapter 2.

2°. Reduction of Cauchy Problem for an Ordinary Linear Differential Equation to a Volterra Integral Equation of the Second Kind. Let us consider the ordinary linear differential equation of the *n*th order

$$\frac{d^n y}{dx^n} + \sum_{k=0}^{n-1} a_k(x) \frac{d^k y}{dx^k} = f(x) \quad (a \leqslant x \leqslant b) \tag{44}$$

whose right member $f(x)$ is continuous and whose coefficients $a_k(x)$ are continuous functions possessing the continuous derivatives $d^k a_k/dx^k$ $(k = 0, 1, \ldots, n-1)$. For this equation we shall consider the *Cauchy problem*

$$\left.\frac{d^k y}{dx^k}\right|_{x=x^0} = y_k^0 \quad (k = 0, \ldots, n-1); \quad a < x_0 < b \tag{45}$$

where y_k^0 $(k = 0, \ldots, n-1)$ are given real constants.

The polynomial

$$z(x) = \sum_{k=0}^{n-1} \frac{1}{k!} y_k^0 (x - x_0)^k$$

satisfies conditions (45), and the function $u(x) = y(x) - z(x)$ is a solution of the ordinary differential equation

$$\frac{d^n u}{dx^n} + \sum_{k=0}^{n-1} a_k(x) \frac{d^k u}{dx^k} = f(x) - \sum_{k=0}^{n-1} a_k(x) \frac{d^k z}{dx^k}$$

and satisfies the initial conditions

$$\left.\frac{d^k u}{dx^k}\right|_{x=x_0} = 0 \quad (k = 0, \ldots, n-1)$$

Therefore, without loss of generality, we can assume that in conditions (45) all the numbers y_k^0 are equal to zero, that is

$$\left.\frac{d^k y}{dx^k}\right|_{x=x_0} = 0 \quad (k = 0, \ldots, n-1) \tag{46}$$

On integrating n times equality (44), we obtain, by virtue of (46), the relation

$$y(x) + \sum_{k=0}^{n-1} \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \ldots \int_{x_0}^{x_{n-1}} a_k(t) \frac{d^k y}{dt^k} dt = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \ldots \int_{x_0}^{x_{n-1}} f(t)\, dt \tag{47}$$

Using the identity

$$\int_{x_0}^{x_{i-1}} dx_i \int_{x_0}^{x_i} F(t)(x_i - t)^{j-1} dt = \int_{x_0}^{x_{i-1}} F(t)\, dt \int_{t}^{x_{i-1}} (x_i - t)^{j-1} dx_i = \frac{1}{j} \int_{x_0}^{x_{i-1}} (x_{i-1} - t)^j F(t)\, dt$$

(which is known from mathematical analysis and holds for any continuous function $F(t)$), we can rewrite equality (47) in the form

$$y(x) + \sum_{k=0}^{n-1} \frac{1}{(n-1)!} \int_{x_0}^{x} (x-t)^{n-1} a_k(t) \frac{d^k y}{dt^k} dt = \frac{1}{(n-1)!} \int_{x_0}^{x} (x-t)^{n-1} f(t)\, dt \tag{48}$$

Next we perform integration by parts in the left-hand side of (48); by virtue of conditions (46) this results in

$$y(x)+\frac{1}{(n-1)!}\sum_{k=0}^{n-1}[(-1)^k\int_{x_0}^{x} y(t)\frac{d^k}{dt^k}[a_k(t)(x-t)^{n-1}]\,dt=$$

$$=\frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{n-1}f(t)\,dt$$

that is

$$y(x)+\int_{x_0}^{x}K(x,t)\,y(t)\,dt=F(x) \tag{49}$$

where

$$K(x,t)=\frac{1}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\frac{d^k}{dt^k}[a_k(t)(x-t)^{n-1}]$$

and

$$F(x)=\frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{n-1}f(t)\,dt$$

are given continuous functions.

Thus, problem (44), (46) has been reduced to the equivalent Volterra integral equation of the second kind of form (49) whose kernel $K(x, t)$ and right member $F(x)$ are continuous functions.

Finally, the existence and the uniqueness of the solution of Volterra's integral equation of the second kind proved in Sec. 3°, § 1 of the present chapter imply *the existence and the uniqueness of the solution of Cauchy problem* (44), (46).

3°. **Boundary-Value Problem for Ordinary Linear Differential Equations of the Second Order.** In the theory of ordinary differential equations, besides the Cauchy problem, an important role is also played by the so-called *first boundary-value problem* (the *Dirichlet problem*). Let us state this problem for the case of a linear equation of the form

$$\frac{d^2y}{dx^2}+\lambda p(x)\,y=f(x),\quad a<x<b \tag{50}$$

It is required to find the solution $y(x)$ of equation (50) *which is regular in the interval $a < x < b$ and continuous for $a \leqslant x \leqslant b$ and satisfies the boundary conditions*

$$y(a) = A, \quad y(b) = B \tag{51}$$

where A and B are given real constants.

Like in the study of problem (44), (45), we can assume, without loss of generality, that $A = B = 0$; in other words, we shall consider boundary conditions (51) of the form

$$y(a) = y(b) = 0 \tag{52}$$

Let us suppose that $p(x)$ and $f(x)$ are given continuous real functions defined for $a \leqslant x \leqslant b$ and that λ is a real parameter.

For our further aims we construct *Green's function*

$$G(t, x) = \begin{cases} \dfrac{(t-b)(x-a)}{b-a} & \text{for} \quad x \leqslant t \\ \dfrac{(t-a)(x-b)}{b-a} & \text{for} \quad x \geqslant t \end{cases}$$

which possesses the following properties: (1) it is continuous in the square $a \leqslant x \leqslant b$, $a \leqslant t \leqslant b$; (2) in the intervals $a < x < b$ and $a < t < b$, for $t \neq x$, it possesses second-order derivatives both with respect to t and with respect to x, the derivatives being equal to zero; (3) there holds the limiting relation

$$\lim_{\substack{t \to x \\ t \geqslant x}} \frac{\partial G}{\partial t} - \lim_{\substack{t \to x \\ t \leqslant x}} \frac{\partial G}{\partial t} = 1, \quad a < x < b$$

On integrating the obvious identity

$$G(t, x) \frac{d^2 y}{dt^2} - y(t) \frac{\partial^2 G(t, x)}{\partial t^2} = \frac{d}{dt} \left(G \frac{dy}{dt} - y \frac{\partial G}{\partial t} \right)$$

over the intervals $a < t < x - \varepsilon$ and $x + \varepsilon < t < b$ where ε is a sufficiently small positive number, adding together the results and taking into account (50), (52) and Properties

(1), (2) and (3) of the function $G(t, x)$, we obtain the relation

$$\int_a^{x-\varepsilon} G(t, x)[-\lambda p(t) y(t)+f(t)]\,dt+\int_{x+\varepsilon}^{b} G(t, x)[-\lambda p(t) y(t)+f(t)]\,dt=$$

$$=G(x-\varepsilon, x)\frac{dy}{dt}\Big|_{t=x-\varepsilon}-G(x+\varepsilon, x)\frac{dy}{dt}\Big|_{t=x+\varepsilon}-$$

$$-y(x-\varepsilon)\frac{\partial G(t, x)}{\partial t}\Big|_{t=x-\varepsilon}+y(x+\varepsilon)\frac{\partial G(t, x)}{\partial t}\Big|_{t=x+\varepsilon}$$

Now, passing to the limit for $\varepsilon \to 0$ in the last relation, we obtain

$$y(x)=-\lambda\int_a^b G(t, x)\,p(t)\,y(t)\,dt+\int_a^b G(t, x)\,f(t)\,dt \tag{53}$$

that is

$$y(x)-\lambda\int_a^b K(x, t)\,y(t)\,dt=F(x) \tag{54}$$

where

$$K(x, t)=-G(t, x)\,p(t) \quad \text{and} \quad F(x)=\int_a^b G(t, x)\,f(t)\,dt$$

It can easily be seen that *the solution $y(x)$ of integral equation* (54), *provided it exists, satisfies differential equation* (50) *and boundary conditions* (52).

Indeed, since $y(x)$ is a solution of integral equation (54), or, which is the same, of equation (53), we can write

$$y(x)=\int_a^b G(t, x)[-\lambda p(t)\,y(t)+f(t)]\,dt \tag{55}$$

whence

$$y(x)=\int_a^x \frac{(t-a)(x-b)}{b-a}[-\lambda p(t)\,y(t)+f(t)]\,dt+$$

$$+\int_x^b \frac{(t-b)(x-a)}{b-a}[-\lambda p(t)\,y(t)+f(t)]\,dt$$

From the last expression we find

$$\frac{dy}{dx}=\int_a^x \frac{t-a}{b-a}\left[-\lambda p(t)\, y(t)+f(t)\right] dt+\int_x^b \frac{t-b}{b-a}\left[-\lambda p(t)\, y(t)+f(t)\right] dt$$

and

$$\frac{d^2y}{dx^2}=\frac{x-a}{b-a}\left[-\lambda p(x)\, y(x)+f(x)\right]-\frac{x-b}{b-a}\left[-\lambda p(x)\, y(x)+f(x)\right]=-\lambda p(x)\, y(x)+f(x)$$

Further, since the function $G(t, x)$ is continuous, the passage to the limit in (55) for $x\to a$ and $x\to b$ results in $y(a)=y(b)=0$ because we have the equalities $G(t, a)=G(t, b)=0$.

Thus, problem (50), (52) has been reduced to the equivalent Fredholm integral equation of the second kind of form (54). Consequently, *the propositions proved in* §§ 1 *and* 2 *of the present chapter make it possible to judge upon whether the problem under consideration is conditionally or unconditionally solvable and upon the number of linearly independent solutions of the problem.*

The theory of integral equations presented above readily shows the essential difference between the Cauchy problem and the Dirichlet problem for ordinary differential equations.

§ 4. Singular Integral Equations

1°. The Notion of a Singular Integral Equation. In the case when the kernel $K(x, y)$ of an integral equation turns into infinity for $x=y$ so that the integral exists only in the sense of Cauchy's principal value, the equation is referred to as a *singular integral equation.*

Let S be a closed or non-closed Lyapunov curve. A one-valued function $K(t, t_0)$ defined on S is said to satisfy *Hölder's condition* if for any two pairs of points t, t_0 and

t', t'_0 belonging to S the inequality

$$|K(t, t_0) - K(t', t'_0)| \leqslant A(|t-t'|^{h_1} + |t_0 - t'_0|^{h_2})$$

holds where A, h_1 and h_2 are positive constants, and $h_1 \leqslant 1$, $h_2 \leqslant 1$.

In applications a singular integral equation of the form

$$\alpha(t_0)\varphi(t_0) + \int_S \frac{K(t, t_0)}{t-t_0}\varphi(t)\,dt = f(t_0), \quad t_0 \in S \tag{56}$$

is often encountered where α, K and f are given functions and φ is the sought-for function, all the functions satisfying Hölder's condition (for a function of one variable Hölder's condition was stated in Sec. 1°, § 5 of Chapter 2).

Generally speaking, Fredholm's theorems do not hold in the case of a singular integral equation. The theory of singular integral equations of form (56) satisfying the requirement that the functions $\alpha(t_0)$ and $\beta(t_0) = K(t_0, t_0)$ should not simultaneously turn into zero at any point $t_0 \in S$ was elaborated by N. I. Muskhelishvili in his book *Singular Integral Equations* (Moscow, 1968, in Russian).

Below we shall consider some special classes of singular integral equations whose solutions can be expressed in quadratures.

2°. Hilbert's Integral Equation. In Sec. 7°, § 3 of Chapter 2 we derived Schwarz' formula (89) representing in the integral form an analytic function

$$f(z) = u(x, y) + iv(x, y)$$

in the circle $|z| < 1$ in terms of the boundary values $u(\varphi)$ of its real part $u(x, y)$ under the assumption that the latter is continuous in the closed circle $|z| \leqslant 1$.

We shall suppose that the function $u(\varphi) = u(t_0)$ ($t_0 = e^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$) satisfies Hölder's condition. Let us write Schwarz' formula in the form of formula (87) of Chapter 2:

$$f(z) = \frac{1}{\pi i}\int_{|t|=1} \frac{u(t)\,dt}{t-z} - u(0, 0) + iv(0, 0) \tag{57}$$

From (57) we conclude that the expression

$$f^+(t_0) = u(\varphi) + iv(\varphi), \quad t_0 = e^{i\varphi}, \quad 0 \leqslant \varphi \leqslant 2\pi$$

exists, and, by virtue of formulas (104) and (88) of that chapter, we have

$$f^{+}(t_0) = u(\varphi) + iv(\varphi) =$$
$$= u(\varphi) + \frac{1}{\pi i} \int_{|t|=1} \frac{u(t)\,dt}{t-t_0} - u(0,\ 0) + iv(0,\ 0) =$$
$$= u(\varphi) + \frac{1}{\pi i} \int_{|t|=1} \frac{u(t)\,dt}{t-t_0} - \frac{1}{2\pi i} \int_{|t|=1} \frac{u(t)\,dt}{t} + iv(0,\ 0) =$$
$$= u(\varphi) + \frac{1}{2\pi i} \int_0^{2\pi} u(\psi) \cot \frac{\psi-\varphi}{2}\, d\psi + iv(0,\ 0) \quad (58)$$

where

$$\cot \frac{\psi-\varphi}{2} d\psi = \frac{t+t_0}{t-t_0} \frac{dt}{t}, \qquad t = e^{i\psi}$$

The property of the limiting values of the Cauchy-type integral of the form $\frac{1}{2\pi i} \int_S \frac{f(t)\,dt}{t-z}$ with a function $f(t)$ satisfying Hölder's condition (see Sec. 4°, § 5 of Chapter 2) allows us to conclude from formula (58) that *if* $u(t_0) = u(\varphi)$ *satisfies Hölder's condition on the circumference* $|t_0| = 1$ *of the circle* $|z| < 1$, *then so does the function* $f^{+}(t_0)$.

According to the mean-value theorem for harmonic functions, we have the equality

$$v(0,\ 0) = \frac{1}{2\pi} \int_0^{2\pi} v(\psi)\, d\psi$$

and therefore formula (58) implies the equality

$$v(\varphi) = -\frac{1}{2\pi} \int_0^{2\pi} u(\psi) \cot \frac{\psi-\varphi}{2}\, d\psi + \frac{1}{2\pi} \int_0^{2\pi} v(\psi)\, d\psi \quad (59)$$

where the first integral on the right-hand side is understood in the sense of Cauchy's principal value.

The function $\frac{1}{i} f(z) = v(x,\ y) - iu(x,\ y)$ analytic in the circle $|z| < 1$ can be represented in terms of the boundary values $v(\varphi)$ of its real part $v(x, y)$:

$$\frac{1}{i} f(z) = \frac{1}{\pi i} \int\limits_{|t|=1} \frac{v(t)\,dt}{t-z} - v(0, 0) - iu(0, 0)$$

Using this representation we obtain the integral representation of the function $f(z)$ in terms of the boundary values $v(\theta)$ of its imaginary part $v(x, y)$:

$$f(z) = \frac{1}{\pi} \int\limits_{|t|=1} \frac{v(t)\,dt}{t-z} - iv(0, 0) + u(0, 0)$$

As above, from the last equality we find the expression

$$f^+(t_0) = u(\varphi) + iv(\varphi) =$$

$$= iv(\varphi) + \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \cot \frac{\theta-\varphi}{2}\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} u(\theta)\, d\theta, \quad t = e^{i\theta}$$

whence

$$u(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \cot \frac{\theta-\varphi}{2}\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} u(\theta)\, d\theta \tag{60}$$

Consequently, *if it is known that the function* $u(\varphi)$ *satisfies Hölder's condition, then the function* $v(\varphi)$ *also satisfies Hölder's condition, and these functions are connected by relation* (59). *Conversely, if* $v(\varphi)$ *satisfies Hölder's condition, then so does the function* $u(\varphi)$, *and the relationship between these functions is expressed by formula* (60). This means that formulas (59) and (60), connecting the boundary values of the real part and of the imaginary part of a function $f(z)$ analytic in the circle $|z| < 1$ and satisfying Hölder's condition in the closed circle $|z| \leqslant 1$, are equivalent.

In the case when it is additionally known that

$$\int_0^{2\pi} [u(\psi) + iv(\psi)]\, d\psi = 0 \tag{61}$$

formulas (59) and (60) take the form

$$v(\varphi) = -\frac{1}{2\pi} \int_0^{2\pi} u(\psi) \cot \frac{\psi-\varphi}{2}\, d\psi \tag{62}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} v(\theta)\cot\frac{\theta-\psi}{2}\,d\theta = u(\psi) \tag{63}$$

respectively.

Let us consider equality (63) under the assumption that $u(\varphi)$ is a given real function and $v(\varphi)$ is the sought-for real function, both functions satisfying Hölder's condition and condition (61). Then this equality is a *singular integral equation of the first kind* whose solution is given by formula (62). Equation (63) is called *Hilbert's integral equation*, and (62) is the corresponding *inversion formula*.

On substituting the expression of $v(\theta)$ given by formula (62) into (63) we obtain the following *composition formula for singular integrals*:

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\cot\frac{\theta-\psi}{2}\,d\theta\int_0^{2\pi} u(\varphi)\cot\frac{\varphi-\theta}{2}\,d\varphi = -u(\psi)$$

The functions $\frac{1}{t-t_0}$ and $\cot\frac{\psi-\varphi}{2}$ are called *Cauchy's kernel* and *Hilbert's kernel* respectively.

