Frances Kirwan

University of Oxford

An introduction to intersection homology theory

Longman Scientific & Technical

Copublished in the United States with
John Wiley & Sons, Inc., New York

Longman Scientific & Technical
Longman Group UK Limited
Longman House, Burnt Mill, Harlow
Essex CM20 2JE, England
and Associated Companies throughout the world.

Copublished in the United States with
John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158

First published 1988

AMS Subject Classifications: 55N99, 14F99

ISSN 0269-3674

British Library Cataloguing in Publication Data
Kirwan, F. (Frances)
An introduction to intersection homology theory
1. Homological algebra
I. Title
512'. 55

ISBN 0-582-02879-5

Library of Congress Cataloging-in-Publication Data
Kirwan, Frances Clare, 1959–
An introduction to intersection homology theory.
(Pitman research notes in mathematics series,
ISSN 0269-3674 ; 187)
Bibliography: p.
Includes index.
1. Homology theory. 2. Intersection theory.
I. Title II. Series.
QA612.3.K57 1988 514'.23 88-12989
ISBN 0-470-21198-9

Printed and bound in Great Britain by
Biddles Ltd, Guildford and King's Lynn

Contents

Preface

These notes are based on a course for graduate students entitled 'A beginner's guide to intersection homology theory' given in Oxford in 1987. The course was intended to be accessible to first year graduate students and to mathematicians from different areas of mathematics. The aim was to give some idea of the power, usefulness and beauty of intersection homology theory while only assuming fairly basic mathematical knowledge. To succeed at all in this it was necessary to give at most briefly sketched proofs of the important theorems and to concentrate on explaining the main ideas and definitions. The result is that these notes do not constitute in any sense an introductory textbook on intersection homology. Rather they are intended to be a piece of propaganda on its behalf. The hope is that mathematicians of very varied backgrounds with interests in singular spaces should find the notes readable and should be stimulated to learn in greater depth about intersection homology and use it in their work.

Over the last century ordinary homology theory for manifolds has been applied with enormous success to all sorts of different parts of mathematics. Often however ordinary homology is not as successful in dealing with problems involving singular spaces as with problems involving manifolds. In such situations it is possible that intersection homology (which coincides with ordinary homology for manifolds) may be more successful. Many examples of this phenomenon have been found since intersection homology was introduced a decade ago. It was because exactly this phenomenon has occurred in my own work in the last few years that I became an enthusiast for intersection homology, and, although by no means an expert on the subject, decided to give this course.

The goal I had in mind was to explain enough of the theory of intersection homology to be able to give a sketch (following Bernstein [1]) of the proof of the Kazhdan-Lusztig conjecture (Kazhdan-Lusztig [1], [2]). This relates the representation theory of complex Lie algebras to the theory of Hecke algebras via $\mathcal{D}$-modules and intersection homology, and was in fact important motivation in the development of intersection homology theory (cf. Brylinski

[1]). It seemed a suitable target at which to aim, though much of the material covered on the way is just as interesting (or more so, depending on one's point of view) in its own right.

This goal influenced the structure of the second half of the course and thus the lecture notes. The first half consists of an elementary introduction to intersection homology theory. The introductory chapter, which is intended as motivation for the reader, describes three situations in which intersection homology is more successful than ordinary homology in dealing with singular spaces. The second chapter describes briefly some standard homology theory and sheaf theory; it would be helpful but not essential for the reader to be already familiar with this material. There are several different ways of defining intersection homology which vary in difficulty and elegance: Chapter 3 gives the most elementary of these and describes some of its basic properties.

The singular spaces given most attention throughout the notes are complex varieties, but intersection homology is defined for more general spaces as well (the most general being topological pseudomanifolds). The fourth chapter discusses the relationship between the intersection homology of singular complex projective varieties and an analytically defined cohomology theory, L^2-cohomology, which is a generalisation of De Rham cohomology for compact manifolds. Chapter 5 describes the important sheaf-theoretic constructions and characterisations of intersection homology, due to Deligne and developed in Goresky and MacPerson [5], which imply that intersection homology is a topological invariant.

The final three chapters lead towards the proof of the Kazhdan-Lusztig conjecture which is described in Chapter 8. The sixth chapter discusses the relationship of intersection homology with the Weil conjectures and the arithmetic of algebraic varieties defined over finite fields, while Chapter 7 describes briefly the theory of $\mathcal{D}$-modules and the Riemann-Hilbert correspondence relating $\mathcal{D}$-modules to intersection homology.

Nothing in these lecture notes is original work. The papers I have used most heavily are those listed in the references by Goresky and MacPherson, Borel, Bernstein, and Beilinson, Bernstein and Deligne. I would like to thank Joseph Bernstein for first suggesting several years ago that I should look at intersection homology, and all those who attended the 'beginner's guide' last year for pointing out many slips and errors. I am also grateful to Valerie Siviter for typing the original manuscript and to Terri Moss for typing the final version.

Frances Kirwan
Balliol College, Oxford.

April 1988.

1 Introduction

Homology theory was introduced by Poincaré nearly a hundred years ago in order to study the topology of manifolds. As he foresaw, it has been of immense importance in many areas of mathematics including algebraic and differential geometry, differential equations and group theory. However in many situations where homology provides good answers to questions involving manifolds one would also like to know what happens when the manifolds are replaced by singular spaces, and then ordinary homology is no longer always so useful. About ten years ago a new sort of homology, called intersection homology, was introduced by Goresky and MacPherson. Many others have helped to develop its theory since then. Intersection homology coincides with ordinary homology for manifolds, but for singular spaces it often gives "better" answers than ordinary homology does. This introductory chapter gives three examples of this phenomenon to whet the reader's appetite before the definition of intersection homology is given.

§1.1 The cohomology of complex projective varieties

Let X be a nonsingular complex projective variety. Then X is a subset of a complex projective space

$$\mathbf{P}_m = \frac{\mathbb{C}^{m+1} - \{0\}}{\mathbb{C} - \{0\}} = \{\text{complex lines in } \mathbb{C}^{m+1}\},$$

and is defined by the vanishing of homogeneous polynomials. Let us write

1.1.1 $(x_0 : x_1 : \ldots : x_m)$

for the complex line in $\mathbb{C}^{m+1}$ spanned by a nonzero vector $(x_0,\ldots,x_m) \in \mathbb{C}^{m+1}$. Then X is of the form

1.1.2 $X = \{(x_0:\ldots:x_m) \in \mathbf{P}_m | f_j(x_0,\ldots,x_m) = 0, \quad 1 \le j \le M\}$

where $f_1,\ldots,f_M$ are homogeneous polynomials in m+1 variables. The homogeneity of f_j implies that the condition

$$f_j(x_o,\ldots,x_m) = 0$$

is independent of the choice of vector $(x_o,\ldots,x_m) \in \mathbb{C}^{m+1} - \{0\}$ representing the point $(x_o:\ldots:x_m)$ of $\mathbf{P}_m$.

$\mathbf{P}_m$ is a complex manifold with local coordinates

$$(x_o:\ldots:x_m) \to (\frac{x_o}{x_j},\ldots,\frac{x_{j-1}}{x_j},\frac{x_{j+1}}{x_j},\ldots,\frac{x_m}{x_j})$$

identifying the open subsets

$$\{(x_o:\ldots:x_m) \in \mathbf{P}_m | x_j \neq 0\}, \quad 0 \leq j \leq m$$

of $\mathbf{P}_m$ with $\mathbb{C}^m$. The statement that X is nonsingular means that locally we can choose $f_1,\ldots,f_M$ in 1.1.2 so that the matrix $(\frac{\partial f_i}{\partial x_j})$ has rank M. Then X becomes a complex submanifold of $\mathbf{P}_m$.

Let $H^i(X)$ be the ordinary ith cohomology group of X with complex coefficients. (See Chapter 2 for definitions of ordinary homology and cohomology). Then $H^i(X)$ is a complex vector space which is a topological invariant of X and has the following properties (Griffiths and Harris, [1, Chapter 0, §4, §7, Chapter 1, §2]).

1.1.3 (i) Hodge decomposition. We can write

$$H^i(X) = \bigoplus_{p+q=i} H^{p,q}$$

where each $H^{p,q}$ is a complex subspace of $H^i(X)$ and

$$H^{p,q} = \overline{H^{q,p}} .$$

Note that for complex conjugation to make sense we need a real structure on $H^i(X)$, i.e. a real subspace V of $H^i(X)$ such that

$$H^i(X) \cong V \otimes_{\mathbb{R}} \mathbb{C}.$$

We take V to be the ith cohomology group of X with real coefficients $H^i(X;\mathbb{R})$.

The Hodge decomposition implies that if i is *odd* then

$$\dim_{\mathbb{C}} H^i(X) = 2 \sum_{\substack{p<q \\ p+q=i}} \dim_{\mathbb{C}} H^{p,q}$$

is *even*.

(ii) Poincaré duality. There is a natural nondegenerate pairing

$$H^i(X) \otimes H^{2n-i}(X) \to \mathbb{C}$$

so that

$$H^i(X) \cong (H^{2n-i}(X))^*$$

where $n = \dim_{\mathbb{C}} X$. In particular $\dim_{\mathbb{C}} H^i(X) = \dim_{\mathbb{C}} H^{2n-i}(X)$.

(iii) Lefschetz hyperplane theorem. Let $H \subseteq \mathbf{P}_m$ be a generic hyperplane. ("Hyperplane" means that H is defined by one linear equation; "generic" means that the property we are interested in will not necessarily hold for every linear equation but it will hold for most - more precisely for those in a dense open subset of the space of all possible linear equations). Then the restriction map

$$H^i(X) \to H^i(X \cap H)$$

is an isomorphism for $i < n - 1$ and is injective for $i = n - 1$.

(iv) Hard Lefschetz theorem. There is an isomorphism

$$L^i : H^{n-i}(X) \to H^{n+i}(X)$$

given by multiplication (with respect to the cup product) by the ith power of the hyperplane class $[H] \in H^2(X)$. This enables us to refine the Hodge decomposition. Let L be the map given by multiplication by [H], and if $p+q = n-i$ where $0 \leq i \leq n$ let

$$H^{p,q}_{prim} = \{\xi \in H^{p,q} | L^{i+1}(\xi) = 0\}.$$

Here "prim" stands for primitive cohomology. Then if $p+q \leq n$ we have

$$H^{p,q} = H^{p,q}_{prim} \oplus L(H^{p-1,q-1}_{prim}) \oplus L^2(H^{p-2,q-2}_{prim}) \oplus \dots \quad .$$

Note that the hard Lefschetz theorem implies that

$$L : H^k(X) \to H^{k+2}(X)$$

is injective if $k < n$ (so that $\dim_{\mathbb{C}} H^k(X) \leq \dim_{\mathbb{C}} H^{k+2}(X)$) and surjective if $k + 2 > n$ (so that $\dim_{\mathbb{C}} H^k(X) \geq \dim_{\mathbb{C}} H^{k+2}(X)$).

(v) Hodge signature theorem. Let p and q be integers between 0 and n, and suppose that $\xi \in H^{p,q}_{prim}(X)$ is nonzero. Then under the Poincare duality pairing

$$H^{p+q}(X) \otimes H^{2n-p-q}(X) \to \mathbb{C}$$

the pairing of $\xi \in H^{p+q}(X)$ with the element

$$(\sqrt{-1})^{p-q}(-1)^{(n-p-q)(n-p-q-1)/2} L^{n-p-q}(\bar{\xi}) \in H^{2n-p-q}(X)$$

is a strictly positive real number.

Of these theorems Hodge decomposition in particular is very useful for studying nonsingular projective varieties. If one allows X to vary in a holomorphic way depending on some continuous parameters then $H^i(X)$ is essentially independent of X but the Hodge filtration

$$H^i(X) = F^0 \supseteq F^1 \supseteq \dots \supseteq F^i$$

of $H^i(X)$ defined by

$$F^p = \bigoplus_{j \geq p} H^{j,i-j}$$

varies holomorphically with X in an interesting way. This leads to Griffiths' theory of the variation of Hodge structures which gives one information about moduli spaces (Griffiths [1]).

The properties 1.1.3 (i) - (v) of the cohomology of nonsingular projective varieties fail in general for singular varieties. Let us consider two simple examples of this.

First recall that the complex projective line $\mathbf{P}_1$ can be identified with the extended complex plane $\mathbb{C} \cup \{\infty\}$. Topologically it is a two-dimensional sphere. Thus

$$H^i(\mathbf{P}_1) = \begin{cases} \mathbb{C} & i = 0,2 \\ 0 & \text{otherwise.} \end{cases}$$

(Spanier, [1, Chapter 4, §6, Theorem 6] or 2.1.5 below). Now let X be the complex projective variety

1.1.4 $\{(x:y:z) \in \mathbf{P}_2 | yz = 0\}$.

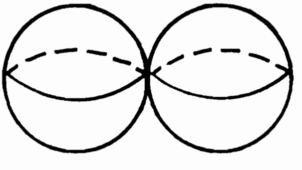
Then X is the union of the two subsets

$$\{(x:y:z) \in \mathbf{P}_2 | y = 0\}$$

and

$$\{(x:y:z) \in \mathbf{P}_2 | z = 0\}$$

of $\mathbf{P}_2$. These subsets are each homeomorphic to $\mathbf{P}_1$ and meet in the single point (1:0:0).



It can easily be shown (using for example the Mayer-Vietoris sequence (Spanier [1, Chapter 4, §6])) that

$H^o(X) = \mathbb{C}$,

$H^1(X) = 0$,

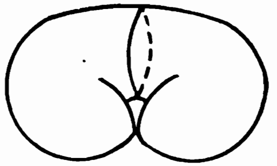
$H^2(X) = \mathbb{C} \oplus \mathbb{C}$.

This means that neither Poincaré duality nor the hard Lefschetz theorem can hold for X.

As a second example let X be the complex projective variety

1.1.5 $\{(x:y:z) \in \mathbf{P}_2 | x^3 + y^3 = xyz\}$.

Then it is not hard to check that topologically X is a two-dimensional sphere with two points identified.



Thus

$H^o(X) = \mathbb{C}$,

$H^1(X) = \mathbb{C}$,

$H^2(X) = \mathbb{C}$.

In particular $\dim_{\mathbb{C}} H^1(X)$ is odd so there cannot be a Hodge decomposition of the cohomology of X.

One remedy for the failure of 1.1.3 (i) - (v) when X is singular is to introduce new cohomology groups $IH^i(X)$ such that if X is nonsingular then $IH^i(X) = H^i(X)$ and such that $IH^i(X)$ has the properties 1.1.3 (i) - (v) even when X is singular. These new cohomology groups $IH^i(X)$ are the intersection cohomology groups of X.

§1.2 De Rham cohomology and L^2-cohomology

When X is a compact manifold the cohomology $H^*(X)$ can be identified with the

De Rham cohomology $H^*_{DR}(X)$ defined as follows (see Bott and Tu [1] for more details).

Let TX be the tangent bundle to X and let T^*X be the cotangent bundle. A differential r-form ω on X is a C^∞ section of the r-fold exterior product $\Lambda^r T^* X$ of the cotangent bundle. In (real) local coordinates $y_1, \ldots, y_m$ we have

$$\omega(y) = \sum_{i_1<\ldots<i_r} a_{i_1 \ldots i_r}(y)\, dy_{i_1} \wedge \ldots \wedge dy_{i_r}$$

where each $a_{i_1\ldots i_r}$ is a smooth real-valued function of $y = (y_1, \ldots, y_m)$.

Let $A^r(X;\mathbb{R})$ be the space of all differential r-forms and let

$$A^r(X) = A^r(X;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

be the space of all complex valued differential r-forms. There is a map

$$d : A^r(X) \to A^{r+1}(X)$$

defined in local coordinates by

$$d\omega = \sum_{i_1<\ldots<i_r} \sum_j (\partial a_{i_1\ldots i_r} / \partial y_j)\, dy_j \wedge dy_{i_1} \wedge \ldots \wedge dy_{i_r}$$

when ω is as above.

Then $d^2 = 0$ (by the symmetry of the second partial derivatives of a C^∞ function). The rth De Rham complex cohomology group of X is by definition the quotient group

1.2.1 $$H^r_{DR}(X) = \frac{\ker d : A^r(X) \to A^{r+1}(X)}{\operatorname{im} d : A^{r-1}(X) \to A^r(X)} .$$

1.2.2 <u>Proposition</u> (Griffiths and Harris [1, p. 43]). $H^r_{DR}(X)$ is canonically isomorphic to $H^r(X)$.

Proposition 1.2.2 together with the famous Hodge theorem (Griffiths and

Harris [1, Chapter 0, §6]) can be used to put a Hodge decomposition on $H^r(X)$ (see 1.1.3) when X is a nonsingular projective variety. The Hodge theorem implies that every De Rham cohomology class in $H^r_{DR}(X)$ contains a unique harmonic differential r-form ω which can be written uniquely as a sum of harmonic (p,q)-forms where p+q = r. A (p,q)-form is one which can be written locally with respect to *complex* local coordinates $z_1,\ldots,z_n$ as a sum of terms of the form

$$\alpha\, dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$$

where $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q$ and α is a smooth function.

One would like to have some sort of analytically defined cohomology when X is singular (at least when X is a singular complex projective variety, perhaps in more general cases too) analogous to De Rham cohomology and canonically isomorphic to intersection cohomology. With luck this could then be used to give analytical proofs that intersection cohomology has a Hodge decomposition and satisfies Poincaré duality and the hard Lefschetz theorem. It should have all sorts of other spin-offs as well, just as the De Rham theorem does.

It is conjectured (and proved in some cases) that there is such a cohomology defined analytically (see Cheeger, Goresky and MacPherson [1]). In fact this is why it was originally hoped that intersection cohomology should satisfy the properties 1.1.3 (i), (ii), (iv). The analytically defined cohomology theory which is conjectured to be the same as intersection cohomology for singular projective varieties is called L^2-cohomology. It is defined as follows.

Let $X \subseteq \mathbf{P}_m$ be a projective variety of complex dimension n. Let Σ be the set of singular points of X. $\mathbf{P}_m$ has a Kähler metric called the Fubini-study metric (Griffiths and Harris [1, p. 31]). The restriction of this to X-Σ gives us a Riemannian metric on the manifold X-Σ, i.e. an inner product g_x on each tangent space $T_x(X-\Sigma) = T_xX$ which varies smoothly with $x \in X-\Sigma$. This inner product g_x induces inner products on the cotangent space $T^*_x(X-\Sigma)$ and its exterior powers $\Lambda^i T^*_x(X-\Sigma)$ for all $i > 0$ and $x \in X-\Sigma$.

Given a smooth i-form ω on X-Σ we have a smooth function $||\omega||^2$ on X-Σ defined by $x \to \|\omega(x)\|^2$ where $\|\ \|$ is the norm on $\Lambda^i T^*_x(X-\Sigma)$ induced by the inner product. The i-form is called *square-integrable* if this function $||\omega||^2$

is integrable over $X-\Sigma$ with respect to the volume form induced by the metric and the natural orientation on $X-\Sigma$. (For more details on differential geometry see e.g. Spivak [1], Sternberg [1], Warner [1]).

Let $L^i(X-\Sigma) \subseteq A^i(X-\Sigma)$ be the space of square-integrable differential i-Σ forms on $X-\Sigma$. The L^2-cohomology of X is defined to be

1.2.3 $$H^i_{(2)}(X) = \frac{\{\omega \in L^i(X-\Sigma) \mid d\omega = 0\}}{\{\eta \in L^i(X-\Sigma) \mid \exists\zeta \in L^{i-1}(X-\Sigma),\ d\zeta = \eta\}}.$$

Note that d may not map $L^{i-1}(X-\Sigma)$ into $L^i(X-\Sigma)$.

Of course if X is nonsingular then $X-\Sigma = X$ is compact so $L^i(X-\Sigma) = A^i(X)$ for all i and

$$H^i_{(2)}(X) = H^i_{DR}(X).$$

1.2.4 Conjecture. (See Cheeger, Goresky and MacPerson [1]). If X is a singular projective variety then $H^*_{(2)}(X)$ is isomorphic to $IH^*(X)$.

It is not even known that $H^i_{(2)}(X)$ is finite-dimensional in general. But the conjecture is known to be true when X has isolated conical singularities (see Chapter 4).

§1.3 Morse theory for singular spaces

Suppose X is a compact manifold. A smooth function $f: X \to \mathbb{R}$ is called a Morse function (Milnor [1]) if the set

$$C(f) = \{x \in X \mid df(x) = 0\}$$

of critical points of f is finite, and for each $x \in C(f)$ the Hessian $H_x(f)$ is nondegenerate. Here $H_x(f)$ is the bilinear form on T_xX given in local coordinates $y_1,\ldots,y_m$ by the matrix

$$\left(\frac{\partial^2 f}{\partial y_i \partial y_j}\right)$$

of the second partial derivatives of f at x. We shall also require for simplicity of notation that if x and y are distinct critical points then $f(x) \neq f(y)$.

The set of Morse functions is open and dense in the set of all smooth functions on X.

If $f: X \to \mathbb{R}$ is a Morse function then for each $y \in \mathbb{R}$ *either*

1.3.1 (i) $y \neq f(x)$ for all critical points $x \in C(f)$, in which case if $\varepsilon > 0$ is small enough the map

$$H_k(X_{y-\varepsilon}) \to H_k(X_{y+\varepsilon})$$

induced by the inclusion of the open set

$$X_{y-\varepsilon} = \{x \in X \mid f(x) < y-\varepsilon\}$$

in

$$X_{y+\varepsilon} = \{x \in X \mid f(x) < y+\varepsilon\}$$

is an isomorphism for all k; *or*

(ii) $y = f(x)$ for some (unique) critical point $x \in C(f)$, in which case there is an integer $I(f;x)$ such that if $\varepsilon > 0$ is small enough the map

$$H_k(X_{y-\varepsilon}) \to H_k(X_{y+\varepsilon})$$

induced by inclusion is an isomorphism except when k is $I(f;x)$ or $I(f;x) - 1$, and for these values of k it fits into an exact sequence

$$0 \to H_{I(f;x)}(X_{y-\varepsilon}) \to H_{I(f;x)}(X_{y+\varepsilon}) \to \mathbb{C}$$

$$\to H_{I(f;x)-1}(X_{y-\varepsilon}) \to H_{I(f;x)-1}(X_{y+\varepsilon}) \to 0.$$

Another way to express this is to say that the relative homology $H_k(X_{y+\varepsilon}, X_{y-\varepsilon})$ is given by

$$H_k(X_{y+\varepsilon}, X_{y-\varepsilon}) = \begin{cases} 0 & \text{if } k \neq I(f;x), \\ \mathbb{C} & \text{if } k = I(f;x). \end{cases}$$

The integer $I(f;x)$ is called the *Morse index* of the critical point x for the function f.

As a consequence of 1.3.1 we obtain the famous *Morse inequalities*, which are most easily written in the following form

1.3.2 $$\sum_{x \in C(F)} t^{I(f;x)} - \sum_{i \geq 0} t^i \dim_{\mathbb{C}} H_i(X) = (1+t)R(t),\ R(t) \geq 0$$

where R(t) is a polynomial in t with *non-negative* integer coefficients. In particular this implies that the dimension of $H_i(X)$ is at most the number of $x \in C(f)$ with $I(f;x) = i$, but the Morse inequalities contain stronger information than this. For example if $I(f;x)$ is even for all $x \in C(f)$ the Morse inequalities can only work if $R(t) = 0$, i.e. if the dimension of $H_i(X)$ is equal to the number of $x \in C(f)$ with $I(f;x) = i$ for all i.

Morse theory can be generalised to the case when X is allowed to be singular provided that intersection homology is used instead of ordinary homology, as follows. (See Goresky and MacPerson [2] and [4] for more details).

Let us assume that X is a subset of a manifold Y defined locally by the vanishing of smooth functions on Y. The set of all functions $f : X \to \mathbb{R}$ which extend to smooth functions on Y contains a dense open subset such that for any $f: X \to \mathbb{R}$ in this subset there exists a finite set $C(f) \subseteq X$ with the following properties.

1.3.3 (i). If $y \in \mathbb{R} - \{f(x) | x \in C(f)\}$ then there is an isomorphism

$$IH_k(X_{y-\varepsilon}) \cong IH_k(X_{y+\varepsilon})$$

for all sufficiently small $\varepsilon > 0$. The isomorphism is induced by the inclusion.

(ii) If $y = f(x)$ for some $x \in C(f)$ then this x is unique, and there exists an integer $I(f;x) \geq 0$, called the Morse index of x for f, and a complex vector space A_x such that if $\varepsilon > 0$ is small enough then

$$IH_k(X_{y-\varepsilon}) \cong IH_k(X_{y+\varepsilon})$$

unless k is $I(f;x)$ or $I(f;x) - 1$, and there is an exact sequence

$$0 \to IH_{I(f;x)}(X_{y-\varepsilon}) \to IH_{I(f;x)}(X_{y+\varepsilon}) \to A_x$$

$$\to IH_{I(f;x)-1}(X_{y-\varepsilon}) \to IH_{I(f;x)-1}(X_{y+\varepsilon}) \to 0.$$

Here A_x depends on x and X but *not* on the function f. In fact A_x is determined by the singularity of X at x. If x is a nonsingular point of X then $A_x = \mathbb{C}$.

From 1.3.3 one gets *generalised Morse inequalities*

1.3.4 $$\sum_{x \in C(F)} t^{I(f;x)} \dim_{\mathbb{C}} A_x - \sum_{i \geq 0} t^i \dim_{\mathbb{C}} IH_i(X) = 1+t)Q(t), \; Q(t) \geq 0,$$

where Q(t) is a polynomial in t with non-negative integer coefficients.

1.3.5 Remark. If X is singular there does not in general exist a Morse index for ordinary homology. As y moves through a critical value the homology of X_y may change in a whole range of dimensions (Goresky and MacPherson [4, §4.5, Example 3]).

This ends the introduction. Its aim was to make the reader sufficiently interested in intersection homology to want to find out how it is actually defined. Next it is necessary to review some ordinary (co)homology theory and sheaf theory.

2 Review of homology, cohomology and sheaf theory

If X is a compact manifold there are several ways of defining the homology and cohomology groups $H^i(X)$ of X which all lead to essentially the same thing in the end: simplicial homology and cohomology; singular homology and cohomology; Čech cohomology of sheaves; sheaf cohomology via derived functors and De Rham cohomology. We shall review the first four of these briefly in this chapter. (For the definition of De Rham cohomology see §1.2. For more details see e.g. Bott and Tu [1], Dold [1], Greenberg [1], Spanier [1]).

§2.1 Simplicial homology

Simplicial homology is the most prosaic and least elegant of these. It is useful for working out examples. We shall need the definition in order to define intersection homology later.

2.1.1 Definition. An *n-simplex* σ in R^N is the convex hull of points $v_0,\ldots,v_n$ such that $v_1-v_0, v_2-v_0,\ldots,v_n-v_0$ are n linearly independent vectors in R^N. Then $v_0,\ldots,v_n$ are the *vertices* of σ and n is the dimension of σ. The *faces* of σ are the (n-1)-simplices whose vertices are also vertices of σ, for example the convex hull of $v_0, v_2, v_3,\ldots,v_n$.

An *orientation* of an n-simplex σ is an ordering of its vertices determined up to even permutation.

A *simplicial complex* in R^N is a set N of simplices such that

2.1.2 (i) if $\sigma \in N$ then every face of σ is in N;

(ii) if $\sigma, \tau \in N$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a simplex whose vertices are also vertices of both σ and τ;

(iii) if $x \in \sigma \in N$ then there is a neighbourhood U of x in R^N such that $U \cap \tau \neq \emptyset$ for only finitely many simplices $\tau \in N$.

2.1.3 Definition. The support

$$|N| = \bigcup_{\sigma \in N} \sigma$$

of a simplicial complex N in $\mathbf{R}^N$ is the union of the simplices which belong to it. A *triangulation* of a topological space X is a homeomorphism T: $|N| \to X$ where N is a simplicial complex.

We shall assume henceforth that X is triangulable, i.e. that X has a triangulation T: $|N| \to X$. Note that N is finite if and only if X is compact.

For each $\sigma \in N$ choose an orientation of σ. Let

$$N^{(i)} = \{\sigma \in N | \sigma \text{ an i-simplex}\}.$$

An i-chain of N with complex coefficients is a formal linear combination

$$\xi = \sum_{\sigma \in N^{(i)}} \xi_\sigma \sigma$$

where the coefficients ξ_σ are complex numbers and only finitely many of them are nonzero. The space $C_i(N)$ of i-chains in N is a complex vector space with basis $N^{(i)}$. The boundary map

$$\partial: C_i(N) \to C_{i-1}(N)$$

is the unique complex linear map such that if $\sigma \in N^{(i)}$ then

$$\partial\sigma = \sum_{\tau \text{ face of } \sigma} \pm \tau$$

where the sign $\pm$ is 1 if the chosen orientation on τ is obtained from the chosen orientation, $v_0,\ldots,v_i$ say, on σ by omitting some v_j where j is even, and is -1 otherwise. Then

$$\partial^2 : C_i(N) \to C_{i-2}(N)$$

is 0, i.e.

$$\mathrm{im}(\partial: C_i(N) \to C_{i-1}(N)) \subseteq \ker(\partial: C_{i-1}(N) \to C_{i-2}(N)).$$

2.1.4 Definition. The ith homology group of N with complex coefficients is

$$H_i(N) = \frac{\ker \partial: C_i(N) \to C_{i-1}(N)}{\operatorname{im} \partial: C_{i+1}(N) \to C_i(N)} .$$

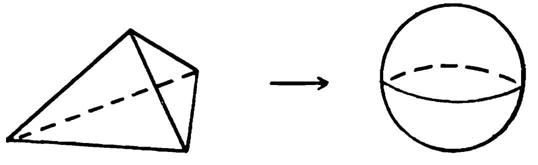
If T: $|N| \to X$ is a triangulation of X then we define the ith homology group of X with respect to T as

$$H_i^T(X) = H_i(N).$$

We also write $C_i^T(X)$ for $C_i(N)$.

In fact $H_i^T(X)$ does not depend on the triangulation T chosen (see 2.2.3 below). It is a definition of homology which is usually easy to calculate in examples.

2.1.5 Example. Let X be the two-dimensional sphere S^2 and let T: $|N| \to X$ be the triangulation indicated by the diagram.



Then

$$\operatorname{im} \partial: C_1(N) \to C_0(N)$$

is spanned by $\{v_i - v_j | 0 \leq i < j \leq 3\}$ so $H_0^T(X) = \mathbb{C}$. Also

$$\ker \partial: C_1(N) \to C_0(N) = \operatorname{im} \partial: C_2(N) \to C_1(N)$$

so $H_1^T(X) = 0$. Finally

$$\ker \partial: C_2(N) \to C_1(N)$$

is spanned by

$$(v_0v_1v_2) - (v_0v_1v_3) + (v_0v_2v_3) - (v_1v_2v_3)$$

so $H_2^T(X) = \mathbb{C}$.

2.1.6 Definition. A triangulation $T\colon |N| \to X$ is a *refinement* of a triangulation $\tilde{T}\colon |\tilde{N}| \to X$ if for each $\sigma \in N$ there exists some $\tilde{\sigma} \in \tilde{N}$ such that $T(\sigma) \subseteq \tilde{T}(\tilde{\sigma})$.

If T is a refinement of $\tilde{T}$ then there is a natural map

$$C_i(\tilde{N}) \to C_i(N)$$

compatible with boundary maps such that if $\tilde{\sigma} \in \tilde{N}^{(i)}$ then

$$\tilde{\sigma} \to \sum_{\sigma \in N^{(i)},\, T(\sigma) \subseteq \tilde{T}(\tilde{\sigma})} \pm \sigma$$

where the sign depends on whether the orientations of σ and $\tilde{\sigma}$ are compatible.

2.1.7 Definition. The space $C_i(X)$ of all *piecewise linear i-chains* is the direct limit of the spaces $C_i^T(X)$ under refinement. That is, a piecewise linear i-chain on X is represented by an element of $C_i^T(X)$ for some triangulation T of X, and two such elements

$$c \in C_i^T(X), \qquad \tilde{c} \in C_i^{\tilde{T}}(X)$$

represent the same piecewise linear i-chain if and only if there exists a common refinement $\bar{T}$ of T and $\tilde{T}$ such that the images of c and $\tilde{c}$ in $C_i^{\bar{T}}(X)$ coincide.

The boundary maps $\partial\colon C_i^T(X) \to C_{i-1}^T(X)$ induce boundary maps

$$\partial\colon C_i(X) \to C_{i-1}(X)$$

such that $\partial^2 = 0$.

2.1.8 Definition. The *simplicial homology* of a triangulable space X is defined by

$$H_i^{simp}(X) = \frac{\ker \partial : C_i(X) \to C_{i-1}(X)}{\operatorname{im} \partial : C_{i+1}(X) \to C_i(X)} .$$

This definition is independent of the choice of triangulation but a priori impossible to compute.

§2.2 Singular homology

Singular homology is the most common first definition of homology. It is not much use for defining intersection homology (see remark 3.4.5 below) but for completeness the definition is included here.

A *singular* i-*simplex* in a topological space X is a continuous map

$$\Sigma : \Delta_i \to X$$

where Δ_i is the standard i-simplex in $\mathbf{R}^i$; that is, Δ_i is the convex hull of the set of points

$$\{(0,\dots,0),(1,0,\dots 0),(0,1,0,\dots 0),\dots,(0,\dots 0,1)\}$$

in $\mathbf{R}^i$. The space $S_i(X)$ of *singular* i-*chains* in X is the complex vector space with the set of singular i-simplices in X as basis. A singular (i-1)-simplex is a face of a singular i-simplex Σ if it is the composition of Σ with one of i+1 standard maps

$$\phi_j : \Delta_{i-1} \to \Delta_i, \qquad 0 \leq j \leq i,$$

which identify Δ_{i-1} with faces of Δ_i. We define

$$\partial\Sigma = \sum_{T \text{ face of } \Sigma} \pm T$$

where the sign depends on orientations (cf. §2.1). If the sign is chosen correctly we get $\partial^2 = 0$ and we define the ith singular homology group of X with complex coefficients to be the quotient

$$2.2.1 \quad H_i^{sing}(X) = \frac{\ker \partial : S_i(X) \to S_{i-1}(X)}{\operatorname{im} \partial : S_{i+1}(X) \to S_i(X)} .$$

If $f:X \to Y$ is a continuous map between topological spaces then f induces linear maps

$$f_*: S_i(X) \to S_i(Y)$$

such that

$$f_*(\sigma) = f \circ \sigma$$

for any singular i-simplex Σ in X. The maps f_* are compatible with the boundary maps, and hence induce

$$2.2.2 \quad f_*: H_*^{sing}(X) \to H_*^{sing}(Y).$$

It is not quite so obvious that if X and Y are triangulable then a continuous map $f: X \to Y$ induces in a natural way a linear map $f_*: H_*^{simp}(X) \to H_*^{simp}(Y)$ on simplicial homology. This follows however from the following important fact.