3°. **Hilbert Transformation.** Let us consider the singular integral equation

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v(t)\,dt}{t-x} = u(x), \quad -\infty < x < \infty \tag{64}$$

where $u(x)$ is a given real function and $v(x)$ is the unknown real function; we shall suppose that $u(x)$ and $v(x)$ satisfy Hölder's condition and that for large values of $|x|$ the inequalities

$$|u(x)| < \frac{A}{|x|^\delta} \quad \text{and} \quad |v(x)| < \frac{A}{|x|^\delta}$$
$$\text{where } A > 0 \text{ and } \delta > 0$$

are fulfilled.

Let us denote by $F(z)$ the Cauchy-type integral involving the function $v(t)$:

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{v(t)dt}{t-z}$$

According to formulas (106) and (107) of Chapter 2, we have

$$F^{+}(x)-F^{-}(x)=v(x) \tag{65}$$

and

$$F^{+}(x)+F^{-}(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{v(t)\,dt}{t-x}, \qquad -\infty<x<\infty \tag{66}$$

From (64) and (66) we conclude that the function $F(z)$ must be the solution of the boundary-value problem

$$F^{+}(x)+F^{-}(x)=\frac{u(x)}{i}$$

By virtue of formula (110) of Chapter 2, the solution of this problem turning into zero at infinity has the form

$$F(z)=\begin{cases}\Phi(z) & \text{for} \quad \operatorname{Im} z>0\\ -\Phi(z) & \text{for} \quad \operatorname{Im} z<0\end{cases} \tag{67}$$

where

$$\Phi(z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u(t)\,dt}{t-z} \tag{68}$$

Since, by (67) and (68), we have

$$F^{+}(x)-F^{-}(x)=\Phi^{+}(x)+\Phi^{-}(x)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)\,dt}{t-x}$$

the sought-for solution $v(x)$ of equation (64) is given by formula (65), that is

$$v(x)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)\,dt}{t-x} \tag{69}$$

Formula (64) expressing the function $u(x)$ in terms of the function $v(x)$ specifies *Hilbert's transformation*; formula (69) giving the solution of integral equation (64) expresses *Hilbert's inverse transformation* (it is also referred to as the *inversion formula for Hilbert's transformation*).

4°. **Integral Equation of the Theory of the Wing of an Airplane.** In the theory of a thin wing of an airplane an important role is played by the integral equation

$$\frac{1}{\pi}\int_{-a}^{a}\frac{v(t)\,dt}{t-x}=u(x), \quad -a<x<a, \quad 0<a<\infty \tag{70}$$

where $v(x)$ and $u(x)$ are real functions satisfying Hölder's condition in the interval $-a<x<a$.

For the limiting values

$$F^{+}(x)=\lim_{z\to x}F(z), \quad \operatorname{Im} z>0,$$

and

$$F^{-}(x)=\lim_{z\to x}F(z), \quad \operatorname{Im} z<0,$$

of the function

$$F(z)=\frac{1}{2\pi i}\int_{-a}^{a}\frac{v(t)\,dt}{t-z}$$

we have the formulas

$$F^{+}(x)-F^{-}(x)=v(x) \tag{71}$$

and

$$F^{+}(x)+F^{-}(x)=\frac{1}{\pi i}\int_{-a}^{a}\frac{v(t)\,dt}{t-x}, \quad -a<x<a \tag{72}$$

which follow from formulas (106) and (107) of Chapter 2. Equalities (70) and (72) imply that the function $F(z)$ must be the solution of the boundary-value problem

$$F^{+}(x)+F^{-}(x)=\frac{u(x)}{i}, \quad -a<x<a \tag{73}$$

$$F^{+}(x)-F^{-}(x)=0, \quad -\infty<x<-a, \quad a<x<\infty \tag{74}$$

According to (73) and (74), the function

$$\Omega(z)=\sqrt{a^2-z^2}\,F(z) \tag{75}$$

satisfies the boundary conditions

$$\Omega^{+}(x)+\Omega^{-}(x)=\begin{cases}\frac{1}{i}\sqrt{a^2-x^2}\,u(x) \text{ for } -a<x<a\\ 0, \quad -\infty<x<-a \text{ for } a<x<\infty\end{cases} \tag{76}$$

As was already shown in Sec. 5°, § 5 of Chapter 2, the solution of problem (76) is given by the formula

$$\Omega(z)=\begin{cases}\Phi(z) & \text{for } \operatorname{Im} z>0\\ -\Phi(z) & \text{for } \operatorname{Im} z<0\end{cases} \tag{77}$$

where

$$\Phi(z)=-\frac{1}{2\pi}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}\,u(t)\,dt}{t-z}+\frac{C_0}{2} \tag{78}$$

and C_0 is an arbitrary constant.

Let us compute the expressions $\Omega^+(x)$ and $\Omega^-(x)$ using formulas (77) and (78):

$$\Omega^+(x)=-i\frac{\sqrt{a^2-x^2}}{2}u(x)-\frac{1}{2\pi}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}\,u(t)\,dt}{t-x}+\frac{C_0}{2}$$

and

$$\Omega^-(x)=-i\frac{\sqrt{a^2-x^2}}{2}u(x)+\frac{1}{2\pi}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}\,u(t)\,dt}{t-x}-\frac{C_0}{2}$$

From these expressions, by virtue of (71) and (75), we directly derive the *inversion formula for integral equation* (70):

$$v(x)=-\frac{1}{\pi}\int_{-a}^{a}\sqrt{\frac{a^2-t^2}{a^2-x^2}}\,\frac{u(t)\,dt}{t-x}+\frac{C}{\sqrt{a^2-x^2}} \tag{79}$$

where $C=\operatorname{Re} C_0$.

On taking as C the value determined by the formula

$$C=-\frac{1}{\pi}\int_{-a}^{a}\sqrt{a^2-t^2}\,\frac{u(t)\,dt}{t-a}$$

we obtain from (79) the solution of equation (70) which is *bounded in the neighbourhood of the end point* a:

$$v(x)=-\frac{1}{\pi}\int_{-a}^{a}\sqrt{\frac{(a+t)(a-x)}{(a-t)(a+x)}}\,\frac{u(t)\,dt}{t-x} \tag{80}$$

Formula (80) expresses *Hilbert's transformation on a finite interval* $(-a, a)$.

5°. Integral Equation with a Kernel Having Logarithmic Singularity. In continuum mechanics an integral equation of the form

$$\frac{1}{\pi}\int_{-a}^{a} \ln|t-x|\, v(t)\,dt = u(x), \quad -a<x<a \tag{81}$$

where $u(x)$ and $v(x)$ are real functions is frequently encountered.

Let us assume that the function $u(x)$ is differentiable and that the function $v(x)$ and the derivative $\frac{du(x)}{dx}$ satisfy Hölder's condition for $-a<x<a$; then we can repeat the argument presented in Sec. 2°, § 5 of Chapter 2, differentiate equation (81) and write it in the form

$$\frac{1}{\pi}\int_{-a}^{a} \frac{v(t)\,dt}{t-x} = -u'(x), \quad -a<x<a \tag{82}$$

where the integral is understood in the sense of Cauchy's principal value.

Formula (79) implies that the *general solution* of equation (82) is the function

$$v(x) = \frac{1}{\pi}\int_{-a}^{a} \sqrt{\frac{a^2-t^2}{a^2-x^2}}\,\frac{u'(t)\,dt}{t-x} + \frac{C}{\sqrt{a^2-x^2}}$$

where C is an arbitrary real constant.

As was already mentioned in the foregoing section, the solution of equation (82) which is *bounded in the neighbourhood of the end point a* is expressed by the formula

$$v(x) = \frac{1}{\pi}\int_{-a}^{a} \sqrt{\frac{(a+t)(a-x)}{(a-t)(a+x)}}\,\frac{u'(t)\,dt}{t-x}$$

In applications it is sometimes necessary to find the solution $v(x)$ of equation (82) which is bounded at both end points of the interval $(-a, a)$. It is evident that this solution can only exist when the equality

$$\int_{-a}^{a} \frac{u'(t)\,dt}{\sqrt{a^2-t^2}} = 0 \tag{83}$$

holds; in this case the solution has the form

$$v(x) = \frac{1}{\pi} \int_{-a}^{a} \sqrt{\frac{a^2-x^2}{a^2-t^2}} \frac{u'(t)\,dt}{t-x}, \quad -a < x < a$$

In particular, condition (83) holds in the case when $u(x)$ is an even function.

CHAPTER 6

BASIC PRACTICAL METHODS FOR THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

§ 1. The Method of Separation of Variables

1°. Solution of Mixed Problem for Equation of Oscillation of a String. In the theory of oscillation of a string an important role is played by the solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

(*the equation of oscillation of a string*) which can be represented in the form

$$u(x, t) = v(x) w(t) \tag{2}$$

Such solutions are called *standing waves*, and their construction lies in the foundation of the *method of separation of variables* (also referred to as *Fourier's method*).

The substitution of expression (2) of $u(x, t)$ into the left-hand side of equation (1) results in

$$v''(x) w(t) - v(x) w''(t) = 0$$

whence

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{w(t)} \tag{3}$$

Since the left-hand side of (3) is independent of t and the right-hand side is independent of x, there must be

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{w(t)} = \text{const} \tag{4}$$

On denoting by $-\lambda$ the constant on the right-hand side of (4), we can rewrite these equalities in the form

$$v''(x) + \lambda v(x) = 0 \tag{5}$$

and

$$w''(t) + \lambda w(t) = 0 \tag{6}$$

Equalities (5) and (6) are ordinary linear differential equations of the second order with constant coefficients.

From the theory of ordinary differential equations it is known that the general solution $v(x)$ of equation (5) has the form

$$v = c_1 x + c_2 \tag{7}$$

for $\lambda = 0$, the form

$$v = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x \tag{8}$$

for $\lambda > 0$, and the form

$$v = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x} \tag{9}$$

for $\lambda < 0$ where c_1 and c_2 are arbitrary real constants.

Similarly, for $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$ the general solution of equation (6) has the form

$$\begin{aligned} w &= c_3 t + c_4 \\ w &= c_3 \cos \sqrt{\lambda}\, t + c_4 \sin \sqrt{\lambda}\, t \\ w &= c_3 e^{\sqrt{-\lambda}\, t} + c_4 e^{-\sqrt{-\lambda}\, t} \end{aligned} \tag{10}$$

respectively where c_3 and c_4 are real arbitrary constants.

Let us find a non-trivial (that is not identically equal to zero) solution $u(x, t)$ of equation (1) which is regular in the half-strip $0 < x < \pi$, $t > 0$ and continuous for $0 \leqslant x \leqslant \pi$, $t \geqslant 0$, and satisfies the boundary conditions

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t \geqslant 0 \tag{11}$$

We shall construct the solution of problem (1), (11) as a standing wave of form (2). Then the functions $v(x)$ and $w(t)$ must satisfy equations (5) and (6) respectively and the conditions $v(0)\, w(t) = v(\pi)\, w(t) = 0$ whence it follows that

$$v(0) = 0, \quad v(\pi) = 0 \tag{12}$$

The problem of determining a non-trivial solution $v(x)$ of equation (5) satisfying conditions (12) is a special case of the so-called *Sturm-Liouville problem*.

A number λ for which equation (5) possesses a non-trivial solution $v(x)$ satisfying conditions (12) is called an *eigenvalue*, and the solution $v(x)$ itself is called an *eigenfunction corresponding to* (or *belonging to* or *associated with*) *the eigenvalue* λ.

Problem (5), (12) possesses no non-trivial solutions of form (7) and (9); indeed, the substitution of expressions (7) and (9) of $v(x)$ into (12) results in $c_2 = 0$, $\pi c_1 + c_2 = 0$ and $c_1 + c_2 = 0$, $e^{\sqrt{-\lambda}\pi}c_1 + e^{-\sqrt{-\lambda}\pi}c_2 = 0$ respectively, that is $c_1 = c_2 = 0$ in both cases. Now let us substitute expression (8) into (12); this yields $c_1 = 0$ and $c_2 \sin\sqrt{\lambda}\,\pi = 0$, whence it follows that problem (5), (12) possesses a non-trivial solution of form (8) if and only if

$$\sin\sqrt{\lambda}\,\pi = 0$$

Thus, the solutions of form (8) of problem (5), (12) exist only when $\lambda = n^2$ where n is a nonzero integer.

Since the functions $\sin nx$ and $\sin(-n)x = -\sin nx$ are linearly dependent, it is natural to confine ourselves to the consideration of the natural values 1, 2, . . . of n.

Thus, we have come to the conclusion that the numbers $\lambda = n^2$ $(n = 1, 2, \ldots)$ are the eigenvalues of problem (5), (12), the corresponding eigenfunctions being $c_n \sin nx$ $(n = 1, 2, \ldots)$ where c_n are arbitrary real constants different from zero.

In what follows we shall suppose, without loss of generality, that $c_n = 1$ $(n = 1, 2, \ldots)$. Accordingly, the system of the eigenfunctions will be written in the form $v_n(x) = \sin nx$ $(n = 1, 2, \ldots)$. Consequently, homogeneous problem (1), (11) possesses an infinitude of linearly independent solutions $u_n(x, t) = \sin nx \cdot w_n(t)$ where, by virtue of (10),

$$w_n(t) = a_n \cos nt + b_n \sin nt$$

and a_n, b_n are arbitrary real constants.

The system of solutions

$$\sin nx\,(a_n \cos nt + b_n \sin nt) \qquad (n = 1, 2, \ldots) \quad (13)$$

of equation (1) we have constructed makes it possible to solve the following *mixed problem* (the *boundary-initial-value problem*): *it is required to find the solution* $u(x, t)$ *of equation* (1) *which is regular in the half-strip* $0 < x < \pi$, $t > 0$ *and con-*

tinuous for $0 \leqslant x \leqslant \pi$, $t \geqslant 0$, *and satisfies boundary conditions* (11) *and the initial conditions*

$$u(x, 0) = \varphi(x), \quad \left.\frac{\partial u(x, t)}{\partial t}\right|_{t=0} = \psi(x) \tag{14}$$

where $\varphi(x)$ *and* $\psi(x)$ *are given sufficiently smooth real functions.*

We shall construct the solution $u(x, t)$ of problem (1), (11) and (14) in the form of a series

$$u(x, t) = \sum_{n=1}^{\infty} \sin nx (a_n \cos nt + b_n \sin nt) \tag{15}$$

It is evident that if the series on the right-hand side of formula (15) is uniformly convergent, the function $u(x, t)$ specified by that formula satisfies boundary conditions (11) For this function to satisfy initial conditions (14) as well, we must have

$$\sum_{n=1}^{\infty} a_n \sin nx = \varphi(x), \quad \sum_{n=1}^{\infty} nb_n \sin nx = \psi(x) \tag{16}$$

whence

$$a_n = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin nx \, dx \text{ and } b_n = \frac{2}{\pi n} \int_0^{\pi} \psi(x) \sin nx \, dx$$

From the theory of Fourier's series it is known that the continuity of the functions $\varphi''(x)$ and $\psi'(x)$ in the interval $0 \leqslant x \leqslant \pi$ and the conditions $\varphi(0) = \varphi(\pi) = \psi(0) = \psi(\pi) = 0$ guarantee the validity of representation (16) and the uniform convergence of the trigonometric series on the right-hand side of (15). Besides, in this case the sum $u(x, t)$ of series (15) is a continuously differentiable function for $0 \leqslant x \leqslant \pi$, $t \geqslant 0$, satisfying conditions (11) and (14).

If it is additionally known that the functions $\varphi(x)$ and $\psi(x)$ are continuous in the interval $0 \leqslant x \leqslant \pi$ together with their derivatives up to the third and the second order inclusive, respectively, and if $\varphi(0) = \varphi''(0) = \varphi(\pi) = \varphi''(\pi) = 0$, $\psi(0) = \psi(\pi) = 0$ then the function $u(x, t)$ represented by formula (15) possesses partial derivatives up to the second order inclusive, and these derivatives can be computed by means of term-by-term differentiation of the series on the right-hand side of (15). It is evident that *under these assumptions the sum* $u(x, t)$ *of series* (15) *is the sought-for solution of the mixed (boundary-initial-value) problem* (1), (11), (14). Each of the summands

$$\sin nx\,(a_n \cos nt + b_n \sin nt) \quad (n = 1, 2, \ldots)$$

on the right-hand side of (15) is called (in the theory of propagation of sound) *natural oscillation* (or a *harmonic*) of the string whose end points (0, 0) and (π, 0) are fixed.

Let us prove the *uniqueness of the solution of mixed problem* (1), (11), (14). To this end it is sufficient to show that for $\varphi(x) = \psi(x) = 0$, $0 \leqslant x \leqslant \pi$ problem (1), (11), (14) possesses only a trivial solution (i.e. the solution identically equal to zero).

As was already proved in Sec. 1°, § 3 of Chapter 3, the solution $u(x, t)$ of the homogeneous Cauchy problem $u(x, 0) = 0$, $\left.\frac{\partial u(x, t)}{\partial t}\right|_{t=0} = 0$, $0 \leqslant x \leqslant \pi$, for equation (1) is identically equal to zero within the right triangle with vertices at the points A (0, 0), B (π, 0) and C ($\pi/2$, $\pi/2$). It can easily be seen that the solution $u(x, t)$ of equation (1) which is equal to zero on the line segments AC and AD where $D = D(0, \pi/2)$ turns into zero everywhere in the triangle ACD. Indeed, the integration of the identity

$$-2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 = 0$$

over the triangular domain $AC_\tau D_\tau$, where $C_\tau = C_\tau(\tau, \tau)$ and $D_\tau = D_\tau(0, \tau)$, for any fixed τ, $0 < \tau < \pi/2$, yields, by virtue of formula (*GO*), the equality

$$\int\limits_{AC_\tau + C_\tau D_\tau + D_\tau A} 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\,dt + \left(\frac{\partial u}{\partial x}\right)^2 dx + \left(\frac{\partial u}{\partial t}\right)^2 dx = 0$$

Since $u = 0$ on the line segments AC_τ and $D_\tau A$, we derive from the last equality the relation

$$\int\limits_{D_\tau C_\tau}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx = 0$$

which means that $\frac{\partial u(x, t)}{\partial x} = \frac{\partial u(x, t)}{\partial t} = 0$ on $D_\tau C_\tau$; it follows that $u(x, t) = 0$ in the triangle ACD. It can similarly be proved that we also have $u(x, t) = 0$ everywhere in the triangle BCD_1 where $D_1 = D_1(\pi, \pi/2)$.

Since the function $u(x, t)$ satisfies the homogeneous initial conditions $u(x, t) = \frac{\partial u(x, t)}{\partial t} = 0$ for $t = \pi/2$, $0 \leqslant x \leqslant \pi$, the consecutive repetition of the above argument leads to the conclusion that $u(x, t) = 0$ at all the points belonging to the strip $0 \leqslant x \leqslant \pi$, $t \geqslant 0$.

2°. **Oscillation of a Membrane.** The oscillation of an elastic membrane whose edge is fixed along a curve C lying in the plane $t = 0$ and bounding a finite domain G in that plane is described by the solution of the wave equation with two spatial variables x and y (*the equation of oscillation of a membrane*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{17}$$

This solution satisfies initial conditions of the form

$$u(x, y, 0) = \varphi(x, y), \quad \left.\frac{\partial u(x, y, t)}{\partial t}\right|_{t=0} = \psi(x, y), \quad (x, y) \in G \tag{18}$$

and the boundary condition

$$u(x, y, t) = 0, \quad t \geqslant 0, \quad (x, y) \in C \tag{19}$$

Using the method of standing waves which was considered in the foregoing section, we conclude, in an analogous manner, that for an expression of the form

$$u(x, y, t) = v(x, y)\, w(t) \tag{20}$$

to satisfy equation (17), the functions $v(x, y)$ and $w(t)$ must be solutions of the equations

$$\Delta v(x, y) + \lambda v(x, y) = 0 \tag{21}$$

and

$$w''(t) + \lambda w(t) = 0 \tag{22}$$

respectively where

$$\lambda = -\frac{\Delta v(x, y)}{v(x, y)} = -\frac{w''(t)}{w(t)} = \text{const}$$

and Δ is Laplace's operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

On substituting the expression of the function $u(x, y, t)$ given by formula (20) into boundary condition (19), we obtain

$$v(x, y)\, w(t) = 0, \quad (x, y) \in C, \quad t \geqslant 0$$

The last relation is equivalent to the boundary condition

$$v(x, y) = 0, \quad (x, y) \in C \tag{23}$$

for the function $v(x, y)$.

A number λ for which Dirichlet's homogeneous problem (23) for Helmholtz' equation (21) possesses a non-trivial solution $v(x, y)$ is called an *eigenvalue*, and the function $v(x, y)$ is called an *eigenfunction corresponding to* (or *associated with*) λ.

Let us suppose that the contour C bounding the plane region G is a piecewise-smooth Jordan curve and that $v(x, y)$ is an eigenfunction of problem (21), (23) associated with an eigenvalue λ.

On integrating the obvious identity

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial v}{\partial y}\right) - v\,\Delta v$$

over the domain G and using formula (GO), we obtain, by virtue of (21) and (23), the relation

$$\int_G (v_x^2 + v_y^2)\, dx\, dy = \int_C v\frac{\partial v}{\partial \nu}\, ds - \int_G v\,\Delta v\, dx\, dy = \lambda \int_G v^2\, dx\, dy$$

From this relation we conclude that *the eigenvalue λ must be positive*. Therefore we can use the notation $\lambda = \mu^2$ where μ is a real number. Accordingly, by virtue of (10), the general solution of equation (22) can be written in the form

$$w(t) = c_3 \cos \mu t + c_4 \sin \mu t \tag{24}$$

Formula (24) implies that $w(t)$ is a periodic function with period $2\pi/\mu$.

For some general assumptions concerning the domain G it can be proved that there exists a countable set of the numbers $\mu_1, \mu_2, \ldots$ and the corresponding countable set of eigenfunctions $v_1(x, y), v_2(x, y), \ldots$. Below we shall prove this fact for the special case when the domain G is a circle.

The solutions of equation (22) having form (24) and corresponding to μ_n ($n = 1, 2, \ldots$) can be written in the form $w_n(t) = a_n \cos \mu_n t + b_n \sin \mu_n t$ where a_n and b_n are arbitrary real constants; the corresponding system of solutions of equation (17) is

$$u_n(x, y, t) = v_n(x, y)(a_n \cos \mu_n t + b_n \sin \mu_n t) \quad (25)$$
$$n = 1, 2, \ldots$$

If v_k and v_m are two eigenfunctions corresponding to λ_k and λ_m ($\lambda_k \neq \lambda_m$) then

$$\int_G v_k(x, y)\, v_m(x, y)\, dx\, dy = 0, \quad k \neq m \quad (26)$$

To prove equality (26) it is sufficient to integrate the identity

$$\frac{\partial}{\partial x}\left(v_k \frac{\partial v_m}{\partial x} - v_m \frac{\partial v_k}{\partial x}\right) + \frac{\partial}{\partial y}\left(v_k \frac{\partial v_m}{\partial y} - v_m \frac{\partial v_k}{\partial y}\right) =$$
$$= v_k \Delta v_m - v_m \Delta v_k$$

over the domain G and then make use of formula (GO) and the equalities

$$\Delta v_k = -\lambda_k v_k, \quad \Delta v_m = -\lambda_m v_m, \quad (x, y) \in G$$
$$v_k(x, y) = v_m(x, y) = 0, \quad (x, y) \in C$$

This results in

$$\int_G (v_k \Delta v_m - v_m \Delta v_k)\, dx\, dy = (\lambda_k - \lambda_m) \int_G v_k v_m\, dx\, dy = 0$$

whence, since $\lambda_k \neq \lambda_m$, follows formula (26).

We shall construct the solution $u(x, y, t)$ of problem (17), (18), (19) in the form of a series

$$u(x, y, t) = \sum_{n=1}^{\infty} v_n(x, y)(a_n \cos \mu_n t + b_n \sin \mu_n t) \quad (27)$$

where a_n and b_n ($n = 1, 2, \ldots$) are some real constants.

Let us suppose that the series on the right-hand side of (27) is uniformly convergent and that it is legitimate to differentiate it twice term-by-term; then the sum $u(x, y, t)$ of that series will satisfy equation (17) and boundary condition (19).

For the function $u(x, y, t)$ to satisfy initial conditions (18) as well, the coefficients a_n and b_n must be such that the equalities

$$\sum_{n=1}^{\infty} a_n v_n(x, y) = \varphi(x, y), \quad \sum_{n=1}^{\infty} \mu_n b_n v_n(x, y) = \psi(x, y) \tag{28}$$

hold, whence, taking into account (26), we find

$$\begin{aligned} a_n &= \frac{1}{N^2(v_n)} \int_G \varphi(x, y)\, v_n(x, y)\, dx\, dy \\ b_n &= \frac{1}{\mu_n N^2(v_n)} \int_G \psi(x, y)\, v_n(x, y)\, dx\, dy \end{aligned} \tag{29}$$

where

$$N(v_n) = \left(\int_G v_n^2(x, y)\, dx\, dy \right)^{1/2} \tag{30}$$

The number $N(v_n)$ determined by formula (30) is called the *norm* of the function $v_n(x, y)$.

3°. The Notion of a Complete Orthonormal System of Functions. Real functions $v_k(x, y)$ $(k = 1, \ldots, n)$ defined in a domain G each of which is not identically equal to zero are said to be *linearly independent* if there are no real constants c_k $(k = 1, \ldots, n)$ among which at least one is different from zero such that

$$\sum_{k=1}^{n} c_k v_k(x, y) = 0, \quad (x, y) \in G$$

An infinite system of functions

$$v_k(x, y) \quad (k = 1, 2, \ldots) \tag{31}$$

is called *linearly independent* if any finite system of functions $v_k(x, y)$ chosen from system (31) is linearly independent.

We shall suppose that the functions $v_k(x, y)$ $(k = 1, 2, \ldots)$ are square integrable in the domain G. A linearly independent system of form (31) is said to be *orthogonal* if

$$\int_G v_k(x, y)\, v_m(x, y)\, dx\, dy = 0, \quad k \neq m \tag{32}$$

An orthogonal system of form (31) is called *orthonormal* if

$$N(v_k) = 1 \quad (k = 1, 2, \ldots)$$

It is obvious that, given an orthogonal system of form (31), we can always make it orthonormal by dividing its every member $v_k(x, y)$ by the corresponding number $N(v_k)$.

Let $\varphi(x, y)$ be an arbitrary function defined and square integrable in G, and let system (31) be orthonormal. Then the numbers

$$a_k = \int_G \varphi(x, y)\, v_k(x, y)\, dx\, dy \quad (k = 1, 2, \ldots) \tag{33}$$

are called *Fourier's coefficients* of the function $\varphi(x, y)$ with respect to orthonormal system (31).

An expression of the form $\sum_{k=1}^{n} \alpha_k v_k(x, y)$ where α_k are real constants will be referred to as a *linear combination* of the functions $v_k(x, y)$ $(k = 1, \ldots, n)$.

The number

$$M = \int_G \left(\varphi - \sum_{k=1}^{n} \alpha_k v_k\right)^2 dx\, dy \tag{34}$$

is called the *mean square deviation* of the linear combination $\sum_{k=1}^{n} \alpha_k v_k(x, y)$ from the function φ.

By virtue of (32) and (33), formula (34) implies that

$$M = N^2(\varphi) + \sum_{k=1}^{n} (\alpha_k - a_k)^2 - \sum_{k=1}^{n} a_k^2 \geqslant 0$$

whence it follows that for a fixed value of n mean square deviation (34) attains its minimum when $\alpha_k = a_k$ $(k = 1, \ldots, n)$.

For the linear combination $\sum_{k=1}^{n} a_k v_k(x, y)$ we have the inequality

$$\int_G \left(\varphi - \sum_{k=1}^{n} a_k v_k\right)^2 dx\, dy \geqslant 0$$

which implies that

$$\sum_{k=1}^{n} a_k^2 \leqslant N^2(\varphi) \tag{35}$$

for any n. Consequently, *the number series whose terms are the squares of Fourier's coefficients of the function* $\varphi(x, y)$ *is convergent, and the inequality*

$$\sum_{k=1}^{\infty} a_k^2 \leqslant N^2(\varphi) \tag{36}$$

is fulfilled. Relation (36) is called *Bessel's inequality*.

An orthonormal system of form (31) is said to be *complete* (in the function space to which $\varphi(x, y)$ and $v_k(x, y)$, $k = 1, 2, \ldots$, belong) if

$$\lim_{n\to\infty} \int_G \left(\varphi - \sum_{k=1}^{n} a_k v_k\right)^2 dx\,dy = 0$$

or, which is the same,

$$\sum_{k=1}^{\infty} a_k^2 = N^2(\varphi) \tag{37}$$

Condition (37) expressing the completeness of system (31) by far not always guarantees that the function $\varphi(x, y)$ can be represented as the sum of the first series (28):

$$\varphi(x, y) = \sum_{k=1}^{\infty} a_k v_k(x, y) \tag{38}$$

If the function $\varphi(x, y)$ *is continuous together with its partial derivatives up to the second order inclusive, representation* (38) *is sure to hold.*

If we assume that the functions $\varphi(x, y)$ and $\psi(x, y)$ are continuous together with their derivatives up to the second order inclusive, the series on the right-hand side of (27) whose coefficients a_n and b_n are determined by formulas (29) is uniformly convergent. To guarantee the possibility of term-by-term differentiation of that series in order to compute the derivatives up to the second order we should additionally assume that the derivatives of $\varphi(x, y)$ and $\psi(x, y)$ up to the fourth and the third order respectively are continuous. *Under these assumptions the sum* $u(x, y, t)$ *of series* (27) *will be a regular solution of equation* (17).

This assertion can be proved using the theory of the Fredholm integral equations, but here we shall not dwell on the proof.

4°. **Oscillation of Circular Membrane.** In the study of the oscillation of a circular membrane we shall assume, without loss of generality, that in the state of equilibrium the membrane occupies the circle $x^2 + y^2 \leqslant 1$ in the xy-plane.

In this section, besides the orthogonal Cartesian coordinates x, y, we shall also use the polar coordinates r, θ which are connected with x and y by the equalities $x = r\cos\theta$ and $y = r\sin\theta$.

When we pass from the orthogonal Cartesian coordinates to the polar coordinates, Laplace's operator is transformed according to the formula

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Therefore in the polar coordinates equation (21) is written as

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \mu^2 v = 0, \quad \mu^2 = \lambda \tag{39}$$

For a function $v(r, \theta)$ of the form

$$v(r, \theta) = R(r)\,\Theta(\theta) \tag{40}$$

to be a solution of equation (39), the functions $R(r)$ and $\Theta(\theta)$ should satisfy the equations

$$r^2 R''(r) + rR'(r) + (\mu^2 r^2 - \omega) R(r) = 0 \tag{41}$$

and

$$\Theta''(\theta) + \omega\Theta(\theta) = 0 \tag{42}$$

respectively where ω is a real constant:

$$\omega = -\frac{\Theta''}{\Theta} = \frac{r^2 R'' + rR' + r^2\mu^2 R}{R} = \text{const}$$

From (40) it follows that for the function $v(r, \theta)$ to be one-valued, the function $\Theta(\theta)$ must be one-valued, that is $\Theta(\theta)$ must be a periodic function of period 2π. It follows that in equation (42) the constant ω must be equal to n^2: $\omega = n^2$ where n is an arbitrary integer. Accordingly, the general solution of equation (42) takes the form

$$\Theta(\theta) = \alpha_n \cos n\theta + \beta_n \sin n\theta \tag{43}$$

where α_n and β_n are arbitrary real constants.

If $R(r)$ is a solution of equation (41) which is regular for $0 \leqslant r < 1$ and continuous for $0 \leqslant r \leqslant 1$, and satisfies the condition

$$R(1) = 0 \tag{44}$$

then, in the case under consideration, the function $v(r, \theta)$ represented by formula (40) is a solution of problem (21), (23).

On changing the variable r with the aid of the formula $\mu r = \rho$ and transforming the unknown function according to the formula $R(r) = R\left(\frac{\rho}{\mu}\right) = J(\rho)$ we bring equation (41) to the form

$$J''(\rho) + \frac{1}{\rho} J'(\rho) + \left(1 - \frac{n^2}{\rho^2}\right) J(\rho) = 0 \tag{45}$$

In what follows we shall assume that the integer n satisfies the condition $n \geqslant 0$.

Let us consider the power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k!\,(n+k)!} \tag{46}$$

Since $(k!)^2 = \prod_{j=1}^{k} j(k-j+1) \geqslant k^k$, we have

$$\frac{1}{2^{2k} k!\,(n+k)!} < \frac{1}{k^k}$$

for $k > 0$, and therefore

$$\lim_{k \to \infty} \sqrt[2k]{\frac{1}{2^{2k} k!\,(n+k)!}} = 0$$

Now, taking into account the Cauchy-Hadamard formula, we conclude that the radius of convergence of power series (46) is infinite, whence it follows that the sum of series (46) is an entire function of the variable z.

The last fact makes it possible to verify directly that the entire functions

$$J_n(\rho) = \sum_{k=0}^{\infty} (-1)^k \frac{\rho^{n+2k}}{2^{n+2k} k!\,(n+k)!} \quad (n = 0, 1, \ldots) \tag{47}$$

satisfy equation (45), that is the functions $R(r) = J_n(\mu r)$ are solutions of equation (41) for $\omega = n^2$.

The functions $J_n(\rho)$ $(n = 0, 1, \ldots)$ determined by formula (47) are called *Bessel's functions* (*of the first kind*); they satisfy equation (45) which is referred to as *Bessel's equation.*

It can be proved that *Bessel's function with a nonnegative integral index n possesses an infinite* (*countable*) *set of real zeros.* We shall denote them as $\rho_{n,m}$ $(m = 1, 2, \ldots)$.

The eigenvalues μ^2 of problem (41), (44) should be found from the equalities

$$R(1) = J_n(\mu) = 0 \quad (n = 0, 1, \ldots)$$

Consequently, *the eigenvalues of problem* (41), (44) *are the squares of the zeros of Bessel's functions, that is*

$$\mu_m^2 = \rho_{n,m}^2 \neq 0$$

From (40) and (43) it follows that the eigenfunctions associated with these eigenvalues have the form

$$v_{n,m}(r, \theta) = J_n(\rho_{n,m} r)(\alpha_n \cos n\theta + \beta_n \sin n\theta) \tag{48}$$

On substituting the expression of $v_{n,m}(r, \theta)$ given by (48) into the right-hand side of formula (25), we obtain the solutions

$$u_{n,m}(x, y, t) = J_n(\rho_{n,m} r)(\alpha_n \cos n\theta + \beta_n \sin n\theta)(a_{n,m} \cos \rho_{n,m} t + b_{n,m} \sin \rho_{n,m} t) \tag{49}$$

$$n = 0, 1, \ldots; \; m = 1, 2, \ldots$$

of equation (17) satisfying boundary condition (19); each of these solutions describes natural oscillation of a circular membrane with a fixed edge.

For the case under consideration, the family of solutions (49) of equation (17) makes it possible to construct the solution of problem (17), (18), (19) using the scheme indicated in Sec. 2°, § 4 of the present chapter.