2.2.3 Theorem. If $T: |N| \to X$ is any triangulation of a topological space X then there are natural isomorphisms

$$H_*^{sing}(X) \cong H_*^{simp}(X) \cong H_*^T(X).$$

The proof of this is based on the simplicial approximation theorem (Spanier [1, Chapter 3 §4, Chapter 4 §6 Theorem 8]), which tells us that any singular p-simplex

$$\Sigma: \Delta_p \to X$$

can be "approximated" by $\tilde{\Sigma}: \Delta_p \to X$ where $\tilde{\Sigma}$ is piecewise-linear with respect to the given triangulation T on X and a refinement of the obvious triangulation on Δ_p. The approximation is such that

$$\Sigma = \tilde{\Sigma} + \partial\bar{\Sigma}$$

for some (p+1)-chain $\bar{\Sigma}$ in $S_{p+1}(X)$. Since $\partial^2 = 0$ we get

$$\partial\Sigma = \partial\tilde{\Sigma}.$$

It follows that the natural map

$$H_*^T(X) \to H_*^{sing}(X)$$

is an isomorphism. Taking direct limits, since $H_*^{sing}(X)$ is independent of T we find that

$$H_*^{simp}(X) \cong H_*^{sing}(X).$$

To define *cohomology* groups instead of homology groups we can use the dual ∂^* of the boundary operator. Thus the ith singular cohomology group of X is

$$H^i_{sing}(X) = \frac{\ker \partial^*: C_i(X)^* \to C_{i+1}(X)^*}{\operatorname{im} \partial^*: C_{i-1}(X)^* \to C_i(X)^*}$$

and the ith simplicial cohomology group $H^i_{simp}(Y)$ is defined similarly.

Because we are working with coefficients in a *field* $\mathbb{C}$, not an arbitrary ring, we have natural isomorphisms

$$H^i_{sing}(X) \cong (H_i^{sing}(X))^*$$

and

$$H^i_{simp}(X) \cong (H_i^{simp}(X))^*$$

between the cohomology groups and the duals of the corresponding homology groups.

§2.3 Homology with closed support

We have defined the simplicial and singular homology groups of a triangulable space X using chains which are *finite* linear combinations of simplices. It is also possible to work with chains which are (formal) infinite linear combinations of simplices (Borel-Moore [1]). We get new homology groups (sometimes called Borel-Moore homology groups or homology groups with closed support). When X is compact the two sorts of homology are canonically isomorphic.

Let $T : |N| \to X$ be a triangulation of X. The space

$$C_i^T((X))$$

of *locally finite i-chains* of X with respect to T is the vector space consisting of all formal linear combinations

$$\xi = \sum_{\sigma \in N^{(i)}} \xi_\sigma \sigma$$

where the coefficients ξ_σ are complex numbers. We do *not* impose the condition that only finitely many of the ξ_σ are nonzero. $C_i^T(X)$ is the subspace of $C_i^T((X))$ spanned by $N^{(i)}$ and we can identify $C_i^T((X))$ with the dual of $C_i^T(X)$ using the basis $N^{(i)}$.

We define the space $C_i((X))$ of locally finite piecewise linear i-chains on X as the direct limit of the spaces $C_i^T((X))$ under refinement.

The support

$$|\xi| = \bigcup_{\xi_\sigma \neq 0} T(\sigma)$$

of a locally finite i-chain

$$\xi = \sum_{\sigma \in N^{(i)}} \xi_\sigma \sigma \in C_i^T((X))$$

is always a closed subset of X (since any simplicial complex N is locally finite). It is easy to see that the support $|\xi|$ is compact if and only if $\xi \in C_i^T(X)$. Thus i-chains $\xi \in C_i(X)$ are sometimes called i-*chains with compact support* (and the groups $H_i^{simp}(X)$ are called homology with compact support) whereas chains $\xi \in C_i((X))$ are called i-*chains with closed support.*

The boundary map $\partial: C_i^T(X) \to C_{i-1}^T(X)$ extends in the obvious way to a boundary map

$$\partial: C_i^T((X)) \to C_{i-1}^T((X))$$

such that $\partial^2 = 0$. There is an induced boundary map

$$\partial: C_i((X)) \to C_{i-1}((X)).$$

2.3.1 Definition. The homology groups with closed support (or Borel-Moore homology groups) $H_i^{c\ell}(X)$ of X are defined to be the quotients

$$H_i^{c\ell}(X) = \frac{\ker \partial: C_i((X)) \to C_{i-1}((X))}{\operatorname{im} \partial: C_{i+1}((X)) \to C_i((X))} .$$

Of course when X is compact then if $T: |N| \to X$ is a triangulation the simplicial complex N is finite, so

$$C_i^T((X)) = C_i^T(X).$$

Thus

2.3.2 $\quad H_i^{c\ell}(X) = H_i^{simp}(X)$

when X is compact.

We can also define singular homology groups with closed support.

§2.4 Sheaves.

We have now considered simplicial and singular homology and cohomology and also De Rham cohomology for compact manifolds. In order to define two more important forms of cohomology we need to review some sheaf theory. For more details see e.g. Godement [1], Serre [1], Hartshorne [1].

2.4.1 Definition. A *presheaf* F on a topological space X is given by the following data:

(a) for every open subset U of X an abelian group $F(U)$,

(b) for every inclusion $U \subseteq V$ of open subsets of X a homomorphism

$$\rho_{VU} : F(V) \to F(U)$$

called the restriction homomorphism, satisfying

(i) $F(\emptyset) = 0$;

(ii) $\rho_{UU} : F(U) \to F(U)$ is the identity;

(iii) if $U \subseteq V \subseteq W$ then $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$.

If $U \subseteq V$ are open subsets of X and $s \in F(V)$ then we write

$$s|_U$$

for

$$\rho_{VU}(s) \in F(U).$$

A presheaf F on X is a *sheaf* if in addition it has the following property.

(iv) Let $\{V_i | i \in I\}$ be a collection of open subsets of X. Suppose that we are given elements $s_i \in F(V_i)$ for all $i \in I$ satisfying

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$

for all $i,j \in I$. Then there exists a *unique* $s \in F(\bigcup_{i \in I} V_i)$ such that

$$s|_{V_i} = s_i$$

for all $i \in I$.

2.4.2. <u>Examples</u>. (1). Let A be any abelian group. The *constant sheaf* A_X on X determined by A is defined by

$$A_X(\emptyset) = \{0\},\ A_X(U) = \{\text{continuous maps } f\colon U \to A\} \text{ if } U \neq \emptyset,$$

with the obvious restriction homomorphisms. Here A is supposed to have the *discrete* topology. (Even if A has a natural topology such as when $A = \mathbb{C}$, we take the discrete topology when defining A_X). Thus if U is non-empty and connected every continuous map $f:U \to A$ is constant so $A_X(U) = A$.

(2) Let $\pi:Y \to X$ be continuous and define F by

$$F(U) = \{\text{continuous } \sigma:U \to Y \mid \pi \circ \sigma(x) = x,\ \forall x \in U\}$$

$$= \{\text{sections of } \pi \text{ over } U\},$$

with the obvious restriction maps. F is called the *sheaf of sections* of $\pi:Y \to X$. For example Y might be the tangent bundle of X when X is a manifold, or Y might be $\Lambda^r T^* X$. In these cases the elements of $F(U)$ are vector fields over U or differential r-forms on U.

2.4.3 Definition. Let F be a presheaf over X. The *stalk* F_x of F at $x \in X$ is the direct limit of abelian groups

$$F_x = \varinjlim \{F(U) \mid x \in U,\ U \text{ open in } X\}.$$

Thus an element of F_x is represented by a pair (U,s) where U is an open subset of X such that $x \in U$ and $s \in F(U)$. Two pairs (U,s) and (V,t) represent the same element of F_x if there exists an open neighbourhood W of x in X such that $W \subseteq U \cap V$ and $s|_W = t|_W$. We write s_x for the element of F_x represented by (U,s).

Elements of $F(U)$ are called *sections* of F over U. Elements of F_x are called *germs* of sections of F at x.

2.4.4 Definition. Let F and G be (pre)sheaves over X. A map of (pre)sheaves ϕ: $F \to G$ is given by homomorphisms

$$\phi(U) : F(U) \to G(U)$$

for all open subsets $U \subseteq X$ such that if $V \subseteq U$ then the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{\phi(U)} & G(U) \\ \text{restriction} \downarrow & & \downarrow \text{restriction} \\ F(V) & \xrightarrow[\phi(V)]{} & G(V) \end{array}$$

commutes. There is then an induced homomorphism

$$\phi_x : F_x \to G_x$$

for all $x \in X$. The map ϕ is called an isomorphism if $\phi(U) : F(U) \to G(U)$ is an isomorphism for all open subsets U of X. If F and G are sheaves this is the case if and only if $\phi_x : F_x \to G_x$ is an isomorphism for all $x \in X$.

2.4.5 Definition. Let F be a presheaf over X. The *sheaf* F^+ *associated to* F is defined as follows. If U is an open subset of X then $F^+(U)$ is the set of functions $f: U \to \coprod_{x \in X} F_x$ satisfying

i) $f(x) \in F_x$ for all $x \in U$; and

ii) if $x \in U$ then there is an open neighbourhood W of x in U and there is some $s \in F(W)$ such that $f(y) = s_y$ for all $y \in W$.

$F^+(U)$ becomes an abelian group under pointwise addition. The obvious restriction homomorphisms make F^+ into a sheaf.

Alternatively if we put an appropriate topology on the disjoint union of stalks

$$Y = \coprod_{x \in X} F$$

then we can define F^+ as the sheaf of sections of $\pi: Y \to X$, where $\pi(s_x) = x$ if $s_x \in F_x$.

There is a natural map of presheaves

$$\phi: F \to F^+$$

such that if U is open in X and $s \in F(U)$ then

$$\phi(U)s : U \to \coprod_{x \in U} F_x$$

sends $x \in U$ to $s_x \in F_x$. This map ϕ is an isomorphism if and only if F is a sheaf. It has the universal property that any map or presheaves $\psi: F \to G$ from F to a *sheaf* G over X factors uniquely as the composition of $\phi: F \to F^+$ and a map of sheaves $\theta: F^+ \to G$.

If $\phi: F \to G$ is a map of sheaves over X the kernel $\ker \phi$ is the sheaf defined by

$$\ker \phi(U) = \ker \{\phi(U): F(U) \to G(U)\}$$

with the restriction maps induced by those of F. However the presheaf whose space of sections over U is

$$\text{im} \{\phi(U): F(U) \to G(U)\}$$

is *not* necessarily a sheaf. Since we are interested in sheaves rather than presheaves we define im ϕ to be the sheaf associated to this presheaf.

Similarly if F is a subsheaf of G (i.e. F is a sheaf over X such that $F(U)$ is a subgroup of $G(U)$ for all open subsets U of X and the restriction maps of F are induced by those of G) then the presheaf

$$U \to \frac{G(U)}{F(U)}$$

is not necessarily a sheaf. We define the quotient sheaf G/F to be the sheaf associated to this presheaf.

Suppose that $f: X \to Y$ is a continuous map and that F and G are sheaves on X and Y. We define a sheaf f_*F on Y by

$$f_*F(U) = F(f^{-1}(U))$$

for every open subset U of Y. We define f^*G to be the sheaf on X associated to the presheaf H defined as follows. If V is open in X then $H(V)$ is the limit with respect to restriction of the abelian groups $G(U)$ where U runs over all open subsets of Y containing f(V). If f: X → Y is the inclusion of a subset X of Y in Y then f^*G is called the restriction of G to X.

§2.5 Čech cohomology of sheaves

Let F be a sheaf on a topological space X and let $\mathcal{U} = \{U_i | i \in I\}$ be an open covering of X.

For each $p \geq 0$ let $I^{(p)}$ be the set of all subsets of I with precisely p+1 elements. If

$$K = \{i_o, \ldots, i_p\} \in I^{(p)}$$

let

$$U_K = U_{i_o} \cap U_{i_1} \cap \ldots \cap U_{i_p}.$$

Let

$$C^p(\mathcal{U}, F) = \prod_{K \in I^{(p)}} F(U_K).$$

Then $C^p(\mathcal{U}, F)$ is an abelian group. An element

$$\alpha \in C^p(\mathcal{U}, F)$$

is determined by giving elements

$$\alpha_K \in F(U_K)$$

for each $K \in I^{(p)}$.

For each $K \in I^{(p)}$ choose an orientation of K, i.e. an ordering of the elements of K up to even permutations. Define a coboundary map

$$d: C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$$

as follows. If $K = \{i_o, \ldots, i_{p+1}\} \in I^{(p+1)}$ set

$$(d\alpha)_K = \sum_{j=0}^{p+1} \pm \alpha_{K-\{i_j\}}|U_K$$

where the sign ± depends on whether or not the orientation chosen for K coincides with the orientation chosen for K - $\{i_j\}$ with i_j placed at the beginning. Then it is easy to check that

$$d^2 = 0$$

so we can define

$$\text{2.5.1} \qquad \check{H}^p(\mathcal{U},\mathcal{F}) = \frac{\ker d\colon C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{L},\mathcal{F})}{\operatorname{im} d\colon C^{p-1}(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{F})} \ .$$

An open covering $\mathcal{V}$ of X is called a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists some $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then there is a map

$$C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{V},\mathcal{F})$$

induced by the restriction maps of $\mathcal{F}$ which commutes with the coboundary maps. We define

$$C^p(X,\mathcal{F})$$

to be the direct limit of the $C^p(\mathcal{U},\mathcal{F})$ with respect to refinement. That is, every element of $C^p(X,\mathcal{F})$ is represented by an element of $C^p(\mathcal{U},\mathcal{F})$ for some open covering $\mathcal{U}$, and elements of $C^p(\mathcal{U},\mathcal{F})$ and $C^p(\mathcal{V},\mathcal{F})$ represent the same element of $C^p(X,\mathcal{F})$ if they map to the same element of $C^p(\mathcal{W},\mathcal{F})$ for some common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$.

The coboundary maps

$$d\colon C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$$

induce coboundary maps

$$d\colon C^p(X,\mathcal{F}) \to C^{p+1}(X,\mathcal{F}).$$

The pth *Čech cohomology* group of X with coefficients in F is by definition the quotient

$$\check{H}^p(X,F) = \frac{\ker d\colon C^p(X,F) \to C^{p+1}(X,F)}{\operatorname{im} d\colon C^{p-1}(X,F) \to C^p(X,F)} .$$

In fact if X is triangulable we can always choose an open cover $\mathcal{U}$ so that

$$\check{H}^p(\mathcal{U},F) = \check{H}^p(X,F).$$

This follows from the proof of the following proposition.

2.5.3 Proposition. If X is triangulable then $H^*_{simp}(X) \cong H^*(X,\mathbb{C}_X)$ where $\mathbb{C}_X$ is the constant sheaf on X determined by $\mathbb{C}$.

Sketch proof. Consider a triangulation $T\colon |N| \to X$ of X. Let V be the set of vertices of N. If $\sigma \in N$ let

$$\sigma^\circ = \sigma - \bigcup_{\substack{\tau \neq \sigma \\ \tau \text{ face of } \sigma}} \tau$$

be the interior of σ, and if $v \in V$ let

$$U_v = \bigcup_{\sigma \in N,\, v \in \sigma} T(\sigma^\circ).$$

Then $\mathcal{U} = \{U_v | v \in V\}$ is an open covering of X.

If $K = \{v_0,\ldots,v_p\} \in V^{(p)}$ then

$$U_K = U_{v_0} \cap \ldots \cap U_{v_p}$$

is nonempty and connected if $v_0,\ldots,v_p$ are the vertices of a p-simplex in N, and is empty otherwise. Thus the constant sheaf $\mathbb{C}_X$ satisfies

$$\mathbb{C}_X(U_K) = \begin{cases} \mathbb{C} & \text{if } K \text{ spans a p-simplex in } N, \\ 0 & \text{otherwise.} \end{cases}$$

So given a Čech cochain $\alpha \in C^p(\mathcal{U},\mathbb{C}_X)$, or equivalently given elements

$$\alpha_K \in \mathbb{C}_X(U_K)$$

for all $K \in I^{(p)}$, we can define a simplicial cochain

$$\phi(\alpha) \in (C_p^T(X))^*$$

by putting

$$\phi(\alpha)\cdot\tau = \pm\alpha_{\{v_0,\ldots,v_p\}} \in \mathbb{C}_X(U_{\{v_0,\ldots,v_p\}}) = \mathbb{C}$$

if τ is the p-simplex with vertices $v_0,\ldots,v_p$, and extending linearly. The sign depends on whether the orientation chosen for τ is the same as that chosen for $K = \{v_0,\ldots,v_p\}$. We thus get an isomorphism

$$\phi : C^p(\mathcal{U},\mathbb{C}_X) \to (C_p^T(X))^*$$

which respects the coboundary maps and hence induces an isomorphism

$$\check{H}^*(\mathcal{U},\mathbb{C}_X) \to H_T^*(X).$$

Since we can refine T to make $\mathcal{U}$ arbitrarily fine we get in the limit an isomorphism

$$\check{H}^*(X,\mathbb{C}_X) \cong H^*_{simp}(X).$$

§2.6 Cohomology of sheaves via derived functors

For more details see e.g. Godement [1], Hilton and Stammbach [1], Grothendieck [1], Cartan and Eilenberg [1].

2.6.1 Definition. A *covariant* (respectively *contravariant*) *functor* F from the category Sh(X) of sheaves on X to the category Ab of abelian groups is a rule which assigns to each sheaf F on X an abelian group $F(F)$ and to each map of sheaves $\phi: F \to G$ over X a homomorphism $F(\phi): F(F) \to F(G)$ (respectively $F(\phi): F(G) \to F(F)$) satisfying

i) $F(1_F) = 1_{F(F)}$ where 1_F and $1_{F(F)}$ are the identity maps on F and $F(F)$; and

ii) $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ (respectively $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$).

The functor F is *additive* if in addition

iii) when ϕ, ψ are both maps of sheaves $F \to G$ then

$$F(\phi + \psi) = F(\phi) + F(\psi)$$

where the map of sheaves $\phi + \psi \colon F \to G$ is defined by

$$(\phi + \psi)(U)(s) = \phi(U)(s) + \psi(U)(s)$$

for all U open in X and $s \in F(U)$.

Here + on the right hand side of this equation denotes the abelian group structure on $G(U)$.

The functor F is *exact* if in addition

iv) given a short exact sequence

$$0 \to F \xrightarrow{\phi} G \xrightarrow{\psi} H \to 0$$

of maps of sheaves over X the sequence

$$0 \to F(F) \xrightarrow{F(\phi)} F(G) \xrightarrow{F(\psi)} F(H) \to 0$$

(or $0 \to F(H) \to F(G) \to F(F) \to 0$ in the contravariant case) is an exact sequence of abelian groups.

Here a sequence

$$\ldots \to A_{i-1} \xrightarrow{\phi_i} A_i \xrightarrow{\phi_{i+1}} A_{i+1} \to \ldots$$

of maps of sheaves over X is called exact if

$$\text{im } \phi_i = \ker \phi_{i+1}$$

in the sense of sheaves for each i. A *short exaxt sequence* is an exact sequence

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0;$$

in other words we have $\ker \phi = 0$, $\operatorname{im} \phi = \ker \psi$ and $\operatorname{im} \psi = \mathcal{H}$.

An additive functor F is called *left* (respectively *right*) *exact* if the sequence of abelian groups obtained from any short exact sequence of sheaves via F is left (respectively right) exact: that is, we drop the condition that the second map should be surjective (respectively that the first map should be injective).

2.6.2 Exercise. The functor Γ_X from Sh(X) to Ab defined by

$$\Gamma_X(\mathcal{F}) = \mathcal{F}(X)$$

$$\Gamma_X(\phi) = \phi(X)$$

is a left exact additive covariant functor.

2.6.3 Remark. Let $\mathcal{F}$ be a sheaf over X and let U be an open subset of X. The space $\mathcal{F}(U)$ of sections of $\mathcal{F}$ over U is commonly written in several different ways, for example

$$\Gamma(U,\mathcal{F}) = \Gamma_U(\mathcal{F}) = \mathcal{F}(U).$$

If $s \in \mathcal{F}(U)$ is a section of $\mathcal{F}$ over U then the *support* $|s|$ of s is the closure in U of the subset

$$\{x \in U | s_x \neq 0\}$$

where s_x is the image of s in the stalk $\mathcal{F}_x$. If $|s|$ is compact then s is said to have compact support. The space of sections of $\mathcal{F}$ over U with compact support is denoted by

$\Gamma_c(U,F)$.

2.6.4 Definition. A sheaf I is called *injective* if the contravariant functor F: Sh(X) → Ab given by

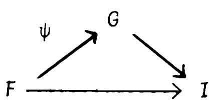
$$F(F) = \{\text{sheaf maps } F \to I\}$$

$$F(\psi) = \text{composition with } \psi$$

is exact. This functor is left exact for any sheaf I, so I is injective if and only if given

$$\psi : F \to G$$

with $\ker \psi = 0$, every sheaf map $F \to I$ extends to a map $G \to I$ such that the diagram



commutes.

2.6.5 Proposition. (Hartshorne [1, Chapter III, 2.3]). If A is a sheaf on X there is an exact sequence

$$0 \to A \to I^0 \xrightarrow{d_0} I^1 \xrightarrow{d_1} \cdots.$$

of sheaves such that I^j is injective for all j.

Such an exact sequence is called an *injective resolution* of A.

2.6.6 Definition. A *complex of sheaves* (or sheaf of cochain complexes) $A^\bullet$ is a set of sheaves $\{A^i | i \in \mathbf{Z}\}$ and sheaf maps $\{d^i: A^i \to A^{i+1} | i \in \mathbf{Z}\}$ satisfying $d^{i+1} \circ d^i = 0$ for all i. If the A^i are specified only in some range, e.g. for $i \geq 0$, then we set $A^i = 0$ for all other $i \in \mathbb{Z}$.

The ith *cohomology sheaf* $\underline{\underline{H}}^i(A^\bullet)$ of the complex $A^\bullet$ is the sheaf

$$\underline{\underline{H}}^i(A^\bullet) = \frac{\ker\ d^i}{\text{im}\ d^{i-1}} .$$

The stalk of the sheaf $\underline{\underline{H}}^i(A^\bullet)$ at any $x \in X$ is the ith cohomology group $H^i(A^\bullet_x)$ of the stalk complex $A^\bullet_x$ at x.

2.6.7 Definition. Let F be a left exact additive covariant functor from the category Sh(X) of sheaves on X to the category Ab of abelian groups. Then the ith right derived functor R^iF of F is defined as follows. For each sheaf A choose an injective resolution

$$0 \to A \to I^o \xrightarrow{d_o} I^1 \to \ldots.$$

of A and apply F to this sequence omitting A to get a complex

$$0 \to F(I^o) \xrightarrow{F(d_o)} F(I^1) \xrightarrow{F(d_1)} F(I^2) \to \ldots.$$

Note that this is a complex since

$$F(d_{i-1}) \circ F(d_i) = F(d_{i-1} \circ d_i) = F(0) = 0.$$

Let $R^iF(A)$ be the ith cohomology sheaf of this complex, i.e.

$$R^iF(A) = \frac{\ker F(d_i)}{\text{im}\ F(d_{i-1})}$$

where $d_{-1} = 0$.

It can be shown that $R^iF(A)$ is independent, up to canonical isomorphism, of the choice of injective resolution. Moreover R^iF is an additive functor from Sh(X) to Ab, and there is a natural isomorphism

$$R^oF(A) \cong F(A).$$

In addition, given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

of sheaves over X there is a natural homomorphism

$$\delta^i: R^iF(A'') \to R^{i+1}F(A')$$

for each i fitting into a long exact sequence

2.6.8 $$\ldots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to \ldots$$

Now we can give our last definition of cohomology.

2.6.9 <u>Definition</u>. If F is a sheaf on X let its ith cohomology group be defined by

$$H^i(X,F) = R^i\Gamma_X(F).$$

If X is a sufficiently nice topological space, for example an open subset of a compact manifold or an open subset of a complex projective variety, and F is a sufficiently nice sheaf on X, for example a constant sheaf, then

2.6.10 $$H^i(X,F) \cong H^i(X,F)$$

for all $i \geq 0$ (see e.g. Hartshorne [1, Chapter III, Exercise 4.11]).

2.6.11 <u>Remark</u>. It is easy to check that

$$H^o(X,F) = \Gamma_X(F) = F(X) = \check{H}^o(X,F).$$

§2.7 <u>Conclusion</u>

If X is an open subset of a complex projective variety we have four different definitions of the cohomology of X:

$X^*_{simp}(X)$ simplicial cohomology

$H^*_{sing}(X)$ singular cohomology

$\check{H}^*(X,\mathbb{C}_X)$ Čech cohomology

$H^*(X, \mathbb{C}_X)$ derived functor cohomology

and these are all canonically isomorphic. We shall denote them all simply by $H^*(X)$. If X is a nonsingular projective variety then $H^*(X)$ is also canonically isomorphic to the De Rham cohomology $H^*_{DR}(X)$ of X.

To complete this chapter it is necessary to mention a few important properties of the cohomology $H^*(X)$ of X. First of all $H^*(X)$ has a natural *ring structure* defined by the *cup product*

2.7.1 $H^i(X) \otimes H^j(X) \to H^{i+j}(X).$

The cup product is easiest to describe when X is nonsingular and $H^*(X)$ is identified with the De Rham cohomology $H^*_{DR}(X)$. Then an element of $H^i(X)$ is represented by a closed i-form α on X (i.e. an i-form α satisfying $d\alpha = 0$). Similarly an element of $H^j(X)$ is represented by a closed j-form β on X. The cup product of these elements of $H^i(X)$ and $H^j(X)$ is the element of $H^{i+j}(X)$ represented by the (i+j)-form $\alpha \wedge \beta$. It is easy to check that this is well-defined by using the formula

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta.$$

Alternatively we can define the cup product using singular cohomology (see e.g. Spanier [1, Chap. 5 §6]). This definition makes the singular cohomology of *any* topological space into a ring.

The existence of a natural ring structure is one of the properties of ordinary cohomology which does *not* carry over to intersection cohomology. Another such property is the homotopy invariance of ordinary cohomology: if $f: X \to Y$ is a homotopy equivalence between topological spaces then the induced map

2.7.2 $f^*: H^*_{sing}(Y) \to H^*_{sing}(X)$

is an isomorphism (see e.g. Spanier [1, Theorem 4.4.9]). We shall see that this is not true in general for intersection cohomology, but that intersection cohomology is a homeomorphism invariant (i.e. if f is a homeomorphism then it induces an isomorphism f^* on intersection cohomology).

A property of cohomology which carries over (though only in special circumstances) to intersection cohomology is the existence of relative cohomology.

Again this can be defined in different ways corresponding to the different definitions of cohomology: let us take singular cohomology. Suppose X is a topological space and Y is a subset of X. Then the space $S_i(Y)$ of singular i-chains in Y is a subspace of the space $S_i(X)$ of singular i-chains in X, so we can define

$$S_i(X,Y) = \frac{S_i(X)}{S_i(Y)} .$$

Then the boundary map $\partial: S_i(X) \to S_{i-1}(X)$ induces a boundary map

$$\partial: S_i(X,Y) \to S_{i-1}(X,Y).$$

We define the ith *relative (singular) homology group* of the pair (X,Y) to be

$$H_i^{sing}(X,Y) = \frac{\ker \partial : S_i(X,Y) \to S_{i-1}(X,Y)}{\operatorname{im} \partial : S_{i+1}(X,Y) \to S_i(X,Y)} .$$

This group fits into a long exact sequence of abelian groups

2.7.3 $$\ldots \to H_i^{sing}(Y) \to H_i^{sing}(X) \to H_i^{sing}(X,Y) \to H_{i-1}^{sing}(Y) \to \ldots$$

(Spanier [1, Chapter 4 §5]). Similarly we can define the ith relative (singular) cohomology groups $H^i_{sing}(X,Y)$ and these fit into a long exact sequence

2.7.4 $$\ldots \to H^{i-1}_{sing}(Y) \to H^i_{sing}(X,Y) \to H^i_{sing}(X) \to H^i_{sing}(Y) \to \ldots .$$

3 The definition of intersection homology

In this chapter we shall define intersection homology. We shall mainly be interested in the intersection homology of complex quasi-projective varieties, though intersection homology can be defined for a much larger class of topological spaces. For further details see Goresky and MacPherson [1] and [5], MacPherson [1], Borel [1].

§3.1 Quasi-projective complex varieties

Recall that a complex projective variety X is a subset

$$X \subseteq \mathbf{P}_N = \frac{\mathbb{C}^{N+1} - \{0\}}{\mathbb{C} - \{0\}}$$

of some complex projective space $\mathbf{P}_N$ which is defined by the vanishing of homogeneous polynomial equations.

A *quasi-projective complex variety* X is a subset of $\mathbf{P}_N$ of the form

$$X = Z - Y$$

where Z and Y are projective subvarieties of $\mathbf{P}_N$. That is, there exist homogeneous polynomials $f_1,\ldots,f_r$ and $g_1,\ldots,g_s$ in N+1 variables such that a point $(x_0:\ldots:x_N) \in \mathbf{P}_N$ belongs to X if and only if $f_j(x_0,\ldots,x_N) = 0$ for all j such that $1 \leq j \leq r$, and $g_j(x_0,\ldots,x_N) \neq 0$ for some j such that $1 \leq j \leq s$.

For example $\mathbb{C}^N$ can be identified with the quasi-projective variety

$$\{(x_0:\ldots:x_N) \in \mathbf{P}_N | x_0 \neq 0\},$$

via the mapping

$$(x_1,\ldots,x_N) \to (1:x_1:\ldots:x_N)$$

with inverse

$$(x_o:\dots:x_N) \to (\frac{x_1}{x_o},\dots,\frac{x_N}{x_o}).$$

Using the same mapping any subset of $\mathbb{C}^N$ defined by the vanishing of polynomials $f_1,\dots,f_m$ of degrees $d_1,\dots,d_m$ in N variables is identified with the quasi-projective variety

$$\{(x_o:\dots:x_N) \in \mathbf{P}_N | x_o \neq 0, \ \ \hat{f}_j(x_o,\dots,x_N) = 0, \ \ 1 \leq j \leq m\}$$

where

$$\hat{f}_j(x_o,\dots,x_N) = x_o^{d_j} f_j \ (\frac{x_1}{x_o},\dots,\frac{x_N}{x_o}).$$

Any quasi-projective variety X is an open subset of its closure in $\mathbf{P}_N$ which is a projective variety.

A point x of X is called *nonsingular* if there is an open neighbourhood U of x in $\mathbf{P}_N$ and homogeneous polynomials $f_1,\dots,f_m$ in N+1 variables such that

$$X \cap U = \{x_o:\dots:x_N) \in U \mid f_j(x_o,\dots,x_N) = 0, \ 1 \leq j \leq m\}$$

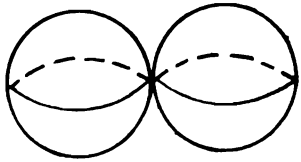
and the matrix of partial derivatives

$$\frac{\partial f_j}{\partial x_i}$$

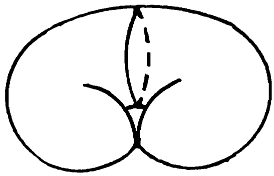
has rank m at x. Otherwise x is called a singular point of X. The set $X_{nonsing}$ of nonsingular points of X is a dense open subset of X, and each connected component is a complex submanifold of $\mathbf{P}_N$. The variety X is said to have *pure dimension* n if each connected component of $X_{nonsing}$ is a complex manifold of complex dimension n.

X is called *irreducible* if it cannot be expressed as the union of two closed subvarieties Y and Z unless either Y or Z is X itself.

3.1.1 Examples. The variety $X = \{(x:y:z) \in \mathbf{P}_2 | yz = 0\}$ is not irreducible.



The variety $X = \{(x:y:z) \in \mathbf{P}_2 | x^3 + y^3 = xyz\}$ is irreducible.



Any quasi-projective variety is the union of finitely many irreducible quasi-projective subvarieties $X_1,\ldots,X_k$ such that $X_i \nsubseteq X_j$ if $i \neq j$. The subvarieties $X_1,\ldots,X_k$ are called the irreducible components of X. It is easy to check that X has pure dimension n if and only if

$$(X_j)_{nonsing} = X_j - \{\text{singular points of } X_j\}$$

is a complex manifold of dimension n for each j.

A variety of pure dimension one is called a *curve*. A variety of pure dimension two is called a *surface*.

We shall mainly be interested in the intersection homology of quasi-projective varieties of pure dimension. However the definition can be extended to a much more general class of topological spaces (see below).

§3.2 Stratifications

What we need to define intersection homology is a suitable stratification. For quasi-projective varieties we use a Whitney stratification.

Let X be a quasi-projective variety of pure dimension n.

3.2.1 Definition. A *Whitney stratification* of X is given by a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$$

of X by closed subvarieties X_j such that for each j the locally closed subvariety

$$X_j - X_{j-1}$$

is either empty or is a *nonsingular* quasi-projective variety of pure dimension j. The connected components S_α of the subvarieties $X_j - X_{j-1}$ are called the *strata* of the stratification and are required to satisfy Whitney's conditions (a) and (b) (Whitney [1]).

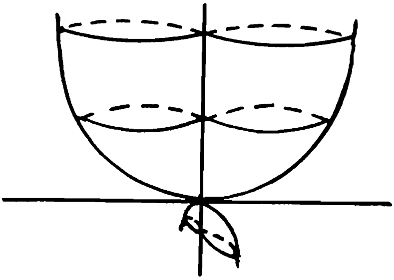
Whitney's condition (a). If a sequence of points $a_i \in S_\alpha$ tends to a point $c \in S_\beta$ then the tangent space $T_c S_\beta$ to c at S_β is contained in the limit of the tangent spaces $T_{a_i} S_\alpha$, provided that this limit exists.

Whitney's condition (b). If a sequence of points $b_i \in S_\beta$ and $a_i \in S_\alpha$ both tend to the same point $c \in S_\beta$ then the limit of the lines joining a_i to b_i is contained in the limit of the tangent spaces to S_α at a_i, provided that both limits exist.

Roughly speaking, the object of these conditions is to ensure that the normal structure to each stratum S_β is constant along S_β. They imply that for any points x and y on S_β there is a homeomorphism of X to itself which preserves all the strata and takes x to y (this follows from 3.3.2 below).

3.2.2 Example Consider the quasi-projective variety

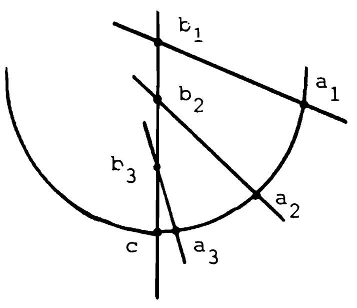
$$X = \{(x,y,z) \in \mathbb{C}^3 \mid x^4 + y^4 = xyz\}.$$



Let $X_2 = X$, let X_1 be the z-axis and let X_o be empty. This defines a stratification of X with two strata

$$S_\alpha = X - X_1 \text{ and } S_\beta = X_1,$$

both nonsingular. This stratification fails Whitney's condition (b). Consider sequences of points a_i and b_i in S_α and S_β chosen as in the diagram below so that the a_i are converging to c much faster than the b_i are. Then the lines joining a_i to b_i will tend to the vertical line through c, while the tangent spaces $T_{a_i}S_\alpha$ tend to the horizontal plane through c.



To obtain a Whitney stratification we must take $c \in X_o$. If X_2 and X_1 are defined as before and $X_o = \{c\}$ then we have a Whitney stratification of X.

3.2.3 Theorem. (Whitney [1, Theorem 19.2]). Any quasi-projective variety X of pure dimension n has a Whitney stratification.

We shall define the intersection homology of X using a fixed Whitney stratification.

§3.3 Topological pseudomanifolds

In fact in order for intersection homology to be defined it suffices that X be a topological pseudomanifold.

If L is a compact Hausdorff topological space then the open cone C(L) on L is the result of identifying the subset $L \times \{0\}$ of $L \times [0,1)$ to a single point (called the vertex of the cone).

3.3.1 Definition. An m-dimensional *topological stratification* of a paracompact Hausdorff topological space Y is given by a filtration

$$Y = Y_m \supseteq Y_{m-1} \supseteq \cdots \supseteq Y_1 \supseteq Y_0$$

of Y by closed subsets Y_j such that if $x \in Y_j - Y_{j-1}$ there exist a neighbourhood N_x of x in Y, a compact Hausdorff space L with an (m-j-1)-dimensional topological stratification

$$L = L_{m-j-1} \supseteq \cdots \supseteq L_1 \supseteq L_0$$

and a homeomorphism

$$\phi: N_x \to \mathbb{R}^j \times C(L),$$

where C(L) is the open cone on L, such that ϕ takes $N_x \cap Y_{j+i+1}$ homeomorphically onto

$$\mathbb{R}^j \times C(L_i) \subseteq \mathbb{R}^j \times C(L)$$

for $m - j - 1 \geq i \geq 0$, and ϕ takes $N_x \cap Y_j$ homeomorphically onto

$$\mathbb{R}^j \times \{\text{vertex of } C(L)\}.$$

This is an inductive definition. When m = 0 we require that Y should be a countable set with the discrete topology.

Y is called a *topological pseudomanifold* of dimension m if it has such a filtration satisfying

$$Y_{m-1} = Y_{m-2}$$

and $Y - Y_{m-1}$ is dense in Y.

Any manifold Y is a topological pseudomanifold with filtration

$$Y \supseteq \emptyset \supseteq \cdots \supseteq \emptyset.$$

3.3.2 Theorem. (Borel [1, IV §2]). Any Whitney stratification

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$$

of a complex quasi-projective variety X of pure dimension n makes X into a topological pseudomanifold of dimension 2n with filtration

$$Y_{2n} \supseteq Y_{2n-1} \supseteq \cdots \supseteq Y_o$$

defined by

$$Y_{2j} = Y_{2j+1} = X_j.$$

For the first definition of intersection homology which we shall give we actually need X to be more than a topological pseudomanifold with filtration Y_j as above. We also require X to have a triangulation which is compatible with the filtration (i.e. each Y_j is a union of simplices).

3.3.3 Theorem. (Lojasiewicz [1], [2], Goresky [1]). Let

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$$

be a Whitney stratification of a complex quasi-projective variety X of pure dimension n. Then there is a triangulation of X compatible with the stratification.

§3.4 Intersection chains and perversities

Now let us assume that X is a complex quasi-projective variety with a fixed Whitney stratification

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o.$$

Let $T: |N| \to X$ be a triangulation of X compatible with the stratification. Recall that $C_i^T(X)$ is the space of all (finite) simplicial i-chains of X with respect to T, and $C_i^T((X))$ is the space of all locally finite simplicial i-chains of X with respect to T.

3.4.1 Definition. The *support* $|\xi|$ of a simplicial i-chain

$$\xi = {}_{\sigma\in N^{(i)}} \xi_\sigma \sigma$$

is given by

$$|\xi| = \bigcup_{\xi_\sigma \neq 0} T(\sigma).$$

We are going to define subspaces $IC_i^{\bar{T}}(X)$ and $IC_i^T(X))$ of $C_i^T(X)$ and $C_i^T(X))$ whose elements will be those i-chains ξ such that the intersection of $|\xi|$ and X_j is "not too big" for each j. To make "not too big" precise we need the concept of a perversity.

3.4.2 Definition. A *perversity* is a finite sequence

$$\bar{p} = (p_2, p_3, \ldots, p_N)$$

of integers satisfying $p_2 = 0$ and $p_{k+1} = p_k$ or $p_{k+1} = p_k + 1$ for all $k \geq 2$.

Examples. The zero perversity $\bar{0} = (0,0,\ldots,0)$.

The top perversity $\bar{t} = (0,1,2,3,\ldots,N-2)$.

If $\bar{p}$ is a perversity, the *complementary perversity* is

$$\bar{t} - \bar{p} = (0, 1-p_1, 2-p_2, 3-p_3 \ldots)$$

Let us fix a perversity $\bar{p} = (p_2, p_3, \ldots, p_{2n})$.

3.4.3 Definition. Let $IC_i^{\bar{p},T}(X)$ be the subspace of $C_i^T(X)$ consisting of all those i-chains $\xi \in C_i^T(X)$ such that

$$\dim_{\mathbb{R}} |\xi| \cap X_{n-k} \leq i - 2k + p_{2k}$$

and

$$\dim_{\mathbb{R}} |\partial\xi| \cap X_{n-k} \leq i - 2k + p_{2k} - 1$$

for all $k \geq 1$. Note that by convention the empty set has dimension $-\infty$. Define $IC_i^{\bar{p},T}((X))$ similarly.

3.4.4 Remark. 2k appears in these inequalities because it is the *real* codimension of X_{n-k} in X. In the more general case of a piecewise linear pseudomanifold Y one requires

$$\dim_{\mathbb{R}} |\xi| \cap Y_{m-k} \leq i - k + p_k$$

and

$$\dim_{\mathbb{R}} |\partial\xi| \cap Y_{m-k} \leq i - k + p_k - 1.$$

For complex varieties it is only the p_{2k} terms which matter, so we could define a perversity for our purposes to be a sequence

$$\bar{p} = (p_2, p_4, p_6, \ldots, p_{2n})$$

satisfying $p_2 = 0$ and $p_{2k+2} = p_{2k}$, $p_{2k}+1$ or $p_{2k}+2$.

In particular for complex varieties we can consider the *middle perversity* $\bar{m}$ given by

$$m_{2k} = k - 1.$$

It does not matter what m_k is when k is odd. The middle perversity is special because it is its own complementary perversity.

3.4.5 Remark. Since the triangulation T is compatible with the stratification, the intersection $|\xi| \cap X_{n-k}$ is a union of simplices and hence has a well-defined (real) dimension. This would not be the case if we worked with singular chains instead of simplicial ones. However there are alternatives to simplicial chains which do work, such as semi-analytic chains (see e.g. Hardt [1]).

It is easy to check that if $\tilde{T}$ is a refinement of the triangulation T then the induced map

$$C_i^T((X)) \to C_i^{\tilde{T}}((X))$$

sends a chain $\xi \in C_i^T((X))$ to a chain with the same support as ξ. Hence it restricts to maps

$$IC_i^{\bar{p},T}((X)) \to IC_i^{\bar{p},\tilde{T}}((X))$$

and

$$IC_i^{\bar{p},T}(X) \to IC_i^{\bar{p},\tilde{T}}(X).$$

3.4.6 Definition. The space $IC_i^{\bar{p}}(X)$ of (finite) *piecewise linear intersection i-chains* is the direct limit of the $IC_i^{\bar{p},T}(X)$ over all triangulations T of X compatible with the stratification. The space $IC_i^{\bar{p}}((X))$ of locally finite piecewise linear intersection i-chains is defined similarly.

Thus a piecewise linear intersection i-chain is represented by an element of $IC_i^{\bar{p},T}(X)$ for some T, and

$$\eta \in IC_i^{\bar{p},T}(X) \quad \text{and} \quad \zeta \in IC_i^{\bar{p},\tilde{T}}(X)$$

represent the same element of $IC_i^{\bar{p}}(X)$ if and only if there is a common refinement $\tilde{\tilde{T}}$ of $\tilde{T}$ and T, compatible with the stratification, such that η and ζ induce the same element of

$$IC_i^{\bar{p},\tilde{\tilde{T}}}(X).$$

It is easy to check from the definition and the fact that $\partial^2 = 0$ that the boundary maps

$$\partial: C_i((X)) \to C_{i-1}((X))$$

induce boundary maps from $IC_i^{\bar{p}}((X))$ to $IC_{i-1}^{\bar{p}}((X))$ and from $IC_i^{\bar{p}}(X)$ to $IC_{i-1}^{\bar{p}}(X)$.

3.4.7 Definition. The ith *intersection homology group* of X with perversity $\bar{p}$ is

$$IH_i^{\bar{p}}(X) = \frac{\ker \partial: IC_i^{\bar{p}}(X) \to IC_{i-1}^{\bar{p}}(X)}{\operatorname{im} \partial: IC_{i+1}^{\bar{p}}(X) \to IC_i^{\bar{p}}(X)} .$$

$IH_i^{\bar{p},T}(X)$ and the intersection cohomology groups $IH_{\bar{p}}^i(X)$ and $IH_{\bar{p},T}^i(X)$ are defined similarly. In fact as for ordinary simplicial homology we have

$$IH_i^{\bar{p},T}(X) \cong IH_i^{\bar{p}}(X)$$

for any triangulation $T : |N| \to X$ compatible with the stratification.

Of course a priori $IH_i^{\bar{p}}(X)$ depends on the choice of stratification of X. We shall see later that in fact it is independent of this choice (Goresky and MacPherson [5, §4]).

The middle perversity

$$\bar{m} = (0,0,1,1,2,2,\ldots,n-1)$$

will be the most important for us so let us put

3.4.8 $\quad IH_i(X) = IH_i^{\bar{m}}(X), \quad IC_i(X) = IC_i^{\bar{m}}(X).$

etc.

3.4.9 Remark. We can also define intersection homology groups with *closed support* by using locally finite intersection chains:

$$IH_i^{c\ell}(X) = \frac{\ker \partial: IC_i((X)) \to IC_{i-1}((X))}{\operatorname{im} \partial: IC_{i+1}((X)) \to IC_i((X))} .$$

The definitions of intersection homology given in the literature are inconsistent, and often the groups $IC_i^{c\ell}(X)$ are called the intersection homology groups of X instead of the groups $IH_i(X)$ defined at 3.4.7. This is because the groups $IH_i^{c\ell}(X)$ fit better with the sheaf-theoretic approach to intersection homology (see Chapter 5) although the groups $IH_i(X)$ fit better with classical homology theory. Of course when X is compact there is a natural identification

$$IH_i(X) \cong IH_i^{c\ell}(X)$$

so it does not matter which definition is used.

§3.5 Simple examples of intersection homology

If X is nonsingular then $IH_*^{\bar{p}}(X) = H_*(X)$ for any perversity $\bar{p}$.

Suppose X is a quasi-projective variety of pure dimension n with one isolated singularity x, so that $X - \{x\}$ is nonsingular. Define a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$$

by $X_j = \{x\}$ if $0 \leq j < n$. This gives a Whitney stratification of X. We have

3.5.1 Proposition.

$$IH_i(X) = \begin{cases} H_i(X) & \text{if } i > n, \\ \text{Im}\,(H_i(X-\{x\}) \to H_i(X)) & \text{if } i = n, \\ H_i(X-\{x\}) & \text{if } i < n. \end{cases}$$

Proof.

$$IC_i(X) = \{\xi \in C_i(X) \mid \dim |\xi| \cap \{x\} \leq i - n - 1,$$
$$\dim |\partial\xi| \cap \{x\} \leq i - n - 2\}.$$

Hence if $i \leq n$ then

$$IC_i(X) = IC_i(X-\{x\}) = C_i(X-\{x\}),$$

whereas if $i \geq n + 2$ then

$$IC_i(X) = C_i(X).$$

Hence $IH_i(X) \cong H_i(X-\{x\})$ if $i \leq n - 1$ and $IH_i(X) = H_i(X)$ if $i \geq n + 2$. Moreover

$$\ker (\partial : IC_{n+1}(X) \to IC_n(X)) = \ker (\partial : C_{n+1}(X) \to C_n(X))$$

so

$$IH_{n+1}(X) \cong H_{n+1}(X).$$

Finally

$$\partial(IC_{n+1}(X)) = (\partial C_{n+1}(X)) \cap IC_n(X)$$

and

$$IC_n(X) = C_n(X-\{x\})$$

so

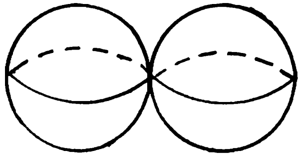
$$IH_n(X) \cong \mathrm{im}(H_n(X-\{x\}) \to H_n(X)).$$

As specific examples consider the curves

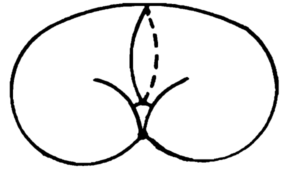
$$X_1 = \{(x,y,z) \in \mathbf{P}_2 \mid yz = 0\}$$

and

$$X_2 = \{(x,y,z) \in \mathbf{P}_2 \mid x^3 + y^3 = xyz\} .$$



X_1



X_2

We have

3.5.2 $\quad IH_o(X_1) \cong \mathbb{C} \oplus \mathbb{C}, \quad IH_o(X_2) \cong \mathbb{C},$

$IH_1(X_1) = 0, \quad IH_1(X_2) = 0,$

$IH_2(X_1) \cong \mathbb{C} \oplus \mathbb{C}, \quad IH_2(X_2) = \mathbb{C}.$

§3.6 Normalisations

Let Y be a topological pseudomanifold with filtration

$$Y = Y_m \supseteq Y_{m-1} = Y_{m-2} \supseteq \cdots \supseteq Y_o.$$

Then Y is called (topologically) *normal* if every $y \in Y$ has an open neighbourhood U in Y such that $U - Y_{m-2}$ is connected. Any manifold is normal.

3.6.1 Remark. A quasi-projective complex variety X is called normal if the stalk at x of the sheaf of regular functions on X is an integrally closed ring for every $x \in X$. It can be shown using Zariski's Main Theorem (Hartshorne [1, V Thm. 5.2]) that if a quasi-projective complex variety X is normal in the algebraic sense then it is topologically normal.

3.6.2 Proposition. (Goresky and MacPherson [1, 4.3]). Let X be a quasi-projective complex variety of pure dimension n. If X is (topologically) normal then there are canonical isomorphisms

$$IH_i^{\bar{t}}(X) \cong H_i(X)$$

and

$$IH_i^{\bar{0}}(X) \cong H^{2n-i}(X),$$

where $\bar{t} = (0,1,2,\dots,2n-2)$ is the top perversity and

$\bar{0} = (0,0,\dots,0)$ is the zero perversity.

Any quasi-projective variety X has a normalisation $\pi: \tilde{X} \to X$. Here $\tilde{X}$ is a normal quasi-projective variety and π is a finite-to-one surjective holomorphic map with a suitable universal property. π restricts to an isomorphism over the nonsingular part $X_{nonsing}$ of X.

3.6.3 Proposition. (Goresky and MacPerson [1, 4.2]). If $\pi: \tilde{X} \to X$ is a normalisation of X then there is a natural isomorphism

$$IH_i^{\bar{p}}(\tilde{X}) \cong IH_i^{\bar{p}}(X)$$

for any perversity $\bar{p}$.

The normalisation $\tilde{X}$ of a *curve* X is always non-singular (Hartshorne III Ex. 5.8), and hence by §3.5 and 3.6.3 we have

3.6.4 $\quad IH_i^{\bar{p}}(X) \cong H_i(\tilde{X})$

for every perversity $\bar{p}$. However this fails in general for higher-dimensional varieties.

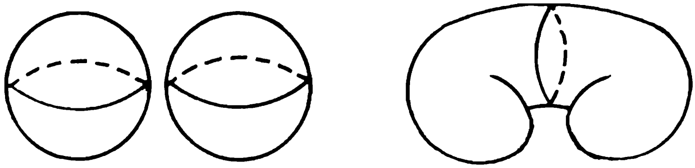
3.6.5 Examples. Consider the curves

$$X_1 = \{(x,y,z) \in \mathbf{P}_2 \mid yz = 0\}$$

and

$$X_2 = \{(x,y,z) \in \mathbf{P}_2 \mid x^3 + y^3 = xyz\}$$

again. The normalisation of X_1 is the disjoint union of two copies of $\mathbf{P}_1$ and the normalisation of X_2 is $\mathbf{P}_1$.



This fits with 3.5.2 and 3.6.4.

§3.7 Relative intersection homology

Suppose that $U \subseteq X$ is an open subset of a quasi-projective variety X. We can restrict the chosen Whitney stratification of X to U. Then any piecewise linear intersection i-chain

$$\xi \in IC_i(U)$$

can be regarded as an element of $IC_i^T(X)$ for a suitable triangulation T of X (depending on ξ). Hence we get an inclusion

$$IC_i(U) \to IC_i(X)$$

which commutes with the boundary maps. Thus there is a natural map

$$IH_i(U) \to IH_i(X).$$

Moreover there is an induced complex

$$IC_\bullet(X,U) = \frac{IC_\bullet(X)}{IC_\bullet(U)} .$$

The ith *relative intersection homology group* of the pair (X,U) is

3.7.1 $$IH_i(X,U) = \frac{\ker \partial: IC_i(X,U) \to IC_{i-1}(X,U)}{\operatorname{im} \partial: IC_{i+1}(X,U) \to IC_i(X,U)} .$$

Just as for ordinary homology there is a long exact sequence

3.7.2 $\quad \ldots \to IH_i(U) \to IH_i(X) \to IH_i(X,U) \to IH_{i-1}(U) \to \ldots$

(Goresky and MacPherson [4, 1.3]).

3.7.3 Warning. The ordinary relative homology groups $H_i(X,A)$ are defined for any subset A of X, but this fails for intersection homology.

§3.8 Intersection homology is not a homotopy invariant

Like ordinary homology, intersection homology will turn out to be a *topological invariant*. In other words any homeomorphism f: X → Y will induce an isomorphism

$$f_*: IH_*(X) \to IH_*(Y).$$

However in contrast to ordinary homology, an arbitrary continuous map f: X → Y does not in general induce a homomorphism $f_*:IH_*(X) \to IH_*(Y)$. Moreover intersection homology is *not* a homotopy invariant. That is, the existence of a homotopy equivalence between X and Y does not necessarily imply that $IH_*(X)$ and $IH_*(Y)$ are isomorphic. This can be seen by doing a local calculation. For any quasi-projective variety X and any x ∈ X we shall give a description of the intersection homology of a neighbourhood of x in X which is contractible (i.e. homotopy equivalent to a point). It will be clear that in general this intersection homology is not the same as the intersection homology of a point.

For any $x \in X_j - X_{j-1}$ we know that there is a compact pseudomanifold L_x (called the *link* of $X_j - X_{j-1}$ at x) and a neighbourhood N_x of x in X which is homeomorphic in a stratification-preserving way to the product

$$\mathbb{C}^j \times C(L_x)$$

of $\mathbb{C}^j$ and the cone

$$C(L_x) = (L_x \times [0,1))/L_x \times \{0\}$$

(see §3.3). We can take N_x to be the intersection of $X \subseteq \mathbf{P}_N$ with any small ball centre x in $\mathbf{P}_N$, and L_x to be the intersection of X with a small sphere centre x in a submanifold of $\mathbf{P}_N$ which meets X_j transversely at x (Borel [1, IV §2]).

3.8.1 Proposition. (Goresky and MacPherson [5, 2.4]).

(a) $$IH_i(N_x) \cong \begin{cases} 0 & \text{if } i \geq n - j, \\ IH_i(L_x) & \text{if } i < n - j. \end{cases}$$

(b) $$IH_i(X,X-\{x\}) = \begin{cases} IH_{i-2j-1}(L_x) & \text{if } i > n + j, \\ 0 & \text{if } i \leq n + j. \end{cases}$$

Sketch proof. One needs special cases of the excision theorem and the Künneth theorem (Goresky and MacPherson [4, 1.5 and 1.6]).

3.8.2 Excision theorem. $IH_i(X,X-\{x\}) \cong IH_i(N_x,N_x - \{x\})$.

3.8.3 Künneth theorem. $IH_i(\mathbb{C}^j \times C(L_x)) \cong IH_i(C(L_x))$ and

$$IH_i(\mathbb{C}^j \times C(L_x),\ \mathbb{C}^j \times C(L_x) - \{(0,v)\})$$

$$\cong IH_{i-2j}(C(L_x),\ C(L_x) - \{v\})$$

where v is the vertex of the cone $C(L_x)$.

The proofs are easy adaptations of the proofs in the case of ordinary homology (Spanier [1, Corollary 4.6.5 and Theorem 5.3.10]).

Then one has

$$IH_i(N_x) \cong IH_i(\mathbb{C}^j \times C(L_x)) \cong IH_i(C(L_x))$$

and

$$\begin{aligned} IH_i(X,X-\{x\}) &\cong IH_i(N_x,N_x-\{x\}) \\ &\cong IH_i(\mathbb{C}^j \times C(L_x),\ \mathbb{C}^j \times C(L_x) - \{(0,v)\}) \\ &\cong IH_{i-2j}(C(L_x),C(L_x) - \{v\}). \end{aligned}$$

Suppose $i \leq n - j$ and $\xi \in IC_i(C(L_x))$.
Then

$$\dim |\xi| \cap \{v\} \leq i - (n-j) - 1 < 0$$

so $v \notin |\xi|$. Hence

$$IC_i(C(L_x), \ C(L_x) - \{v\}) = 0$$

when $i \leq n - j$. On the other hand if $i \geq n - j$ and $\xi \in IC_i(C(L_x))$ then one can form an intersection $i+1$-chain $c(\xi) \in IC_{i+1}(C(L_x))$, the "cone on ξ", such that ξ is the boundary of $c(\xi)$. Hence

$$IH_i(C(L_x)) = 0$$

if $i \geq n - j$. We have shown that if $i \leq n - j$ then

$$IH_i(C(L_x), C(L_x) - \{v\}) = 0$$

and so by the long exact sequence 3.7.2 if $i < n - j$ then

$$IH_i(C(L_x)) \cong IH_i(C(L_x) - \{v\}).$$

But

$$IH_i(C(L_x) - \{v\}) \cong IH_i(L_x \times (0,1))$$

$$\cong IH_i(L_x)$$

by the Künneth theorem. Finally because $IH_i(C(L_x)) = 0$ for $i \geq n - j$ the long exact sequence 3.7.2 shows that

$$IH_i(C(L_x), C(L_x) - \{v\}) \cong IH_{i-1}(C(L_x) - \{v\})$$

$$\cong IH_{i-1}(L_x)$$

when $i > n - j$. The proof of the proposition follows from combining these results.

§3.9 Intersection homology with local coefficients

Let X be any topological space.

3.9.1 Definition. A (complex) *local coefficient system* L on X is given by data consisting of a finite dimensional complex vector space L_x for each $x \in X$ and an isomorphism

$$\phi^* : L_{\phi(0)} \to L_{\phi(1)}$$

for any continuous path ϕ: $[0,1] \to X$ in X, satisfying

(i) $\phi^* = \psi^*$ when ϕ and ψ are homotopic relative to fixed end points, and

(ii) $(\phi.\psi)^* = \psi^* \circ \phi^*$ if $\phi(1) = \psi(0)$ and $\phi\cdot\psi$ is the composite path from $\phi(0)$ to $\psi(1)$.

3.9.2 Remark. Equivalently (if X is connected) L is given by a sheaf on X which is locally isomorphic to a constant sheaf defined by a finite dimensional vector space, or a representation of the fundamental group $\pi_1(X)$ on a finite dimensional vector space, or alternatively by a complex vector bundle on X with a flat connection.

We define a flat section of L to be a map

$$g: X \to \coprod_{x \in X} L_x$$

such that $g(x) \in L_x$ for all $x \in X$, and if ϕ is a path from x to y in X then

$$\phi^*(g(x)) = g(y).$$

The restriction of L to any simply connected subset Y of X is trivial, in the sense that the isomorphism $\phi^* : L_x \to L_y$ induced by a path ϕ from x to y in Y is independent of the choice of path. In particular if T: $|N| \to X$ is a triangulation the restriction of L to any i-simplex $T(\sigma)$ where $\sigma \in N^{(i)}$ is trivial, so if L_σ is the space of all flat sections of L over $T(\sigma)$ then the restriction maps

$$\rho_x^\sigma : L_\sigma \to L_x$$

are isomorphisms for all $x \in T(\sigma)$. Moreover if $\tilde{\sigma}$ is a face of σ and $x \in \tilde{\sigma}$ then the composition

$$\rho_{\tilde{\sigma}}^\sigma = (\rho_x^{\tilde{\sigma}})^{-1} \circ \rho_x^\sigma : L_\sigma \to L_{\tilde{\sigma}}$$

is independent of x. Let $C_i^T(X,L)$ be the vector space consisting of all formal expressions of the form

$$\xi = \sum_{\sigma \in N^{(i)}} \ell_\sigma \, \sigma$$

with $\ell_\sigma \in L_\sigma$ and only finitely many ℓ_σ nonzero. Define

$$\partial : C_i^T(X,L) \to C_{i-1}^T(X,L)$$

by

$$\partial(\xi) = \sum_{\sigma \in N^{(i)},\ \tilde{\sigma} \text{ face of } \sigma} \pm \rho_{\tilde{\sigma}}^\sigma (\ell_\sigma) \tilde{\sigma}$$

where the sign $\pm$ is defined as before depending on a fixed choice of orientations.

Taking direct limits over triangulations we get the space of simplicial i-chains with coefficients in L

$$C_i(X,L).$$

The ith homology group of X with coefficients in L is by definition the quotient

3.9.3 $$H_i(X,L) = \frac{\ker \partial : C_i(X,L) \to C_{i-1}(X,L)}{\operatorname{im} \partial : C_{i+1}(X,L) \to C_i(X,L)} .$$

Now suppose that X is a quasi-projective variety with a fixed Whitney stratification

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o.$$

To make this procedure work for intersection homology we only need the local coefficient system L to be defined on the nonsingular open subset $X - X_{n-1}$ of X, not on X itself. This is because the allowability conditions on intersection i-chains ξ mean that if $\sigma \in N^{(i)}$ and the coefficient of ξ indexed by σ is non-zero then

$$\sigma^o \subseteq X - X_{n-1}$$

and

$$\tilde{\sigma}^o \subseteq X - X_{n-1}$$

for any face $\tilde{\sigma} \in N^{(i-1)}$ of σ. Thus we can use this procedure to define the intersection homology groups $IH_i(X,L)$ of X with coefficients in L for any local coefficient system L on $X - X_{n-1}$.

§3.10 Generalised Poincaré duality.

One of the reasons that the middle perversity is particularly important is the generalised Poincaré duality theorem for intersection homology.

3.10.1 Generalised Poincaré duality. (Goresky and MacPherson [1, §3.3]). Suppose that X is a projective variety of pure dimension n. Then if $\bar{p}$ and $\bar{q}$ are complementary perversities and i+j = 2n there is a nondegenerate pairing

$$IH_i^{\bar{p}}(X) \otimes IH_j^{\bar{q}}(X) \to \mathbb{C}.$$

Taking $\bar{p} = \bar{q} = \bar{m}$ we get a nondegenerate pairing

$$IH_i(X) \otimes IH_j(X) \to \mathbb{C}.$$

More precisely any $a \in IH_i(X)$ and $b \in IH_j(X)$ can be represented by $\xi \in IC_i(X)$ and $\eta \in IC_j(X)$ such that the supports $|\xi|$ and $|\eta|$ meet only in $X-X_{n-1}$ (which is nonsingular) and they meet in finitely many points. The number of these points counted with appropriate weights depending on the coefficients of the chains ξ and η is a complex number which is independent of the choice of ξ and η and is denoted $a \cap b$. Moreover if $a \neq 0$ there exists some b such that $a \cap b \neq 0$.

Poincaré duality for $IH_*(X)$ can be interpreted as the statement that there is a natural isomorphism

3.10.2 $$IH_i(X) \cong (IH_{2n-i}(X))^* \cong IH^{2n-i}(X)$$

for all i. Thus it is equivalent to the existence of a natural nondegenerate pairing between intersection cohomology groups

3.10.3 $$IH^i(X) \otimes IH^{2n-i}(X) \to \mathbb{C}.$$

In fact if X is normal then for any i and j there are natural intersection pairings

3.10.4 $$IH_i(X) \otimes IH_j(X) \to H_{i+j-2n}(X)$$

and

3.10.5 $$IH^i(X) \otimes IH^j(X) \to H_{2n-i-j}(X)$$

(Goresky and MacPherson [1, §2.3]). However we cannot replace the homology groups on the right hand side by intersection homology or cohomology groups. There is *no natural ring structure* on $IH^*(X)$.

3.10.6 Remark. If X is topologically normal (see §3.6) then

$$IH_i^{\bar{t}}(X) \cong H_i(X)$$

and

$$IH_i^{\bar{o}}(X) \cong H^{2n-i}(X)$$

and generalised Poincaré duality between $IH_*^{\bar{t}}(X)$ and $IH_*^{\bar{o}}(X)$ becomes the ordinary duality

$$H_i(X) \otimes H^i(X) \to \mathbb{C}.$$

3.10.7 Definition. The intersection cohomology of X with *compact supports* is defined as

$$IH_c^*(X) = \varinjlim_{K \subseteq X,\, K \text{ compact}} IH^*(X, X-K).$$

It is the cohomology theory corresponding to intersection homology with closed supports (see 3.4.9). That is, there are natural isomorphisms

$$IH_c^i(X) \cong (IH_i^{c\ell}(X))^*$$

for all i. Of course when X is compact itself then

$$IH_c^*(X) \cong IH^*(X).$$

Poincaré duality can be generalised from the case when X is compact (i.e. projective) to the case when X is any quasi-projective complex variety by using intersection cohomology with compact supports.

3.10.8 (Poincaré duality). If X is a quasi-projective complex variety of dimension n then there is a natural perfect pairing

$$IH^i(X) \otimes IH_c^{2n-i}(X) \to \mathbb{C}$$

for all $0 \le i < 2n$ (Goresky and MacPherson [5, §5.3], Borel [1,I 4.3]).

3.10.9 Remark. We noted in §1.1 that the cohomology of a nonsingular complex projective variety satisfies

(i) Hodge decomposition,
(ii) Poincaré duality,
(iii) Lefschetz hyperplane theorem,
(iv) Hard Lefschetz theorem,
(v) Hodge signature theorem.

The intersection cohomology (with respect to the middle perversity) of a singular complex projective variety is believed to have the same five properties. There is not yet a published proof of (i) or (v), though see Saito [1] and [2]. However the diligent reader may find proofs of (ii), (iii), (iv) in Goresky and MacPherson [1], [4], [5] and Beilinson, Bernstein and Deligne [1].

4 L^2-cohomology and intersection cohomology

Recall from §1.2 that the L^2-cohomology of a projective variety $X \subseteq \mathbf{P}_n$ is defined by

$$H^i_{(2)}(X) = \frac{\{\omega \in L^i(X-\Sigma) \mid d\omega = 0\}}{\{\eta \in L^i(X-\Sigma) \mid \exists \zeta \in L^{i-1}(X-\Sigma),\ d\zeta = \eta\}}$$

where Σ is the set of singular points of X and

$$L^i(X-\Sigma) = \{\omega \in A^i(X-\Sigma) \mid \int_{X-\Sigma} ||\omega||^2 < \infty\}$$

is the space of square-integrable differential i-forms on $X-\Sigma$. Here the norm $||\omega||$ of ω is defined using the restriction to $X-\Sigma$ of the standard Kähler metric (the Fubini-Study metric) on $\mathbf{P}_n$.

When the singularities of X are particularly simple it is possible to show that

$$H^*_{(2)}(X) \cong IH^*(X)$$

by doing a local calculation for L^2-cohomology and comparing it with the local calculation (3.8.1) for intersection cohomology. It is conjectured that such an isomorphism holds in general. (For further details see Cheeger [1], [2] and [3] and Cheeger, Goresky and MacPherson [1].)