Equation (21) is called the *metaharmonic equation*, and its regular solutions are termed *metaharmonic functions.*

On substituting the solutions $J_n(\mu r)$ and $\cos n\theta$, $\sin n\theta$, $\mu^2 = \lambda$, $n = 0, 1, \ldots$, of equations (41) and (42) into the right-hand side of (40) we obtain the metaharmonic functions $J_n(\mu r) \cos n\theta$ and $J_n(\mu r) \sin n\theta$. The first of these functions turns into zero on the circles $r = \frac{\rho_{n,m}}{\mu}$ and on the rays $\theta = \left(\pi k + \frac{\pi}{2}\right)/n$ while the second function turns into zero on the circles $r = \frac{\rho_{n,m}}{\mu}$ and on the rays $\theta = \pi k/n$, $m = 1, 2, \ldots$; $k = 0, 1, \ldots, n-1$; these circles and rays on which the functions turn into zero are called *nodal lines*.

As we know, the Dirichlet problem for Laplace's equation is always solvable and its solution is unique; for metaharmonic equation (21) an analogous assertion may not hold. For instance, as was mentioned above, homogeneous Dirichlet problem (23) for equation (21) possesses the linearly independent solutions $J_n(\rho_{n,m} r) \cos n\theta$ and $J_n(\rho_{n,m} r) \times \sin n\theta$ in the circle $r < 1$ for $\lambda = \rho_{n,m}^2$ whereas, as it turns out, in some cases the non-homogeneous Dirichlet problem has no solutions.

5°. Some General Remarks on the Method of Separation of Variables. The Fourier method of separation of variables can be successfully used for constructing solutions of a wide class of partial differential equations.

Let us consider an equation of the form

$$\sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x) \frac{\partial u}{\partial x_i} + C(x) u = \alpha(t) \frac{\partial^2 u}{\partial t^2} + \beta(t) \frac{\partial u}{\partial t} + \gamma(t) u \quad (50)$$

For a function $u(x, t)$ of the form

$$u(x, t) = v(x) w(t) \quad (51)$$

to satisfy equation (50) the functions $v(x)$ and $w(t)$ must satisfy the equations

$$\sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x) \frac{\partial v}{\partial x_i} + [C(x) + \lambda] v(x) = 0 \quad (52)$$

and

$$\alpha(t) w''(t) + \beta(t) w'(t) + [\gamma(t) + \lambda] w(t) = 0 \quad (53)$$

respectively where $\lambda = \text{const}$.

When the number of spatial variables is $n = 1$, that is in the case of the equation

$$A(x)\frac{\partial^2 u}{\partial x^2}+B(x)\frac{\partial u}{\partial x}+C(x)u=$$
$$=\alpha(t)\frac{\partial^2 u}{\partial t^2}+\beta(t)\frac{\partial u}{\partial t}+\gamma(t)u \quad (54)$$

the corresponding equation of form (52) for the function $v(x)$ is written as

$$A(x)v''+B(x)v'+[C(x)+\lambda]v=0 \quad (55)$$

Both equations (53) and (55) are ordinary linear differential equations whose solutions can be investigated in a rather simple way. However, the construction of the complete system of solutions of form (51) for equation (54) and the proof of the possibility of the representation of the solution $u(x, t)$ of mixed problem (18), (19) for this equation in the form of the sum of a series with respect to solutions (51) cannot be performed without resorting to the spectral theory of linear operators.

The investigation of this problem becomes very complicated when there are separate points in the interval of variation of the independent variable x (or t) at which the function $A(x)$ (or the function $\alpha(t)$) turns into zero. However, it is these cases that are most frequently encountered in applications. To investigate problems of this kind it becomes necessary to consider the so-called *special functions.*

In the case when $A(x) = x^2$, $B(x) = x$, $C(x) = x^2$, $\lambda = -\mu^2$ equation (55) is *Bessel's equation*

$$x^2v''+xv'+(x^2-\mu^2)v=0$$

which we considered earlier. The solutions of this equation are called *Bessel's* (or *cylindrical*) *functions.* Bessel's functions $J_n(x)$ with nonnegative integral indices n were used in the foregoing section.

In the case when $A(x) = 1 - x^2$, $B(x) = -x$, $C(x) = 0$, $\lambda = n^2$ equation (55) is *Chebyshev's equation*

$$(1-x^2)v''-xv'+n^2v=0$$

Chebyshev's functions

$$T_n(x)=\frac{1}{2}\left[\left(x+i\sqrt{1-x^2}\right)^n+\left(x-i\sqrt{1-x^2}\right)^n\right]$$

and

$$u_n(x) = \frac{1}{2i}\left[\left(x + i\sqrt{1-x^2}\right)^n - \left(x - i\sqrt{1-x^2}\right)^n\right]$$

($n = 0, 1, \ldots$) are solutions of Chebyshev's equation; it is obvious that the functions $T_n(x)$ are polynomials (they are called *Chebyshev's polynomials*).

Laguerre's equation

$$xv'' + (1 - x)v' + \lambda v = 0$$

is also a special case of equation (55).

6°. **Solid and Surface Spherical Harmonics.** By virtue of formula (26) of Introduction, the homogeneous polynomials

$$u_\alpha^m(x, y, z) = \sum_{n \geqslant 0} (-1)^n \frac{z^{2n}}{(2n)!} \Delta^n (x^\alpha y^{m-\alpha}) \qquad (56)$$

$$\alpha = 0, \ldots, m$$

and

$$u_{m+\beta+1}^m(x, y, z) = \sum_{n \geqslant 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \Delta^n (x^\beta y^{m-\beta-1}) \quad (57)$$

$$\beta = 0, \ldots, m-1$$

of the mth degree $\left(\text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ are harmonic functions; they are called *solid spherical harmonics.*

Formulas (56) and (57) give all linearly independent solid spherical harmonics of degree m, their total number being $2m + 1$.

For instance, putting $m = 1$ in formulas (56) and (57) we obtain

$$u_0^1 = y, \quad u_1^1 = x, \quad u_2^1 = z$$

and putting $m = 2$ we obtain

$$u_0^2 = y^2 - z^2, \quad u_1^2 = xy, \quad u_2^2 = x^2 - z^2,$$

$$u_3^2 = zy, \quad u_4^2 = zx$$

If we pass to the spherical coordinates r, φ, θ which are connected with the orthogonal Cartesian coordinates x, y, z by means of the relations

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta$$

the solid spherical harmonics $u_k^m(x, y, z)$ take the form

$$u_k^m(x, y, z) = r^m Y_m^k(\varphi, \theta), \quad k = 0, \ldots, 2m \tag{58}$$

where the expressions $Y_m^k(\varphi, \theta)$ are called *Laplace's surface spherical harmonics.*

As was already mentioned in Sec. 1°, § 1 of Chapter 1, the harmonicity of functions (58) implies the harmonicity of the functions

$$\frac{1}{r} u_k^m\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right) = \frac{1}{r^{m+1}} Y_m^k(\varphi, \theta), \quad k = 0, \ldots, 2m \tag{59}$$

Let us write Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ in spherical coordinates:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0$$

If we require that a function $u(x, y, z)$ of the form

$$u(x, y, z) = Y(\varphi, \theta)\, w(r)$$

should be harmonic, the separation of variables yields

$$\frac{d}{dr}\left(r^2 \frac{dw}{dr}\right) - \lambda w = 0 \tag{60}$$

and

$$\frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial \varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \lambda Y = 0 \tag{61}$$

where $\lambda = \text{const}$.

In particular, for $\lambda = m(m+1)$ equations (60) and (61) take the form

$$\frac{d}{dr}\left(r^2 \frac{dw}{dr}\right) - m(m+1)\, w = 0 \tag{62}$$

and

$$\frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial \varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + m(m+1)\, Y = 0 \tag{63}$$

The factors $\frac{1}{r^{m+1}}$ and $Y_m^k(\varphi, \theta)$ in expressions (59) for the harmonic functions $\frac{1}{r} u_k^m\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$ are solutions of equations (62) and (63) respectively.

For a function $Y(\varphi, \theta)$ of the form

$$Y(\varphi, \theta) = \Phi(\varphi)\Theta(\theta)$$

to be a solution of equation (63), the functions Φ and Θ must satisfy the equations

$$\Phi'' + \mu\Phi = 0, \tag{64}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[m(m+1) - \frac{\mu}{\sin^2\theta}\right]\Theta = 0 \tag{65}$$

respectively where $\mu = \text{const}$.

The condition that the function $Y(\varphi, \theta)$ is periodic with respect to φ with period 2π implies that the constant μ in equation (64) must be equal to the square of an integer: $\mu = n^2$ where n is an integral number. Accordingly, using the notation $\cos\theta = t$ and $\Theta(\theta) = \Theta(\arccos t) = v(t)$ we rewrite equation (65) in the form

$$(1-t^2)v'' - 2tv' + \left[m(m+1) - \frac{n^2}{1-t^2}\right]v = 0 \tag{66}$$

From equation (66), for $n = 0$, we obtain the equation

$$(1 - t^2)v'' - 2tv' + m(m+1)v = 0$$

known as *Legendre's equation.* The linearly independent solutions of this equation are called *Legendre's functions of the first* and *of the second kind* and are denoted $P_m(\cos\theta)$ and $Q_m(\cos\theta)$ respectively.

As to the linearly independent solutions $P_m^n(\cos\theta)$ and $Q_m^n(\cos\theta)$ of equation (66), they are called *Legendre's associated functions of the first* and *of the second kind.*

7°. Forced Oscillation. The non-homogeneous equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = f(x, t) \tag{67}$$

where $f(x, t)$ is a given real continuous function, is referred to as the *equation of forced oscillation of a string.*

In the case when $f(x, t) = f_n(t)\sin nx$ it is natural to construct the solution $u_n(x, t)$ of equation (67) in the form $u_n(x, t) = w_n(t)\sin nx$. From (67) we obtain the ordinary linear differential equation

$$\ddot{w}_n(t) + n^2 w_n(t) = -f_n(t)$$

for the determination of $w_n(t)$; it is obvious that the function

$$w_n^0(t) = -\frac{1}{n}\int_0^t f_n(\tau)\sin n(t-\tau)\,d\tau$$

is a particular solution of this equation. Consequently, the general solution of this equation has the form

$$w_n(t) = a_n\cos nt + b_n\sin nt - \frac{1}{n}\int_0^t f_n(\tau)\sin n(t-\tau)\,d\tau$$

where a_n and b_n are arbitrary real constants.

The system of solutions

$$u_n(x, t) = \\ = \sin nx\left[a_n\cos nt + b_n\sin nt - \frac{1}{n}\int_0^t f_n(\tau)\sin n(t-\tau)\,d\tau\right]$$

$$n = 1, 2, \ldots$$

of equation (67) makes it possible to investigate mixed problem (67), (11), (14).

Let the functions $\varphi(x)$ and $\psi(x)$ be the sums of series (16) and let $f(x, t) = \sum_{n=1}^{\infty} f_n(t)\sin nx$. Under the assumption that these series can be differentiated and integrated termwise, the solution $u(x, t)$ of mixed problem (67), (11), (14) can be written in the form

$$u(x, t) = \sum_{n=1}^{\infty}\sin nx\left[a_n\cos nt + b_n\sin nt - \frac{1}{n}\int_0^t f_n(\tau)\sin n(t-\tau)\,d\tau\right]$$

It should be noted that if the boundary conditions are non-homogeneous, that is if instead of (11) we have $u(0, t) = \alpha(t)$ and $u(\pi, t) = \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are twice continuously differentiable functions, the transformation

$u(x, t) = v(x, t) + \alpha(t) + \frac{x}{\pi}[\beta(t) - \alpha(t)]$ of the unknown function $u(x, t)$ in equation (67) results in the equation

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} = f(x, t) + \alpha''(t) + \frac{x}{\pi}[\beta''(t) - \alpha''(t)]$$

for the new unknown function $v(x, t)$; the new boundary conditions are homogeneous: $v(0, t) = v(\pi, t) = 0$. The initial conditions for the function $v(x, t)$ are obtained by changing in the corresponding manner the initial conditions for $u(x, t)$.

Forced oscillations of a membrane can be investigated in a similar way.

§ 2. The Method of Integral Transformation

1°. Integral Representation of Solutions of Ordinary Linear Differential Equations of the Second Order. The class of differential equations whose solutions can be expressed in terms of elementary functions is rather narrow.

In the foregoing section we used the method of separation of variables in order to construct solutions of partial differential equations as sums of infinite series. However, it is sometimes convenient to represent the solution of a differential equation under consideration in the form of an integral involving some known functions and also solutions of some simpler equations.

Let us consider an ordinary homogeneous linear differential equation of the second order of the form

$$L(y) = p(z)\,y'' + q(z)\,y' + r(z)\,y = 0 \tag{68}$$

whose coefficients are analytic functions defined throughout the whole complex plane of the variable z.

We shall seek the solution $y(z)$ of equation (68) in the form of the integral

$$y(z) = \int_C K(z, \zeta)\, v(\zeta)\, d\zeta \tag{69}$$

where C is a piecewise smooth contour, $v(\zeta)$ is an analytic function (yet unknown), and $K(z, \zeta)$ is an analytic function with respect to the variables z, ζ satisfying an equation of the form

$$p(z)\frac{\partial^2 K}{\partial z^2}+q(z)\frac{\partial K}{\partial z}+r(z)K=a(\zeta)\frac{\partial^2 K}{\partial \zeta^2}+b(\zeta)\frac{\partial K}{\partial \zeta}+c(\zeta)K \tag{70}$$

the coefficients $a(\zeta)$, $b(\zeta)$ and $c(\zeta)$ being some given analytic functions.

The calculations below are carried out under the assumption that all the operations we perform are legitimate. From (69) we obtain

$$L(y)=\int_C L(K)v(\zeta)\,d\zeta$$

Taking into account (70) we can rewrite this relation in the form

$$L(y)=\int_C M(K)v(\zeta)\,d\zeta \tag{71}$$

where the symbol M under the integral sign denotes the differential operator on the right-hand side of (70): $M = a(\zeta)\frac{\partial^2}{\partial\zeta^2}+b(\zeta)\frac{\partial}{\partial\zeta}+c(\zeta)$.

Let us perform integration by parts in (71). Then, assuming that in the resultant expression all the terms not involving integrals turn into zero, we obtain

$$L(y)=\int_C K(z, \zeta)M^*(v)\,d\zeta \tag{72}$$

where

$$M^*(v)=\frac{d^2}{d\zeta^2}(av)-\frac{d}{d\zeta}(bv)+c(\zeta)v$$

is the *(Lagrange) adjoint differential operator of M*.

In case the function v satisfies the equation

$$M^*(v)=0 \tag{73}$$

the function $y(z)$ expressed by formula (69) is obviously a solution of differential equation (68).

To demonstrate the application of this technique let us consider Bessel's equation (45) which we shall write in the form

$$z^2 y'' + zy' + (z^2 - n^2)\, y = 0 \tag{74}$$

We shall construct the solution $y(z)$ of equation (74) using formula (69) in which we put

$$K(z,\ \zeta) = \mp \frac{1}{\pi} e^{-iz \sin \zeta}$$

In this case we have

$$L(K) = z^2 K_{zz} + zK_z + (z^2 - n^2)\, K = -K_{\zeta\zeta} - n^2 K$$

and therefore equalities (71) and (72) are written in the form

$$L(y) = -\int_C (K_{\zeta\zeta} + n^2 K)\, v(\zeta)\, d\zeta =$$

$$= \pm \frac{1}{\pi} \int_C (v_{\zeta\zeta} + n^2 v)\, e^{-iz \sin \zeta}\, d\zeta$$

From the last relation we conclude that $M^* = -\frac{d^2}{d\zeta^2} - n^2$, that is equation (73) has the form

$$\frac{d^2 v}{d\zeta^2} + n^2 v = 0$$

The solutions of this equation are the functions $e^{\pm in\zeta}$.

Consequently, the functions determined by formula (69) in which

$$K(z,\ \zeta) = \mp \frac{1}{\pi} e^{-iz \sin \zeta} \quad \text{and} \quad v(\zeta) = e^{in\zeta}$$

are solutions of Bessel's equation (74).