§4.1 Isolated conical singularities

4.1.1 Definition. Two Riemannian metrics g and h on a manifold Y are called *quasi-isometric* if there exists a positive constant K such that at every point y of Y the inner products g_y and h_y on the tangent space T_yY satisfy the inequalities:

$$K^{-1} g_y \le h_y \le K g_y.$$

The norms defined by the metrics g and h then satisfy corresponding inequalities at each point. In particular if ω is a differential i-form on Y then ω is square-integrable in the sense that

$$\int_Y \|\omega\|^2 < \infty$$

with respect to the norm defined by the metric g, if and only if ω is square-integrable with respect to the norm defined by the metric h. Thus the L^2-cohomology groups of Y defined using two quasi-isometric metrics are the same.

4.1.2 Definition. If Y is a compact manifold with Riemannian metric g_Y let $C^*(Y)$ be the punctured cone

$$C^*(Y) = C(Y) - \{\text{vertex}\} = (0,1) \times Y$$

with Riemannian metric

$$g = dt \otimes dt + t^2 \, \pi^* g_Y$$

where t is the standard coordinate on the open interval (0,1) and $\pi:(0,1) \times Y \to Y$ is the projection. (Recall that the cone C(Y) is obtained from the product $[0,1) \times Y$ by identifying the points of $\{0\} \times Y$ to give a single point which is the vertex of the cone).

Note that any differential i-form ξ on $C^*(Y)$ can be written uniquely as

4.1.3 $\quad \xi = \eta + dt \wedge \zeta$

where η and ζ are differential forms which do not involve dt. In other words with respect to (real) local coordinates $(y_1,\ldots,y_m)$ on Y we can write

4.1.4 $\quad \eta(t,y) = \sum_{\alpha \in I(i)} \eta_\alpha(t,y) \, dy^\alpha$

where I(i) is the set of all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_i)$ such that $1 \leq \alpha_1 < \ldots < \alpha_i \leq m$, where

$$dy^{\alpha} = dy^{\alpha_1} \wedge \dots \wedge dy^{\alpha_i}$$

and where η_{α} is a smooth function on $(0,1) \times Y$. Similarly

$$4.1.5 \quad \zeta(t,y) = \sum_{\alpha \in I(i-1)} \zeta_{\alpha}(t,y)\, dy^{\alpha}.$$

Thus for fixed $t \in (0,1)$ we can regard $\eta(t,y)$ and $\zeta(t,y)$ as defining differential forms on Y. The Riemannian metric on $C^*(Y)$ is defined in such a way that the norm of ξ is given by

$$4\ 1.6 \quad \|\xi(t,y)\|^2 = t^{-2i}\|\eta(t,y)\|_Y^2 + t^{-2(i-1)}\|\zeta(t,y)\|_Y^2$$

where $\|\cdot\|_Y$ is the norm induced by the metric g_Y on Y. The factor t^{-2i} occurs because $\eta(t,y)$ lies in the ith exterior power of the dual of the tangent space to $C^*(Y)$ at the point (t,y).

4.1.7 Definition. Let $X \subseteq \mathbf{P}_n$ be a quasi-projective variety with isolated singularities. Let

$$\Sigma = \{x_1,\dots,x_q\}$$

be the set of singular points of X. We say that X has *isolated conical singularities* if there exist compact Riemannian manifolds $Y_1,\dots,Y_q$ and disjoint open neighbourhoods $U_1,\dots,U_q$ of $x_1,\dots,x_q$ in X such that U_j is homeomorphic to the cone $C(Y_j)$ and $U_j-\{x_j\}$ is quasi-isometric to the punctured cone $C^*(Y_j)$ for $1 \leq j \leq q$. Here $U_j-\{x_j\}$ is given the restriction of the Fubini-Study metric on $\mathbf{P}_n$ and $C^*(Y_j)$ is given the metric defined at 4.1.2.

§4.2 The L^2-cohomology of a punctured cone

In order to find the L^2-cohomology of a variety with isolated conical singularities we first need to calculate the L^2-cohomology of a punctured cone $C^*(Y)$ in terms of the cohomology of the compact Riemannian manifold Y.

4.2.1 Proposition. (Cheeger [2]). Let Y be a compact Riemannian manifold of dimension m and let $C^*(Y)$ be the punctured cone on Y with the metric defined at 4.1.2. Then

$$H^i_{(2)}(C^*(Y)) \cong \begin{cases} H^i(Y) & \text{if } i \leq m/2, \\ 0 & \text{if } i > m/2. \end{cases}$$

4.2.2 Remark. Note that the L^2-cohomology of Y is the same as its De Rham cohomology (since Y is compact) and hence there is a natural isomorphism

$$H^i_{(2)}(Y) \cong H^i(Y).$$

Sketch proof of Proposition 4.2.1. As in §4.1 let

$$\pi: C^*(Y) = (0,1) \times Y \to Y$$

be the projection. If $\omega \in A^i(Y)$ is a differential i-form on Y then with respect to local coordinates $(y_1,\ldots,y_m)$ we can write

$$\omega(y) = \sum_{\alpha \in I(i)} \omega_\alpha(y)\, dy^\alpha .$$

The i-form $\pi^*\omega$ on $C^*(Y)$ is then defined in local coordinates $(t,y_1,\ldots,y_m)$ by the same formula

$$\pi^*\omega(t,y) = \sum_{\alpha \in I(i)} \omega_\alpha(y)\, dy^\alpha.$$

By 4.1.6 we have

$$\|\pi^*\omega(t,y)\|^2 = t^{-2i} \|\omega(y)\|^2_Y.$$

Moreover the volume form on $C^*(Y)$ at a point (t,y) differs from the volume form on Y at y by a factor of t^m so

$$\int_{C^*(Y)} \|\pi^*\omega\|^2 = \int_0^1 \int_Y t^{-2i} \|\omega\|^2\, t^m\, dt.$$

Since Y is compact it follows that $\pi^*\omega$ is square integrable if and only if $\omega = 0$ or

$$\int_0^1 t^{m-2i} dt < \infty \quad .$$

Therefore if $m - 2i > - 1$ or equivalently

$$i \leq m/2$$

then π^* restricts to a map

$$\pi^* : L^i(Y) \to L^i(C^*(Y))$$

which commutes with d and hence induces a natural map

4.2.3 $\pi^* : H^i(Y) \cong H^i_{(2)}(Y) \to H^i_{(2)}(C^*(Y))$.

We shall show that this map is an isomorphism for all $i \leq m/2$.

Given a differential i-form ξ on $C^*(Y)$ write

$$\xi = \eta + dt \wedge \zeta$$

as at 4.1.3. There is an i-form $\partial\eta/\partial t$ on $C^*(Y)$ defined in local coordinates $(y_1,\ldots,y_m)$ by

$$\frac{\partial\eta}{\partial t}(t,y) = \sum_{\alpha\in I(i)} \frac{\partial\eta_\alpha}{\partial t}(t,y)\, dy^\alpha$$

in the notation of 4.1.4. Similarly there is an (i-1)-form $\partial\zeta/\partial t$ given by

$$\frac{\partial\zeta}{\partial t}(t,y) = \sum_{\alpha\in I(i-1)} \frac{\partial\zeta_\alpha}{\partial t}(t,y)\, dy^\alpha .$$

We can define

$$d_Y : A^i(C^*(Y)) \to A^{i+1}(C^*(Y))$$

in local coordinates $(y_1,\ldots,y_m)$ by

$$d_Y\xi(t,y) = \sum_{1\le j\le m} \sum_{\alpha\in I(i)} \frac{\partial\eta_\alpha}{\partial y_j}(t,y)\, dy_j \wedge dy^\alpha$$

$$+ \sum_{1\le j\le m} \sum_{\alpha\in I(i-1)} \frac{\partial\zeta_\alpha}{\partial y_j}(t,y)\, dy_j \wedge dt \wedge dy^\alpha.$$

Then

$$d_Y\xi = d_Y\eta - dt \wedge d_Y\zeta,$$

and

$$d\xi = d_Y\xi + dt \wedge \frac{\partial\eta}{\partial t}$$

$$= d_Y\eta + dt \wedge (\frac{\partial\eta}{\partial t} - d_Y\zeta).$$

Now fix $s \in (0,1)$ and define

$$H: A^i(C^*(Y)) \to A^{i-1}(C^*(Y))$$

in local coordinates $(y_1,\ldots,y_m)$ by

$$(H\xi)(t,y) = \sum_{\alpha\in I(i-1)} (\int_s^t \zeta_\alpha(\tau,y)d\tau)dy^\alpha,$$

where $\xi = \eta + dt \wedge \zeta$ as before. We shall write this more conveniently as

$$H\xi = \int_s^t \zeta.$$

Then

$$dH\xi = d_Y \int_s^t \zeta + dt \wedge \frac{\partial}{\partial t} \int_s^t \zeta$$

$$= \int_s^t d_Y\zeta + dt \wedge \zeta.$$

Also

$$H d\xi = H(d_Y\eta + dt \wedge (\frac{\partial\eta}{\partial t} - d_Y\zeta))$$

$$= \int_s^t (\frac{\partial\eta}{\partial t} - d_Y\zeta)$$

$$= \eta - \pi^*(\eta^{(s)}) - \int_s^t d_Y\zeta$$

where $\eta^{(s)} \in A^i(Y)$ is given in local coordinates $(t,y_1,\ldots,y_m)$ by

$$\eta^{(s)}(y) = \sum_{\alpha \in I(i)} \eta_\alpha(s,y)\, dy^\alpha.$$

Thus

4.2.4 $$dH\xi + Hd\xi = dt \wedge \zeta + \eta - \pi^*(\eta^{(s)})$$

$$= \xi - \pi^*(\eta^{(s)}).$$

Now if ξ is a square-integrable i-form then

$$\int_{C^*(Y)} \|\xi\|^2 = \int_0^1 \int_Y (t^{-2i} \|\eta^{(t)}\|_Y^2 + t^{-2(i-1)} \|\zeta^{(t)}\|_Y^2)\, t^m\, dt$$

is finite. Since $H\xi$ is an (i-1)-form

$$\int_{C^*(Y)} \|H\xi\|^2 = \int_0^1 \int_Y t^{-2(i-1)} \|\int_s^t \zeta^{(\tau)} d\tau\|_Y^2\, t^m\, dt.$$

Using the Cauchy-Schwarz inequality and reversing the order of integration we find that if $i \leq m/2$ then

$$\int_{C^*(Y)} \|H\xi\|^2 \leq \int_0^1 \int_Y t^{m-2i+2} \left| \int_s^t \|\zeta^{(\tau)}\|_Y^2\, d\tau \right| dt$$

$$= \int_0^s \int_Y \|\zeta^{(\tau)}\|_Y^2 \int_0^\tau t^{m-2i+2}\, dt\, d\tau + \int_s^1 \int_Y \|\zeta^{(\tau)}\|_Y^2 \int_\tau^1 t^{m-2i+2}\, dt d\tau$$

$$\le \frac{1}{m-2i+3}\left(\int_0^s \int_Y \|\zeta^{(\tau)}\|_Y^2 \tau^{m-2i+3}\, d\tau \right.$$

$$\left. + \int_s^1 \int_Y \|\zeta^{(\tau)}\|_Y^2\, d\tau \right)$$

$$\le \left(\frac{1 + s^{-m+2i-2}}{m-2i+3}\right) \int_{C^*(Y)} \|\xi\|^2 < \infty .$$

Hence $H\xi$ is square-integrable.

We have shown that if ξ is a square-integrable i-form on $C^*(Y)$ and $i \le m/2$ then $H\xi$ is square-integrable and

4.2.5 $\quad \xi = dH\xi + Hd\xi + \pi^*(\eta^{(s)}).$

Therefore if $d\xi = 0$ then

$$\xi \in d(L^{i-1}(C^*(Y))) + \pi^*(L^i(Y))$$

so

$$\pi^* : H^i(Y) \to H^i_{(2)}(C^*(Y))$$

is surjective for $i \le m/2$. Moreover since $d^2 = 0$ we have by 4.2.5

$$d\xi = d(Hd\xi) + d\pi^*(\eta^{(s)})$$

$$= d(Hd\xi) + \pi^*(d\eta^{(s)}).$$

It comes straight from the definition of H that $Hd\xi = 0$ if $d\xi \in \pi^*(L^i(Y))$, and hence it follows easily that $\pi^* : H^i(Y) \to H^i_{(2)}(C^*(Y))$ is injective for $i \le m/2$.

It remains to show that $H^i_{(2)}(C^*(Y)) = 0$ for $i > m/2$. The Cauchy-Schwarz inequality tells us that if ϕ is a square-integrable i-form on $C^*(Y)$ and $0 < a < b < 1$ then

$$\left(\int_a^b \int_Y \|\phi^{(t)}\|_Y^2 \, dt\right)^2$$

$$\leq \left(\int_0^1 \int_Y t^{m-2i} \|\phi^{(t)}\|_Y^2 dt\right) \left(\int_a^b \int_Y t^{2i-m} \, dt\right)$$

$$= \left(\int_{C*(Y)} \|\phi\|^2\right) \left(\int_Y 1\right) \left(\frac{b^{2i-m+1} - a^{2i-m+1}}{2i-m+1}\right).$$

Therefore the integral $\int_0^1 \int_Y \|\phi^{(t)}\|_Y \, dt$ exists if $i \geq m/2$, and so for almost all $y \in Y$ the integral

$$\int_0^t \phi = \int_0^t \phi^{(\tau)} \, d\tau$$

exists for all $t \in (0,1)$. The idea is now that if $\xi = \eta + dt \wedge \zeta$ is a square-integrable i-form and $i-1 \geq m/2$ then we define

$$H^o\xi = \int_0^t \zeta.$$

The argument used above can be easily modified to show that $H^o\xi$ is square-integrable and that

$$\xi = dH^o\xi + H^o d\xi.$$

In particular if $d\xi = 0$ then $\xi = dH^o\xi$.
From this it can be deduced that $H^i_{(2)}(C^*(Y)) = 0$ when $i-1 \geq m/2$, though technical difficulties arise because $H^o\xi$ is not necessarily differentiable.

The only case we have not yet covered is when m is odd and $i = (m+1)/2$. This case is more delicate but it can be shown $H^i_{(2)}(C^*(Y)) = 0$ in this case also (see Cheeger [2]).

§4.3 The natural map $H^*_{(2)}(X) \to IH^*(X)$.

Let X be a projective variety, and let β be a square-integrable differential i-form on

$$X_{nonsing} = X - \Sigma$$

such that $d\beta$ is also square-integrable. Then one can show that for almost all intersection chains $\xi \in IC_i(X)$ the integral

$$\int_\xi \beta$$

exists and Stokes' theorem

$$\int_\xi d\beta = \int_{\partial\xi} \beta$$

is satisfied. (Note that the support of an intersection chain $\xi \in IC_i(X)$ is never contained in the singular set Σ where β is not defined: it meets Σ in a subset of dimension at most i-2). In this way integration can be used to define a natural pairing

4.3.1 $\quad H^i_{(2)}(X) \otimes IH_i(X) \to \mathbb{C}$

or equivalently a natural map

4.3.2 $\quad H^i_{(2)}(X) \to (IH_i(X))^* = IH^i(X).$

When X is nonsingular this map is the De Rham isomorphism

$$H^i_{DR}(X) \to H^i(X)$$

(Griffiths and Harris [1, p. 44]).

Now suppose that X is a projective variety with isolated conical singularities, and let $n = \dim_{\mathbb{C}} X$.

4.3.3 Lemma. Every $x \in X$ has arbitrarily small open neighbourhoods U in X such that the natural maps

$$H^*_{(2)}(U) \to IH^*(U)$$

are isomorphisms.

Proof. It is easy to check that if $x \in X$ is a nonsingular point of X then x has arbitrarily small open neighbourhoods U in

$$X_{nonsing} = X - \Sigma$$

which are quasi-isometric to cones on a sphere. A simpler version of the argument used to prove 4.2.1 shows that the L^2-cohomology of such a neighbourhood is trivial, i.e.

$$H^i_{(2)}(U) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand

$$IH^i(U) \cong H^i(U) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

since U is nonsingular and contractible. It is clear from the definition of 4.3.1 that the natural map

$$H^i_{(2)}(U) \to IH^i(U)$$

is nonzero when $i = 0$ and hence is an isomorphism for all $i \geq 0$.

Now suppose x is a singular point of X. Then since X has isolated conical singularities there is a compact Riemannian manifold Y and an open neighbourhood U of x in X such that U is homeomorphic to the cone C(Y) and U-{x} is quasi-isometric to the punctured cone $C^*(Y)$. It is easy to see that U may be chosen arbitrarily small. Then by 4.2.1 since the real dimension of Y is 2n-1 we have

4.3.4 $$H^i_{(2)}(U) \cong \begin{cases} H^i(Y) & \text{if } i \leq n-1, \\ 0 & \text{if } i \geq n. \end{cases}$$

On the other hand since U has a single isolated singularity at x it follows from 3.5.1 that

$$IH_i(U) \cong \begin{cases} H_i(U-\{x\}) & \text{if } i \leq n-1, \\ \mathrm{Im}(H_i(U-\{x\}) \to H_i(U)) & \text{if } i = n, \\ H_i(U) & \text{if } i \geq n+1. \end{cases}$$

Moreover since U is contractible we have

$$H_i(U) = 0$$

if $i \geq 1$, and since $U-\{x\}$ is homeomorphic to $C^*(Y) = (0,1) \times Y$ we have

$$H_i(U-\{x\}) \cong H_i((0,1) \times Y) \cong H_i(Y)$$

for all i. Thus

4.3.5 $$IH_i(U) \cong \begin{cases} H_i(Y) & \text{if } i \leq n-1, \\ 0 & \text{if } i \geq n. \end{cases}$$

Taking duals and comparing with 4.3.4 we find that

$$H^i_{(2)}(U) \cong IH^i(U)$$

for all i.

In order to check that this isomorphism corresponds to the natural map

$$H^i_{(2)}(U) \to IH^i(U)$$

it suffices to consider the case $i \leq n - 1$. Then the isomorphism

$$H^i_{(2)}(U) \cong H^i(Y)$$

of 4.3.4 is the composition of the inverse of the map

$$\pi^* : H^i_{(2)}(Y) \to H^i_{(2)}(U-\{x\}) = H^i_{(2)}(U)$$

induced by the projection

$$\pi : U-\{x\} = (0,1) \times Y \to Y$$

with the natural isomorphism

$$H^i_{(2)}(Y) = H^i_{DR}(Y) \to H^i(Y).$$

On the other hand the isomorphism

$$IH_i(U) \cong H_i(Y)$$

of 4.3.5 is the composition of the identification

$$IH_i(U) = IH_i(U-\{x\}) = H_i(U-\{x\})$$

with the isomorphism

$$\pi_* : H_i(U-\{x\}) \to H_i(Y).$$

The result follows.

Let X be a projective variety with isolated conical singularities. It turns out (cf. 5.3.6) that the existence of the natural map

$$H^*_{(2)}(X) \to IH^*(X)$$

together with Lemma 4.3.3 implies the following theorem.

4.3.6 Theorem. (Cheeger [2]). The natural map

$$H^*_{(2)}(X) \to IH^*(X)$$

from the L^2-cohomology of X to its intersection cohomology is an isomorphism.

It is conjectured that this theorem holds without the hypothesis that X has isolated conical singularities.

5 The intersection sheaf complex $\underline{\underline{IC}}^{\cdot}_X$

Let X be a quasi-projective variety with a fixed Whitney stratification defined by the filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o.$$

We have defined the intersection homology groups $IH_i(X)$ as the homology groups of a chain complex $IC_{\cdot}(X)$. The group $IC_i(X)$ of intersection i-chains is the subspace of $C_i(X)$ consisting of all those i-chains $\xi \in C_i(X)$ satisfying

$$\dim |\xi| \cap X_{n-k} \leqq i - k - 1$$

and

$$\dim |\partial\xi| \cap X_{n-k} \leqq i - k - 2$$

for all $k \geq 1$. In this chapter we shall give a sheaf-theoretic description of $IH_*(X)$ which leads to a proof that $IH_*(X)$ is a topological invariant of X. For more details see Goresky and MacPherson [5], Borel [1], Brylinski [1].

§5.1 Definition of the sheaf complex $\underline{\underline{C}}^{\cdot}_X$ and $\underline{\underline{IC}}^{\cdot}_X$

We noted in §3.7 that if U is an open subset of X then $IC_{\cdot}(U)$ is a subcomplex of $IC_{\cdot}(X)$. However the complexes $IC_{\cdot}(U)$ do not define a complex of sheaves on X because if U and V are open subsets such that $U \subseteq V$ the natural map goes from $IC_i(U)$ to $IC_i(V)$, not the other way round. To get a complex of sheaves we shall use the complexes $IC_i((U))$ instead.

Recall that if N is a locally finite simplicial complex in $\mathbb{R}^N$ and $T: |N| \to X$ is a triangulation of X then an element $\xi \in C_i^T((X))$ is a formal linear combination

$$\xi = \sum_{\sigma \in N^{(i)}} \xi_\sigma \, \sigma$$

where the coefficients ξ_σ are complex numbers. The chain ξ belongs to $C_i^T(X)$ if only finitely many of the coefficients ξ_σ are nonzero. Since T is a locally finite triangulation of X the support

$$|\xi| = \bigcup_{\xi_\sigma \neq 0} T(\sigma)$$

of any $\xi \in C_i^T((X))$ is a closed subset of X, and it is compact if and only if $\xi \in C_i^T(X)$. If the triangulation of T is compatible with the stratification of X then $\dim_{\mathbb{R}} |\xi| \cap X_j$ is well defined for all j, and $IC_i^T((X))$ is the subspace of $C_i^T((X))$ consisting of all those $\xi \in C_i^T((X))$ satisfying

$$\dim_{\mathbb{R}} |\xi| \cap X_{n-k} \leq i - k - 1$$

and

$$\dim_{\mathbb{R}} |\partial\xi| \cap X_{n-k} \leq i - k - 2$$

for all $k \geq 1$. The boundary map

$$\partial\colon C_i^T((X)) \to C_{i-1}^T((X))$$

restricts to a boundary map

$$\partial\colon IC_i^T((X)) \to IC_{i-1}^T((X))$$

which makes $IC_\bullet^T((X))$ into a complex, and $IC_\bullet((X))$ is the direct limit of these complexes under refinement.

Now suppose that U and V are open subsets of X and that $V \subseteq U$. If

$$T\colon |N| \to U$$

is a locally finite triangulation of U then there is a locally finite triangulation

$$S\colon |M| \to V$$

of V such that if $\sigma \in M^{(i)}$ there is a unique $\tau(\sigma) \in N^{(i)}$ such that

$$S(\sigma) \subseteq T(\tau(\sigma)).$$

Thus to any chain $\xi \in C_i^T((U))$ we can associate a chain $\rho(\xi) \in C_i^S((V))$ defined by

5.1.1 $$\rho(\xi) = \sum_{\sigma \in M^{(i)}} \xi_{\tau(\sigma)}\sigma .$$

Then

$$|\rho(\xi)| = |\xi| \cap V.$$

Thus if $\xi \in IC_i^T(U)$ then $\rho(\xi) \in IC_i^S(V)$. Taking limits we get well defined restriction maps

$$\rho : IC_i((U)) \to IC_i((V))$$

which commute with the boundary maps. These restriction maps define a sheaf on X whose space of sections over U is $IC_i((U))$. By convention, because people like to work with sheaves of cochain complexes, not sheaves of chain complexes, the sheaf defined by the $IC_i((U))$ is denoted by

$$\underline{\underline{IC}}_X^{-i}.$$

5.1.2 Warning. Unfortunately there is inconsistency in the literature in the indexing of the sheaf complex $\underline{\underline{IC}}_X^{\bullet}$. The only consistency is in working with sheaves of cochain complexes not sheaves of chain complexes. Sometimes the index -i is replaced by 2n-i or n-i where n is the complex dimension of X (see Goresky and MacPherson [5, §2.3]).

The boundary maps define a sheaf map

$$\partial: \underline{\underline{IC}}_X^{-i} \to \underline{\underline{IC}}_X^{-i+1}$$

so $\underline{\underline{IC}}_X^{\bullet}$ becomes a sheaf of cochain complexes on X. Similarly we can define a sheaf of cochain complexes $\underline{\underline{C}}_X^{\bullet}$ on X.

5.1.3 Remark. Recall from 2.5.3 that if U is an open subset of X then

$$\Gamma_c(U, \underline{\underline{IC}}_X^i)$$

is the space of sections of $\underline{\underline{IC}}_X^i$ over U with compact support. We have

$$\Gamma_c(U, \underline{\underline{IC}}_X^{-i}) = IC_i(U)$$

and under this identification the sheaf map

$$\partial\colon \underline{\underline{IC}}_X^{-i} \to \underline{\underline{IC}}_X^{-i+1}$$

induces the original boundary map

$$\partial\colon IC_i(U) \to IC_{i-1}(U).$$

5.1.4 Remark. Of course if L is a local coefficient system over X then we can define complexes of sheaves

$$\underline{\underline{C}}^\bullet_{(X,L)} \quad \text{and} \quad \underline{\underline{IC}}^\bullet_{(X,L)}$$

over X in the obvious way (cf. §3.9). Indeed to define $\underline{\underline{IC}}^\bullet_{(X,L)}$ we only need a local coefficient system L over the nonsingular open subset $X - X_{n-1}$ of X, not over X itself.

§5.2 The cohomology of the sheaves $\underline{\underline{C}}_X^j$ and $\underline{\underline{IC}}_X^j$

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be any open cover of X. Then there is a triangulation

$$T\colon |\mathcal{N}| \to X$$

of X such that for every $\sigma \in \mathcal{N}$ there is some $i \in I$ such that

$$T(\sigma) \subseteq U_i .$$

For each σ we choose some such i and denote it by $i(\sigma)$.

5.2.1 Remark. We call T a triangulation *subordinate* to the open cover $\mathcal{U}$.

Now suppose that V is any open subset of X. We shall use the triangulation T to define maps

$$\rho_i^V : \underline{\underline{C}}_X^{-j}(U_i \cap V) \to \underline{\underline{C}}_X^{-j}(V)$$

such that if $\xi \in \underline{\underline{C}}_X^{-j}(U_i \cap V)$ then the support of $\rho_i^V(\xi)$ is contained in $U_i \cap V$ and if $\zeta \in \underline{\underline{C}}_X^{-j}(V)$ then

$$\sum_{i \in I} \rho_i^V (\zeta|_{U_i \cap V}) = \zeta.$$

This infinite sum will make sense because it is *locally finite*, i.e. its restriction to any sufficiently small open subset of V involves only finitely many nonzero terms. Such a collection of maps ρ_i^V is called a *partition of unity* for the sheaf $\underline{\underline{C}}_X^{-j}$ subordinate to the open cover $\mathcal{U}$.

In order to define the partition of unity we first note that we can represent any element of

$$\underline{\underline{C}}_X^{-j}(U_i \cap V) = C_j((U_i \cap V))$$

by an element

$$\xi = \sum_{\sigma \in M^{(j)}} \xi_\sigma \sigma$$

of $C_j^S((U_i \cap V))$ where

$$S: |M| \to U_i \cap V$$

is a triangulation of $U_i \cap V$ such that for every $\sigma \in M^{(j)}$ there is a unique $\tau \in N^{(j)}$ with

$$S(\sigma) \subseteq T(\tau) \subseteq U_{i(\tau)}.$$

We write $\tau = \tau(\sigma)$. Then if $\tau \in N^{(j)}$ we have

$$\bigcup_{\sigma\in M^{(j)},\tau(\sigma)=\tau} S(\sigma) = T(\tau) \cap V.$$

Since T is a triangulation of X the subset

$$\bigcup_{\tau\in N^{(j)},i(\tau)=i} T(\tau)$$

of U_i is a *closed* subset of X. Therefore the subset

$$\bigcup_{\sigma\in M^{(j)},i(\tau(\sigma))=i} S(\sigma)$$

of $U_i \cap V$ is a *closed* subset of V. This means that we can choose a triangulation

$$R: |L| \to V$$

of V such that if $\sigma \in M^{(j)}$ and $i(\tau(\sigma)) = i$ then $\sigma \in L^{(j)}$ and $R(\sigma) = S(\sigma)$. Then we can define

5.2.2 $$\rho_i^V(\xi) = \sum_{\sigma\in M^{(j)},i(\tau(\sigma))=i} \xi_\sigma \sigma$$

as an element of $C_j^R((V))$ and hence as an element of

$$\underline{\underline{C}}_X^j(V) = C_j((V)).$$

It is easy to check that these maps

$$\rho_i^V: \underline{\underline{C}}_X^{-j} (U_i \cap V) \to \underline{\underline{C}}_X^{-j}(V)$$

form a partition of unity for the sheaf $\underline{\underline{C}}_X^{-j}$ subordinate to the open cover $\mathcal{U}$. Moreover the same maps restrict to give a partition of unity for the sheaf $\underline{\underline{IC}}_X^{-j}$ as well.

5.2.3 Lemma. If $p \geq 1$ then

$$H^p(X, \underline{\underline{C}}_X^{-j}) = 0 = H^p(X, \underline{\underline{IC}}_X^{-j})$$

for all j.

Proof. We shall work with Čech cohomology (see §2.5), and we shall consider only the case of the sheaf $\underline{\underline{C}}_X^{-j}$; the argument for $\underline{\underline{IC}}_X^{-j}$ is just the same.

Suppose $a \in C^p(X, \underline{\underline{C}}_X^{-j})$ satisfies $da = 0$. Then there is an open cover $\mathcal{U} = \{U_i | i \in I\}$ of X such that a is represented by some $\alpha \in C^p(\mathcal{U}, \underline{\underline{C}}_X^{-j})$ satisfying $d\alpha = 0$. That is, for each

$$K = \{i_o, \ldots, i_{p+1}\} \in I^{(p+1)}$$

we have

$$\sum_{j=0}^{p+1} \pm \alpha_{K-\{i_j\}}|_{U_K} = 0$$

where the sign ± depends on whether or not the orientation chosen for K coincides with the orientation chosen for $K-\{i_j\}$ with i_j placed at the beginning.

It suffices to show that there exists $\beta \in C^{p-1}(\mathcal{U}, \underline{\underline{C}}_X^{-j})$ such that $\alpha = d\beta$; i.e. that for each

$$K = \{i_o, \ldots, i_p\} \in I^{(p)}$$

we have

$$\sum_{j=0}^{p} \pm \beta_{K-\{i_j\}}|_{U_K} = \alpha_K.$$

Choose a partition of unity for the sheaf $\underline{\underline{C}}_X^{-j}$ subordinate to the cover $\mathcal{U}$. For every open subset V of X this gives us maps

$$\rho_i^V : \underline{\underline{C}}_X^{-j}(U_i \cap V) \to \underline{\underline{C}}_X^{-j}(V)$$

such that if $\xi \in \underline{\underline{C}}_X^{-j}(U_i \cap V)$ then the support of $\rho_i^V(\xi)$ is contained in $U_i \cap V$ and if $\zeta \in \underline{\underline{C}}_X^{-j}(V)$ then

$$\sum_{i \in I} \rho_i^V (\zeta|_{U_i \cap V}) = \zeta.$$

We can now define β as follows. If

$$K = \{i_0, \ldots, i_{p-1}\} \in I^{(p-1)}$$

let $\beta_K \in \underline{\underline{C}}_X^{-j}(U_K)$ be defined by the locally finite sum

$$\beta_K = \sum_{i \in I-K} \pm \rho_i^{U_K} (\alpha_{K \cup \{i\}})$$

where the sign $\pm$ depends on whether or not the orientation chosen for $K \cup \{i\}$ coincides with the orientation chosen for K with i placed at the beginning. It is now messy but straightforward to check, using the fact that $d\alpha = 0$, that $d\beta = \alpha$ as required.

5.2.4 A sheaf F on X is called a *fine sheaf* if for every open cover $\mathcal{U}$ of X there is a partition of unity for F subordinate to $\mathcal{U}$. The proof of Lemma 5.2.3 shows that if F is a fine sheaf then

$$H^p(X,F) = 0$$

for all $p \geq 1$.

§5.3 Spectral sequences and hypercohomology

By a spectral sequence we shall mean a collection of complex vector spaces

$$\{E_r^{p,q} \mid p,q,r \in \mathbb{Z}, \quad r \geq r_0\}$$

where $r_0 = 0,1$ or 2, together with linear maps

$$d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$$

satisfying $d_r^2 = 0$ and

$$E_{r+1}^{p,q} = \frac{\ker d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}}{\operatorname{im} d_r: E_r^{p-r,q+r-1} \to E_r^{p,q}} .$$

If for all $p,q \in \mathbb{Z}$ there exists some $r(p,q) \in \mathbb{Z}$ and a complex vector space

$E_\infty^{p,q}$ such that

$$E_r^{p,q} = E_\infty^{p,q}$$

for all $r \geq r(p,q)$ then we say that the spectral sequence has limit $\{E_\infty^{p,q} | p,q \in \mathbf{Z}\}$.

Let $\{F^pK^\cdot | 0 \leq p \leq n\}$ be a filtered complex of vector spaces, in the sense that $F^oK^\cdot = K^\cdot$ is a complex

$$\cdots \to F^oK^o \xrightarrow{d} F^oK^1 \xrightarrow{d} F^oK^2 \to \cdots$$

of vector spaces and if $0 < p \leq n$ then $F^pK^\cdot$ is a subcomplex of $F^{p-1}K^\cdot$ with $F^nK^\cdot = 0$. There is an associated spectral sequence $\{E_r^{p,q} | p,q.r \in \mathbf{Z},\ r \geq 0\}$ with

$$E_r^{p.q} = \frac{\{a \in F^pK^{p+q}: da \in F^{p+r}K^{p+q+1}\}}{d(F^{p-r+1}\ K^{p+q-1}) + F^{p+1}\ K^{p+q}} \ .$$

If $\alpha \in E_r^{p,q}$ is represented by $a \in F^p\ K^{p+q}$ then $d\alpha \in E_r^{p+r,q-r+1}$ is represented by $da \in F^{p+r}\ K^{p+q+1}$.

Now let $A^\cdot$ be a complex of sheaves on a topological space X and let $\mathcal{U}$ be an open cover of X. Let

$$C^p(\mathcal{U},A^q)$$

be the space of Čech p-cochains over $\mathcal{U}$ with coefficients in A^q. We have a boundary map

$$\delta_1 : C^p(\mathcal{U},A^q) \to C^{p+1}(\mathcal{U},A^q)$$

and the sheaf complex differential $d: A^q \to A^{q+1}$ induces

$$\delta_2 : C^p(\mathcal{U},A^q) \to C^p(\mathcal{U},A^{q+1})$$

satisfying

$$\delta_2^2 = 0 = \delta_1^2,\ \delta_1\delta_2 = \delta_2\delta_1.$$

Passing to the limit with respect to refinement of open covers $\mathcal{U}$ of X we obtain a complex vector space

$$C^{p,q} = \varinjlim C^p(\mathcal{U}, A^q)$$

with boundary maps $\delta_1\colon C^{p,q} \to C^{p+1,q}$ and $\delta_2\colon C^{p,q} \to C^{p,q+1}$. We define the *hypercohomology* $\mathcal{H}^*(X,A^\bullet)$ of $A^\bullet$ to be the cohomology of the complex $(K^\bullet,d)$ where

$$K^n = \bigoplus_{p+q=n} C^{p,q}$$

and $d = \delta_1 + (-1)^p\delta_2$ on $C^{p,q}$. That is

5.3.1 $$\mathcal{H}^n(X,A^\bullet) = \frac{\ker d\colon K^n \to K^{n+1}}{\operatorname{im} d\colon K^{n-1} \to K^n} .$$

We have a filtered complex $\{F^pK^\bullet\}$ defined by setting

$$F^pK^n = \bigoplus_{\substack{p'+q=n \\ p' \geq p}} C^{p',q}$$

which makes $F^pK^\bullet$ a subcomplex of $K^\bullet$. There is an associated spectral sequence $\{E_r^{p,q} | r \geq 0\}$ with

$$E_2^{p,q} = H^p(X, \underline{\underline{H}}^q(A^\bullet))$$

where $\underline{\underline{H}}^q(A^\bullet)$ is the qth cohomology sheaf of $A^\bullet$, and

$$E_\infty^{p,q} = \operatorname{Gr}^p \mathcal{H}^{p+q}(X,A^\bullet)$$

where

5.3.2 $$\operatorname{Gr}^p\mathcal{H}^{p+q}(X,A^\bullet) = \frac{\ker d\colon F^pK^n \to F^pK^{n+1}}{\operatorname{im} d\colon F^pK^{n-1} \to F^pK^n} .$$

(see e.g. Griffiths and Harris [1, Chapter 3, §5], Bott and Tu [1, §14]). This implies that there is an isomorphism

$$H^n(X,A^{\bullet}) \cong \bigoplus_{p+q=n} E_\infty^{p,q} .$$

But this isomorphism is not canonical whereas the associated filtration of $H^n(X,A^{\bullet})$ by the subspaces

$$\bigoplus_{\substack{p'+q=n \\ p' \geq p}} E_\infty^{p',q}$$

is canonical. In this situation we say that the spectral sequence $\{E_r^{p,q}\}$ *abuts* to

$$H^{p+q}(X,A^{\bullet}).$$

By reversing the roles of p and q we get another filtered complex $\{\tilde{F}^qK^{\bullet}\}$ defined by

$$\tilde{F}^qK^n = \bigoplus_{\substack{p+q'=n \\ q' \geq q}} C^{p,q'}$$

with an associated spectral sequence $\{\tilde{E}_r^{p,q} | r \geq 0\}$ such that $\tilde{E}_2^{p,q}$ is the qth cohomology group of the complex $H^p(X,A^{\bullet})$. By symmetry this spectral sequence also abuts to the hypercohomology

$$H^{p+q}(X,A^{\bullet}).$$

In particular it follows that if

$$H^p(X,A^q) = 0$$

for all q and all p > 0 then

$$\tilde{E}_\infty^{p,q} = 0$$

for all q and all p > 0 so

5.3.3 $\quad H^n(X,A^{\bullet}) \cong \tilde{E}_\infty^{o,n}$

is the nth cohomology group of the complex

$$H^0(X,A^\bullet) = \Gamma(X,A^\bullet).$$

5.3.4 Example. We have seen (Lemma 5.2.3) that if X is a quasi-projective variety with a fixed Whitney stratification then the complexes of sheaves $\underline{\underline{C}}_X^\bullet$ and $\underline{\underline{IC}}_X^\bullet$ on X satisfy

$$H^p(X,\underline{\underline{C}}_X^q) = 0 = H^p(X,\underline{\underline{IC}}_X^q)$$

for all q and all $p > 0$. Thus their hypercohomology groups are canonically isomorphic to the cohomology groups of the complexes $H^0(X,\underline{\underline{C}}_X^\bullet)$ and $H^0(X,\underline{\underline{IC}}_X^\bullet)$. Since

$$H^0(X,\underline{\underline{C}}_X^j) = C_{-j}((X))$$

and

$$H^0(X,\underline{\underline{IC}}_X^j) = IC_{-j}((X))$$

these cohomology groups are the same as the homology groups of the complexes $C_\bullet((X))$ and $IC_\bullet((X))$. Thus we have canonical isomorphisms

$$\mathcal{H}^{-n}(X,\underline{\underline{C}}_X^\bullet) \cong H_n^{c\ell}(X).$$

and

$$\mathcal{H}^{-n}(X,\underline{\underline{IC}}_X^\bullet) \cong IH_n^{c\ell}(X).$$

Of course when X is compact then

$$H_n^{c\ell}(X) = H_n(X)$$

and

$$IH_n^{c\ell}(X) = IH_n(X).$$

5.3.5 Definition. A sheaf map $\phi\colon \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ between two sheaves of cochain complexes on X which commutes with boundary maps is called a *quasi-isomorphism* if the induced maps

$$\underline{\underline{H}}^i(A^\bullet) \to \underline{\underline{H}}^k(B^\bullet)$$

of cohomology sheaves are isomorphisms for all i.
Equivalently the maps on stalks

$$H^i(A_x^\bullet) \to H^i(B_x^\bullet)$$

are all isomorphisms where $A_x^\bullet$ and $B_x^\bullet$ are the stalk complexes. (Recall from 2.5.6 that the stalk of $\underline{\underline{H}}^i(A^\bullet)$ at x is $H^i(A_x^\bullet)$). A *generalised quasi-isomorphism* $A^\bullet \to B^\bullet$ is a sequence

$$A^\bullet \to A_1^\bullet \leftarrow A_2^\bullet \to A_3^\bullet \leftarrow A_4^\bullet \to \dots \to B^\bullet$$

of quasi-isomorphisms.

It follows from the existence of the spectral sequence described above that a generalised quasi-isomorphism $\phi: A^\bullet \to B^\bullet$ induces an isomorphism from the hypercohomology of $A^\bullet$ to the hypercohomology of $B^\bullet$.

5.3.6 Remark. Let X be a projective variety. We have seen that we can idetnify the intersection homology groups of X with the hypercohomology groups of a sheaf complex $\underline{\underline{IC}}_X^\bullet$ over X.

Thus if $A^\bullet$ is another sheaf complex over X and $\phi: A^\bullet \to \underline{\underline{IC}}_X^\bullet$ is a quasi-isomorphism, then ϕ induces isomorphisms between the hypercohomology groups of $A^\bullet$ and the intersection homology groups of X.

§5.4 Towards the topological invariance of intersection homology

Let X be a projective variety with a fixed Whitney stratification.

5.4.1 Definition. The ith *local intersection homology sheaf* $\underline{\underline{H}}^{-i}(\underline{\underline{IC}}_X^\bullet)$ is the -ith cohomology sheaf of the complex $\underline{\underline{IC}}_X^\bullet$. That is, it is the quotient

$$\frac{\operatorname{Ker} \partial : \underline{\underline{IC}}_X^{-i} \to \underline{\underline{IC}}_X^{-(i-1)}}{\operatorname{Im} \partial : \underline{\underline{IC}}_X^{-(i+1)} \to \underline{\underline{IC}}_X^{-i}}$$

in the sense of sheaves. The stalk of $\underline{H}^{-i}(\underline{\underline{IC}}_X^{\bullet})$ at $x \in X$ is the relative intersection homology group $IH_i(X, X-\{x\})$.

We saw at 5.3.4 that the intersection homology groups of X are canonically isomorphic to the hypercohomology groups of the complex $\underline{\underline{IC}}_X^{\bullet}$. Therefore there is a spectral sequence with E_2 term given by

5.4.2 $\quad E_2^{pq} = H^p\ (X;\ \underline{H}^q(\underline{\underline{IC}}_X^{\bullet}))$

which abuts to $IH_{-p-q}(X)$ (see §5.3). In particular we have the following very important fact (see 5.3.6).

5.4.3 $\quad IH_*(X)$ is determined by $\underline{\underline{IC}}_X^{\bullet}$ up to generalised quasi-isomorphism.

We want to know that $IH_*(X)$ is independent of the choice of Whitney stratification on X. For this it suffices to show that the sheaf complex $\underline{\underline{IC}}_X^{\bullet}$ is independent up to genéralised quasi-isomorphism of the choice of Whitney stratification.

5.4.4 Definition. A complex $A^{\bullet}$ of sheaves over X is *bounded* if there is an integer m such that

$$A^i = 0 \quad \text{when} \quad |i| \geq m.$$

$A^{\bullet}$ is *constructible* if there is a stratification $\{S_\alpha | \alpha \in A\}$ by X by quasi-projective subvarieties S_α such that the cohomology sheaf $\underline{H}^i(A^{\bullet})|_{S_\alpha}$ is locally constant for all i and α.

5.4.5 Remark. The restriction to S_α of a sheaf F on X is the sheaf $F|_{S_\alpha}$ on S_α defined as follows. If U is an open subset of S then $F|_{S_\alpha}(U)$ is the direct limit with respect to restriction of the abelian groups F(V) for V open in X such that $U \subseteq V$. A sheaf F on X is called *locally constant* if every $x \in X$ has a neighbourhood U in X such that for every $y \in U$ the restriction map

$$F(U) \to F_y$$

to the stalk at y is an isomorphism.

It is easy to check that $\underline{\underline{IC}}^{\bullet}_X$ is a bounded constructible complex of sheaves on X by using the local calculation of intersection homology 3.8.1.

The fact that $IH_*(X)$ is independent of the choice of Whitney stratification follows from the next theorem which characterises $\underline{\underline{IC}}^{\bullet}_X$ uniquely up to generalised quasi-isomorphism by properties which do not depend on the choice of stratification.

5.4.6 <u>Theorem</u>. (Goresky and MacPherson [5, §4.1 and §4.3]). The sheaf of cochain complexes $\underline{\underline{IC}}^{\bullet}_X$ is uniquely characterised up to a canonical generalised quasi-isomorphism by the fact that it is a bounded constructible complex of sheaves on X satisfying the following properties.

(a) There is a subvariety $\Sigma \subseteq X$ of complex codimension at least 1 such that

$$\underline{\underline{IC}}^{\bullet}_X|_{X-\Sigma}$$

is generalised quasi-isomorphic to the trivial complex $\mathbb{C}_{X-\Sigma}[2n]$, which is the constant sheaf $\mathbb{C}_{X-\Sigma}$ in dimension $i = -2n$ and 0 in other dimensions.

(b) For all $x \in X$ the cohomology $H^{-i}(\underline{\underline{IC}}^{\bullet}_{X,x})$ of the stalk complex $\underline{\underline{IC}}^{\bullet}_{X,x}$ is a finite-dimensional complex vector space for all i and is 0 when $i > 2n$.

(c) For all $i < 2n$

$$\dim_{\mathbb{C}}\{x \in X | H^{-i}(\underline{\underline{IC}}^{\bullet}_{X,x}) \neq 0\} < i-n.$$

(d) $\underline{\underline{IC}}^{\bullet}_X$ is self dual in the sense of Verdier duality (Goresky and MacPherson [5, §1.12], Borel and Moore [1], Verdier [1], [2]).

The Verdier self duality of $\underline{\underline{IC}}^{\bullet}_X$ simply comes down to Poincaré duality - the existence of a natural perfect pairing

$$IH^i(U) \otimes IH_c^{2n-i}(U) \to \mathbb{C}$$

for all open subsets U of X. One checks that $\underline{\underline{IC}}^\bullet_X$ satisfies the conditions 5.4.6 (b) and (c) by using the local calculation 3.8.1 of intersection homology and the fact that

$$H^{-i}(\underline{\underline{IC}}^\bullet_{X,x}) = IH_i(X, X-\{x\}).$$

To see that $\underline{\underline{IC}}^\bullet_X$ satisfies condition 5.4.6 (a) take $\Sigma = X_{n-1}$. Then

$$\underline{\underline{IC}}^\bullet_X|_{X-\Sigma} = \underline{\underline{C}}^\bullet_{X-\Sigma}$$

and

$$\underline{\underline{H}}^i(\underline{\underline{C}}^\bullet_{X-\Sigma}) = \begin{cases} 0 & \text{if } i \neq -2n \\ \mathbb{C}_{X-\Sigma} & \text{if } i = -2n \end{cases}$$

because its stalk at any $x \in X - \Sigma$ is

$$H_{-i}(X, X-\{x\}) = \begin{cases} 0 & \text{if } i \neq -2n \\ \mathbb{C} & \text{if } i = -2n \end{cases}$$

since x is a nonsingular point of X. Using this it is easy to find a quasi-isomorphism

$$\mathbb{C}_{X-\Sigma}[2n] \to \underline{\underline{IC}}^\bullet_X|_{X-\Sigma}.$$

5.4.7 Remark. The condition 5.4.6 (d) may be replaced by the "dual" of condition (c), namely

(d') For all $i < 2n$

$$\dim_{\mathbb{C}}\{x \in x \mid H_x^{-j}(\underline{\underline{IC}}^\bullet_X) \neq 0\} < i-n,$$

where $j = 2n-i$ and $H_x^k(\underline{\underline{IC}}^\bullet_X)$ denotes the kth hypercohomology group with compact supports of $\underline{\underline{IC}}^\bullet_X$ restricted to a small open neighbourhood N_x of x of the form described in §3.8.

§5.5 Deligne's construction of the intersection sheaf complex

It has been indicated why $\underline{\underline{IC}}^{\bullet}_X$ satisfies the conditions 5.4.6 (a) - (d) but not why it is uniquely characterised by them up to generalised quasi-isomorphism. The proof of this uses Deligne's construction of a complex which is canonically generalised quasi-isomorphic to $\underline{\underline{IC}}^{\bullet}_X$.

5.5.1 Definition. If $A^{\bullet}$ is a complex of sheaves on X and $p \in \mathbb{Z}$ we define the *truncated complex* $\tau_p A^{\bullet}$ to be the complex which in degree i is

$$\begin{cases} A^i & \text{if } i < p, \\ \ker(d\colon A^p \to A^{p+1}) & \text{if } i = p, \\ 0 & \text{if } i > p. \end{cases}$$

Suppose we have a fixed Whitney stratification

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$$

of X as before. Let

$$i_k : X - X_{n-k} \to X - X_{n-k-1}$$

be the inclusion.

Recall from §2.4 that if F is a sheaf on $X - X_{n-k}$ then $(i_k)_*F$ is the sheaf on $X - X_{n-k-1}$ satisfying

5.5.2 $$((i_k)_*F)(V) = F(V \cap (X - X_{n-k})) = F(i_k^{-1}(V))$$

for any open subset V of $X - X_{n-k-1}$.

5.5.3 Theorem. (Deligne's construction of $\underline{\underline{IC}}^{\bullet}_X$, Goresky and MacPherson [5, §3]). The complex of sheaves

$$\tau_{-n-1}R(i_n)_* \cdots \tau_{-2n+1}R(i_2)_*\tau_{-2n}R(i_1)_*\mathbb{C}_{X-X_{n-1}}[2n]$$

satisfies conditions (a) - (d) of Theorem 5.4.6. (As in §2.6 $R(i_k)_*$ is the right derived functor of $(i_k)_*$).

Once it has been shown that the conditions (a) - (d) of 5.4.6 uniquely characterise $\underline{\underline{IC}}^{\bullet}_X$ up to generalised quasi-isomorphism it will follow that the complex defined in 5.5.3 is generalised quasi-isomorphic to $\underline{\underline{IC}}^{\bullet}_X$. This is very important because this construction can be used to define intersection homology in any situation where there is a good sheaf theory and good stratifications, for example algebraic geometry in characteristic $p > 0$ (see Chapter 6). Moreover this construction does not require the filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$$

of X to be an algebraic one. We shall see that there is a "canonical filtration"

$$X = X_n^{can} \supseteq X_{n-1}^{can} \supseteq \cdots \supseteq X_o^{can}$$

of X by closed subsets X_j^{can} such that Deligne's construction applied to this filtration gives a complex which is generalised quasi-isomorphic to $\underline{\underline{IC}}^{\bullet}_X$. The X_j^{can} will *not* necessarily be subvarieties of X but they *will* be uniquely determined by X without any need of choice.

5.5.4 Definition of the canonical filtration. Let U_1 be the largest open subset of X such that the cohomology sheaves of the complex $\underline{\underline{C}}^{\bullet}_X$ restricted to U_1 are all locally constant. (Equivalently U_1 is the union of all open subsets U of X such that the cohomology sheaves of $\underline{\underline{C}}^{\bullet}_X$ restricted to U are locally constant).

Since the -pth cohomology sheaf of $\underline{\underline{C}}^{\bullet}_X$ has stalk $H_p(X,X-\{x\})$ at x, and

$$H_p(X,X-\{x\}) = \begin{cases} \mathbb{C} & \text{if } p = 2n \\ 0 & \text{if } p \neq 2n \end{cases}$$

if x is a nonsingular point of X, it follows that

$$X - X_{n-1} \subseteq U_1$$

for any Whitney stratification $X \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$ of X.

Let

$$X^{can}_{n-1} = X - U_1;$$

then X^{can}_{n-1} is a closed subset of X and

$$X^{can}_{n-1} \subseteq X_{n-1}.$$

Define X^{can}_{n-k} inductively for $1 < k \leq n$ as follows. If $1 \leq j < k$ let

$$i^{can}_j : X - X^{can}_{n-j} \to X - X^{can}_{n-j-1}$$

and

$$h_k : X - X^{can}_{n-k} \to X$$

be the inclusions. Then let X^{can}_{n-k-1} be the complement in X^{can}_{n-k} of the largest open subset of X^{can}_{n-k} on which the cohomology sheaves of both the complexes

$$\underline{\underline{C}}^{\bullet}_{X^{can}_{n-k}}$$

and

$$R(h_k)_* \tau_{k-2n-2} R(i^{can}_{k-1})_* \cdots R(i^{can}_2)_* \tau_{-2n} R(i^{can}_1)_* \ \underline{\underline{C}}^{\bullet}_{U_1}$$

are locally constant. By induction each X^{can}_{n-k} is a closed subset of X and the filtration

$$X = X^{can}_n \supseteq X^{can}_{n-1} \supseteq \cdots \supseteq X^{can}_o$$

is coarser than any Whitney stratification

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_o$$

in the sense that

$$X^{can}_j \subseteq X_j$$

for $0 \leq j \leq n$.

The complex

$$\tau_{-n-1}R(i_n^{can})_* \tau_{-n-2} \cdots R(i_2^{can})_* \tau_{-2n}R(i_1^{can})_* \underline{\underline{C}}^{\bullet}_{U_1}[2n]$$

on X is independent of the choice of stratification on X, and is indeed a topological invariant of X. One can show by induction (Goresky and MacPherson [5, §4]) that any bounded constructible complex of sheaves on X which satisfies the conditions (a) - (d) of 5.4.6 is canonically generalised quasi-isomorphic to this complex. Since $\underline{\underline{IC}}^{\bullet}_X$ satisfies 5.4.6 (a) - (d) it follows that $\underline{\underline{IC}}^{\bullet}_X$ is uniquely characterised by 5.4.6 (a) - (d) up to canonical generalised quasi-isomorphism. Thus $IH_*(X)$ is independent of the choice of Whitney stratification and is a topological invariant of X in the sense that a homeomorphism $f: X \to Y$ induces an isomorphism

$$f_* : IH_*(X) \to IH_*(Y).$$

6 Characteristic p and the Weil conjectures

In 1973 Deligne completed the proof of the famous Weil conjectures which relate the arithmetic of projective varieties defined over finite fields and the homology of nonsingular complex projective varieties (Deligne [1], [2]). The conjectures were stated by André Weil in the 1940s (Weil [1]) and progress (leading to partial proof) was made by Grothendieck and others in the early 1960s. (See the survey articles Katz [1] and Serre [2] for more details).

§6.1 Statement of the Weil conjectures

Let $X \subseteq \mathbf{P}_N$ be a nonsingular m-dimensional complex projective variety defined over an algebraic number ring R (e.g. $R = \mathbf{Z}$). That is, X can be defined by the vanishing of homogeneous polynomials with coefficients in R.

6.1.1 Example. The Fermat curve of degree n is defined in $\mathbf{P}_2$ by the equation

$$x^n + y^n = z^n$$

over $\mathbf{Z}$.

Let π be a maximal ideal of R. (For example if $R = \mathbf{Z}$ then $\pi = p\mathbf{Z}$ where p is a prime). Then R/π is a finite field. Let p be the characteristic of R/π. Then

$$R/\pi = \mathbf{F}_q$$

is a field with $q = p^s$ elements for some positive integer s.

We can define a projective variety

$$X \subseteq \mathbf{P}_N(\mathbf{F}_q) = \frac{\mathbf{F}_q^{N+1} - \{0\}}{\mathbf{F}_q - \{0\}}$$

by reducing modulo π the equations with coefficients in R which define X.

If we choose π so that the characteristic p of R/π is not one of finitely many "bad" primes for X then X_π is a nonsingular m-dimensional projective variety over the field $\mathbf{F}_q$.

Let $\bar{X}_\pi$ be the corresponding variety defined over the algebraic closure $\bar{\mathbf{F}}_q$ of $\mathbf{F}_q$ by the same equations as X_π. For each $r \geq 1$ there is a unique subfield $\mathbf{F}_{q^r}$ of $\bar{\mathbf{F}}_q$ such that $\mathbf{F}_{q^r}$ has q^r elements. Moreover

$$\mathbf{F}_{q^r} \subseteq \mathbf{F}_{q^t}$$

if and only if r divides t.

Let N_r be the number of points in $\bar{X}_\pi$ of the form $(x_0:\ldots:x_N)$ where each x_j lies in $\mathbf{F}_{q^r}$. Define Z(t) by

6.1.2 $$Z(t) = \exp\Big(\sum_{r\geq 1} N_r \frac{t^r}{r}\Big) \in \mathbb{Q}[[t]].$$

6.1.3 Example. If $X = \mathbf{P}_m$, $R = \mathbf{Z}$, $\pi = p\mathbf{Z}$ then

$$N_r = 1 + p^r + p^{2r} + \ldots + p^{mr}$$

and

$$Z(t) = \exp\Big(\sum_{r\geq 1} (1 + p^r + \ldots + p^{mr}) \frac{t^r}{r}\Big)$$

$$= \frac{1}{(1-t)(1-pt)(1-p^2t)\ldots(1-p^m t)} .$$

The *Weil conjectures* relate the numbers N_r to the *Betti numbers* $\dim H_j(X)$ of X. They can be expressed in terms of the function Z(t) as follows.

6.1.4 (1) $$Z(t) = \frac{P_1(t)P_3(t)\ldots P_{2m-1}(t)}{P_0(t)P_2(t)\ldots P_{2m}(t)}$$

where $P_0(t) = 1-t$, $P_{2m}(t) = 1 - q^m t$ and if $1 \leq j \leq 2m-1$ then $P_j(t)$ is a polynomial in t with integer coefficients satisfying

$$P_j(t) = \prod_{1\leq i\leq \dim H_j(X)} (1 - \alpha_{ji} t)$$

where each α_{ji} is an algebraic integer and $|\alpha_{ji}| = q^{j/2}$.

(Note that these conditions mean that $Z(t)$ uniquely determines the polynomials $P_j(t)$ and hence the Betti numbers of X since $\dim H_j(X) = \deg P_j(t)$).

(2) Let $E = \sum_j (-1)^j \dim H_j(X)$ be the Euler characteristic of X. Then $Z(t)$ satisfies a functional equation

$$Z(1/q^m t) = \pm\, q^{mE/2}\, t^E\, Z(t).$$

6.1.5 Remark. The statement "$|\alpha_{ji}| = q^{j/2}$" is called the *Riemann hypothesis* by analogy with the Riemann zeta function as follows. Put $t = q^{-s}$ in $Z(t)$ to get

$$Z(q^{-s}) = \exp\Big(\sum_{r\geq 1} N_r \frac{q^{-rs}}{r}\Big).$$

Define a prime divisor $\mathfrak{p}$ of X to be an equivalence class of points of $\bar{X}_\pi$ modulo conjugation over $\mathbf{F}_q$, and let its norm be

$$\operatorname{Norm} \mathfrak{p} = q^{\deg \mathfrak{p}}$$

where $\deg \mathfrak{p}$ is the number of points in the equivalence class.
Then since $\mathbf{F}_{q^i} \subseteq \mathbf{F}_{q^j}$ if and only if i divides j the number of points of $\bar{X}_\pi$ defined over $\mathbf{F}_{q^r}$ is

$$N_r = \sum_{\deg \mathfrak{p} | r} \deg \mathfrak{p}.$$

Hence

$$\begin{aligned} Z(q^{-s}) &= \exp \sum_{r\geq 1} \sum_{\deg \mathfrak{p}|r} \frac{\deg \mathfrak{p}\, (\operatorname{Norm} \mathfrak{p})^{-sr/\deg \mathfrak{p}}}{r} \\ &= \exp \sum_{\mathfrak{p}} \sum_i \frac{(\operatorname{Norm} \mathfrak{p})^{-si}}{i} \\ &= \prod_{\mathfrak{p}} \exp(-\log(1 - (\operatorname{Norm} \mathfrak{p})^{-s}) \\ &= \prod_{\mathfrak{p}} (1 - (\operatorname{Norm} \mathfrak{p})^{-s})^{-1}. \end{aligned}$$

Recall that the classical zeta function is given by

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

The classical Riemann hypothesis says that the zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ in $\mathbb{C}$. When $\dim_{\mathbb{C}} X = 1$ the statement that $|\alpha_{ji}| = q^{j/2}$ where

$$\begin{aligned} Z(t) &= \prod_j \Bigl(\prod_{1\leq i\leq \dim H_j(X)} (1 - \alpha_{ji}t) \Bigr)^{(-1)^{j+1}} \\ &= \Bigl(\prod_{1\leq i\leq \dim H_1(X)} (1 - \alpha_{1i}t) \Bigr) (1-t)^{-1}(1-qt)^{-1} \end{aligned}$$

is equivalent to the statement that if $Z(t) = 0$ then $|t| = q^{-\frac{1}{2}}$, i.e. that if $Z(q^{-s}) = 0$ then $\mathrm{Re}(s) = \frac{1}{2}$.

Weil proved some special cases of his conjectures and realised that the general case followed if one could define a suitable cohomology theory for varieties over fields of nonzero characteristic analogous to ordinary cohomology for varieties over $\mathbb{C}$. Grothendieck was able to define such a cohomology theory, ℓ-adic cohomology, using the theory of étale topology (due to himself and Artin) and thus proved part of the conjectures (the rationality of $Z(t)$ and the functional equation). Deligne finished the proof in 1973, by proving the analogue of the Riemann hypothesis. Before defining ℓ-adic cohomology let us see how its properties lead to a proof of the Weil conjectures.

§6.2 Basic properties of ℓ-adic cohomology

Let Y be a quasi-projective variety over an algebraically closed field k of characteristic $p \geq 0$. Let ℓ be a prime number different from p. Let

$$\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^r\mathbb{Z}$$

be the ring of ℓ-adic integers, and let $\mathbb{Q}_\ell$ be its field of fractions. The ith ℓ-adic cohomology group of Y is written $H^i(Y,\mathbb{Q}_\ell)$. It has the following properties (see e.g. Milne [1]).

6.2.1 (a) ℓ-adic cohomology is a contravariant functor from the category of quasi-projective varieties over k to the category of vector spaces over $\mathbb{Q}_\ell$.

(b) $H^i(Y,\mathbb{Q}_\ell) = 0$ unless $0 \leq i \leq 2m$ where m is the dimension of Y. The dimension of $H^i(Y,\mathbb{Q}_\ell)$ is finite for all i if Y is projective (and conjecturally for any quasi-projective Y).

(c) Poincaré duality. If Y is nonsingular and projective then there is a natural perfect pairing

$$H^i(Y,\mathbb{Q}_\ell) \otimes H^{2m-i}(Y,\mathbb{Q}_\ell) \to H^{2m}(Y,\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$$

for $0 \leq i \leq 2m$.

(d) Lefschetz fixed point formula. If Y is nonsingular and projective of dimension m over k and f: $Y \to Y$ has only isolated fixed points each of multiplicity one (i.e. 1 is not an eigenvalue of the derivative of f at any $y \in Y$ such that $f(y) = y$) then the Lefschetz number L(f) of f defined by

$$L(f) = \sum_{j=0}^{2m} (-1)^j \operatorname{Tr}(f^*: H^j(Y,\mathbb{Q}_\ell) \to H^j(Y,\mathbb{Q}_\ell))$$

is equal to the number of fixed points. More generally when f has isolated fixed points of multiplicities possibly greater than one then L(f) is the number of fixed points counted according to the multiplicities.

(e) Comparison and change of base field. If X is a complex projective variety then $H^j(X,\mathbb{Q}_\ell)$ is the ordinary cohomology of X with coefficients in $\mathbb{Q}_\ell$, so

$$\dim_{\mathbb{C}} H^j(X) = \dim_{\mathbb{Q}_\ell} H^j(X,\mathbb{Q}_\ell).$$

Moreover if X is defined over an algebraic number ring R, as in §6.1, then

$$H^j(X,\mathbb{Q}_\ell) = H^j(\bar{X}_\pi,\mathbb{Q}_\ell).$$

These are the properties of ℓ-adic cohomology which we shall need. ℓ-adic cohomology also satisfies most of the familiar properties of cohomology, such as the existence of relative cohomology, long exact sequences, spectral sequences and so on.

Let X be a nonsingular complex projective variety defined over an algebraic number ring R, and define $\bar{X}_\pi$ as in §6.1. The properties 6.2.1 (a) - (e) of ℓ-adic cohomology can be used to prove the Weil conjectures 6.1.4. The crucial ingredient is the definition of the Frobenius mapping.

6.2.2 Definition. The *Frobenius mapping* $f: \bar{X}_\pi \to \bar{X}_\pi$ is given by

$$f(x_0: x_1:\dots:x_N) = (x_0{}^q:\dots:x_N{}^q).$$

This makes sense because the equations defining $\bar{X}_\pi$ as a subset of $\mathbf{P}_N(\bar{\mathbf{F}}_q)$ have coefficients in the field $\mathbf{F}_q$, and if $p(X_0,\dots,X_N)$ is a polynomial with coefficients in $\mathbf{F}_q$ then

$$p(X_0^q,\dots,X_N^q) = (p(X_0,\dots,X_N))^q.$$

A point $x \in \bar{X}_\pi$ is fixed by the rth iterate f^r of f if and only if it has coordinates in $\mathbf{F}_{q^r}$. Hence the number N_r of points in $\bar{X}_\pi$ with coordinates in $\mathbf{F}_{q^r}$ is the same as the number of fixed points of f^r. One can check that all the fixed points of f^r have multiplicity one. Thus by the Lefschetz fixed point formula 6.2.1 (d) we have

6.2.3 $$N_r = L(f^r)$$

for all $r \geq 1$. This means that

$$\begin{aligned}
Z(t) &= \exp \sum_{r\geq 1} \frac{L(f^r)t^r}{r} \\
&= \exp \sum_{r\geq 1} \sum_{0\leq j\leq 2m} (-1)^j \mathrm{Tr}((f^r)^*: H^j(\bar{X}_\pi,\mathbb{Q}_\ell)\to H^j(\bar{X}_\pi,\mathbb{Q}_\ell)) \frac{t^r}{r} \\
&= \prod_{j=0}^{2m} (\exp \sum_{r\geq 1} (-1)^j \mathrm{Tr}((f^*)^r: H^j(\bar{X}_\pi,\mathbb{Q}_\ell)\to H^j(\bar{X}_\pi,\mathbb{Q}_\ell)) \frac{t^r}{r}) \\
&= \prod_{j=0}^{2m} \det(1-tf^*: H^j(\bar{X}\ ,\mathbb{Q}_\ell)\to H^j(\bar{X}_\pi,\mathbb{Q}_\ell))^{(-1)^{j+1}} \\
&= \frac{P_1(t)P_3(t)\dots P_{2m-1}(t)}{P_0(t)P_2(t)\dots P_{2m}(t)}
\end{aligned}$$

where

$$P_j(t) = \det(1 - tf^*:H^j(\bar{X}_\pi,\mathbb{Q}_\ell) \to H^j(\bar{X}_\pi,\mathbb{Q}_\ell)).$$

Then

$$P_j(t) = \prod_{1 \leq i \leq \dim H^j(\bar{X}_\pi,\mathbb{Q}_\ell)} (1-\alpha_{ji}t)$$

where the α_{ji} are the eigenvalues of the action of the Frobenius map on $H^j(\bar{X}_\pi;\mathbb{Q}_\ell)$. Thus the Riemann hypothesis is equivalent to the eigenvalues of the Frobenius action on $H^j(\bar{X}_\pi,\mathbb{Q}_\ell)$ being algebraic integers of modulus $q^{j/2}$.

The functional equation 6.1.4 (2) for $Z(t)$ comes straight from Poincaré duality and the fact that if $\alpha \in H^i(\bar{X}_\pi,\mathbb{Q}_\ell)$ and $\beta \in H^{2m-i}(\bar{X}_\pi,\mathbb{Q}_\ell)$ then the Poincaré pairing of $f^*\alpha$ and $f^*\beta$ is q^m times the Poincaré pairing of α and β. This is because of the naturality of the Poincaré pairing and because the Frobenius map

$$f^* : H^{2m}(\bar{X}_\pi,\mathbb{Q}_\ell) \to H^{2m}(\bar{X}_\pi,\mathbb{Q}_\ell)$$

is multiplication by q^m.