All the operations we have performed are legitimate if we suppose that $\operatorname{Re} z > 0$ and if, for instance, we take as the contour of integration C in formula (69) the broken line

$$\xi = 0,\ -\infty < \eta \leqslant 0;\ \eta = 0,\ -\pi \leqslant \xi \leqslant 0;$$
$$\xi = -\pi,\quad 0 \leqslant \eta < \infty \tag{75}$$

(see Fig. 24) or the broken line

$$\xi = 0,\ -\infty < \eta \leqslant 0;\ \eta = 0,\ 0 \leqslant \xi \leqslant \pi;$$
$$\xi = \pi,\ 0 \leqslant \eta < \infty,\ \zeta = \xi + i\eta \tag{76}$$

Let the path of integration C be broken line (75) and let $K(z, \zeta) = -\frac{1}{\pi} e^{-iz \sin \zeta}$. We shall denote by $H_n^1(z)$ the cor-

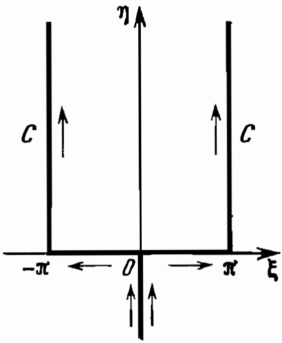

Fig. 24

responding solution of equation (74) defined by formula (69) in the half-plane Re $z > 0$:

$$H_n^1(z) = -\frac{i}{\pi} \int_{-\infty}^{0} \exp(-iz \sin i\eta - n\eta)\, d\eta - -\frac{1}{\pi} \int_{0}^{-\pi} \exp(-iz \sin \xi + n\xi i)\, d\xi - -\frac{i}{\pi} \int_{0}^{\infty} \exp(iz \sin i\eta - n\eta - \pi n i)\, d\eta \quad (77)$$

In the case when the path C coincides with broken line (76) and $K(z, \zeta) = \frac{1}{\pi} e^{-iz \sin \zeta}$ we shall denote the corresponding solution as $H_n^2(z)$:

$$H_n^2(z) = \frac{i}{\pi} \int_{-\infty}^{0} \exp(-iz \sin i\eta - n\eta)\, d\eta + +\frac{1}{\pi} \int_{0}^{\pi} \exp(-iz \sin \xi + n\xi i)\, d\xi +$$

$$+\frac{i}{\pi}\int_{0}^{\infty}\exp(iz\sin i\eta-n\eta+\pi ni)\,d\eta \qquad (78)$$

Taking into account the relation

$$\sin i\eta=\frac{e^{-\eta}-e^{\eta}}{2i}=-\frac{\sinh\eta}{i}$$

we can rewrite formulas (77) and (78) (after some simple transformations) in the form

$$H_n^1(z)=\frac{1}{\pi i}\int_{-\infty}^{0}\exp(z\sinh\eta-n\eta)\,d\eta+$$

$$+\frac{1}{\pi}\int_{-\pi}^{0}\exp(-iz\sin\xi+in\xi)\,d\xi+$$

$$+\frac{1}{\pi i}\int_{0}^{\infty}\exp(-z\sinh\eta-n\eta-\pi ni)\,d\eta \qquad (79)$$

and

$$H_n^2(z)=-\frac{1}{\pi i}\int_{-\infty}^{0}\exp(z\sinh\eta-n\eta)\,d\eta+$$

$$+\frac{1}{\pi}\int_{0}^{\pi}\exp(-iz\sin\xi+in\xi)\,d\xi-$$

$$-\frac{1}{\pi i}\int_{0}^{\infty}\exp(-z\sinh\eta-n\eta+\pi ni)\,d\eta \quad (80)$$

respectively.

The solutions of Bessel's equation (74) specified by formulas (79) and (80) are called *Hankel's functions* (they are also referred to as *Bessel's functions of the third kind*). Their linear combination

$$J_n(z)=\frac{1}{2}[H_n^1(z)+H_n^2(z)] \qquad (81)$$

is *Bessel's function* (*of the first kind*) and the linear combination

$$N_n(z)=\frac{1}{2i}[H_n^1(z)-H_n^2(z)] \qquad (82)$$

is called *Neumann's function* (also referred to as *Bessel's function of the second kind*).

From (79), (80) and (81) we obtain

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-iz \sin \xi + in\xi)\, d\xi - \\ - \frac{\sin n\pi}{\pi} \int_0^{\infty} \exp(-z \sinh \eta - n\eta)\, d\eta \quad (83)$$

In the case when n is an integer the second summand on the right-hand side of (83) vanishes, and the expression for $J_n(z)$ takes the form

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-iz \sin \xi + in\xi)\, d\xi = \\ = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin \xi - n\xi)\, d\xi \quad (84)$$

From formula (84) it follows that *for an integral value of the index n the expression $J_n(z)$ is an entire function of the complex variable z, and, besides,*

$$J_{-n}(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin \xi + n\xi)\, d\xi = \\ = \frac{1}{\pi} \int_0^{\pi} \cos[z \sin(\pi - t) + n(\pi - t)]\, dt = \\ = (-1)^n \frac{1}{\pi} \int_0^{\pi} \cos(z \sin t - nt)\, dt = (-1)^n J_n(z)$$

It can be proved that *the solutions of Bessel's equation* (74) *determined by formulas* (79) *and* (80) *in the half-plane* Re $z >$ > 0 *can be continued analytically to the half-plane* Re $z < 0$ *for any value of the index n* and that *when the number n is non-integral the points $z = 0$ and $z = \infty$ of the complex variable z are branch points for the analytically continued functions.*

Besides, *the functions* $J_n(z)$ *and* $N_n(z)$ *are linearly independent, and they can be represented as the sums of the series*

$$J_n(z)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k} \qquad (85)$$

and

$$N_n(z)=\frac{J_n(z)\cos n\pi-J_{-n}(z)}{\sin \pi n}$$

respectively.

The fact that the function $J_n(z)$ determined by formula (85) satisfies equation (74) can easily be verified directly if we take into account the well-known property of Euler's gamma function: $\Gamma(k+1)=k\Gamma(k)$.

2°. **Laplace, Fourier and Mellin Transforms.** Let $f(t)$ be a real or a complex function of the real variable t, $0\leqslant t<\infty$, satisfying the following conditions: (1) the function $f(t)$ is continuous everywhere except, possibly, a finite number of points of discontinuity of the first kind, and (2) there exist constants $M>0$ and $\xi_0>0$ such that $|f(t)|<Me^{\xi_0 t}$ for all t.

Under these assumptions the integral

$$F(\zeta)=\int_0^{\infty} f(t)\,e^{-\zeta t}\,dt \qquad (86)$$

exists for all values of ζ whose real parts satisfy the inequality $\operatorname{Re}\zeta>\xi_0$ and is an analytic function of the complex variable $\zeta=\xi+i\eta$ in the half-plane $\operatorname{Re}\zeta>\xi_0$.

The function $F(\zeta)$ determined by formula (86) is called the *Laplace transform* (or *image*) of the function $f(t)$, and the function $f(t)$ itself is referred to as the *original* (or the *Laplace inverse transform* or the *inverse image of* $F(\zeta)$). The transformation from $f(t)$ to $F(\zeta)$ is called the *Laplace transformation*.

In applications we frequently encounter the problem of inverting equality (86); in other words, it is sometimes necessary to express the original $f(t)$ in terms of its Laplace transform $F(\zeta)$ (this operation is referred to as the *Laplace inverse transformation*).

It can be proved that *if* (1) *the function* $F(\zeta)$ *is analytic in the half-plane* $\operatorname{Re}\zeta > \xi_0$, (2) *for* $\operatorname{Re}\zeta \geqslant a$ *where* a *is any number exceeding* ξ_0 *the function* $F(\zeta)$ *tends to zero as* $\zeta \to \infty$ *uniformly with respect to* $\arg \zeta$:

$$\lim_{\zeta \to \infty} F(\zeta) = 0$$

and (3) *the integral*

$$\int_{-\infty}^{\infty} F(a + i\eta)\, d\eta$$

is absolutely convergent, then the Laplace inverse transform of the function $F(\zeta)$ *exists, and the transformation inverse to* (86) *has the form*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(a + i\eta)\, e^{(a+i\eta)t}\, d\eta \tag{87}$$

where the integral on the right-hand side is understood in the sense of Cauchy's principal value.

Using the notation

$$g(t) = f(t)\, e^{-at} \quad \text{and} \quad G(\eta) = \frac{1}{\sqrt{2\pi}} F(a + i\eta)$$

we can rewrite formulas (86) and (87) in the following way:

$$G(\eta) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-i\eta t} g(t)\, dt \tag{88}$$

and

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta t} G(\eta)\, d\eta \tag{89}$$

The function $G(\eta)$ defined by formula (88) is called the *Fourier transform* of the function $g(t)$ (formula (88) itself expresses the *Fourier transformation* from $g(t)$ to $G(\eta)$). In case $g(t) = 0$ for $-\infty < t < 0$ the lower limit of integration on the right-hand side of (88) can obviously be taken equal to $-\infty$.

Accordingly, formula (89) expresses the *Fourier inverse transformation* from $G(\eta)$ to $g(t)$, and $g(t)$ is the *original* (or the *Fourier inverse transform* or the *inverse image*) of $G(\eta)$.

If the function $g(t)$ is defined everywhere for $-\infty < t < \infty$ (it must not necessarily be equal to zero for $-\infty < t < 0$), by the Fourier transform of this function is meant the integral

$$G(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta t} g(t)\, dt \tag{90}$$

For Fourier transform (90) to exist it is sufficient that the function $g(t)$ should satisfy the following conditions: (a) $g(t)$ *has a finite number of extrema*, (b) *it is continuous everywhere except, possibly, a finite number of points of discontinuity of the first kind, and* (c) *the integral*

$$\int_{-\infty}^{\infty} g(t)\, dt$$

is absolutely convergent; in this case the transformation inverse to (90) *is expressed by formula* (89).

Although the proof of this fact does not require intricate mathematical techniques, we shall not dwell on it here.

The replacement of the variable of integration η by $-\eta$ readily shows that *formula* (89) *can be taken as the original definition of the Fourier transformation for which transformation* (90) *plays the role of its inversion.*

The *Mellin transformation* from a function $f(t)$ defined for $0 \leqslant t < \infty$ to a new function $F(\zeta)$ of the complex variable ζ is specified by the integral

$$F(\zeta) = \int_0^{\infty} t^{\zeta-1} f(t)\, dt \tag{91}$$

where ζ is a complex variable and $t^{\zeta-1}$ is understood as the one-valued function

$$t^{\zeta-1} = e^{(\zeta-1)\ln t}$$

in whose expression by $\ln t = \log_e t$ we mean the principal value of the logarithmic function.

For $\zeta = a - i\tau$, the change of the variable of integration $t = e^\xi$ brings formula (91) to the form

$$F(a - i\tau) = \int_{-\infty}^{\infty} e^{a\xi} e^{-i\tau\xi} f(e^\xi)\, d\xi \tag{92}$$

Under the assumption that the function $e^{a\xi} f(e^\xi)$ satisfies conditions guaranteeing the existence of its Fourier transform, we derive from (92), using (89), the equality

$$e^{a\xi} f(e^\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(a - i\tau)\, e^{i\xi\tau}\, d\tau$$

Further, on returning to the variable $t = e^\xi$, we obtain the expression

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(a - i\tau)\, t^{-(a - i\tau)}\, d\tau \tag{93}$$

Consequently, the transformation inverse to (91) which expresses $f(t)$ (the *Mellin inverse transform* or the *original*) in terms of $F(\zeta)$ (the *Mellin transform* or *image* of the function $f(t)$) is given by formula (93) expressing the *Mellin inverse transformation*. This formula can also be written as

$$f(t) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} t^{-\zeta} F(\zeta)\, d\zeta \tag{94}$$

The theory of the Laplace, the Fourier and the Mellin integral transformations (and of integral transformations of other kinds) is the subject of one of the divisions of applied mathematics which is called the *operational calculus*.

3°. Application of the Method of Integral Transformations to Partial Differential Equations. In integral representation (69) of the solution $y(z)$ of ordinary differential equation (68) the kernel $K(z, \zeta)$ satisfies linear partial differential equation (70).

In Sec. 1° we took a definite solution $K(z, \zeta)$ of equation (70) and used it to construct solutions of equation (68). In the present section we shall consider a procedure which is in a certain sense reverse to the above.

Let it be required to find a solution $u(x, t)$ of a linear partial differential equation of the second order of the form

$$a(x)\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial^2 u}{\partial t^2}+c(x)\frac{\partial u}{\partial t}+d(x)\frac{\partial u}{\partial t}+e(x)u=0 \quad (95)$$

which is regular in the half-strip $0<x<l$, $t>0$ and continuous for $0 \leqslant x \leqslant l$, $t \geqslant 0$, and satisfies the initial conditions

$$u(x, 0)=\varphi(x), \quad \left.\frac{\partial u(x, t)}{\partial t}\right|_{t=0}=\psi(x) \quad (96)$$

and the boundary conditions

$$u(0, t)=f_1(t), \quad u(l, t)=f_2(t) \quad (97)$$

the coefficients of the equation depending solely on the spatial variable x.

Let us suppose that the class of solutions of equation (95) we deal with and the complex parameter ζ are such that the integrals

$$v(x, \zeta)=\int_0^\infty u(x, t)e^{-\zeta t}\,dt \quad (98)$$

and

$$F_1(\zeta)=\int_0^\infty u(0, t)e^{-\zeta t}\,dt, \quad F_2(\zeta)=\int_0^\infty u(l, t)e^{-\zeta t}\,dt \quad (99)$$

exist and that the operations

$$\frac{dv(x, \zeta)}{dx}=\int_0^\infty \frac{\partial u(x, t)}{\partial x}e^{-\zeta t}\,dt \quad (100)$$

$$\frac{\partial^2 v(x, \zeta)}{dx^2}=\int_0^\infty \frac{\partial^2 u(x, t)}{\partial x^2}e^{-\zeta t}\,dt \quad (101)$$

$$\int_0^\infty \frac{\partial u(x, t)}{\partial t}e^{-\zeta t}\,dt=\zeta\int_0^\infty u(x, t)e^{-\zeta t}\,dt+u(x, t)e^{-\zeta t}\Big|_0^\infty=$$

$$=\zeta v(x, \zeta)-u(x, 0) \quad (102)$$

and

$$\int_0^\infty \frac{\partial^2 u(x, t)}{\partial t^2} e^{-\zeta t}\, dt = \zeta^2 v(x, \zeta) - \zeta u(x, 0) - \left.\frac{\partial u(x, t)}{\partial t}\right|_{t=0} \quad (103)$$

are legitimate.

On multiplying both sides of equation (95) by $e^{-\zeta t}$ and performing the integration with respect to t from $t = 0$ to $t = \infty$ we obtain, by virtue of (96), (97), (98), (99), (100), (101), (102) and (103), the equalities

$$av'' + cv' + (e + d\zeta + b\zeta^2)\, v = \\ = b\varphi(x)\, \zeta + b\psi(x) + d\varphi(x) \quad (104)$$

and

$$v(0, \zeta) = F_1(\zeta), \quad v(l, \zeta) = F_2(\zeta) \quad (105)$$

Thus, we have reduced the solution of (mixed) boundary-initial-value problem (95), (96), (97) to the determination of the solution $v(x, \zeta)$ of boundary-value problem (105) for ordinary differential equation (104).

It should be noted that the existence of the solution of problem (104), (105) by far not always guarantees the possibility of the inversion of Laplace transformation (98).

If problem (104), (105) is solvable and its solution $v(x, \zeta)$ is unique, and if the inverse transform corresponding to Laplace transform (98) exists, that is

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x, a + i\eta)\, e^{(a+i\eta)t}\, d\eta \quad (106)$$

it is evident that problem (95), (96), (97) cannot possess more than one solution. In the case when the function $u(x, t)$ specified by formula (106) is continuous together with its partial derivatives up to the second order inclusive, it will be the sought-for solution of problem (95), (96), (97).

The determination of the function $u(x, t)$ with the aid of formula (106) involves rather lengthy calculations; therefore the Fourier transformation is more preferable for solving partial differential equations encountered in concrete physical problems. Besides, the conditions sufficient for the existence of the Fourier inverse transform are more often naturally fulfilled.

4°. Application of Fourier Transformation to the Solution of Cauchy Problem for the Equation of Oscillation of a String. Let it be required to find the solution $u(x, t)$ of the equation of oscillation of a string

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{107}$$

satisfying the initial conditions

$$u(x, 0) = \varphi(x), \quad \left.\frac{\partial u(x, t)}{\partial t}\right|_{t=0} = \psi(x) \tag{108}$$

which is regular in the half-plane $t > 0$, the functions $\varphi(x)$ and $\psi(x)$ being given and sufficiently smooth.

Let us suppose that the function $u(x, t)$ and its partial derivatives up to the second order inclusive are continuous and tend to zero for $x^2 + t^2 \to \infty$ sufficiently fast so that the Fourier transformation

$$v(t, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)\, e^{-ix\xi}\, dx \tag{109}$$

makes sense. We shall also suppose that the operations below are all legitimate:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x, t)}{\partial x^2} e^{-ix\xi}\, dx =$$

$$= -\frac{\xi^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)\, e^{-ix\xi}\, dx = -\xi^2 v(t, \xi) \tag{110}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x, t)}{\partial t^2} e^{-ix\xi}\, dx = \frac{d^2 v(t, \xi)}{dt^2} \tag{111}$$

$$v(0, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0)\, e^{-i\xi x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x)\, e^{-i\xi x}\, dx = \Phi(\xi) \tag{112}$$

and

$$\left.\frac{dv(t,\ \xi)}{dt}\right|_{t=0}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left.\frac{\partial u(x,\ t)}{\partial t}\right|_{t=0}e^{-ix\xi}\,dx=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,e^{-ix\xi}\,dx=\Psi(\xi)\quad(113)$$

where $\Phi(\xi)$ and $\Psi(\xi)$ are the Fourier transforms of the functions $\varphi(x)$ and $\psi(x)$ respectively.

Let us multiply both sides of equation (107) by $e^{-ix\xi}$ and perform integration with respect to x from $-\infty$ to ∞; by virtue of (110), (111), (112) and (113), this results in

$$v_{tt}(t,\ \xi)+\xi^2 v(t,\ \xi)=0\quad(114)$$

and

$$v(0,\ \xi)=\Phi(\xi),\ v_t(0,\ \xi)=\Psi(\xi)\quad(115)$$

Let us write the general solution $v(t, \xi)$ of ordinary differential equation (114) in the form

$$v=c_1(\xi)\,e^{i\xi t}+c_2(\xi)\,e^{-i\xi t}\quad(116)$$

where c_1 and c_2 are arbitrary expressions independent of t and dependent solely on the parameter ξ.

From (115) and (116) we obtain

$$c_1+c_2=\Phi(\xi)\quad\text{and}\quad c_1-c_2=\frac{\Psi(\xi)}{i\xi}$$

whence it follows that

$$c_1=\frac{1}{2}\Phi(\xi)+\frac{1}{2i\xi}\Psi(\xi)\quad\text{and}\quad c_2=\frac{1}{2}\Phi(\xi)-\frac{1}{2i\xi}\Psi(\xi)$$

The substitution of the values of c_1 and c_2 we have found into the right-hand side of (116) yields the solution

$$v(t,\ \xi)=\frac{1}{2}\Phi(\xi)\,(e^{i\xi t}+e^{-i\xi t})+\frac{1}{2i\xi}\Psi(\xi)\,(e^{i\xi t}-e^{-i\xi t})\quad(117)$$

of equation (114) satisfying initial conditions (115).