Having seen why ℓ-adic cohomology is useful for proving the Weil conjectures, let us consider how it is defined. For more details see Milne [1].

§6.3 <u>Étale topology and cohomology</u>

Let Y be a quasi-projective variety defined over an algebraically closed field k. The Zariski topology on Y is the topology whose closed subsets are the subsets defined by the vanishing of homogeneous polynomials (i.e. the closed subvarieties of Y). This topology reflects the algebraic structure of Y. However it is too coarse for many purposes. Of course when k is the complex field $\mathbb{C}$ we can also give Y the usual complex topology, by regarding it as a subset of a complex projective space, but in general this topology is not available.

The *étale topology* on Y plays a role similar to that of the complex topology. It is not a topology at all in the usual sense but it behaves in much the same way as a topology. Instead of open subsets of Y one works with étale morphisms $g: U \to Y$. Roughly speaking these are unbranched coverings of Zariski open subsets of Y.

More precisely, if U is a quasi-projective variety over k then $g: U \to Y$ is an étale morphism if and only if every $x_0 \in U$ satisfies the following condition. There are Zariski open neighbourhoods V of x_0 in U and W of $f(x_0)$ in Y, functions

$$a_j: W \to k, \qquad 1 \leq j \leq n,$$

such that each a_j is a rational function in the homogeneous coordinates on W and for each $x \in W$ the polynomial

$$p(T,x) = T^n + a_1(x)T^{n-1} + \ldots + a_n(x)$$

in T has simple roots, and an isomorphism

$$V \to \{(t,x) \in k \times W | p(t,x) = 0\}$$

whose projection onto W is g. An étale morphism $g: U \to Y$ is of finite type if and only if $g^{-1}(y)$ is finite for every y in Y.

If $g: U \to Y$ and $f: V \to U$ are étale morphisms of finite type then so is $g \circ f : V \to Y$. Moreover if $g: U \to Y$ and $f: V \to Y$ are étale morphisms of finite type then there is a commutative diagram of étale morphisms of finite type (called a pullback diagram),

$$\begin{array}{ccc} W & \xrightarrow{\ a\ } & U \\ b \downarrow & & \downarrow g \\ V & \xrightarrow[\ f\ ]{} & Y \end{array}$$

with the universal property that if

$$\begin{array}{ccc} W' & \xrightarrow{\ \tilde{a}\ } & U \\ \tilde{b} \downarrow & & \downarrow g \\ V & \xrightarrow[\ f\ ]{} & Y \end{array}$$

is another commutative diagram then there is a unique $\tau: W' \to W$ such that $\tilde{a} = a \circ \tau$ and $\tilde{b} = b \circ \tau$. The composition $g \circ a = f \circ b : W \to Y$ plays the role for the etale topology of the intersection of $g: U \to Y$ and $f: V \to Y$.

The definition of a *sheaf* F on Y with respect to the étale topology is closely analogous to the definition of a sheaf for a genuine topology on Y. For each étale morphism of finite type $g: U \to Y$ there is an abelian group $F(g)$ satisfying the following conditions.

6.3.1 (i) If $g: U \to Y$ and $f: V \to U$ are étale morphisms of finite type then there is a restriction map

$$F(g) \to F(g \circ f)$$

$$s \to s|_{g \circ f}$$

with the usual functorial properties (Milne [1, Chapter 2 §1]).

(ii) If $g: U \to Y$ and $g_i: U_i \to U$ are étale morphisms of finite type such that $U = \bigcup_{i \in I} g_i(U_i)$ and if $s_i \in F(g \circ g_i)$ satisfy

$$s_i|_{g \circ g_{ij}} = s_j|_{g \circ g_{ij}}$$

for all i and j where $g_{ij}: U_{ij} \to U$ fits into the pullback diagram

$$\begin{array}{ccc} U_{ij} & \longrightarrow & U_i \\ \downarrow & \overset{g_{ij}}{\searrow} & \downarrow g_i \\ U_j & \underset{g_j}{\longrightarrow} & U \end{array}$$

then there exists a unique $s \in F(g)$ such that $s_i = s|_{g \circ g_i}$ for all $i \in I$.

Sheaf maps are defined in the obvious way and we get a category $\acute{E}t\,Sh(Y)$ of etale sheaves on Y.

The definition of right derived functors given in §2.6 for left exact additive functors from the category $Sh(Y)$ to the category Ab of abelian groups can be adapted directly to define the right derived functors of functors from $\acute{E}t\,Sh(Y)$ to Ab. There is a functor $\Gamma_Y: \acute{E}t\,Sh(Y) \to Ab$

defined by

6.3.2 $\Gamma_Y(F) = F(1_Y\colon Y \to Y)$,

where 1_Y is the identity map on Y, and Γ_Y is a left exact additive functor. The étale cohomology groups of F are defined to be the right derived functors of Γ_Y applied to F:

6.3.3 $H^i_{ét}(Y,F) = R^i\,\Gamma_Y(F)$.

6.3.4 Remark. Alternatively one can adapt the definition of Čech cohomology and define étale Čech cohomology groups

$$\check{H}^i_{ét}\,(Y,F)$$

which for sufficiently well behaved sheaves F are canonically isomorphic to the groups

$$H^i_{ét}\,(Y,F).$$

Now suppose that ℓ is any prime number different from the characteristic p of k. The constant sheaf $(\mathbb{Q}_\ell)_Y$ on Y is defined by

$$(\mathbb{Q}_\ell)_Y(g\colon U \to Y) = \{\text{continuous maps } h\colon U \to \mathbb{Q}_\ell\}$$

where U is given the Zariski topology and $\mathbb{Q}_\ell$ has the discrete topology (so that a continuous map $h\colon U \to \mathbb{Q}$ is constant on every connected component of U). The restriction map

$$(Q_\ell)_Y(g) \to (\mathbb{Q}_\ell)_Y(g \circ f)$$

is given by composition with f.

6.3.5 Definition. The ℓ-adic cohomology of Y is defined to be

$$H^*(Y,\mathbb{Q}_\ell) = H^*_{ét}(Y,(\mathbb{Q}_\ell)_Y).$$

§6.4 The Weil conjectures for singular varieties and ℓ-adic intersection cohomology

Suppose that $Y = \bar{X}_\pi$ where X_π is the reduction modulo a prime ideal π of a complex projective variety $X \subseteq \mathbb{P}_N$ defined over an algebraic number ring R and $\bar{X}_\pi$ is the extension of X to a variety defined over the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q = R/\pi$. We have seen that if X is nonsingular and π is chosen appropriately then properties of the ℓ-adic cohomology of Y can be used to prove the Weil conjectures for X. What happens when X is allowed to be singular? The Weil conjectures certainly fail as they stand, but they can be made to work if one uses intersection cohomology throughout instead of ordinary cohomology. To see why this might be true we must define ℓ-adic intersection cohomology.

By enlarging the algebraic number ring R if necessary, we can assume that X has a Whitney stratification given by a filtration

$$X = X_m \supseteq X_{m-1} \supseteq \cdots \supseteq X_0$$

where each X_j is defined over R. Moreover we can assume that if Y_j is the extension to $\bar{\mathbb{F}}_q$ of the reduction of X_j modulo π then

$$Y = Y_m \supseteq Y_{m-1} \supseteq \cdots \supseteq Y_0$$

is a filtration of

$$Y = \bar{X}_\pi$$

by closed subvarieties Y_j such that $Y_j - Y_{j-1}$ is either empty or is nonsingular of dimension j for each j. We can now use Deligne's construction of intersection homology (§5.5) to define the ℓ-adic intersection cohomology

$$IH^*(Y,\mathbb{Q}_\ell)$$

of Y as follows. Let $i_k: Y - Y_{m-k} \to Y - Y_{m-k-1}$ be the inclusion. Define a complex of sheaves $\underline{\underline{IC}}^{\bullet}_Y$ on Y in the étale topology by

6.4.1 $$\underline{\underline{IC}}^{\bullet}_Y = \tau_{-m-1}R(i_m)_*\cdots\tau_{-2m+1}R(i_2)_*\tau_{-2m}R(i_1)_*(\mathbb{Q}_\ell)_{Y-Y_{m-1}}[2m]$$

where

$$(\mathbb{Q}_\ell)_{Y-Y_{m-1}}\,[2m]$$

is the complex on $Y-Y_{m-1}$ which is the constant sheaf $(\mathbb{Q}_\ell)_{Y-Y_{m-1}}$ in degree $i = -2m$ and 0 in other degrees (all with respect to the étale topology). Define

$$IH_*(Y,\mathbb{Q}_\ell)$$

to be the hypercohomology of this complex $\underline{\underline{IC}}_Y$ and let $IH^*(Y,\mathbb{Q}_\ell)$ be its dual. For more details see Beilinson, Bernstein and Deligne [1], Brylinski [1].

ℓ-adic intersection cohomology thus defined has the following properties.

6.4.2 (i) Comparison and change of base field. With the notation above we have

$$IH^*(Y,\mathbb{Q}_\ell) \cong IH^*(X,\mathbb{Q}_\ell)$$

where $Y = \bar{X}_\pi$. Moreover

$$\dim_{\mathbb{Q}_\ell} IH^i(X,\mathbb{Q}_\ell) = \dim_{\mathbb{C}} IH^i(X).$$

(ii) Poincaré duality. There is a perfect pairing

$$IH^i(Y,\mathbb{Q}_\ell) \otimes IH^{2m-i}(Y,\mathbb{Q}_\ell) \to \mathbb{Q}_\ell.$$

(iii) Lefschetz fixed point formula. If $f\colon Y \to Y$ is an isomorphism with isolated fixed points then the Lefschetz number

$$L(f) = \sum_{j=0}^{2m} (-1)^j \operatorname{Tr}(f^*\colon IH^j(Y,\mathbb{Q}_\ell) \to IH^j(Y,\mathbb{Q}_\ell))$$

of f is equal to the number of fixed points of f counted according to multiplicity. Unfortunately the definition of multiplicity becomes more complicated when the fixed point is a singular point of Y (cf. Goresky and MacPherson [6]).

As in §6.2 we consider the Frobenius map $f: \bar{X}_\pi \to \bar{X}_\pi$ defined by

$$f(x_0:\ldots:x_N) = (x_0^q:\ldots:x_N^q).$$

The Lefschetz number $L(f^r)$ of the rth iterate of f is the number of points of $\bar{X}_\pi$ defined over the field $\mathbf{F}_{q^r}$, but counted according to multiplicity. The multiplicity of a nonsingular point is one, but in general the multiplicity depends on the singularity of $\bar{X}_\pi$ at the point in question.

Just as in the nonsingular case the Frobenius map acts trivially on $IH^0(\bar{X}_\pi,\mathbb{Q}_\ell)$ and as multiplication by q^m on $IH^{2m}(\bar{X}_\pi,\mathbb{Q}_\ell)$. Moreover the eigenvalues of its action on

$$IH^j(\bar{X}_\pi,\mathbb{Q}_\ell)$$

for any j between 0 and 2m are algebraic integers α_{ji} with modulus

6.4.3 $\quad |\alpha_{ji}| = q^{j/2}.$

This fact, sometimes called the Riemann hypothesis as in §6.1, is very important. (Its proof makes use of Poincaré duality: once it has been shown that $|\alpha_{ji}| \leq q^{j/2}$ for all j then Poincaré duality gives the reverse inequality $|\alpha_{ji}| \geq q^{j/2}$). Its importance is not merely that it can be used to generalise the Weil conjectures to apply to singular projective varieties, provided one uses intersection homology and counts points according to multiplicities depending on the singularity of the points. Its main importance is that in any reasonable cohomology theory such as ℓ-adic intersection cohomology there are natural boundary maps and degeneracy maps appearing in long exact sequences, spectral sequences etc. These maps often go from the cohomology of one space to the cohomology of another space in a *different* dimension. Because these maps are natural, in the case of the ℓ-adic intersection cohomology of subvarieties of $\mathbf{P}_N(\bar{\mathbf{F}}_q)$ defined over $\mathbf{F}_q$ they must commute with the Frobenius maps. But since the eigenvalues of the Frobenius maps acting on ℓ-adic intersection cohomology groups of projective varieties in different dimensions are different this means that the boundary and degeneracy maps between such intersection cohomology groups must vanish. Using the comparison theorem one finds that the corresponding boundary maps and degeneracy maps for the ordinary intersection cohomology of complex projective varieties must vanish also. This enables one to prove important theorems about the ordinary intersection cohomology of complex varieties (see Beilinson, Bernstein and Deligne [1]).

6.4.4 Remark. At first sight this argument only applies when the complex varieties involved are defined over algebraic number rings. However a finite set of equations defining any complex projective variety can be "deformed" slightly without altering the intersection cohomology so that the equations become equations with coefficients in an algebraic number field.

§6.5 The Decomposition Theorem

One of the most important theorems about intersection cohomology which can be proved via the ℓ-adic intersection cohomology of varieties defined over fields of nonzero characteristic is the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (see Beilinson, Bernstein and Deligne [1], Goresky and MacPherson [3], MacPherson [1]).

6.5.1 Definition. Let $X \subseteq \mathbf{P}_n$ and $Y \subseteq \mathbf{P}_n$ be quasi-projective complex varieties. A regular map

$$\phi: X \to Y$$

is a map such that for each $x = (x_0:\ldots:x_n) \in X$ there exist homogeneous polynomials $f_0,\ldots,f_m$ in n+1 variables, all of the same degree and not all vanishing on $(x_0,\ldots,x_n)$, such that

$$\phi(y_0:\ldots:y_n) = (f_0(y_0,\ldots,y_n):\ldots:f_m(y_0,\ldots,y_n))$$

for all $(y_0:\ldots:y_n)$ in some neighbourhood U of x in X. ϕ is called *projective* if it can be factored as

$$X \xrightarrow{\psi} \mathbf{P}_N \times Y \xrightarrow{\chi} Y$$

for some N, where ψ is an isomorphism (i.e. a regular map with a regular inverse) of X onto a closed subvariety of $\mathbf{P}_N \times Y$ and χ is the projection of $\mathbf{P}_N \times Y$ onto Y. Note that every fibre $\phi^{-1}(y)$ of ϕ is a projective subvariety of P_N.

6.5.2 Decomposition theorem. Let ϕ: X → Y be a projective map between complex quasi-projective varieties. Then there exist closed subvarieties V_α of Y and local systems L_α on the nonsingular parts $(V_\alpha)_{\text{nonsing}}$ of V_α and integers ℓ_α such that

$$IH_j(X) \cong \bigoplus_\alpha IH_{j-\ell_\alpha}(V_\alpha, L_\alpha)$$

for all $j \geq 0$.

6.5.3 Remark. In fact this decomposition comes from a decomposition up to generalised quasi-isomorphism of complexes of sheaves over Y

$$\phi_* \underline{\underline{IC}}^\bullet_X \cong \bigoplus_\alpha i^*_\alpha \ \underline{\underline{IC}}^\bullet_{(V_\alpha, L_\alpha)} [\ell_\alpha]$$

where i_α: V_α → Y is the inclusion.

Two special cases are important.

6.5.4 (1) Suppose that ϕ:X → Y is a resolution of singularities of Y. That is, X is nonsingular and ϕ is a surjective projective map which restricts to an isomorphism from a dense open subset of X to the nonsingular part Y_{nonsing} of Y. Then there is a unique α, say α_0, such that V_{α_0} = Y, and moreover $\ell_{\alpha_0} = 0$ and L_{α_0} is the constant local system $\mathbb{C}$. Thus

$$H_j(X) \cong IH_j(X) \cong IH_j(Y) \oplus \Big(\bigoplus_{\alpha \neq \alpha_0} IH_{j-\ell_\alpha}(V_\alpha, L_\alpha)\Big).$$

In other words the intersection homology of any quasi-projective variety Y is a direct summand of the ordinary homology of any resolution of singularities X of Y.

(2) Suppose that ϕ:X → Y is a projective map which is topologically a fibration, with fibre V (a projective variety). That is, every y ∈ Y has an open neighbourhood U in Y such that there is a homeomorphism

$$\phi^{-1}(U) \to U \times V$$

whose projection onto U is $\phi: \phi^{-1}(U) \to U$. In such a situation there is a spectral sequence $\{E_r^{p,q}\}$ called the *Leray spectral sequence* which abuts to

$$H^{p+q}(X)$$

and has E_2 term given by

$$E_2^{p,q} = H^p(Y, H^q(V)),$$

where $H^q(V)$ denotes the local system L on Y such that

$$L_y = H^q(V_y)$$

where $V_y = \phi^{-1}(y) \cong V$ for each $y \in Y$ (Bott and Tu [1, p. 169], Griffiths and Harris [1, p. 463]). Similarly there is a spectral sequence of intersection cohomology abutting to

$$IH^{p+q}(X)$$

with E_2 term $IH^p(Y; IH^q(V))$. The decomposition theorem for ϕ is equivalent to the degeneration of this spectral sequence at the E_2 term. That is, it says that

$$IH^j(X) \cong \bigoplus_{p+q=j} IH^p(Y, IH^q(V))$$

or equivalently

$$IH_j(X) \cong \bigoplus_{p+q=j} IH_p(Y, IH_q(V)).$$

6.5.5 Example. (Cheeger, Goresky and MacPherson [1, §5.2]). If $a \leq b$ are positive integers the Grassmann variety

$$Gr(a,\mathbb{C}^b) = \{a\text{-dimensional subspaces of } \mathbb{C}^b\})$$

is a nonsingular projective variety of dimension

$a(b-a)$.

If M is a fixed subspace of $\mathbb{C}^b$ and $c \leq a$ is a positive integer then

$$S = \{V \in Gr(a,\mathbb{C}^b) | \dim V \cap M \geq c\}$$

is a projective subvariety of $Gr(a,\mathbb{C}^b)$ called a *single condition Schubert variety*. There is a resolution of singularities

$$\phi: \tilde{S} \to S$$

where

$$\tilde{S} = \{(V,W) \in Gr(a,\mathbb{C}^b) \times Gr(c,\mathbb{C}^b) \mid W \subseteq V \cap M\}$$

and

$$\phi(V,W) = V.$$

If we choose an isomorphism of M with $\mathbb{C}^d$ where $d = \dim M$ then we can define

$$\rho: \tilde{S} \to Gr(c,\mathbb{C}^b)$$

by

$$\rho(V,W) = W.$$

It is easy to check that ρ is a projective fibration with fibre

$$Gr(a-c, \mathbb{C}^{b-c}).$$

Thus the decomposition theorem applied to ϕ and ρ tells us that $IH_*(S)$ is a direct summand of

$$IH_*(\tilde{S}) \cong H_*(\tilde{S})$$

and that

$$H_j(\tilde{S}) \cong \bigoplus_{p+q=j} H_p(Gr(c,\mathbb{C}^b),\ H_q(Gr(a-c,\mathbb{C}^{b-c})).$$

Since Grassmann varieties are simply connected any local system over $Gr(c,\mathbb{C}^b)$ is trivial, so we get

$$H_j(\tilde{S}) \cong \bigoplus_{p+q=j} H_p(Gr(c,\mathbb{C}^b)) \otimes H_q(Gr(a-c,\mathbb{C}^{b-c})).$$

The homology of Grassmann varieties is well known (see e.g. Griffiths and Harris [1, Chapter 1 §5]). The Betti numbers

$$B_j = \dim H_j(Gr(a,\mathbb{C}^b))$$

of $Gr(a,\mathbb{C}^b)$ are given by the formula

$$\sum_{j\geq 0} B_j t^j = \frac{\prod_{a\leq i<b} (1+t^2+t^4+\ldots+t^{2i})}{\prod_{1\leq j<b-a} (1+t^2+t^4+\ldots+t^{2j})} .$$

6.5.6 <u>Definition</u>. A resolution of singularities $\phi: X \to Y$ is called *small* if for every $r > 0$

$$\operatorname{codim} \{x \in Y \mid \dim \phi^{-1}(x) \geq r\} > 2r.$$

6.5.7 <u>Theorem</u> (Goresky and MacPherson [5, §6.2]). If $\phi: X \to Y$ is a small resolution then

$$IH_*(X) \cong IH_*(Y).$$

This theorem can be proved by showing that the sheaf complex $\phi_* \underline{\underline{IC}}^{\bullet}_X$ on Y satisfies the criteria 5.4.6 (a) - (d) which characterise $\underline{\underline{IC}}^{\bullet}_Y$ up to generalised quasi-isomorphism, and that its hypercohomology is $IH_*(X)$.

6.5.8 <u>Exercise</u>. The resolution $\phi:\tilde{S} \to S$ of the Schubert variety S defined at 6.5.5 is a small resolution.

Thus

$$IH_*(S) \cong H_*(\tilde{S}) \cong H_*(Gr(c,\mathbb{C}^b)) \otimes H_*(Gr(a-c,\mathbb{C}^{b-c})).$$

6.5.9 Remark. Every quasi-projective complex variety has a resolution of singularities but not every quasi-projective complex variety has a small resolution. Some varieties have more than one small resolution. One can show, for example, that intersection cohomology has no natural ring structure generalising the cup product on ordinary cohomology by exhibiting a variety with two small resolutions whose cohomology rings are not isomorphic.

7 D-modules and the Riemann-Hilbert correspondence

This chapter contains a brief sketch of the theory of $\mathcal{D}$-modules and their relationship to intersection homology. For further details see e.g. Borel [2], Deligne [3], Bernstein [1], Malgrange [1], Kashiwara, Kawai and Kimura [1], Kashiwara [1], [2], [3], Le Dung Trang and Mebkhout [1], Oda [1].

§7.1 The Riemann-Hilbert problem

Consider the system of m first-order differential equations

7.1.1 $$\frac{df_i}{dz} = \sum_{j=1}^{m} a_{ij}(z)\, f_j(z), \qquad 1 \leq i \leq m$$

in m complex-valued functions of one complex variable z, where each $a_{ij}(z)$ is a meromorphic function of z defined on a connected open subset U of $\mathbf{P}_1 = \mathbb{C} \cup \{\infty\}$.

7.1.2 Example. A single mth order differential equation

$$\frac{d^m f}{dz^m} + a_1(z)\frac{d^{m-1}f}{dz^{m-1}} + \ldots + a_m(z)\, f(z) = 0$$

is equivalent to the system of equations

$$\frac{df_i}{dz} = f_{i+1}, \qquad 1 \leq i \leq m-1,$$

$$\frac{df_m}{dz} = -a_1(z)\, f_m(z) - \ldots - a_m(z)\, f_1(z).$$

If each meromorphic function $a_{ij}(z)$ is holomorphic on U then the solutions of the system 7.1.1 are multivalued holomorphic functions of $z \in U$ and the space Σ of solutions is a vector space of dimension m. However if at least one of the coefficients $a_{ij}(z)$ has a singularity at some $b \in U$ then in general the solutions have branch points at b and b is called a *singular point* of the system.

7.1.3 Definition. A singular point $b \in U$ is called a *regular singular point* of the system 7.1.1 if whenever

$$(f_1(z), \ldots, f_m(z))$$

is a multivalued solution of the system near b then there is some positive integer r such that

$$|z-b|^r f_j(z) \to 0$$

for each j as $z \to b$. Otherwise b is called an *irregular* singular point.

When all the functions $a_{ij}(z)$ are rational (i.e. they are meromorphic on $\mathbf{P}_1$) then the system 7.1.1 is said to be of *Fuchsian type* if all the singular points are regular. The mth order equation 7.1.2 is said to be of Fuchsian type if the corresponding system of m first order differential equations is of Fuchsian type.

7.1.4 Example. Let α be a fixed complex number. The equation

$$\frac{df}{dz} = \frac{\alpha}{z} f(z)$$

has solutions $f(z) = cz^{\alpha}$ for $c \in \mathbb{C}$. If $\alpha \notin \mathbf{Z}$ these solutions are multivalued with branch points at 0, and 0 is a regular singular point of the system.

In fact the system 7.1.1 has a regular singular point at 0 if and only if it is equivalent to a system of the same form such that the coefficients $a_{ij}(z)$ have poles of order at most one at 0 (see e.g. Borel [2, III 1.3.1]).

Now let $b_o, \ldots, b_k$ be the points of U which are singular points for the system 7.1.1. If γ is a closed path in

$$U - \{b_o, \ldots, b_k\}$$

then analytic continuation along γ induces a linear transformation $\phi(\gamma): \Sigma \to \Sigma$ of the space of solutions. If we choose a basis of Σ we get a representation

7.1.5 $\phi: \pi_1(U - \{b_o,\dots,b_k\}) \to GL(m,\mathbb{C})$

of the fundamental group of $U-\{b_o,\dots,b_k\}$. This representation ϕ is called the *monodromy* of the system (with respect to the chosen basis of Σ). Note that up to a choice of basis such a representation ϕ corresponds exactly to a *local system* L on $U-\{b_o,\dots,b_k\}$ with $L_x \cong \mathbb{C}^m$ for all $x \in U - \{b_o,\dots,b_k\}$ (cf. §3.9).

In 1857 Riemann posed the following problem. Given points $b_o,\dots,b_k \in \mathbf{P}_1$ and a faithful representation

$$\phi: \pi_1(\mathbf{P}_1 - \{b_o,\dots,b_k\}) \to GL(m,\mathbb{C})$$

find all systems of Fuchsian type whose singular points are $b_o,\dots,b_k$ and whose monodromy (with respect to some basis of the space of solutions) is ϕ. Riemann showed that when $m = k = 2$ there is a unique system of Fuchsian type with given singular points b_o,b_1,b_2 and given monodromy

$$\phi: \pi_1(\mathbf{P}_1 - \{b_o,b_1,b_2\}) \to GL(2;\mathbb{C}).$$

When the singular points are 0, 1, ∞ this system is given by the hypergeometric equation

7.1.6 $$z(1-z)\,\frac{d^2f}{dz^2} + (\gamma - (\alpha+\beta+1)z)\,\frac{df}{dz} - \alpha\beta f = 0$$

where α, β, γ are constants depending on the monodromy ϕ.

When in 1900 Hilbert listed twenty three problems as targets for mathematicians in the twentieth century he included a generalisation of Riemann's question. It is easy to extend to arbitrary compact Riemann surfaces the definitions of systems of first order differential equations with meromorphic coefficients, systems of Fuchsian type and monodromy. (One way to identify functions on a fixed compact Riemann surface S with multi-valued functions on $\mathbf{P}_1$). Suppose we are given a compact Riemann surface S, points $b_o,\dots,b_k$ of S and a representation

$$\phi: \pi_1(S - \{b_o,\dots,b_k\}) \to GL(m,\mathbb{C})$$

of the fundamental group $\pi_1(S - \{b_0,\ldots,b_k\})$. Hilbert's twenty first problem (often called the *Riemann-Hilbert problem*) was to find those systems of differential equations of Fuchsian type over S whose monodromy is ϕ.

Many mathematicians worked on this problem and it was finally shown a hundred years after Riemann posed his original question that there is an exact correspondence between systems of Fuchsian type and their monodromy representations (see Röhrl [1]). This correspondence is often called the *Riemann-Hilbert correspondence*. However systems with irregular singularities are not determined by their monodromy representations.

So far we have been considering systems of differential equations in *one* complex variable, i.e. over a one-dimensional complex manifold. In this chapter we shall discuss a more general form of the Riemann-Hilbert correspondence which relates differential systems (or $\mathcal{D}$-modules) on a complex quasi-projective variety X to the intersection sheaf complexes of subvarieties of X with coefficients in local systems.

§7.2 Differential systems over $\mathbb{C}^n$.

Fix $n \geq 1$ and let $\mathcal{O}$ denote either the ring of holomorphic functions on $\mathbb{C}^n$ or the ring of polynomial functions on $\mathbb{C}^n$. The choice we make depends on whether we wish later to study holomorphic $\mathcal{D}$-modules or algebraic $\mathcal{D}$-modules. We shall mainly be interested in algebraic $\mathcal{D}$-modules, but the theories are very closely related. Let $\mathcal{D}$ be the ring of differential operators generated by the ring $\mathcal{O}$ together with

$$D_1, D_2, \ldots, D_n$$

(which are to be thought of as $\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}$ where $z_1,\ldots,z_n$ are complex coordinates on $\mathbb{C}^n$), satisfying the relations

$$D_iD_j = D_jD_i,$$

$$D_ig = g\,D_i + \frac{\partial g}{\partial z_i} \quad \text{if} \quad g \in \mathcal{O}.$$

Then $\mathcal{D}$ acts on $\mathcal{O}$ via $g.f = gf$, $D_if = \frac{\partial f}{\partial z_i}$ for $f \in \mathcal{O}$.

7.2.1 Definition. A *differential system* on $\mathbb{C}^n$ is a left $\mathcal{D}$-module M such that there is an exact sequence of left $\mathcal{D}$-modules

$$\mathcal{D}^p \to \mathcal{D}^q \to M \to 0$$

where $\mathcal{D}^p = \mathcal{D} \oplus \ldots \oplus \mathcal{D}$ is the direct sum of $\mathcal{D}$ with itself p times. A *solution* of the differential system M with values in a left $\mathcal{D}$-module F is a map of left $\mathcal{D}$-modules

$$\phi: M \to F$$

7.2.2 Motivation. If M is a differential system with an exact sequence

$$\mathcal{D}^p \to \mathcal{D}^q \to M \to 0$$

then M is generated as a left $\mathcal{D}$-module by the images $f_1,\ldots,f_q$ under the surjection $\mathcal{D}^q \to M$ of the usual basis $e_1,\ldots,e_q$ of $\mathcal{D}^q$ as a left $\mathcal{D}$-module. Moreover the kernel of this surjection is the image of the map $\mathcal{D}^p \to \mathcal{D}^q$. Hence it is generated as a left $\mathcal{D}$-module by the images $r_1,\ldots,r_p$, say, of the standard basis of $\mathcal{D}^p$. We can write

$$r_i = \sum_{1 \leq j \leq q} d_{ij} e_j$$

where each d_{ij} is an element of $\mathcal{D}$. Then the generators $f_1,\ldots,f_q$ of M satisfy the relations

$$\sum_{1 \leq j \leq q} d_{ij}\, f_j = 0, \qquad 1 \leq i \leq p.$$

A solution ϕ of M with values in F is uniquely determined by the images $\phi(f_1),\ldots,\phi(f_q)$ of the generators $f_1,\ldots,f_q$ of M in F. If $\phi_1,\ldots,\phi_q$ are elements of F then there is a solution $\phi: M \to F$ such that $\phi(f_j) = \phi_j$ for $1 \leq j \leq q$ if and only if the ϕ_j satisfy the equations

$$\sum_{1 \leq j \leq q} d_{ij}\, \phi_j = 0, \qquad 1 \leq i \leq p.$$

Thus a differential system M on $\mathbb{C}^n$ together with a choice of generators and relations for M is "equivalent" to a finite set of partial differential equations in a finite number of unknown functions in the variables $z_1,\ldots,z_n$.

7.2.3 Examples. (1) The equations

$$\frac{\partial f}{\partial z_1} = 0, \quad z_1 \frac{\partial f}{\partial z_2} + z_2 \frac{\partial f}{\partial z_3} + \ldots + z_{n-1} \frac{\partial f}{\partial z_n} = 0$$

define a differential system M_1 on $\mathbb{C}^n$ with one generator f and two relations $D_1 f$ and $(z_1 D_2 + z_2 D_3 + \ldots + z_{n-1} D_n) f$. Thus there is an exact sequence of left $\mathcal{D}$-modules

$$\mathcal{D}^2 \to \mathcal{D} \to M_1 \to 0.$$

(2) The equations $\frac{\partial f}{\partial z_1} = 0, \ \frac{\partial f}{\partial z_2} = 0, \ldots, \frac{\partial f}{\partial z_n} = 0$

define a differential system M_2 on $\mathbb{C}^n$ with one generator f and n relations $D_1 f,\ldots,D_n f$. Recall that the *commutator* of two elements δ_1,δ_2 of $\mathcal{D}$ is

$$[\delta_1,\delta_2] = \delta_1\delta_2 - \delta_2\delta_1.$$

The differential systems M_1 and M_2 are isomorphic (as left $\mathcal{D}$-modules) because

$$[D_j, z_1 D_2 + z_2 D_3 + \ldots + z_{n-1} D_n] = D_{j+1}$$

for $1 \leq j < n$, so the left ideal of $\mathcal{D}$ generated by D_1 and $z_1 D_2 + \ldots + z_{n-1} D_n$ is the same as the left ideal generated by $D_1,\ldots,D_n$.

(3) When n = 1 write z for z_1 and D for D_1.

Consider the differential system M_3 with one generator f and one relation

$$D^m f + a_1 D^{m-1} f + \ldots + a_m f$$

where $a_1,\ldots,a_m \in \mathcal{O}$, corresponding to the differential equation

$$\frac{\partial^m f}{\partial z^m} + a_1(z)\frac{\partial^{m-1} f}{\partial z^{m-1}} + \ldots + a_m(z)f = 0.$$

It was observed at 7.1.2 that M_3 is isomorphic to the differential system M_4 with m generators $f_1,\ldots,f_m$ and m relations

$$Df_1 - f_2 ,\ldots, Df_{m-1} - f_m$$

and

$$Df_m + a_1 f_m + \ldots + a_m f_1.$$

§7.3 $\mathcal{D}_X$-modules and intersection homology

We can globalise the definition of a differential system. Let X be either a complex manifold or a complex quasi-projective variety. Denote by $\mathcal{O}_X$ the sheaf of holomorphic (respectively regular) functions on X. That is, if U is an open subset of X (in either the complex topology or the Zariski topology) then

$$\mathcal{O}_X(U) = \{\text{holomorphic functions } h: U \to \mathbb{C}\}$$

or

$$\mathcal{O}_X(U) = \{\text{regular functions } h: U \to \mathbb{C}\}.$$

Recall that a regular function h is one which can be expressed locally with respect to homogeneous coordinates on the ambient projective space as the quotient P/Q of a homogeneous polynomial P by a locally nonvanishing homogeneous polynomial Q of the same degree. A regular function $\mathbb{C}^n \to \mathbb{C}$ is just a polynomial function.