Applying inversion formula (89) to Fourier transformation (109) we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t, \xi) e^{i\xi x} \, d\xi$$

whence, taking into account (117), we derive the formula

$$u(x, t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} [e^{i\xi(x+t)} + e^{i\xi(x-t)}] \Phi(\xi) \, d\xi +$$

$$+ \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} [e^{i\xi(x+t)} - e^{i\xi(x-t)}] \frac{1}{i\xi} \Psi(\xi) \, d\xi \quad (118)$$

By virtue of inversion formula (89), the rightmost equalities (112) and (113) take the form

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\tau) e^{i\tau\xi} \, d\tau$$

and

$$\psi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(\tau) e^{i\tau\xi} \, d\tau$$

respectively, and hence

$$\left.\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\tau(x+t)} \Phi(\tau) \, d\tau = \varphi(x+t) \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\tau(x-t)} \Phi(\tau) \, d\tau = \varphi(x-t) \end{aligned}\right\} \quad (119)$$

and

$$\frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\Psi(\tau)}{\tau} [e^{i\tau(x+t)} - e^{i\tau(x-t)}] \, d\tau =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(\tau) \, d\tau \int_{x-t}^{x+t} e^{i\tau\xi} \, d\xi = \int_{x-t}^{x+t} \psi(\xi) \, d\xi \quad (120)$$

From (119) and (120), using formula (118), we readily derive D'Alembert's formula for the solution of problem (107), (108):

$$u(x, t) = \frac{1}{2}[\varphi(x+t) + \varphi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi \qquad (121)$$

(this solution was already obtained in Sec. 3°, § 1, Chapter 3).

5°. **Convolution.** By the *convolution* (or German *faltung*) $f * \varphi$ of two functions $f(x)$ and $\varphi(x)$ defined in the interval $-\infty < x < \infty$ is meant the integral

$$f * \varphi = \int_{-\infty}^{\infty} f(t)\, \varphi(x-t)\, dt \qquad (122)$$

dependent on x as a parameter.

If the Fourier transforms

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt, \quad \Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t)\, e^{-it\xi}\, dt \qquad (123)$$

and the inverse transforms

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)\, e^{i\xi t}\, d\xi, \quad \varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\xi)\, e^{i\xi t}\, d\xi \qquad (124)$$

exist, convolution (122) *can be written in the form*

$$f * \varphi = \int_{-\infty}^{\infty} F(\xi)\, \Phi(\xi)\, e^{i\xi x}\, d\xi \qquad (125)$$

Indeed, the substitution of the expression $\varphi(x-t)$ found from (124) into the right-hand side of formula (122) results in

$$f * \varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} \Phi(\xi)\, e^{i(x-t)\xi}\, d\xi$$

whence, assuming that it is legitimate to change the order of integration, we find

$$f * \varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\xi) e^{ix\xi} d\xi \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt$$

Finally, on replacing the second integral on the right-hand side of the last equality by $F(\xi)$, we obtain, according to (123), formula (125).

Using the Fourier transformation and formulas (122) and (125) expressing the convolution we can easily obtain the solution

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(\xi - x)^2}{4t}} d\xi \tag{126}$$

of the Cauchy-Dirichlet problem

$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty \tag{127}$$

for the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \tag{128}$$

in the half-plane $t > 0$ (this solution was already derived in Sec. 1°, § 2, Chapter 4).

Indeed, let us suppose that the functions $u(x, t)$ and $\varphi(x)$ are sufficiently smooth and that for $x^2 + t^2 \to \infty$ they decrease so fast that the Fourier transforms

$$v(t, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\xi x} dx \tag{129}$$

and

$$\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-i\xi x} dx \tag{130}$$

exist and the operations

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u(x, t)}{\partial t} e^{-i\xi x} dx = \frac{dv(t, \xi)}{dt} \tag{131}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x, t)}{\partial x^2} e^{-i\xi x}\, dx =$$

$$= -\frac{\xi^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)\, e^{-i\xi x}\, dx = -\xi^2 v(t, \xi) \qquad (132)$$

are legitimate.

On multiplying equation (128) by $\frac{1}{\sqrt{2\pi}} e^{-i\xi x}$ and integrating with respect to x from $-\infty$ to ∞, we obtain, by virtue of (127), (129), (130), (131) and (132), the equalities

$$\frac{dv}{dt} + \xi^2 v = 0 \qquad (133)$$

and

$$v(0, \xi) = \Phi(\xi) \qquad (134)$$

Further, we write equation (133) in the form

$$\frac{dv}{v} = -\xi^2\, dt$$

and integrate it, which immediately yields the general solution

$$v(t, \xi) = ce^{-\xi^2 t} \qquad (135)$$

where c is an arbitrary expression dependent solely on ξ.

On substituting expression (135) of $v(t, \xi)$ into (134), we find $c = \Phi(\xi)$. Consequently, the solution of ordinary differential equation (133) satisfying condition (134) is the function

$$v(t, \xi) = \Phi(\xi)\, e^{-\xi^2 t}$$

Knowing the function $v(t, \xi)$ we can rewrite (129) in the form

$$\Phi(\xi)\, e^{-\xi^2 t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)\, e^{-i\xi x}\, dx \qquad (136)$$

Finally, applying inversion formula (89) to (136) we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\xi)\, e^{-\xi^2 t + i\xi x}\, d\xi \qquad (137)$$

The Fourier transform of the function $e^{-\xi^2 t}$ (with respect to the variable ξ) for the positive values of t is the function

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 t} e^{ix\xi}\, d\xi = \frac{1}{\sqrt{2t}}\, e^{-\frac{x^2}{4t}}$$

(the last formula can easily be verified; by the way, there exist extensive tables of the Fourier transforms of the functions most frequently encountered in applications in which the Fourier transform we are interested in can be found).

Next, on applying formulas (122) and (125) to the convolution $f * \varphi$, we derive from (137) the expression

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\xi)\, e^{-\xi^2 t} e^{ix\xi}\, d\xi =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi)\, f(x-\xi, t)\, d\xi = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi$$

which is what we intended to prove.

6°. **Dirac's Delta Function.** In Sec. 2° § 2 of the present chapter we imposed certain conditions on the function $g(x)$ which guaranteed the existence of Fourier transform (90).

Unfortunately, Fourier's transformation does not make sense for a rather wide class of functions. For instance, even for the function $G(x) = \text{const} \neq 0$ the integral on the right-hand side of (89) is divergent. However, we can formally consider the Fourier transform of the constant $G = 1/\sqrt{2\pi}$. By definition, this transform is called *Dirac's delta function* (*δ-function*):

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi}\, d\xi \qquad (138)$$

We can also (formally) consider the Fourier inverse transform corresponding to (138):

$$\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(\xi)\, e^{-ix\xi}\, d\xi \tag{139}$$

The formal equality (139) can be written in the equivalent form

$$\int_{-\infty}^{\infty} \delta(\xi)\, e^{-ix\xi}\, d\xi = 1$$

Let us suppose that $f(x)$ is a function defined for $-\infty < x < \infty$ and satisfying conditions which guarantee the existence of the (mutually inverse) transforms

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{-ix\xi}\, d\xi \tag{140}$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)\, e^{ix\xi}\, d\xi \tag{141}$$

Taking into account formulas (122), (125), (139), (140) and (141) we obtain for the convolution $f * \delta$ the expression

$$f * \delta = \int_{-\infty}^{\infty} f(t)\, \delta(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)\, e^{ix\xi}\, d\xi = f(x)$$

Thus, we have arrived at a very important conclusion: *the convolution $f * \delta$ is equal to the value of the function f at the point x:*

$$f * \delta = f(x) \tag{142}$$

Formula (142) makes it possible to considerably simplify some lengthy calculations, particularly those encountered in quantum mechanics.

In the special case when $f(x) = 1$ $(-\infty < x < \infty)$ formulas (142) and (122), after a simple change of the variable of integration, lead to the formula

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad (143)$$

Dirac's δ-function is sometimes (formally) defined as a function which is equal to zero for all the values of t different from zero and is equal to ∞ for $t = 0$ with the additional requirement that equality (143) should hold.

This definition of the δ-function is inconsistent from the point of view of classical mathematical analysis. A rigorous definition of the δ-function and of the above operations on it is given in the modern theory of generalized functions.

§ 3. The Method of Finite Differences

1°. Finite-Difference Approximation of Partial Differential Equations. In applications it is sometimes necessary to find an approximate, in a certain sense, solution of a concrete problem of mathematical physics. Below we briefly discuss one of the methods of constructing approximate solutions of partial differential equations known as the *method of finite differences* (or, simply, the *finite-difference method*).

Let us consider a linear partial differential equation of the second order in two independent variables

$$a(x, y)\frac{\partial^2 u}{\partial x^2} + b(x, y)\frac{\partial^2 u}{\partial y^2} + c(x, y)\frac{\partial u}{\partial x} + d(x, y)\frac{\partial u}{\partial y} + e(x, y)u = f(x, y) \qquad (144)$$

The variables x and y will be interpreted as orthogonal Cartesian coordinates in the plane. Let us cover the xy-plane with the square grid of points $x = m\cdot h$, $y = n\cdot h$ ($m, n = 0, \pm 1, \ldots$), where h is a given positive number. The vertices of each square cell are referred to as *grid-points*, and the number h is called the *grid-size*.

On the basis of the definition of partial derivatives, we can write for every grid-point (x, y) the following approximate expressions for the derivatives $\frac{\partial u(x, y)}{\partial x}$, $\frac{\partial u(x, y)}{\partial y}$, $\frac{\partial^2 u(x, y)}{\partial x^2}$

and $\frac{\partial^2 u(x, y)}{\partial y^2}$ (under the assumption that the five points (x, y), $(x-h, y)$, $(x+h, y)$, $(x, y-h)$ and $(x, y+h)$ belong to the domain of definition D of equation (144):

$$\begin{aligned}
\frac{\partial u(x, y)}{\partial x} &\approx \frac{u(x, y)-u(x-h, y)}{h}\\
\frac{\partial u(x, y)}{\partial y} &\approx \frac{u(x, y)-u(x, y-h)}{h}\\
\frac{\partial^2 u(x, y)}{\partial x^2} &\approx \frac{u(x+h, y)+u(x-h, y)-2u(x, y)}{h^2}\\
\frac{\partial^2 u(x, y)}{\partial y^2} &\approx \frac{u(x, y+h)+u(x, y-h)-2u(x, y)}{h^2}
\end{aligned} \tag{145}$$

This makes it possible to replace partial differential equation (144) at each grid-point by its finite-difference approximation which is an algebraic linear equation of the form

$$\begin{gathered}
a(x, y)[u(x+h, y)+u(x-h, y)-2u(x, y)]+\\
+b(x, y)[u(x, y+h)+u(x, y-h)-\\
-2u(x, y)]+c(x, y)h[u(x, y)-u(x-h, y)]+\\
+d(x, y)h[u(x, y)-u(x, y-h)]+\\
+h^2 e(x, y)u(x, y)=h^2 f(x, y)
\end{gathered} \tag{146}$$

involving $u(x, y)$, $u(x-h, y)$, $u(x+h, y)$, $u(x, y-h)$ and $u(x, y+h)$ as the unknowns.

Making the variable point (x, y) range over the set of the grid-points belonging to D, we obtain an algebraic linear system of equations of type (146) with respect to the values of $u(x, y)$ at the grid-points. Some of these unknown values either can be determined independently of system (146) on the basis of the initial and boundary conditions or are contained in the algebraic linear equations generated by these conditions, and these additional linear equations together with system (146) form the finite-difference approximation to the whole original problem. The solution of the system of algebraic linear equations obtained in this way is taken as an approximation to the exact solution of the problem in question.

2°. Dirichlet Problem for Laplace's Equation. Let us consider the application of the finite-difference method to the approximate solution of the Dirichlet problem

$$u(x, y) = \varphi(x, y), \quad (x, y) \in S \tag{147}$$

for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{148}$$

in a domain D with boundary S.

In this case the system of equations of form (146) is written as

$$u(x+h, y) + u(x-h, y) + \\ + u(x, y+h) + u(x, y-h) - 4u(x, y) = 0 \tag{149}$$

Let us denote by Q_δ the set of the square cells of the grid lying within the domain D such that at least one of the

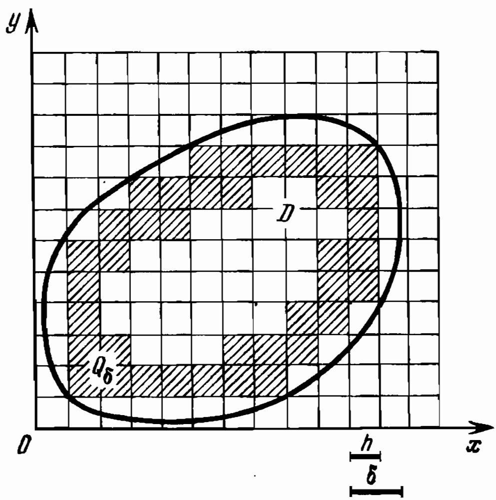

Fig. 25

vertices of each square lies at a distance not exceeding a given number $\delta > h$ from the boundary S of the domain D (see Fig. 25).

For each grid-point which is a vertex of a square cell belonging to Q_δ we take as $u(x, y)$ the value of the sought-for harmonic function prescribed by condition (147) at the point belonging to S which is the nearest to that grid-point (x, y) (there can be several such points on S; in this case we arbitrarily choose one of the values of φ prescribed at these points and take it as $u(x, y)$).

As to the values of $u(x, y)$ at the other grid-points belonging to D, it can be proved that *system* (149) *can be resolved with respect to these unknown values and that the solution is unique and tends to the sought-for solution of problem* (147), (148) *for* $\delta \to 0$.

3°. First Boundary-Value Problem for Heat Conduction Equation. Using formulas (145) we can write the finite-difference analog of the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad (150)$$

in the form

$$u(x+h, y) + u(x-h, y) - \\ -2u(x, y) - hu(x, y) + hu(x, y-h) = 0 \qquad (151)$$

Let D be the domain in the plane of the variables x and y bounded by the line segments OA and BN lying on the

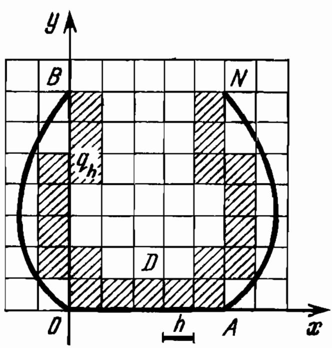

Fig. 26

straight lines $y = 0$ and $y = H$ $(H > 0)$ and by two smooth curves OB and AN each of which meets every straight line $y =$ const at not more than one point (see Fig. 26). We shall denote by S the part of the boundary of the domain D consisting of OB, OA and AN.

Next we shall discuss how the boundary condition

$$u(x, y) = f(x, y), \quad (x, y) \in S$$

should be taken into account in the determination of the approximate solution of equation (150) in the domain D. Let us denote as Q_h the set of the square cells of the grid which do not fall outside the closed domain $\bar{D}$, and let ∂Q_h be the boundary of Q_h. By q_h we shall denote the collection of those squares belonging to Q_h which do not lie inside the uppermost row adjoining the line segment BN and such that at least one of the vertices of each square lies on ∂Q_h (see Fig. 26).

For a grid-point (x, y) which is a vertex of a square belonging to q_h we take as $u(x, y)$ the value of the function f assumed at the point lying on S which is the nearest to that grid-point. The unknown values of $u(x, y)$ at the other grid-points lying in D are found by solving the corresponding algebraic linear system of equations of type (151).

4°. Some General Remarks on Finite-Difference Method. Let us consider an arbitrary (non-linear, in the general case) partial differential equation:

$$F(x, y, u, u_x, \ldots) = 0$$

On replacing the partial derivatives contained in the equation by their approximate values (145) we readily pass to the finite-difference approximation to the given equation:

$$F\left(x, y, u(x, y), \frac{u(x, y) - u(x-h, y)}{h}, \ldots\right) = 0$$

However, when the boundary and the initial conditions are replaced by their finite-difference approximations, there may arise some difficulties, particularly when the conditions contain partial derivatives of the sought-for solution; these difficulties are by far not always easy to overcome.

Further, after the finite-difference scheme has been elaborated, the approximate solution is found using modern electronic computers. These computers, however perfect, have limited computation speed, and even in the case of a linear equation, the number of algebraic linear equations becomes very large for sufficiently small h; therefore the appropriate choice of the finite-difference approximation to the boundary and initial conditions plays an extremely important role.

§ 4. Asymptotic Expansions

1°. Asymptotic Expansion of a Function of One Variable. Let $f(z)$ and $S_n(z)$ ($n = 0, 1, \ldots$) be functions defined in a neighbourhood of a point z_0 in the complex plane of the variable z, and let E be a set of points, belonging to this neighbourhood, for which z_0 is a limit point.

If the functions $S_n(z)$ ($n = 0, 1, \ldots$) satisfy the conditions

$$\lim_{z \to z_0} [S_n(z) - S_{n-1}(z)] = 0, \quad S_n \neq S_{n-1}$$

and

$$\lim_{z \to z_0} \frac{f(z) - S_n(z)}{S_n(z) - S_{n-1}(z)} = 0, \qquad z \in E$$

and if the behaviour of these functions in the neighbourhood of the point z_0 is known, then, for every fixed n, we obtain certain information on the behaviour of the function $f(z)$ near the point z_0.