A differential operator on X is a sheaf map

$$\delta: \mathcal{O}_X \to \mathcal{O}_X$$

such that in local coordinates δ is given by a differential operator on a subset of $\mathbb{C}^n$ with either holomorphic or regular coefficients. Equivalently for some positive integer k and every open $U \subseteq X$

$$\delta(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$$

satisfies

$$[\hat{f}_k[\hat{f}_{k-1}\ldots[\hat{f}_1,[\hat{f}_o,\delta]]\ldots]] = 0$$

for any $f_o,\ldots,f_k \in \mathcal{O}_X(U)$ where $\hat{f}_j$ is the operator on $\mathcal{O}_X(U)$ given by multiplication by f_j.

The sheaf $\mathcal{D}_X$ on X is defined by

$$\mathcal{D}_X(U) = \{\text{differential operators on } U\}$$

for U open in X. Note that both $\mathcal{O}_X$ and $\mathcal{D}_X$ are sheaves of rings on X, in the sense that if U is open in X then $\mathcal{O}_X(U)$ and $\mathcal{D}_X(U)$ are rings and the restriction maps preserve the ring structure.

If A is a sheaf of rings on X then a left A-module is a sheaf M on X such that for each open U in X the abelian group $M(U)$ is an $A(U)$-module and the restriction maps respect the module structure.

7.3.1 Definition. A sheaf of rings A on X is called *coherent* if given any map of left A-modules

$$\theta:A^p \to A^q$$

then for all $x \in X$ there exist open neighbourhoods U of x in X and finitely many sections $\sigma_1,\ldots,\sigma_r$ of $\ker\theta$ over U such that $\sigma_1,\ldots,\sigma_r$ generate $\ker\theta|_U$ as an $A|_U$-module. That is, the map

$$(A|_U)^r \to \ker\theta|_U$$

given by sending $(\alpha_1,\ldots,\alpha_r) \in A(V)^r$ to

$$\alpha_1\sigma_1|_V + \ldots + \alpha_r\sigma_r|_V \in \ker\theta(V)$$

for $V \subseteq U$ is surjective. Equivalently there is an exact sequence of left $A|_U$-modules

$$A^r|_U \to A^p|_U \xrightarrow{\theta} A^q|_U.$$

7.3.2 Theorem. (see e.g. Borel [2, II § 3]). $\mathcal{O}_X$ and $\mathcal{D}_X$ are coherent sheaves of rings on X.

7.3.3 Definition. If A is a coherent sheaf of rings on X then a left A-module M is called *coherent* if every $x \in X$ has an open neighbourhood U in X such that there is an exact sequence

$$A^p|_U \to A^q|_U \to M|_U \to 0$$

of left $A|_U$-modules.

Coherent A-modules on X are better behaved than A-modules in general. Some pathological examples are avoided by imposing the condition of coherence (cf. Pham [1, § 2.6]).

The natural way to globalise the definition of a differential system is now the following.

7.3.4 Definition. A *differential system* on X is a coherent left $\mathcal{D}_X$-module M.

We shall see that differential systems are closely related to intersection homology as follows. Let X be a nonsingular complex projective variety of dimension n. Let Ω_X^r be the sheaf of holomorphic sections of the bundle $\Lambda^r T^* X$ where $T^* X$ is the *complex* cotangent bundle of X. Then Ω_X^r is a left $\mathcal{O}_X$-module. A local section ω of Ω_X^r is given in local coordinates $z_1,\ldots,z_n$ on X by

$$\omega(z) = \sum_{i_1<\ldots<i_r} a_{i_1\ldots i_r}(z)\, dz_{i_1} \wedge \ldots \wedge dz_{i_r}$$

where the coefficients $a_{i_1\ldots i_r}$ are holomorphic functions of $z = (z_1,\ldots,z_n)$. We define $d\colon \Omega_X^r \to \Omega_X^{r+1}$ in local coordinates by

$$d\omega(z) = \sum_{i_1<\ldots<i_r} \sum_{1\le k\le n} (\partial a_{i_1\ldots i_r}/\partial z_k)\, dz_k \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_r}.$$

Given a coherent $\mathcal{D}_X$-module M we define the *De Rham complex* DR(M) of M to be the complex

$$0 \to M \xrightarrow{d_M} \Omega_X^1 \otimes_{\mathcal{O}_X} M \xrightarrow{d_M} \Omega_X^2 \otimes_{\mathcal{O}_X} M \dots \Omega_X^n \otimes_{\mathcal{O}_X} M \to 0$$

where in local coordinates $z_1,\dots,z_n$ the sheaf map d_M is given by

$$d_M(\omega \otimes m) = d\omega \otimes m + \sum_{1\leq k\leq n} (dz_k \wedge \omega) \otimes D_k\omega.$$

(cf. Pham [1, §2.14.2]).

The *Riemann-Hilbert correspondence* will tell us that, under the De Rham functor DR, irreducible holonomic $\mathcal{D}_X$-modules with regular singularities correspond exactly to the intersection sheaf complexes of irreducible subvarieties of X with coefficients in local systems, up to generalised quasi-isomorphism.

In order to explain the Riemann-Hilbert correspondence we must define holonomic $\mathcal{D}_X$-modules with regular singularities.

§7.4 The characteristic variety of a $\mathcal{D}_X$-module

Let $P \in \mathcal{D}$ be a differential operator on $\mathbb{C}^n$. Then we can write

$$P = \sum_{|\alpha|\leq m} c_\alpha(z)\, D^\alpha$$

where $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$ and $c_\alpha(z) \in \mathcal{O}$. If m is chosen as small as possible then m is called the *order* of P, and the *principal symbol* of P is

$$\sigma(P) = \sum_{|\alpha|=m} c_\alpha(z)\, \xi^\alpha \in \mathcal{O}[\xi_1,\dots,\xi_n]$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ and $\mathcal{O}[\xi_1,\dots,\xi_n]$ is the polynomial ring in $\xi_1,\dots,\xi_n$ with coefficients in $\mathcal{O}$. For any $m \in \mathbb{N}$ we define the *mth symbol* $\sigma_m(P)$ of P by the same formula.

Let $\mathcal{D}^{(m)} \subseteq \mathcal{D}$ be the set of differential operators of order at most m. (By convention the operator 0 has order $-\infty$). Let Σ be the ring

$$\Sigma = \bigoplus_{m\geq 0} \mathcal{D}^{(m)} / \mathcal{D}^{(m-1)}$$

with multiplication

$$(\mathcal{D}^{(m)}/\mathcal{D}^{(m-1)}) \otimes (\mathcal{D}^{(\ell)}/\mathcal{D}^{(\ell-1)}) \to \mathcal{D}^{(\ell+m)}/\mathcal{D}^{(\ell+m-1)}$$

defined by the composition of differential operators

$$\mathcal{D}^{(m)} \otimes \mathcal{D}^{(\ell)} \to \mathcal{D}^{(\ell+m)}.$$

There is an isomorphism

$$\Sigma \to \mathcal{O}[\xi_1,\ldots,\xi_n]$$

whose restriction to $\mathcal{D}^{(m)}/\mathcal{D}^{(m-1)}$ is induced by the symbol σ_m. We shall use this isomorphism to identify Σ with the polynomial ring $\mathcal{O}[\xi_1,\ldots,\xi_n]$.

Now consider a differential system $\mathcal{M}$ on $\mathbb{C}^n$ with a given exact sequence

$$\mathcal{D}^p \to \mathcal{D}^q \to \mathcal{M} \to 0.$$

Let $\mathcal{M}^{(m)}$ be the image of $(\mathcal{D}^{(m)})^q$ in $\mathcal{M}$, and let

$$\mathrm{Gr}\,\mathcal{M} = \bigoplus_{m\geq 0} \frac{\mathcal{M}^{(m)}}{\mathcal{M}^{(m-1)}} .$$

Then $\mathrm{Gr}\,\mathcal{M}$ is a coherent Σ-module. Let I be the ideal in Σ which is the annihilator of $\mathrm{Gr}\,\mathcal{M}$ and let $\sqrt{I}$ be its radical. Then

$$I = \{p \in \mathcal{O}[\xi_1,\ldots,\xi_n] \mid pu = 0, \ \forall u \in \mathrm{Gr}\,\mathcal{M}\}$$

and

$$\sqrt{I} = \{p \in \mathcal{O}[\xi_1,\ldots,\xi_n] \mid \exists\ k \geq 1,\ p^k \in I\}.$$

7.4.1 Theorem. $\sqrt{I}$ depends only on $\mathcal{M}$, not on the choice of exact sequence

$$\mathcal{D}^p \to \mathcal{D}^q \to \mathcal{M} \to 0.$$

Sketch proof. (For more details see e.g. Pham [1, §2.8]). One shows that

$\sqrt{T}$ is a homogeneous ideal (that is, it is generated by homogeneous polynomials) and that the following two statements hold.

(i) If $P \in \mathcal{D}^{(m)}$ and $p = \sigma_m(P)$ then $p \in \sqrt{T}$ if and only if

$$p^s M^{(\ell)} \subseteq M^{(\ell+ms-r(s))} \quad \forall\ \ell \in \mathbb{N},\ \ s \in \mathbb{N}$$

where $r(s) \to \infty$ as $s \to \infty$.

(ii) If $\{\tilde{M}^{(m)} | m \geq 0\}$ is another such filtration of M then there exist $\lambda, \mu \in \mathbb{N}$ such that

$$M^{(\ell)} \subseteq \tilde{M}^{(\ell+\lambda)}, \quad \tilde{M}^{(\ell)} \subseteq M^{(\ell+\mu)}$$

for all $\ell \geq 0$.

This is enough to prove the theorem.

We can globalise Theorem 7.4.1. Let X be a complex manifold or a non-singular quasi-projective complex variety as before. Then the sheaf of rings $\mathcal{D}_X$ is graded by a filtration $\mathcal{D}_X^{(m)}$ in the obvious way and the sheaf of rings

$$\mathrm{Gr}\ \mathcal{D}_X = \bigoplus_{m \geq 0} \frac{\mathcal{D}_X^{(m)}}{\mathcal{D}_X^{(m-1)}}$$

can be naturally identified with the sheaf of holomorphic (or regular) functions on T^*X which are polynomial in the variables in the fibre direction.

If U is an open subset of X and

$$P \in \mathcal{D}_X^{(m)}(U)$$

is a differential operator of order m over U then the symbol $\sigma(P)$ of P is the image of P under the composition

$$\mathcal{D}_X^{(m)}(U) \to \frac{\mathcal{D}_X^{(m)}(U)}{\mathcal{D}_X^{(m-1)}(U)} \to \mathrm{Gr}\ \mathcal{D}_X(U).$$

Now let M be a coherent $\mathcal{D}_X$-module. One can always find *locally* a "good filtration" (Borel [2, II §4], Pham [1, §2.8])

$$M^{(0)} \subseteq M^{(1)} \subseteq \ldots \subseteq M^{(m)} \subseteq \ldots$$

of M; that is, a filtration which satisfies the conditions

7.4.2 (i) $\mathcal{D}^{(r)}M^{(m)} \subseteq M^{(r+m)}$ with equality when m is sufficiently large;

(ii) $M^{(m)}$ is a coherent $\mathcal{O}_X$-module.

Then (locally)

$$\mathrm{Gr}\, M = \bigoplus_{m \geq 0} \frac{M^{(m)}}{M^{(m-1)}}$$

is a coherent sheaf of Gr $\mathcal{D}_X$-modules, and its annihilator I is a coherent sheaf of ideals in Gr $\mathcal{D}_X$. Moreover locally $\sqrt{I}$ is a coherent sheaf of ideals in Gr $\mathcal{D}_X$ which is *independent* of the choice of filtration. This means that $\sqrt{I}$ is well defined globally as a sheaf of ideals in Gr $\mathcal{D}_X$, which is a sheaf of functions on T^*X. The set of zeros of this sheaf of ideals is a closed analytic (or quasi-projective) subvariety of T^*X called the *characteristic variety*

7.4.3 Ch(M)

of M. Since $\sqrt{I}$ is generated by elements which are homogeneous polynomials in the variables $\xi_1,\ldots,\xi_n$ of the fibre directions of T^*X it follows that Ch(M) is a *conical* subvariety of T^*X, i.e. it is invariant under scalar multiplication in the fibres of T^*X.

§7.5 Holonomic differential systems

It is easy to prove the following lemma.

7.5.1 Lemma. If $P \in \mathcal{D}^{(m)}$ and $Q \in \mathcal{D}^{(\ell)}$ then the commutator $[P,Q]$ of P and Q is an element of $\mathcal{D}^{(\ell+m-1)}$ and

$$\sigma_{\ell+m-1}([P,Q]) = \sum_{1\leq i\leq n} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \xi_i}$$

where $f = \sigma_m(P)$ and $g = \sigma_\ell(Q)$.

7.5.2 Definition. If f and g lie in $[\xi_1,\ldots,\xi_n]$ then the Poisson bracket $\{f,g\} \in \mathcal{O}[\xi_1,\ldots,\xi_n]$ of f and g is defined by

$$\{f,g\} = \sum_{1\leq i\leq n} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \xi_i} .$$

7.5.3 Theorem. (cf. Pham [1, §2.9], Sato, Kawai and Kashiwara [1]). If $\mathcal{M}$ is a differential system on $\mathbb{C}^n$ and $\sqrt{I}$ is defined as in §7.4 then $\sqrt{I}$ is *involutive*, in the sense that if $f \in \sqrt{I}$ and $g \in \sqrt{I}$ then $\{f,g\} \in \sqrt{I}$.

This theorem can be globalised in the obvious way.

If X is a complex manifold then T^*X has a holomorphic symplectic form ω defined in local coordinates by

$$\omega = \sum_{1\leq i\leq n} d\xi_i \wedge dz_i .$$

That is, ω is a holomorphic section of $\Lambda^2T^*(T^*X)$ such that

$$d\omega = 0$$

and when elements of the fibres of $\Lambda^2T^*(T^*X)$ are identified with skew-symmetric bilinear forms on the tangent spaces to T^*X the skew-symmetric bilinear form defined by ω on each tangent space is nondegenerate.

Theorem 7.5.3 has an infinitesimal counterpart in terms of this symplectic form ω.

7.5.4 Theorem. Let η be a nonsingular point of the characteristic variety $Ch(\mathcal{M})$ of a differential system $\mathcal{M}$ on X. Then

$$(T_\eta\, Ch(\mathcal{M}))^\perp \subseteq T_\eta Ch(\mathcal{M})$$

where $\perp$ denotes the orthogonal complement in $T_\eta(T^*X)$ with respect to the holomorphic symplectic form ω.

It follows immediately from 7.5.4 that if the characteristic variety $Ch(\mathcal{M})$ is nonempty then its dimension satisfies

$$\dim \mathrm{Ch}(\mathcal{M}) \geq \dim T^*X - \dim \mathrm{Ch}(\mathcal{M}),$$

i.e.

7.5.5 $\dim \mathrm{Ch}(\mathcal{M}) \geq \dim X.$

7.5.6 Definition. A differential system $\mathcal{M}$ on X is called holonomic if $\mathcal{M} = 0$ or

$$\dim \mathrm{Ch}(\mathcal{M}) = n$$

where n is the dimension of X. Equivalently for every nonsingular point η of $\mathrm{Ch}(\mathcal{M})$ we have

$$(T_\eta \mathrm{Ch}(\mathcal{M}))^\perp = T_\eta \mathrm{Ch}(\mathcal{M}).$$

Such a subvariety of T^*X is called a *Lagrangian* subvariety of T^*X.

Holonomic differential systems used to be called "maximally overdetermined"; "overdetermined" because the number of independent equations is greater than or equal to the number of unknowns (otherwise $\sqrt{I} = 0$ so the system is not holonomic) and "maximally" because $\sqrt{I}$ is as large as possible (equivalently the characteristic variety is as small as possible).

§7.6 Examples of characteristic varieties

Consider the differential system $\mathcal{M}_1$ on $\mathbb{C}$ defined by one generator u and one equation

7.6.1 $(zD - \alpha)^q u = 0$

where $\alpha \in \mathbb{C}$, $q \in \mathbb{N}$, z is the coordinate on $\mathbb{C}$ and $D = \frac{\partial}{\partial z}$. The filtration

$$\mathcal{M}_1^{(0)} \subseteq \mathcal{M}_1^{(1)} \subseteq \ldots$$

of $\mathcal{M}_1$ defined by this choice of generator u is given by

$$M_1^{(m)} = \mathcal{D}^{(m)}u$$

$$= \{a_m(z)D^m u + \ldots + a_o(z)u \mid a_j(z) \in \mathcal{O},\ 0 \leq j \leq m\}$$

Hence $M_1^{(m)} / M_1^{(m-1)}$ is generated as an $\mathcal{O}$-module by the image of $D^m u$ in $M_1^{(m)}/M_1{}^{(m-1)}$. The Gr $\mathcal{D}$-module structure on

$$\text{Gr } M_1 = \oplus\, M_1^{(m)}/M_1^{(m-1)}$$

is such that if $p \in \mathcal{D}^{(m)}/\mathcal{D}^{(m-1)} \subseteq \text{Gr } \mathcal{D}$ is represented by $P \in \mathcal{D}^{(m)}$ and if $x \in M_1^{(\ell)}/M_1^{(\ell-1)} \subseteq \text{Gr } M$ is represented by X then px is the image of PX in

$$M_1^{(\ell+m)}/M_1^{(\ell+m-1)} \subseteq \text{Gr } M.$$

Thus when the symbol is used to identify Gr $\mathcal{D}$ with $\mathcal{O}[\xi]$ the image in $M_1^{(m)}/M_1^{(m-1)}$ of $D^m u$ is $\xi^m u$. Hence Gr M is generated as an $\mathcal{O}[\xi]$-module by u. Moreover if

$$a_m(z)\ \xi^m + \ldots + a_o(z) \in \mathcal{O}[\xi]$$

then

$$a_j(z)\ \xi^j u \in M_1^{(j)}/M_1^{(j-1)} \quad \text{for} \quad 0 \leqq j \leqq m$$

so

$$(a_m(z)\xi^m + \ldots + a_o(z))u = 0$$

if and only if

$$a_j(z)\ \xi^j u = 0$$

for $0 \leqq j \leqq m$, and this happens if and only if

$$a_j(z)\ D^j u \in M_1^{(j-1)}$$

for $0 \leq j \leq m$. But $a_j(z)\, D^j u \in M_1^{(j-1)}$ if and only if there exist $b_o(z),\dots,b_{j-1}(z) \in \mathcal{O}$ such that

$$a_j(z)\, D^j u = b_{j-1}(z)\, D^{j-1} u + \dots + b_o(z)u,$$

i.e. if and only if $a_j(z)\, \xi^j$ is the symbol of some $P \in \mathcal{D}$ such that $Pu = 0$. In our case $Pu = 0$ if and only if

$$P \in \mathcal{D}(zD - \alpha)^q$$

so $a_j(z)\, \xi^j u = 0$ if and only if $(z\xi)^q$ divides $a_j(z)\, \xi^j$. Thus $\mathrm{Gr}\, M_1$ is generated as an $\mathcal{O}[\xi]$-module by one generator u with relation

$$(z\xi)^q u = 0.$$

Thus the annihilator I of $\mathrm{Gr}\, M_1$ in $\mathcal{O}[\xi]$ is the ideal generated by $(z\xi)^q$ and $\sqrt{I}$ is generated by $z\xi$. Thus

$$\mathrm{Ch}(M) = \{(z,\xi) \in T^*\mathbb{C} \mid z\xi = 0\}$$

$$= \{(z,\xi) \in T^*\mathbb{C} \mid z = 0 \text{ or } \xi = 0\}.$$

We can choose a different set of generators and relations for M_1 as follows. Let

$$u_j = (zD-\alpha)^{j-1} u \quad \text{for} \quad 1 \leq j \leq q.$$

Then $u_1 = u$ so $u_1,\dots,u_q$ generate M_1 with relations

$$z\, Du_j = u_{j+1} + \alpha u_j, \qquad 1 \leq j < m,$$

$$z\, Du_m = \alpha u_m.$$

With this set of generators $\mathrm{Gr}\, M_1$ becomes the $\mathcal{O}[\xi]$-module generated by $u_1,\dots,u_q$ with relations

$$z\xi u_j = 0, \qquad 1 \le j \le m.$$

Thus in this case both I and $\sqrt{I}$ are the ideal generated by $z\xi$.

$\mathcal{M}_1$ extends to a differential system $\mathcal{M}$ on $\mathbf{P}_1 = \mathbb{C} \cup \{\infty\}$ as follows. Let w be the local coordinate on $\mathbf{P}_1 - \{0\}$ given by

$$w = z^{-1}$$

for $z \in \mathbb{C} - \{0\}$ and such that w takes the value 0 at ∞. Then on $\mathbb{C} - \{0\}$

$$zD = z\frac{d}{dz} = z\frac{dw}{dz}\frac{d}{dw} = -\frac{1}{z}\frac{d}{dw} = -wD_w$$

where $D_w = \frac{d}{dw}$. Thus if $\mathcal{M}_2$ is the differential system on $\mathbf{P}_1 - \{0\}$ defined by one generator u and one relation

$$(wD_w + \alpha)^q u = 0$$

then there is an obvious isomorphism between the restrictions of $\mathcal{M}_1$ and $\mathcal{M}_2$ to $\mathbb{C} - \{0\}$. Hence there is a $\mathcal{D}_X$-module $\mathcal{M}$ on $X = \mathbf{P}_1$ such that

$$\mathcal{M}|_{\mathbb{C}} \cong \mathcal{M}_1, \qquad \mathcal{M}|_{\mathbf{P}_1 - \{0\}} \cong \mathcal{M}_2.$$

The characteristic variety $Ch(\mathcal{M})$ is the subvariety of $T^*\mathbf{P}_1$ which is the union of the zero section and the fibres over 0 and ∞. Since $\dim Ch(\mathcal{M}) = 1 = \dim \mathbf{P}_1$, $\mathcal{M}$ is holonomic.

7.6.2 Remark. Of course when X is one-dimensional a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if and only if its charactersitic variety $Ch(\mathcal{M})$ is not equal to T^*X, or equivalently the sheaf of ideals $\sqrt{I}$ is nonzero. In particular a nonzero differential system on X defined locally by one generator and one nonzero equation is always holonomic.

The differential system $\mathcal{M}$ on $\mathbb{C}^2$ defined by one generator u and one equation

7.6.3 $\quad D_1^2 u + D_2^2 u + a(z_1, z_2)u = 0$

has characteristic variety Ch(M) defined by

$$\xi_1^2 + \xi_2^2 = 0,$$

in $T^*\mathbb{C}^2$. Thus dim Ch(M) = 3 > dim $\mathbb{C}^2$ so M is not holonomic.

§7.7 Left and right $\mathcal{D}_X$-modules.

The sheaf of holomorphic (respectively regular) vector fields on a non-singular complex variety X (i.e. holomorphic or regular sections of TX) can be regarded as an $\mathcal{O}_X$-submodule of the sheaf of rings $\mathcal{D}_X$. In local coordinates a vector field

$$a_1(z)\frac{\partial}{\partial z_1} + \ldots + a_n(z)\frac{\partial}{\partial z_n},$$

where $a_1(z),\ldots,a_n(z)$ are holomorphic or regular functions of $z = (z_1,\ldots,z_n)$, is identified with the differential operator

$$a_1(z)D_1 + \ldots + a_n(z)D_n$$

given by differentiation along the vector field. As a sheaf of rings $\mathcal{D}_X$ is generated by these vector fields together with $\mathcal{O}_X$. Thus a $\mathcal{D}_X$-module structure on an $\mathcal{O}_X$-module M is determined by the action on M of these vector fields.

We have been working with left $\mathcal{D}_X$-modules but we can go freely between left $\mathcal{D}_X$-modules and right $\mathcal{D}_X$-modules. (For more details see e.g. Pham [1, §2.13]). If v is a holomorphic vector field on an open subset U of X and if ω is a holomorphic n form on U (i.e. a holomorphic section of $\Lambda^n T^* X$ over U) then we can contract v and ω using the dual pairing between TX and T^*X to get a holomorphic (n-1) form $i_v\omega$ on U. Clearly $i_v\omega$ is regular if both v and ω are regular. If n = dim X then the Lie derivative of ω along v is the n-form

$$\mathrm{Lie}_v(\omega) = d(i_v\omega).$$

Given a left $\mathcal{D}_X$-module M we can put a right $\mathcal{D}_X$-module structure on the tensor product

$$\Omega(M) = \Omega_X^n \otimes_{\mathcal{O}_X} M.$$

where Ω_X^n is the sheaf of holomorphic (or regular) sections of $\Lambda^n T^* X$, as follows. If U is an open subset of X and if $\omega \in \Omega_X^n(U)$, $u \in M(U)$, $f \in \mathcal{O}_X(U) \subseteq \mathcal{D}_X(U)$ and v is a holomorphic (or regular) vector field on U define

7.7.1 $(\omega \otimes u)f = \omega \otimes fu,$

$$(\omega \otimes u)v = -\mathrm{Lie}_v(\omega) \otimes u - \omega \otimes vu.$$

This defines a right $\mathcal{D}_X$-module structure on $\Omega(M)$.

The motivation for this definition is the usual process of identifying functions with distributions by multiplying by a fixed differential form of top degree. It is not hard to check (see e.g. Bernstein [1, Lecture 1 §4], Borel [2, VI §3], Pham [1, §2.13]) that Ω is a functor which induces an equivalence of the category of left $\mathcal{D}_X$-modules with the category of right $\mathcal{D}_X$-modules.

§7.8 Restriction of $\mathcal{D}_X$-modules

Let Y be a nonsingular subvariety of a nonsingular variety X, let $i: Y \to X$ be the inclusion and let M be a $\mathcal{D}_X$-module. One would like to be able to restrict the $\mathcal{D}_X$-module M to give a $\mathcal{D}_Y$-module in some sensible way. If Y is an open subset of X then this is easy (because then open subsets of Y are open subsets of X) so we may as well assume Y is a closed subvariety of X.

It is not hard to restrict M as a *sheaf* on X to a sheaf $M|_Y$ on Y. If U is an open subset of Y we set

$$M|_Y(U) = \lim_{\to} M(V)$$

where the limit is over all open subsets V of X containing U. Then if $y \in Y$ the stalk of $M|_Y$ at y is the same as the stalk of M at y.

The sheaf $M|_Y$ is an $\mathcal{O}_X|_Y$-module, and so is the sheaf $\mathcal{O}_Y$. The tensor product

7.8.1 $i^\circ M = \mathcal{O}_Y \otimes_{\mathcal{O}_X|_Y} M|_Y$

has a natural $\mathcal{O}_Y$-module structure given by $f(g \otimes u) = fg \otimes u$. In order to make $i^\circ M$ a $\mathcal{D}_Y$-module it is necessary to define the action of holomorphic vector fields (cf. §7.7).

Suppose that U is an open subset of Y and $u \in M|_Y(U)$. Then there is an open subset of V of X such that $U = V \cap Y$ and

$$u = \tilde{u}|_Y$$

for some $\tilde{u} \in M(V)$. By choosing U and V small enough we can assume that there are coordinates $z_1,\dots,z_n$ on V and $y_1,\dots,y_m$ on U. If

$$v = \sum_{1 \le i \le m} a_i(y) \frac{\partial}{\partial y_i}$$

is a holomorphic vector field on U and $f \in \mathcal{O}_Y(U)$ is a holomorphic function on U then v(f) is the holomorphic function on U given by

$$v(f) = \sum_{1 \le i \le m} a_i(y) \frac{\partial f}{\partial y_i} .$$

Now define the action of v on the element $f \otimes u$ of $i^\circ M(U)$ by

7.8.2 $$v(f \otimes u) = v(f) \otimes u + \sum_{1 \le i \le n} f v(z_i) \otimes \left.\frac{\partial \tilde{u}}{\partial z_i}\right|_Y .$$

It can be checked (see e.g. Borel [2, VI 4.1]) that this action is independent of the choice of coordinates and defines a $\mathcal{D}_Y$-module structure on $i^\circ M$. In a similar way given any holomorphic map $\pi: Y \to X$ and a $\mathcal{D}_X$-module M we can define a $\mathcal{D}_Y$-module $\pi^\circ M$.

7.8.3 Example. Suppose $M = \mathcal{D}_X$. Locally we can choose coordinates $z_1,\dots,z_n$ such that Y is defined by

$$z_1 = z_2 = \dots = z_d = 0.$$

Then $i^\circ M$ is the locally free $\mathcal{D}_Y$-module with local basis the set of all monomials in $D_1,\dots,D_d$ where

$$D_i = \frac{\partial}{\partial z_i} .$$

7.8.4 Example. Let $X = \mathbb{C}^2$ and let

$$Y = \{(z_1,z_2) \in \mathbb{C}^2 | z_2 = z_1^2\}.$$

We can identify Y with $\mathbb{C}$ via the isomorphism

$$z \in \mathbb{C} \mapsto (z,z^2) \in Y.$$

Let M be the $\mathcal{D}_X$-module with one generator u and two relations

$$(D_1^2 + D_2^2)u = 0,$$

$$z_1 u = 0.$$

We can change coordinates on $\mathbb{C}^2$ from (z_1,z_2) to (z,w) where $z = z_1$ and $w = z_2 - z_1^2$. Then $z_1 = z$ and $z_2 = w + z^2$ so

$$\frac{\partial}{\partial z} = D_1 + 2z\, D_2, \quad \frac{\partial}{\partial w} = D_2.$$

Hence

$$D_2^j(D_1^2 + D_2^2) = D_2^j\, (\frac{\partial}{\partial z} - 2z\, D_2)^2 + D_2^{j+2}$$

$$= (\frac{\partial}{\partial z})^2\, D_2^j - (4z\, (\frac{\partial}{\partial z}) + 2)\, D_2^{j+1} + (1 + 4z^2)\, D_2^{j+2}$$

and

$$D_2^j\, z_1 = z_1\, D_2^j.$$

Thus $i^\circ(M)$ is the quotient of the free $\mathcal{D}_Y$-module with basis

$$\{D_2^j\, u \mid j \geq 0\}$$

by the submodule generated by

$$\{D^2 D_2^j u - (2 + 4zD)D_2^{j+1}u + (1 + 4z^2)D_2^{j+2}u \mid j \geq 0\}$$

and

$$\{z\ D_2^j\ u \mid j \geq 0\}$$

where z is the standard coordinate on $Y \cong \mathbb{C}$ and

$$D = \frac{d}{dz}\ .$$

Equivalently $i^\circ(M)$ is generated as a $\mathcal{D}_Y$-module by u and $v = D_2 u$ subject to the relations

$$zu = 0 = zv\ ,\ Du = 0 = Dv.$$

7.8.5 Remark. Suppose that F is a left exact covariant functor from the category of quasi-coherent $\mathcal{D}_X$-modules to the category of quasi-coherent $\mathcal{D}_Y$-modules. (Quasi-coherence is a technical condition which is weaker than coherence; for the definition see Bernstein [1, §1.1], Borel [2, VI 1.4], Hartshorne [1, II §5]). We can modify the definition of right derived functors given in §2.6 as follows. Given a $\mathcal{D}_X$-module M choose an injective resolution

$$0 \to M \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \to \cdots$$

of M (as a quasi-coherent $\mathcal{D}_X$-module) and define $RF(M)$ to be the *complex*

$$0 \to F(I_0) \xrightarrow{F(d_0)} F(I_1) \xrightarrow{F(d_1)} F(I_2) \to \cdots\ .$$

If F is left exact then this complex is independent of the choice of injective resolution up to quasi-isomorphism. Similarly if F is right exact then we can define the left derived functor LF of F by reversing all arrows in the definitions. $LF(M)$ is a complex of $\mathcal{D}_Y$-modules defined up to quasi-isomorphism.

Sometimes it is convenient to think of the restriction of a $\mathcal{D}_X$-module M to Y as a complex of $\mathcal{D}_Y$-modules rather than a single $\mathcal{D}_Y$-module. We can regard i° as a right exact covariant functor from the category of quasi-coherent $\mathcal{D}_X$-modules to the category of quasi-coherent $\mathcal{D}_Y$-modules, and thus we can consider its left derived functor $Li°$. It turns out to be convenient to make a dimension shift and so we define

7.8.6 $\quad i^!(M) = Li°(M)\ [d]$

where

$$d = \dim X - \dim Y$$

(Borel [2, VI §4.2], Bernstein [1, §1.8]). It is often useful to regard either the complex $i^!M$ or the $\mathcal{D}_Y$-module $\underline{\underline{H}}°\ i^!M$ as the restriction of M to Y. From our point of view the main reason for this is the following theorem.

7.8.7 Theorem. (Kashiwara). Let $i: Y \to X$ be a closed embedding of non-singular varieties. Then the functor

$$M \to \underline{\underline{H}}°\ i^!\ M$$

is an equivalence between the category of holonomic $\mathcal{D}_X$-modules with support in Y and the category of holonomic $\mathcal{D}_Y$-modules.

For the proof of this theorem see Bernstein [1, §1.10, §3.1] or Borel [2, VI §7.11].

§7.9 Regular singularities

7.9.1 Definition. A $\mathcal{D}_X$-module M on a nonsingular variety X is called a *connection* if it is a coherent locally free $\mathcal{O}_X$-module for the $\mathcal{O}_X$-structure coming from the embedding $\mathcal{O}_X \to \mathcal{D}_X$.

7.9.2 Remark. In fact any $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module is locally free as an $\mathcal{O}_X$-module (Bernstein [1, §2.1(a)], Borel [2, IV, §1.1]).