As the point z_0 is often taken the point at infinity in the complex z-plane, and as $S_n(z)$ are taken the functions

$$S_n(z) = \sum_{k=0}^{n} \frac{a_k}{z^k} \quad (n = 1, 2, \ldots)$$

where a_k are some given numbers.

If

$$\lim_{z \to \infty} z^n [f(z) - S_n(z)] = 0, \quad z \in E \tag{152}$$

for any fixed n, the series

$$a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + \ldots \tag{153}$$

is said to represent the *asymptotic expansion of the function* $f(z)$ *on* E irrespective of whether it is convergent or not; in this case (153) is called an *asymptotic series*, and we write

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k}$$

If condition (152) holds, we obtain from (152) the following expressions for the coefficients a_k of series (153):

$$a_0 = \lim_{z \to \infty} f(z)$$

$$a_n = \lim_{z \to \infty} z^n [f(z) - S_{n-1}(z)] \quad (n = 1, 2, \ldots) \qquad (154)$$

We have thus established *the uniqueness of the asymptotic expansion of the function $f(z)$ provided that it exists.*

This conclusion does not contradict the fact that a given series of form (153) can serve as an asymptotic expansion for different functions on one and the same set E. For instance, according to (154), the asymptotic series for the function $f(z) = e^{-z}$ on the set E: $\{0 < z < \infty\}$ has all its coefficients equal to zero: $a_k = 0$ $(k = 0, 1, \ldots)$. At the same time, it is evident that this series represents the asymptotic expansion of the function $f(z) = 0$ as well.

This example shows that *even when the asymptotic series of a function $f(z)$ is convergent the sum of the series must not necessarily coincide with $f(z)$.*

Let us consider the function $f(z)$ expressed by the integral

$$f(z) = \int_z^{\infty} e^{z-t} \frac{dt}{t}, \quad 0 < z < \infty \qquad (155)$$

where the integration goes along the part $z < t < \infty$ of the real axis.

The integration by parts in (155) yields

$$f(z) = \frac{1}{z} - \int_z^{\infty} e^{z-t} \frac{dt}{t^2}$$

On repeating this process n times we obtain

$$f(z) = \sum_{k=0}^{n-1} (-1)^k \frac{k!}{z^{k+1}} + (-1)^n n! \int_z^{\infty} e^{z-t} \frac{dt}{t^{n+1}} \qquad (156)$$

From (156) it readily follows that *the series*

$$\frac{1}{z}-\ldots+(-1)^{n-1}\frac{(n-1)!}{z^n}+\ldots \qquad (157)$$

gives the asymptotic expansion for the function $f(z)$ *determined by formula* (155), *the set* E *coinciding with the positive real axis.*

Indeed, let us denote by $S_n(z)$ the sum of the first n terms of series (157):

$$S_n(z)=\sum_{k=1}^{n}(-1)^{k-1}\frac{(k-1)!}{z^k}$$

By virtue of (156), we obtain

$$f(z)-S_n(z)=(-1)^n n!\int_z^\infty e^{z-t}\frac{dt}{t^{n+1}}=$$

$$=(-1)^n n!\left[\frac{1}{z^{n+1}}-(n+1)\int_z^\infty e^{z-t}\frac{dt}{t^{n+2}}\right] \qquad (158)$$

Now, taking into account the inequalities

$$0<\int_z^\infty e^{z-t}\frac{dt}{t^{n+2}}<\frac{1}{n+1}\cdot\frac{1}{z^{n+1}}$$

we conclude that

$$\lim_{z\to\infty} z^n[f(z)-S_n(z)]=0$$

for any n on the indicated set E.

Although series (157) *is divergent for any* z, $0<z<\infty$, *by virtue of* (158) *it follows that the values of* $S_n(z)$ *are very close to the values of* $f(z)$ *when* z *is sufficiently large for any fixed* $n\geqslant 1$.

As is known, if the point at infinity in the complex z-plane is a removable singularity of an analytic function $f(z)$, then in the neighbourhood of the point at infinity there holds the expansion

$$f(z)=\sum_{k=0}^{\infty}\frac{a_k}{z^k} \qquad (159)$$

It is evident that in this case the expression on the right-hand side of equality (159) is an asymptotic series for the function $f(z)$ on any set E of points, belonging to the neighbourhood of the point $z = \infty$, for which $z = \infty$ is a limit point.

Using the symbolic notation $\varphi(z) = o(z^{-n})$ for the relation $\lim\limits_{z\to\infty} z^n\varphi(z) = 0$ we can easily prove that *if*

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k} \quad \text{and} \quad g(z) \sim \sum_{k=0}^{\infty} \frac{b_k}{z^k} \tag{160}$$

on one and the same set E then

$$f(z) \pm g(z) \sim \sum_{k=0}^{\infty} \frac{a_k \pm b_k}{z^k} \tag{161}$$

and

$$f(z)\cdot g(z) \sim \sum_{k=0}^{\infty} \frac{c_k}{z^k} \tag{162}$$

on that set E where $c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0$.

Indeed, let

$$S'_n(z) = \sum_{k=0}^{n} \frac{a_k}{z^k}, \qquad S''_n(z) = \sum_{k=0}^{n} \frac{b_k}{z^k}$$

$$S_n(z) = S'_n(z) \pm S''_n(z), \qquad \sigma_n(z) = \sum_{k=0}^{n} \frac{c_k}{z^k}$$

It is obvious that

$$S'_n(z)\cdot S''_n(z) = \sigma_n(z) + o(z^{-n}) \tag{163}$$

By virtue of (160), we have

$$f(z) = S'_n(z) + o(z^{-n}) \text{ and } g(z) = S''_n(z) + o(z^{-n}) \tag{164}$$

From (163) and (164) we conclude that

$$f(z) \pm g(z) = S_n(z) + o(z^{-n})$$

and

$$f(z)\cdot g(z) = \sigma_n(z) + o(z^{-n})$$

which proves the validity of asymptotic expansions (161) and (162).

Let us prove that *if* $f(z)$ *is an integrable function for* $0 < z < \infty$ *and if*

$$f(z) \sim \sum_{k=2}^{\infty} \frac{a_k}{z^k} \tag{165}$$

on the set E: $\{0 < z < \infty\}$, *then*

$$\int_z^{\infty} f(t)\,dt \sim \sum_{k=1}^{\infty} \frac{a_{k+1}}{kz^k}$$

on that set where the path of integration coincides with the part $z \leqslant t < \infty$ *of the real axis.*

Indeed, from (165) we conclude that, given an arbitrary $\varepsilon > 0$, there exists a number $z_0 > 0$ such that

$$\left| f(t) - \sum_{k=2}^{n+1} \frac{a_k}{t^k} \right| < \varepsilon t^{-n-1} \tag{166}$$

for all $t > z_0$.

Further, under the assumption that $z > z_0$, using inequality (166), we obtain

$$\left| \int_z^{\infty} \left[f(t) - \sum_{k=2}^{n+1} \frac{a_k}{t^k} \right] dt \right| < \frac{\varepsilon}{nz^n}$$

whence

$$\lim_{z\to\infty} z^n \left[\int_z^{\infty} f(t)\,dt - \sum_{k=1}^{n} \frac{a_{k+1}}{kz^k} \right] = 0$$

which is what we had to prove.

Generally speaking, the existence of an asymptotic expansion

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k}$$

does not guarantee the existence of an asymptotic expansion for the derivative $f'(z)$. For example, the function $f(z) =$

$= e^{-z} \sin e^z$ has an asymptotic expansion of form (153) for $0 < z < \infty$ whose all coefficients are equal to zero: $a_k = 0$ $(k = 0, 1, \ldots)$. At the same time the derivative

$$f'(z) = -e^{-z} \sin e^z + \cos e^z, \quad 0 < z < \infty$$

of this function possesses no asymptotic expansion because it does not have a limit for $z \to \infty$.

It should be noted that all that was said above remains true when in the definition of an asymptotic expansion, instead of series (153), we take a series of the form $\sum_{k=0}^{\infty} a_k z^{-\alpha_k}$ where $\{\alpha_k\}$ is an arbitrary increasing sequence of nonnegative real numbers (which must not necessarily be integers).

Below we present some methods which can be successfully applied for the construction of asymptotic expansions for some classes of functions.

2°. Watson's Method for Asymptotic Expansion. Let us consider the function

$$F(z) = \int_0^N t^m \varphi(t) e^{-zt^\alpha} dt \tag{167}$$

where $0 < N \leqslant \infty$, $\alpha > 0$, $m > -1$, $z > 0$ and the path of integration coincides with the line segment $0 \leqslant t \leqslant N$.

Watson's lemma: if the function $\varphi(t)$ can be represented as the sum of a power series

$$\varphi(t) = \sum_{k=0}^{\infty} c_k t^k, \quad c_0 \neq 0 \tag{168}$$

for an interval $0 \leqslant t \leqslant h_1 \leqslant N$ and if

$$\int_0^N t^m |\varphi(t)| e^{-z_0 t^\alpha} dt < M \tag{169}$$

for a fixed value $z = z_0 > 0$, then

$$F(z) \sim \sum_{k=0}^{\infty} \frac{c_k}{\alpha} \Gamma\left(\frac{m+k+1}{\alpha}\right) z^{-\frac{m+k+1}{\alpha}} \tag{170}$$

for $z \to \infty$, $0 < z < \infty$.

Indeed, let us represent the function $F(z)$ determined by formula (167) in the form

$$F(z)=\int_0^h t^m\varphi(t)\,e^{-zt^\alpha}\,dt+\int_h^N t^m\varphi(t)\,e^{-zt^\alpha}\,dt,\quad 0<h<h_1$$

Since

$$e^{-(z-z_0)t^\alpha}<e^{-(z-z_0)h^\alpha}$$

for $z>z_0$ and $t>h$, by virtue of (169) we have

$$\left|F(z)-\int_0^h t^m\varphi(t)\,e^{-zt^\alpha}\,dt\right|=$$

$$=\left|\int_h^N t^m\varphi(t)\,e^{-zt^\alpha}\,dt\right|<Me^{z_0h^\alpha}e^{-zh^\alpha}\qquad(171)$$

On the basis of (168),

$$\int_0^h t^m\varphi(t)\,e^{-zt^\alpha}\,dt=\sum_{k=0}^{n}c_k\int_0^h t^{m+k}e^{-zt^\alpha}dt+$$

$$+\int_0^h t^{m+n+1}\varphi_1(t)\,e^{-zt^\alpha}\,dt$$

where $\varphi_1(t)=\sum_{k=0}^{\infty}c_{n+k+1}t^k$, the expression $\max_{0\leqslant t\leqslant h}|\varphi_1(t)|$ being finite.

Let us introduce the new variable of integration $\tau=zt^\alpha$; this results in

$$\int_0^h t^{m+k}e^{-zt^\alpha}\,dt=\frac{1}{\alpha}z^{-\frac{m+k+1}{\alpha}}\int_0^{zh^\alpha}\tau^{\frac{m+k+1}{\alpha}-1}e^{-\tau}\,d\tau=$$

$$=\frac{1}{\alpha}z^{-\frac{m+k+1}{\alpha}}\left[\Gamma\left(\frac{m+k+1}{\alpha}\right)-\int_{zh^\alpha}^{\infty}\tau^{\frac{m+k+1}{\alpha}-1}e^{-\tau}\,d\tau\right]\qquad(172)$$

Further, taking into account the fact that for $z \to \infty$ the expression

$$\int_{zh^\alpha}^{\infty} \tau^{p-1} e^{-\tau}\, d\tau = \int_0^{\infty} (\xi + zh^\alpha)^{p-1} e^{-\xi} e^{-zh^\alpha}\, d\xi <$$

$$< 2^p e^{-zh^\alpha} \int_0^{zh^\alpha} (zh^\alpha)^{p-1} e^{-\xi}\, d\xi + e^{-zh^\alpha} 2^p \int_0^{\infty} \xi^{p-1} e^{-\xi}\, d\xi <$$

$$< 2^p e^{-zh^\alpha} [(zh^\alpha)^{p-1} + \Gamma(p)]$$

tends to zero faster than any power $z^{-\omega}$, we conclude, on the basis of (171) and (172), that

$$\lim_{z \to \infty} z^{\frac{m+n+1}{\alpha}} \left[F(z) - \sum_{k=0}^{n} \frac{c_k}{\alpha} \Gamma\left(\frac{m+k+1}{\alpha}\right) z^{-\frac{m+k+1}{\alpha}} \right] = 0$$

for any n, whence follows the validity of asymptotic expansion (170).

Now let us consider the function

$$F(z) = \int_{-A}^{N} \varphi(t)\, e^{-\frac{1}{2} z t^2}\, dt, \quad A > 0, \quad N > 0, \quad z > 0 \quad (173)$$

We shall suppose that in some neighbourhood $-h_1 < t < h_1$, $h_1 < N$, of the point $t = 0$ there holds representation (168) and that the integral on the right-hand side of (173) is absolutely convergent for $z = z_0 > 0$.

Writing the function $\Phi(z) = F(2z)$ in the form

$$\Phi(z) = \int_0^N \varphi(t)\, e^{-zt^2}\, dt - \int_0^{-A} \varphi(t)\, e^{-zt^2}\, dt =$$

$$= \int_0^N \varphi(t)\, e^{-zt^2}\, dt + \int_0^A \varphi(-t)\, e^{-zt^2}\, dt$$

and taking into account the fact that

$$\varphi(t) + \varphi(-t) = 2 \sum_{k=0}^{\infty} c_{2k} t^{2k}$$

and

$$\Gamma\left(\frac{2k+1}{2}\right)=\sqrt{\pi}\,\frac{1\cdot 3\ldots(2k-1)}{2^k}$$

for $0\leqslant t\leqslant h$, $h<h_1$, $h<A$, we obtain, on the basis of (170), the relation

$$\Phi(z)\sim\sum_{k=0}^{\infty}c_{2k}\Gamma\left(\frac{2k+1}{2}\right)z^{-\frac{2k+1}{2}}=$$

$$=c_0\sqrt{\pi}\,z^{-\frac{1}{2}}+\sum_{k=1}^{\infty}\sqrt{\pi}\,\frac{1\cdot 3\ldots(2k-1)}{2^k}\,c_{2k}z^{-\frac{2k+1}{2}}$$

whence it follows that

$$F(z)=\Phi\left(\frac{z}{2}\right)\sim c_0\sqrt{\pi}\,z^{-\frac{1}{2}}+\sqrt{2\pi}\sum_{k=1}^{\infty}1\cdot 3\ldots$$

$$\ldots(2k-1)\,c_{2k}z^{-\frac{2k+1}{2}}\qquad(174)$$

3°. Saddle-Point Method. Let it be required to find the asymptotic expansion of the function

$$F(z)=\int_C\varphi(t)\,e^{zf(t)}dt\qquad(175)$$

for $z\to\infty$, $z>0$, where $\varphi(t)$ and $f(t)$ are analytic functions of the complex variable $t=x+iy$ and the path of integration C recedes to infinity in both positive and negative directions.

If the function $u(t)=\operatorname{Re}f(t)$ attains its maximum value on C at a point $t_0\in C$ and if for t tending to infinity in both directions from t_0 this function tends to minus infinity, then, since the function $e^{izv(t)}$, where $v(t)=\operatorname{Im}f(t)$, is bounded, the main part of the value of $F(z)$ for large values of z corresponds to the integration in (175) over a small part C_1 of the path C containing the point t_0 inside.

Since the function $u(t)$ cannot have a local maximum at any point of the domain of its harmonicity, the path of integration C in expression (175) may not possess the indicated property. However, according to Cauchy's theorem, we can change the path of integration C in an appropriate manner without falling outside the domain D of analyticity of the functions $f(t)$ and $\varphi(t)$, the value of $F(z)$ remaining unchanged.

In order to find in the domain D a curve C and a point t_0 possessing the indicated properties we shall use the following procedure. Let us consider the level line $u(t) = u(t_0)$ of the harmonic function $u(t)$ passing through the point t_0. The curve C passing through the point t_0 must have at the point t_0 the direction coinciding with that of the vector $\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ because it is this vector that determines the direction of the fastest change of the function $u(t)$ whose maximum on C is attained at the point t_0.

According to the Cauchy-Riemann conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, the scalar product $\operatorname{grad} u \operatorname{grad} v$ is equal to zero:

$$\operatorname{grad} u \operatorname{grad} v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

Further, the vector $\operatorname{grad} v$ coincides with the normal to the level line $v(t) = v(t_0)$, whence we conclude that at the point t_0 the curve C must go in the direction of the tangent line to the curve $v(t) = v(t_0)$. Since $\frac{dv}{ds} = \frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds} = 0$ everywhere along the curve $v(t) = v(t_0)$ and since at the point t_0 where the function $u(t)$ attains its maximum on C the equality $\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = 0$ must hold, we have $f'(t_0) = 0$.

Thus, *the point t_0 must be such that $f'(t_0) = 0$ and the curve C possessing the required properties should be sought among the curves $v(t) = v(t_0)$.*

Let us suppose that the point t_0 and the curve C possessing these properties are found. Since in a neighbourhood $|t - t_0| < h_1$ of the point t_0 on C there holds inequality

$$f(t) - f(t_0) = \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n + \ldots < 0, \quad n \geqslant 2 \qquad (176)$$

the change of the variable of integration $t = t(\xi)$ where $t(\xi)$ is determined by the equality

$$f(t) - f(t_0) = -\frac{1}{2}\xi^2 \tag{177}$$

brings expression (175) to the form

$$F(z) = e^{zf(t_0)} \int_C \varphi[t(\xi)]\, e^{-\frac{1}{2}z\xi^2} \frac{dt}{d\xi}\, d\xi \tag{178}$$

Now, using the expansions of the analytic functions $\varphi[t(\xi)]$ and $dt/d\xi = \psi(\xi)$ in the vicinity of the point $\xi = 0$ into power series, we directly obtain the sought-for asymptotic expansion of the function $F(z)$ with the aid of formula (174).

The point t_0 at which $f'(t_0) = 0$ is a saddle point of the surface $u = u(x, y)$; that is why the method discussed above is referred to as the *saddle-point method*.

The saddle-point method involves the determination of t as function of ξ from equation (177), and therefore the practical application of this method is connected with considerable difficulties even after the appropriate point t_0 and the appropriate path of integration C are found.

However, in applications it is often sufficient to find only the first (principal) term of the asymptotic expansion which can easily be constructed.

Indeed, let us suppose that $f''(t_0) \neq 0$ and let us limit ourselves to the consideration of the first term in expansion (176). Instead of (177) we introduce the variable

$$\xi^2 = -(t - t_0)^2 f''(t_0) \tag{179}$$

In the vicinity of the point t_0 let us replace the part of the path of integration C by the line segment $t - t_0 = se^{i\theta}$ where $2\theta = \pi - \arg f''(t_0)$ so that the inequality

$$(t - t_0)^2 f''(t_0) = s^2 e^{2i\theta} f''(t_0) < 0$$

holds along that segment. Then from (179) we obtain

$$\xi = \pm s\sqrt{|f''(t_0)|} \tag{180}$$

that is

$$\frac{d\xi}{dt} = \pm\frac{ds}{dt}\sqrt{|f''(t_0)|} = \pm e^{-i\theta}\sqrt{|f''(t_0)|}$$

From the two values of θ determined by the last relation (they differ by π) we shall choose the one for which the variable ξ in (180) becomes positive when the point t passes through t_0 in the integration process along C, that is we take the value of θ specified by the equality $\frac{d\xi}{dt}=e^{-i\theta}\sqrt{|f''(t_0)|}$.

Hence, by virtue of (174) and (178), for the first term of the sought-for asymptotic expansion we obtain the expression

$$\frac{\varphi(t_0)}{\sqrt{|f''(t_0)|}}\sqrt{2\pi}\, e^{zf(t_0)}e^{i\theta}\frac{1}{z^{1/2}} \tag{181}$$

As an example, let us consider the Hankel function

$$H_n^1(z)=-\frac{1}{\pi}\int_C e^{-iz\sin t}e^{int}\,dt \tag{182}$$

where the integral is taken along broken line (75) considered in Sec. 1° of § 2.

The only point on that broken line at which $f'(t_0)=-i\times\times\cos t_0=0$ is $t_0=-\frac{\pi}{2}$; there are two level lines passing through that point: $\operatorname{Im} f(t)=-\sin x\cosh y=1$. From them, as a new path of integration C in (182), we shall choose the one for which $\theta=\frac{3}{4}\pi$ and $\arg f''\left(-\frac{\pi}{2}\right)=\arg(-i)=-\frac{\pi}{2}$ because it is this level line along which the function $u(t)=\operatorname{Re}(-i\sin t)=\cos x\sinh y$ tends to minus infinity when the point t recedes to infinity.

Since $e^{zf\left(-\frac{\pi}{2}\right)}=e^{iz}$ and $\varphi\left(-\frac{\pi}{2}\right)=-\frac{1}{\pi}e^{-in\frac{\pi}{2}}$, formula (181) implies the expression

$$\sqrt{\frac{2}{\pi}}\,e^{iz}e^{-i\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}z^{-\frac{1}{2}}$$

for the first term of the asymptotic expansion of the function $H_n^1(z)$ for $z\to\infty$, $0<z<\infty$.

§ 5. Variational Methods

1°. Dirichlet Principle. In many cases encountered in applications the partial differential equations under consideration are *Euler's equations* for some *variational problems.* For instance, as was mentioned in Sec. 2°, § 5 of Introduction, Laplace's equation $\Delta u(x, y) = 0$ serves as Euler's equation corresponding to the minimum problem for the *Dirichlet integral*

$$D(u) = \int_D (u_x^2 + u_y^2)\, dx\, dy \tag{183}$$

taken over a domain D with boundary S.

The continuous functions defined in $D \cup S$ and possessing piecewise continuous first derivatives in D for which the Dirichlet integral is finite and which assume given values described by a continuous function $\varphi(x, y)$ on S as the variable point (x, y) approaches the boundary S from the interior of D will be referred to as *admitted functions.*

There is a close relationship between the Dirichlet problem on the determination of the function $u(x, y)$ harmonic in the domain D, continuous in $D \cup S$ and satisfying the boundary condition

$$u(x, y) = \varphi(x, y), \quad (x, y) \in S \tag{184}$$

and the so-called *first variational problem* on the determination of the function, belonging to the class of the admitted functions, for which Dirichlet integral (183) attains its minimum.

If the function $\varphi(x, y)$ defined on S is such that the class of the admitted functions is not void, the Dirichlet problem and the first variational problem are equivalent.

We shall prove this assertion under some additional assumptions.

Let $u(x, y)$ be the solution of the first variational problem. We shall represent the class of the admitted functions in the form $u(x, y) + \varepsilon h(x, y)$ where ε is an arbitrary constant and $h(x, y)$ is an arbitrary function belonging to the class of the admitted functions and satisfying the condition

$$h(x, y) = 0, \quad (x, y) \in S \tag{185}$$

It is evident that

$$D(u+\varepsilon h)=D(u)+2\varepsilon D(u,h)+\varepsilon^2 D(h)\geqslant 0 \qquad (186)$$

where

$$D(u,h)=\int_D (u_x h_x+u_y h_y)\,dx\,dy$$

Since $u(x, y)$ is the minimizing function and ε is an arbitrary constant, relation (186) implies that

$$D(u,h)=0 \qquad (187)$$

We shall suppose that the functions $u(x, y)$ and $h(x, y)$ and the contour S are sufficiently smooth so that the identities

$$u_x h_x+u_y h_y=(u_x h)_x+(u_y h)_y-h\,\Delta u$$

and

$$D(u,h)=\int_S h\frac{\partial u}{\partial \nu}\,ds-\int_D h\,\Delta u\,dx\,dy \qquad (188)$$

hold for them where ν is the outer normal to S.

By (185) and (187), we obtain from (188) the equality

$$\int_D h\,\Delta u\,dx\,dy=0$$

whence, under the assumption that Δu is a continuous function in D, since $h(x, y)$ is arbitrary, we conclude that $\Delta u(x, y)=0$. Consequently, under the assumptions we have made, the solution of the first variational problem is the solution of the Dirichlet problem.

Now let us suppose that $u(x, y)$ is the solution of the Dirichlet problem with boundary condition (184) for Laplace's equation, and let, as above, $u(x, y)+\varepsilon h(x, y)$ be the class of the admitted functions, formula (188) holding for the functions $u(x, y)$ and $h(x, y)$. From formula (188), by virtue of (185) and by the harmonicity of $u(x, y)$, follows equality (187). Therefore from (186) we obtain

$$D(u)\leqslant D(u+\varepsilon h)$$

which means that the function $u(x, y)$ minimizes the Dirichlet integral, and hence it is the solution of the first variational problem.

There are also a number of other boundary-value problems for Laplace's equation to which correspond equivalent variational problems for the Dirichlet integral. For instance, among them we can mention the Neumann problem.

The idea of the reduction of a boundary-value problem for Laplace's equation to the variational problem equivalent to the former was suggested by G.F.B. Riemann. This idea is usually referred to as the *Dirichlet principle.*

2°. Eigenvalue Problem. In Sec. 2°, § 1 of the present chapter we considered the *eigenvalue problem: for a bounded domain D with piecewise smooth boundary S it is required to find the eigenvalues and the eigenfunctions of the equation*

$$\Delta u + \lambda u = 0, \quad (x, y) \in D, \quad \lambda = \text{const} \tag{189}$$

that is to determine the values of λ *for which this equation possesses non-trivial solutions in the domain D satisfying the boundary condition*

$$u(x, y) = 0, \quad (x, y) \in S \tag{190}$$

and to construct these solutions.

The solution of this problem can be obtained when the solution of the following *second variational problem* is known: *it is required to find among the admitted functions satisfying condition* (190) *the one for which the functional*

$$J(u) = \frac{D(u)}{H(u)}$$

assumes its minimum value where

$$H(u) = \int_D u^2\, dx\, dy$$

Indeed, let us suppose that $u(x, y)$ is the solution of the second variational problem, the minimum value of $J(u)$ being positive:

$$J(u) = \frac{D(u)}{H(u)} = \lambda > 0 \tag{191}$$

For the class of the admitted functions $u(x, y) + \varepsilon h(x, y)$ where ε is an arbitrary constant and $h(x, y)$ is an arbitrary admitted function satisfying condition (185) we have

$$F(\varepsilon)=\frac{D(u+\varepsilon h)}{H(u+\varepsilon h)}=\frac{D(u)+2\varepsilon D(u,\ h)+\varepsilon^2 D(h)}{H(u)+2\varepsilon H(u,\ h)+\varepsilon^2 H(h)}\geqslant\lambda$$

where

$$H(u,\ h)=\int_D uh\,dx\,dy$$

Since the function $F(\varepsilon)$ attains its minimum for $\varepsilon=0$, we have

$$F'(0)=2\frac{H(u)\,D(u,\ h)-D(u)\,H(u,\ h)}{H^2(u)}=0$$

whence, by virtue of (191), it follows that

$$H(u)\,[D(u,\ h)-\lambda H(u,\ h)]=0$$

Since $H(u)\neq 0$, this implies

$$D(u,\ h)-\lambda H(u,\ h)=0 \qquad (192)$$

Let us suppose that the functions $u(x,\ y)$ and $h(x,\ y)$ and the contour S bounding the domain D are sufficiently smooth so that formula (188) applies to them; then we can rewrite equality (192) in the form

$$H(\Delta u+\lambda u,\ h)=0$$

whence, like in the foregoing section, it follows that the function $u(x,\ y)$ satisfies equation (189).

If λ^* is an eigenvalue of problem (189), (190) distinct from λ and $u^*(x,\ y)$ is an eigenfunction corresponding to λ^*, then, by virtue of (188), we have

$$H(\Delta u^*+\lambda^* u^*,\ u^*)=-D(u^*)+\lambda^* H(u^*)=0$$

This equality shows that among the eigenvalues of problem (189), (190) the number λ is the minimum one.

All that was said above remains true when instead of the Dirichlet integral we consider a general quadratic functional of the form

$$E(u)=\int_D [p(u_x^2+u_y^2)+2auu_x+2buu_y+cu^2]\,dx\,dy$$

$$p>0,\quad c\leqslant 0$$

whose integrand is a quadratic form with sufficiently smooth coefficients satisfying the condition

$$p\,(\xi^2 + \eta^2) + c\zeta^2 + 2a\xi\zeta + 2b\eta\zeta \geqslant \mu^2\,(\xi^2 + \eta^2)$$

where μ is a real constant.

Euler's equation corresponding to the function $E\,(u)$ has the form

$$2\,(pu_x)_x + 2\,(pu_y)_y + 2\,(au)_x + 2\,(bu)_y - 2au_x - 2bu_y - 2cu = 0$$

It can be written as

$$(pu_x)_x + (pu_y)_y - c^*u = 0$$

where

$$c^* = c - a_x - b_y$$

3°. Minimizing Sequence. If the class of admitted functions $\{u\}$ is not void, the corresponding set of the values of the Dirichlet integral $D\,(u)$ has an infimum d. Although in the general case we do not know whether the functional $D\,(u)$ attains its infimum d for an admitted function belonging to that class, it is evident that there exists a sequence u_n $(n = 1, 2, \ldots)$ of admitted functions such that

$$\lim_{n\to\infty} D\,(u_n) = d \tag{193}$$

A sequence u_n $(n = 1, 2, \ldots)$ for which relation (193) holds is called a *minimizing sequence.*

What has been said also applies to the functional $J\,(u)$.

The existence of a minimizing sequence does not necessarily mean that the variational problem in question is solvable. In this connection the following questions should be subject to further investigation:

(1) *How can a minimizing sequence be constructed?*

(2) *Is the minimizing sequence (provided it exists) convergent?*

(3) *Is the limit of the minimizing sequence* $u = \lim\limits_{n\to\infty} u_n$ *an admitted function?*

A thorough investigation of these questions requires the introduction of some function spaces whose elements include, in particular, the members of the minimizing sequence. After the convergence of the minimizing sequence with respect to the metric of these spaces has been established, it is desirable either to show that the corresponding limit is the solution of the variational problem stated above or to generalize in an appropriate manner the notion of the solution itself. In each such case it is necessary to prove that the solution of the variational problem is the solution of the boundary-value problem either in the ordinary or in the generalized sense.

In variational calculus there are various methods for constructing minimizing sequences. In the application to the problems concerning partial differential equations these methods are usually referred to as *variational* or *direct* methods. It is important to note that some of the variational methods make it possible to construct approximate solutions of the problems under consideration. Below we discuss two such methods.

4°. Ritz Method. The idea of the variational method suggested by W. Ritz is the following. Let us consider the minimum problem for a functional $\Phi(u)$. We shall denote by v_n $(n = 1, 2, \ldots)$ a complete system of admitted functions for the functional $\Phi(u)$ and consider the sequence $u_n = \sum_{k=1}^{n} c_k v_k$ $(n = 1, \ldots)$ where c_k are some constants yet unknown.

Let us determine the coefficients c_k $(k = 1, \ldots, n)$ so that the expression $\varphi_n = \Phi(u_n)$ considered as a function of $c_1, \ldots, c_n$ attains its minimum.

For some classes of functionals W. Ritz proved that $\{u_n\}$ is a minimizing sequence which is convergent and whose limit is the solution of the problem under consideration.

As an example, let us consider the second variational problem on the minimization of the functional $J(u)$ for the case when the domain D is the square $0 < x < \pi$, $0 < y < \pi$; without loss of generality we shall assume that

$$H(u) = 1 \tag{194}$$

As the complete system of admitted functions indicated above we can take the system

$$\sin kx \sin ly \quad (k, l = 1, 2, \ldots)$$

Let

$$u_{mn} = \sum_{k=1}^{m} \sum_{l=1}^{n} c_{kl} \sin kx \sin ly \quad (m, n = 1, 2, \dots)$$

The functions $u_{mn}(x, y)$ obviously satisfy condition (190). Besides,

$$d_{mn} = D(u_{mn}) = \frac{\pi^2}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} c_{kl}^2 (k^2 + l^2) \tag{195}$$

and

$$H(u_{mn}) = \frac{\pi^2}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} c_{kl}^2$$

According to the Ritz method, by virtue of (194), we must find the minimum of expression (195) on condition that

$$\sum_{k=1}^{m} \sum_{l=1}^{n} c_{kl}^2 = \frac{4}{\pi^2} \tag{196}$$

Solving conditional extremum problem (196), (195) we find that all the numbers c_{kl}, except c_{11}, are equal to zero for any m and n and that

$$c_{11} = \frac{2}{\pi}, \quad d_{mn} = 2$$

whence

$$\lim_{\substack{m \to \infty \\ n \to \infty}} u_{mn} = u(x, y) = \frac{2}{\pi} \sin x \sin y$$

and

$$\lim_{\substack{m \to \infty \\ n \to \infty}} d_{mn} = D(u) = \lambda = 2$$

5°. Approximate Solution of Eigenvalue Problem. Bubnov-Galerkin Method. The Ritz method makes it possible to construct an approximate solution of eigenvalue problem (189), (190). Indeed, as an approximation to the solution of the second variational problem on the minimization of the functional $J(u)$ under the condition

$$H(u) = 1$$

we can take the function

$$u_n(x, y) = \sum_{k=1}^{n} c_k v_k(x, y)$$

indicated in the foregoing section where the coefficients c_k $(k = 1, \ldots, n)$ are found by solving the following problem on conditional minimum:

$$d_n(c_1, \ldots, c_n) = D(u_n) = \min$$
$$h_n(c_1, \ldots, c_n) = H(u_n) = 1$$

Hence, it is natural to take the function $u_n(x, y)$ thus constructed as an *approximation* to the eigenfunction of problem (189), (190), and the formula

$$\lambda_n = D(u_n)$$

will give an *approximation* to the eigenvalue of the same problem.

Here the *Bubnov-Galerkin method* should also be mentioned which may be successfully used to construct an approximate solution of an eigenvalue problem. In this method as an approximate expression of the eigenfunction of problem (189), (190) the function

$$u_n(x, y) = \sum_{k=1}^{n} c_k v_k(x, y)$$

is taken whose coefficients c_k $(k = 1, \ldots, n)$ are found from the equalities

$$\sum_{k=1}^{n} H(\Delta v_k + \lambda v_k, v_m) c_k = 0 \quad (m = 1, \ldots, n) \qquad (197)$$

which form a homogeneous linear system of algebraic equations. As is known from linear algebra, this system possesses non-trivial solutions if and only if λ satisfies the equation

$$\det \begin{Vmatrix} H((\Delta v_1 + \lambda v_1, v_1) & \ldots & H(\Delta v_n + \lambda v_n, v_1) \\ \ldots & \ldots & \ldots \\ H(\Delta v_1 + \lambda v_1, v_n) & \ldots & H(\Delta v_n + \lambda v_n, v_n) \end{Vmatrix} = 0 \qquad (198)$$

The values of λ found from equation (198) are taken as approximations to the eigenvalues of problem (189), (190). As was already mentioned, the approximate expressions for the eigenfunctions corresponding to λ are given by the formula

$$u_n(x, y) = \sum_{k=1}^{n} c_k v_k(x, y)$$

where c_k $(k = 1, \ldots, n)$ are the solutions of system (197)

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