7.9.3 Remark. Let M be a coherent locally free $\mathcal{O}_X$-module. Then locally M is freely generated as an $\mathcal{O}_X$-module by finitely many sections $u_1,\ldots,u_m$, say. This means that M can be identified with the sheaf of holomorphic (or regular) sections of a complex vector bundle V of rank m over X. If moreover M is a $\mathcal{D}_X$-module then there exist local sections Γ_{ij}^k of $\mathcal{O}_X$ such that

$$D_i u_j = \sum_{1\leq k\leq m} \Gamma_{ij}^k u_k.$$

The local sections Γ_{ij}^k define a flat connection on V in the sense of differential geometry. Flatness corresponds to the commutativity condition

$$[D_i, D_j] = 0.$$

7.9.4 Remark. Let M be a connection on X generated locally as an $\mathcal{O}_X$-module by a basis of sections $\{u_1,\ldots,u_m\}$. Then Gr M is generated locally as an $\mathcal{O}_X[\xi_1,\ldots,\xi_n]$-module by $u_1,\ldots,u_m$ with relations

$$\xi_i u_j = 0, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m.$$

Thus $\sqrt{I}$ is locally the sheaf of ideals generated by $\xi_1,\ldots,\xi_n$ in $\mathcal{O}_X[\xi_1,\ldots,\xi_n]$. Hence the characteristic variety Ch(M) of M is the zero section in T^*X. In particular any connection is a holonomic $\mathcal{D}_X$-module.

7.9.5 Example. Consider the differential system M on $\mathbb{C}$ defined by one generator u and one equation

$$(zD - \alpha)^m u = 0,$$

where $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$. Then $M|_{\mathbb{C}-\{0\}}$ is generated by the global sections

$$u_1 = u, \quad u_j = (zD - \alpha)^{j-1} u, \qquad 1 < j \leq m,$$

with relations

7.9.6 $$Du_j = \frac{1}{z} u_{j+1} + \frac{\alpha}{z} u_j, \qquad 1 \leq j < m$$

$$Du_m = \frac{\alpha}{z} u_m.$$

If $\alpha \notin \mathbb{N}$ the space of solutions of M in $\mathcal{O}_X$ over any simply connected subset of $\mathbb{C} - \{0\}$ is spanned by the solutions

$$u = z^{\alpha} (\log z)^{j}, \qquad 0 \leq j \leq m-1.$$

$M|_{\mathbb{C}-\{0\}}$ is a connection. But M has no sections over any neighbourhood of 0 in $\mathbb{C}$ so M itself is not a connection. It is the presence of z^{-1} factors in the relations 7.9.6 which prevents M from being a connection over $\mathbb{C}$.

If X is quasi-projective and nonsingular of dimension one (i.e. X is a nonsingular curve) then we can choose an embedding of X in a nonsingular projective curve X^+ such that $X^+ \setminus X$ is a finite set of points. Fix $s \in X^+$ and choose a local coordinate z on a neighbourhood U of s in X^+ such that z vanishes at s. Let $D = d/dz$ be the corresponding differential operator on U.

7.9.7 Definition. Let M be a holonomic $\mathcal{D}_X$-module. Then M has a *regular singularity* at s if U and z can be chosen such that

$$M|_{U-\{s\}}$$

is a connection on U-{s} and is generated as a $\mathcal{D}_{U-\{s\}}$-module by a finitely generated $\mathcal{O}_U$-module which is invariant under the action of zD. That is, on U-{s} the module M is defined by a system of equations in variables $u_1, \dots, u_p$ such that for all i we can write

$$z\, Du_i = \sum_{1 \leq j \leq p} a_{ij}\, u_j$$

where $a_{ij} \in \mathcal{O}_X(U)$. Equivalently

$$D\, u_i = \sum_{1 \leq j \leq p} \frac{a_{ij}}{z} u_j$$

where the a_{ij} extend to holomorphic functions of z on U.

In fact a system M on $\mathbb{C}$ has a regular singularity at 0 if and only if near 0 it is isomorphic to a finite direct sum of $\mathcal{D}$-modules of the form

$$\mathcal{D}/\mathcal{D}(zD - \alpha)^{m}$$

for $m \in \mathbb{N}$ and $\alpha \in \mathbb{C} - \mathbb{N}$ (see e.g. Pham [1, §2.11.6]).

7.9.8 Definition. The holonomic $\mathcal{D}_X$-module M has *regular singularities* if it has a regular singularity at each $s \in X^+$. It is not hard to check that this definition is independent of the choice of X^+.

7.9.9 Definition. A complex of $\mathcal{D}_X$-modules is holonomic (and has regular singularities) if all its cohomology sheaves are holonomic (and have regular singularities).

Finally if M is a holonomic $\mathcal{D}_X$-module on a nonsingular variety X of any dimension we make the following definition.

7.9.10 Definition. M has regular singularities if the restriction $i^! M$ of M to any closed nonsingular curve C in X with inclusion map $i: C \to X$ has regular singularities.

For more details on holonomic $\mathcal{D}_X$-modules with regular singularities see e.g. Borel [1], Bernstein [1], Kashiwara and Kawai [1], Mebkhout [2].

§7.10 The Riemann-Hilbert correspondence

7.10.1 Definition. Let A be a nonsingular subvariety of a nonsingular variety X. The *conormal bundle* to A in X is

$$T^*_A X = \{y \in T^* X \mid \pi(y) \in A,\ y \in (T_{\pi(y)} A)^o\}$$

where $\pi: T^* X \to X$ is the projection and $(T_{\pi(y)} A)^o$ is the annihilator of the tangent space $T_{\pi(y)} A$ to A at $\pi(y)$ in the dual $T^*_{\pi(y)} X$ of $T_{\pi(y)} X$.

If A is a singular subvariety of X with

$$\tilde{A} = \{\text{nonsingular points of } A\}$$

then we define $T^*_A X$ to be the closure in $T^* X|_A$ of

$$T^*_{\tilde{A}} X.$$

Note that T_X^*X is the zero section of T^*X.

7.10.2 Proposition. (see e.g. Pham [1, §2.10.1]). Let $V \subseteq T^*X$ be an irreducible Lagrangian conical closed subvariety of T^*X. Then the image $\pi(V)$ of V under $\pi: T^*X \to X$ is an irreducible subvariety of X and

$$V = T^*_{\pi(V)}X$$

is the conormal bundle to V in X.

Recall from §7.4 and §7.5 that if M is a holonomic $\mathcal{D}_X$-module then the characteristic variety $Ch(M)$ of M is a closed conical Lagrangian subvariety of T^*X.

7.10.3 Corollary. If M is a holonomic $\mathcal{D}_X$-module then every irreducible component V of $Ch(M)$ is of the form

$$V = T^*_S X$$

where S is an irreducible subvariety of X.

7.10.4 Lemma. (Pham [1, §2.10.3]). Let M be a holonomic $\mathcal{D}_X$-module. If $V_1,\ldots,V_p$ are the irreducible components of $Ch(M)$ and if

$$V_i = T^*_{S_i}X$$

let

$$S = \bigcup_{S_i \neq X} S_i \; .$$

Then the restriction of M to X-S is a connection (possibly zero).

Let M be a connection on a quasi-projective variety X. The *sheaf of horizontal sections* of M is the sheaf F on X such that if $(z_1,\ldots,z_n)$ are local coordinates on an open subset V of X then

$$F(V) = \{u \in M(V) \,|\, D_i u = 0, \quad 1 \leq i \leq n\}.$$

It is easy to check that this is a good definition, independent of the choice of local coordinates. Now suppose that $\mathcal{M}$ is freely generated as an $\mathcal{O}_X$-module over V by sections $u_1,\ldots,u_m$ of $\mathcal{M}(V)$ with

$$D_i u_j = \sum_{1\leq k\leq m} \Gamma_{ij}^k u_k.$$

Then a general element u of $\mathcal{F}(V)$ can be written in the form

$$u = f_1u_1 + \ldots + f_mu_m$$

where $f_1,\ldots,f_m \in \mathcal{O}_X(V)$ are functions on V satisfying

$$\frac{\partial f_j}{\partial z_i} + \sum_{1\leq k\leq m} \Gamma_{ik}^j f_k = 0$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$. The theory of existence and uniqueness of local solutions of differential equations implies that if V is simply connected then the restriction to V of $\mathcal{F}$ is isomorphic to the constant sheaf $\mathbb{C}_V^m$, or equivalently the restriction map

$$\mathcal{F}(V) \to \mathcal{F}_x$$

is an isomorphism for all $x \in V$. This means that $\mathcal{F}$ defines a local system L on X with $L_x = \mathcal{F}_x$ (cf. §3.9).

We can now give a classification of irreducible holonomic $\mathcal{D}_X$-modules which is itself sometimes called the Riemann-Hilbert correspondence (cf. Borel [2, IV], Deligne [3], Bernstein [1, §3.14 and §4.1]).

7.10.5 <u>Theorem</u>. (i) Let Y be a closed irreducible subvariety of a non-singular variety X, and let L be an irreducible local system on a dense open nonsingular subvariety U of Y. Then there is a unique irreducible holonomic $\mathcal{D}_X$-module with regular singularities, denoted $\mathcal{M}(Y,L)$, whose support is contained in Y and whose restriction to U is a connection such that the local system defined by its sheaf of horizontal sections is L.

(ii) Any irreducible holonomic $\mathcal{D}_X$-module with regular singularities is isomorphic to $M(Y,L)$ for some Y and L as in (i).

(iii) $M(Y',L')$ is isomorphic to $M(Y,L)$ if and only if $Y = Y'$ and the restrictions of L and L' to some nonempty open subset of Y, on which they are both defined, are isomorphic.

In order to relate this form of the Riemann-Hilbert correspondence to one which involves intersection cohomology, we need the following theorem (cf. Bernstein [1, §5.9]).

7.10.6 Theorem. Let Y be a closed irreducible subvariety of a nonsingular n-dimensional variety X and let L be a local system on a dense open nonsingular subvariety of Y. Then the De Rham complex $DR(M(Y,L))$ of $M(Y,L)$

$$0 \to M(Y,L) \to \Omega_X^1 \otimes_{\mathcal{O}_X} M(Y,L) \to \ldots \to \Omega_X^j \otimes_{\mathcal{O}_X} M(Y,L) \to \ldots$$

has support in Y and its restriction to Y is generalised quasi-isomorphic to the intersection sheaf complex $\underline{\underline{IC}}^{\cdot}_{(Y,L)}$ with a shift in degree.

The idea of the proof of Theorem 7.10.6 is to check that after a shift in degree $DR(M(Y,L))$ satisfies the conditions which uniquely characterise $\underline{\underline{IC}}^{\cdot}_{(Y,L)}$ up to generalised quasi-isomorphism (cf. §5.4). These conditions are

7.10.7 (a) There is a subvariety $\Sigma \subseteq Y$ of complex codimension at least 1 such that L is defined over $Y - \Sigma$ and

$$\underline{\underline{IC}}^{\cdot}_{(Y,L)}\Big|_{Y-\Sigma}$$

is generalised quasi-isomorphic to the complex which is

$$L\Big|_{Y-\Sigma}$$

in dimension $i = -2n$ and 0 in other dimensions, where n is dim Y.

(b) For all $x \in Y$ the cohomology $H^{-i}(\underline{\underline{IC}}^{\cdot}_{(Y,L),x})$ of the stalk complex of $\underline{\underline{IC}}^{\cdot}_{(Y,L)}$ at x is a finite-dimensional vector space for all i and is 0 when $i > 2n$.

(c) For all $i < 2n$

$$\dim_{\mathbb{C}} \{x \in Y \mid H^{-i}(\underline{\underline{IC}}^{\bullet}_{(Y,L),x}) \neq 0\} < i-n.$$

(d) For all $i < 2n$

$$\dim_{\mathbb{C}} \{x \in Y \mid H^{-j}_{x}(\underline{\underline{IC}}^{\bullet}_{(Y,L)}) \neq 0\} < i-n$$

where $j = 2n-i$ and $H^{k}_{x}(\underline{\underline{IC}}^{\bullet}_{(X,L)})$ denotes the kth hypercohomology group with compact supports of $IC^{\bullet}_{(Y,L)}$ restricted to a small open neighbourhood N_x of x of the form described in §3.8.

Using these Theorems 7.10.5 and 7.10.6 one can obtain the Riemann-Hilbert correspondence in the following form, first proved by Kashiwara [1] and Mebkhout [1] in the holomorphic case and Beilinson and Bernstein in the algebraic case (see Bernstein [1] and Borel [2, VIII]).

7.10.8 Theorem . The De Rham functor DR induces a one-to-one correspondence between isomorphism classes of irreducible holonomic $\mathcal{D}_X$-modules with regular singularities and generalised quasi-isomorphism classes of intersection sheaf complexes of irreducible closed subvarieties of X with coefficients in irreducible local systems.

7.10.9 Remark. Bounded constructible sheaf complexes on X which satisfy the conditions 7.10.7 (b), (c), (d) above with strict inequality in (c) and (d) replaced by weak inequality and without the normalising condition (a) are called *perverse sheaves*. The Riemann-Hilbert correspondence can be generalised if one uses the category of perverse sheaves. In fact the De Rham functor gives an equivalence of categories between the category of holonomic $\mathcal{D}$-modules with regular singularities on X and the derived category of perverse sheaves of X (Borel [2, VIII 14.4], Bernstein [1, §5.9]). (The *derived category* of perverse sheaves is obtained from the category of perverse sheaves by formally inverting all quasi-isomorphisms, so that they become isomorphisms.) By considering only the irreducible objects in each category this gives the Riemann-Hilbert correspondence in the form above.

8 The Kazhdan-Lusztig conjecture

To complete this introduction to intersection homology we shall describe briefly the proof of a conjecture of Kazhdan and Lusztig (Kazhdan and Lusztig [1], [2]) concerning the representation theory of Lie algebras. The proof (following Bernstein [1]) involves translating the problem first into the language of $\mathcal{D}$-modules, then via the Riemann-Hilbert correspondence into a problem involving intersection cohomology and finally using ℓ-adic intersection cohomology into the theory of Hecke algebras.

First it is necessary to review some basic facts about the representation theory of complex Lie groups and Lie algebras. For more details see e.g. Atiyah [1], Bourbaki [1], Chevalley [1], Jacobson [1], Kac [1], Springer [1].

§8.1 Verma modules

Let K be a compact Lie group. For simplicity let us assume that K is connected and simply connected. Let $\mathfrak{k}$ be the Lie algebra of K and let

$$\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$$

be its complexified Lie algebra. The Lie bracket [,] on $\mathfrak{g}$ is the unique complex bilinear extension of the Lie bracket on $\mathfrak{k}$.

There is a unique connected, simply connected complex Lie group G whose Lie algebra is $\mathfrak{g}$. We shall assume for simplicity that G is semisimple; that is, that its Lie algebra $\mathfrak{g}$ has no nonzero abelian ideals. For many reasons mathematicians have long been interested in the complex representations of such complex Lie groups G and their Lie algebras $\mathfrak{g}$.

Let T be a maximal torus of G and let $N_G(T)$ be its normaliser in G. Then

$$W = N_G(T)/T$$

is a finite group called the Weyl group of G.

8.1.1 Example. We can take

$$K = SU(n),$$

$$G = SL(n;\mathbb{C}),$$

$$T = \{\text{diagonal matrices in } SL(n;\mathbb{C})\},$$

$$W \cong \Sigma_n,$$

where Σ_n denotes the symmetric group.

Let h be the Lie algebra of T and let h^* be its dual vector space. Then $\alpha \in h^*-\{0\}$ is called a *root* of g if there exists some nonzero $\xi \in g$ such that

$$[h,\xi] = \alpha(h)\xi$$

for all $h \in h$. Let g^α be the set of all $\xi \in h$ such that

$$[h,\xi] = \alpha(h)\xi$$

for all $h \in h$. Let Σ be the set of roots of g. Then

$$g = h \oplus (\bigoplus_{\alpha\in\Sigma} g^\alpha).$$

The Weyl group W acts on h and h^* and permutes the roots. W is generated by elements which act as reflections in hyperplanes. We can choose a fundamental domain (called a Weyl chamber) for the action on W on h^* which is a cone in h^* bounded by hyperplanes.

Let h_+^* be the chosen Weyl chamber (called the *positive Weyl chamber*). Then $\alpha \in \Sigma$ is called a positive root if

$$\alpha(x) > 0$$

for all x in the interior of h_+^*. Let Σ^+ be the set of positive roots. Then Σ is the disjoint union of Σ^+ and $-\Sigma^+$. Let

$$N = \bigoplus_{\alpha \in \Sigma^+} g^{\alpha} .$$

Then N is a nilpotent subalgebra of g. There is a partial order on h^* defined by

$$\alpha \geq \beta \Longleftrightarrow \alpha(x) \geq \beta(x) \ \forall x \in h_+ .$$

Now let M be a g-module, that is, a complex representation of g. Then M is a complex vector space (possibly infinite-dimensional) with an action of g given by a complex linear map

$$\rho: g \to \mathrm{End}(M) = \{\alpha : M \to M \mid \alpha \text{ complex linear}\}$$

which takes the Lie bracket on g to the usual Lie bracket

$$[\alpha,\beta] = \alpha\beta - \beta\alpha$$

on End(M). Assume that M is finitely generated, i.e. that there exist $m_1,\ldots,m_k \in M$ such that the only g-submodule of M containing $m_1,\ldots,m_k$ is M itself.

If we restrict the representation ρ of g to the abelian subalgebra h (or equivalently think of M as an h-module) then M decomposes as a direct sum

$$M = \bigoplus_{\chi \in h^*} M^{\chi}$$

where

$$M^{\chi} = \{m \in M \mid hm = \chi(h)m, \quad \forall h \in h\}.$$

χ is called a *weight* of M if $M^{\chi} \neq 0$ and a *highest weight* if in addition $\eta \leq \chi$ whenever $M^{\eta} \neq 0$. If $\alpha \in \Sigma$ and $\xi \in g^{\alpha}$ and $m \in M^{\chi}$ and $h \in h$ then

$$\begin{aligned} h(\xi m) &= [h,\xi]m + \xi(hm) \\ &= \alpha(h)m + \xi(\chi(h)m) \\ &= (\alpha+\chi)(h)\xi m \end{aligned}$$

so

$$g^{\alpha}M^{\chi} \subseteq M^{\alpha+\chi} .$$

But if $\alpha \in \Sigma^+$ then $\alpha + \chi > \chi$. Thus if χ is a highest weight and $\alpha \in \Sigma^+$ then $M^{\alpha+\chi} = 0$ so

$$g^{\alpha}M^{\chi} = 0.$$

Hence if χ is a highest weight then

$$N\, M^{\chi} = 0 .$$

M is called a *highest weight g-module* if it is generated by a single element $m \in M^{\chi}$ where χ is a highest weight. Any finitely generated g-module has a filtration

$$M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_q = 0$$

by g-submodules M_j such that the quotient g-modules M_j/M_{j+1} are all highest weight modules.

For each $\chi \in h^*$ there exists a unique (usually infinite-dimensional) g-module M_χ generated by one element m_χ satisfying

8.1.2 (i) $\xi m_\chi = 0$ for all $\xi \in N$,

(ii) $hm_\chi = \chi(h)m_\chi$ for all $h \in h$,

with the universal property that every other g-module M generated by one element m satisfying (i) and (ii) is a quotient module of M_χ via a map which sends m_χ to m. The module M_χ is called the *Verma module* for g with highest weight χ.

If P is a proper submodule of M_χ then every weight η of P satisfies $\eta < \chi$. From this it is easy to see that any sum of proper submodules of M_χ is again a proper submodule, so M_χ has a *unique* maximal proper submodule. Equivalently M_χ has a unique irreducible quotient module called L_χ. This module is the unique irreducible g-module with highest weight χ.

A Verma module M_χ has a filtration by submodules

$$M_\chi = M_{\chi,o} \supseteq M_{\chi,1} \supseteq \cdots \supseteq M_{\chi,q} = 0$$

such that the quotient modules $M_{\chi,j} / M_{\chi,j+1}$ are all irreducible. This filtration is not necessarily unique but the modules $M_{\chi,j} / M_{\chi,j+1}$ are uniquely determined by M_χ up to isomorphism and change of order. It turns out that these modules are all of the form L_ϕ where $\phi \in h^*$ and $\phi \leq \chi$, and moreover $\phi + \rho$ lies in the same Weyl group orbit as $\chi + \rho$ where

$$\rho = \frac{1}{2} \sum_{\alpha\in\Sigma^+} \alpha$$

is half the sum of the positive roots of g. The module L_χ occurs exactly once in the list. An important problem in the study of Verma modules (and hence of all representations of g) is to determine how many times L_ϕ occurs in the list when $\phi \neq \chi$.

This problem can be rephrased using the *Grothendieck group* of g-modules. This is the abelian group generated by isomorphism classes [M] of finitely generated g-modules M with relations

$$[M_2] = [M_1] + [M_3]$$

for every exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of g-modules. In the Grothendieck group we can formally write

8.1.3 $$[M_\chi] = \sum_{\phi+\rho\in W(\chi+\rho)} b_{\chi\phi}[L_\phi]$$

for some integer coefficients $b_{\chi\phi}$. Our problem then becomes that of determining these coefficients. The coefficient $b_{\chi\chi}$ is always 1, but the other coefficients are more mysterious.

The matrix $(b_{\chi\phi})$ where $\chi+\rho$ and $\phi+\rho$ run over a fixed Weyl group orbit in h^* is lower triangular with respect to the partial order $\leq$ on h^* and has ones on the diagonal. Hence this matrix is invertible. It is more convenient to work with the inverse matrix $(a_{\chi\phi})$ defined by the equation

$$8.1.4 \quad [L_\chi] = \sum_{\phi+\rho\in W(\chi+\rho)} a_{\chi\phi}[M_\phi]$$

in the Grothendieck group.

The *Kazhdan-Lusztig conjecture* (see Kazhdan and Lusztig [1], [2], Brylinski and Kashiwara [1], Beilinson and Bernstein [1]) identifies the coefficients $a_{\chi\phi}$ in the special case when $\chi + \rho$ and $\phi + \rho$ lie in the Weyl group orbit of $-\rho$. If w and v lie in the Weyl group W let us write a_{wv} for $a_{\chi\phi}$, L_w for L_χ and M_w for M_χ where

$$\chi = w(-\rho) - \rho$$

and

$$\phi = v(-\rho) - \rho.$$

Then the Kazhdan-Lusztig conjecture is concerned with the coefficients a_{wv} satisfying

$$8.1.5 \quad [L_w] = \sum_{v\in W} a_{wv}\,[M_v]$$

in the Grothendieck group. Following Bernstein [1] we shall first identify these coefficients in terms of $\mathcal{D}_X$-modules for a suitable X.

8.2 $\mathcal{D}$-modules over flag manifolds

Recall from §8.1 that the Lie algebra g of G can be decomposed as

$$g = h \oplus (\bigoplus_{\alpha\in\Sigma} g^\alpha).$$

Let B be the Borel subgroup of G whose Lie algebra is

$$b = h \oplus (\bigoplus_{\alpha\in\Sigma^+} g^\alpha).$$

Then

$$X = G/B$$

is the *flag manifold* of G.

8.2.1 Examples. If $G = SL(n;\mathbb{C})$ as in 8.1.1 then we can take B to be the subgroup of $SL(n;\mathbb{C})$ consisting of upper triangular matrices. Then X can be naturally identified with the space of all flags

$$0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_n = \mathbb{C}^n$$

such that V_j is a j-dimensional subspace of $\mathbb{C}^n$ for each j.

The flag manifold X is a nonsingular complex projective variety and G acts transitively on X. Hence if M is a $\mathcal{D}_X$-module then the space $\Gamma(M) = M(X)$ of global sections of M has a natural g-module structure defined as follows.

Given any $\xi \in g$ the infinitesimal action of G on X induces a vector field $x \mapsto \xi_x$ on X. Here ξ_x is the tangent at x to the smooth path

$$t \mapsto \exp(t\xi).x \qquad (t \in \mathbb{R})$$

in X where $\exp: g \to G$ is the exponential mapping (see e.g. Warner [1]). In local coordinates $z_1,\dots,z_n$ we can write

$$\xi_x = a_1(z) \frac{\partial}{\partial z_1} + \dots + a_n(z) \frac{\partial}{\partial z_n} .$$

We can define a differential operator D_ξ on X by

$$D_\xi = a_1(z)\, D_1 + \dots + a_n(z)\, D_n$$

in local coordinates. This gives a Lie algebra homomorphism from g to the space $\mathcal{D}_X(X)$ of differential operators on X defined by

$$\xi \mapsto D_\xi .$$

Hence there is a g-module structure on $\Gamma(M) = M(X)$ defined by

$$\xi.\sigma = D_\xi \sigma, \qquad \xi \in g \;, \sigma \in \Gamma(M).$$

The transitive action of G on X = G/B restricts to an action of B which has finitely many orbits, corresponding to the finitely many double cosets in $B\backslash G/B$. The Bruhat decomposition tells us that these orbits are indexed by the Weyl group W. If $w \in W = N_G(T)/T$ is represented by $\tilde{w} \in N_G(T)$ then the B-orbit X_w of X indexed by w is the B-orbit of the coset $\tilde{w}B$ in X, i.e. the image in X of the double coset $B\tilde{w}B$ in G. The closure $\bar{X}_w$ of any B-orbit X_w in X is a union of B-orbits.

We wish to find $\mathcal{D}_X$-modules λ_w and μ_w supported on $\bar{X}_w$ such that the associated $\mathfrak{g}$-modules $\Gamma(\lambda_w)$ and $\Gamma(\mu_w)$ are naturally isomorphic to L_w and the Verma module M_w. How can we describe these $\mathcal{D}_X$-modules λ_w and μ_w? We can use the Riemann-Hilbert correspondence (§7.10) between $\mathcal{D}_X$-modules and intersection sheaf complexes of subvarieties of X.

Consider the intersection sheaf complex $\underline{\underline{IC}}^{\bullet}_{\bar{X}_w}$ of the irreducible closed subvariety $\bar{X}_w$ of X. By the Riemann-Hilbert correspondence (7.10.5 and 7.10.6) there exists a unique irreducible holonomic $\mathcal{D}_X$-module λ_w with regular singularities such that the De Rham complex $DR(\lambda_w)$ of λ_w is generalised quasi-isomorphic to $\underline{\underline{IC}}^{\bullet}_{\bar{X}_w}$ with a dimension shift.

Let $T^{\bullet}_w$ be the sheaf complex on X which is the extension by zero of the trivial sheaf complex $\mathbb{C}^{\bullet}_{X_w}$ on X_w. In other words T^i_w is zero when i is nonzero, and when i is zero its restriction to X_w is the constant sheaf defined by $\mathbb{C}$ and its stalk at any $x \notin X_w$ is zero. There is a $\mathcal{D}_X$-module μ_w on X supported on $\bar{X}_w$ whose De Rham complex $DR(\mu_w)$ is generalised quasi-isomorphic to the sheaf complex $T^{\bullet}_w$ with a dimension shift.

8.2.2 <u>Theorem</u>. (Bernstein [1], Beilinson and Bernstein [1], Brylinski and Kashiwara [1]). The $\mathfrak{g}$-modules $\Gamma(\mu_w)$ and $\Gamma(\lambda_w)$ are isomorphic to M_w and L_w.

It follows that for suitable integers d(v,w) coefficients a_{wv} defined by equation 8.1.5 can also be defined by the equation

8.2.3 $$\underline{\underline{IC}}^{\bullet}_{\bar{X}_w} \sim \sum_{v \in W} a_{wv}\, T^{\bullet}_v \quad [d(v,w)]$$

where $\sim$ denotes the equivalence relation on the free abelian group of generalised quasi-isomorphism classes of bounded constructible complexes of sheaves on X given by quotienting by the subgroup generated by all elements of the form

$$A^{\bullet} + C^{\bullet} - B^{\bullet}$$

such that there is a short exact sequence

$$0 \to A^{\bullet} \to B^{\bullet} \to C^{\bullet} \to 0.$$

Since $A^{\bullet}[n] \sim (-1)^n A^{\bullet}$ for any complex $A^{\bullet}$ we can replace 8.2.3 by the equation

8.2.4 $$\underline{\underline{IC}}^{\bullet}_{\bar{X}_w} \sim \sum_{v \in W} (-1)^{d(v,w)} a_{wv} T^{\bullet}_v.$$

The *Euler characteristic* of a complex $C^{\bullet}$ of abelian groups with only finitely many nonzero homology groups is by definition

$$\chi(C^{\bullet}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H_i(C^{\bullet}).$$

It is easy to check that if

$$0 \to A^{\bullet} \to B^{\bullet} \to C^{\bullet} \to 0$$

is a short exact sequence of complexes then

$$\chi(A^{\bullet}) + \chi(C^{\bullet}) = \chi(B^{\bullet}).$$

Thus by restricting 8.2.4 to the orbit X_v and taking Euler characteristics of stalk complexes one finds that

8.2.5 $$a_{wv} = 0$$

unless $X_v \subseteq \bar{X}_w$ and

8.2.6 $$a_{wv} = (-1)^{\dim X_v - \dim X_w} \sum_{i \geq 0} (-1)^i \dim IH^i_{X_v}(\bar{X}_w)$$

if $X_v \subseteq \bar{X}_w$ where

$$\dim IH^{i}_{X_v}(\bar{X}_w)$$

denotes the dimension of the stalk of the (-i)th cohomology sheaf of $\underline{\underline{IC}}^{\bullet}_{\bar{X}_w}$ at any point in X_v.

So the question arises whether we can work out the dimensions of these intersection cohomology groups. The answer is that in general we cannot give explicit formulas for their dimensions but we can express them in terms of some interesting polynomials related to Hecke algebras, which can be computed by recursive formulas, given enough time and patience.

The first step is to consider the whole set up in characteristic p where p is a prime number, as in Chapter 6.

§8.3 Characteristic p

Let us assume that G is an algebraic group defined over an algebraic number field R (see Springer [1]) and that π is a prime ideal in R such that R/π is isomorphic to the finite field $\mathbf{F}_q$ with $q = p^m$ elements. Let us assume that the reduction of G modulo π is an algebraic group G_q defined over $\mathbf{F}_q$. As in Chapter 6 when we were considering the Weil conjectures we assume that π is not one of finitely many "bad" primes for G. Then we can assume that the reductions modulo π of the Borel subgroup B, the flag manifold X = G/B and each orbit X_w are respectively a Borel subgroup B_q of G_q, the flag manifold

$$X_q = G_q / B_q$$

and an orbit $(X_w)_q = X_{w,q}$ of B_q on X_q. Then if ℓ is a prime different from p the ℓ-adic intersection cohomology sheaf complex

$$\underline{\underline{IC}}^{\bullet}_{\bar{X}_{w,q}}$$

and the sheaf complex $T^{\bullet}_{w,q}$ given by extending the trivial sheaf complex

$$(\mathbb{Q}_\ell)^{\bullet}_{X_{w,q}}$$

on $X_{w,q}$ by zero satisfy

8.3.1 $\underline{\underline{IC}}^{\bullet}_{\bar{X}_{w,q}} \sim \sum_{v \in W} (-1)^{\dim X_w - \dim X_v} a_{wv} T^{\bullet}_{v,q}$

where the equivalence relation is defined as in §8.2.

The Frobenius mapping (6.2.2) lifts naturally to actions on $\underline{\underline{IC}}^{\bullet}_{\bar{X}_{w,q}}$ and on $T^{\bullet}_{w,q}$. Let us modify the equivalence relation $\sim$ by considering bounded constructible complexes of ℓ-adic sheaves together with distinguished "Frobenius" endomorphisms which lift the Frobenius mapping, and quotienting by the subgroup generated by expressions of the form

$$A^{\bullet} + C^{\bullet} - B^{\bullet}$$

for each short exact sequence

$$0 \to A^{\bullet} \to B^{\bullet} \to C^{\bullet} \to 0$$

which *respects* the Frobenius actions. Let us also write

$$A^{\bullet} \sim q^{j/2} B^{\bullet}$$

if $A^{\bullet}$ is the tensor product of $B^{\bullet}$ with a one-dimensional vector space over $\mathbb{Q}_{\ell}$ on which the Frobenius endomorphism acts as multiplication by an algebraic integer of modulus $q^{j/2}$. Then

8.3.2 $\underline{\underline{IC}}^{\bullet}_{\bar{X}_{w,q}} \sim \sum_{v \in W} p_{wv}(q) T^{\bullet}_{v,q}$

where $p_{wv}(q)$ is a polynomial in $q^{\frac{1}{2}}$ such that

$$p_{ww}(q) = 1$$

and

$$p_{wv} = 0$$

if $X_v \not\subseteq \bar{X}_w$ while

8.3.3 $\quad p_{wv}(q) = \sum_{i\geq 0} (-1)^i q^{i/2} \dim IH^i_{X_v}(\bar{X}_w)$

if $X_v \subseteq \bar{X}_w$. This can be deduced from the Riemann Hypothesis (6.4.3) (see Kazhdan-Lusztig [2]).

In particular it follows from the local calculation 3.8.1 that if $w \neq v$ then $p_{wv}(q)$ is a polynomial in $q^{\frac{1}{2}}$ of degree less than $\dim X_w - \dim X_v$. In fact

$$IH^i_{X_v}(\bar{X}_w) = 0$$

when i is odd, so $p_{wv}(q)$ is a polynomial in q of degree less than

$$\tfrac{1}{2}(\dim X_w - \dim X_v).$$

Note that if we formally put q = 1 then by comparing 8.3.1 and 8.3.2 we get

8.3.4 $\quad p_{wv}(1) = (-1)^{\dim X_w - \dim X_v} a_{wv}$

for all $w, v \in W$.

§8.4 Hecke algebras and the Kazhdan-Lusztig polynomials.

The Hecke algebra H of the Weyl group W of G with parameter q is an algebra over the ring $\mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of polynomials with integer coefficients in $q^{\frac{1}{2}}$ and $q^{-\frac{1}{2}}$. As a module over $\mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ it has a basis consisting of 1 and one element τ_w for each $w \in W$. Its multiplication is uniquely determined by the rules

8.4.1 $\quad \tau_w\tau_v = \tau_{wv}$

if $w,v \in W$ and $\dim X_{wv} = \dim X_w + \dim X_v$, and

8.4.2 $\quad (\tau_\sigma + 1)(\tau_\sigma - q) = 0$

if $\sigma \in W$ acts as a reflection on h (see e.g. Bourbaki [1, Chapter IV, §2, Exercises 22, 24]).

There is a unique involution $D: H \to H$ satisfying

8.4.3 $D(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$

and

8.4.4 $D(\tau_\sigma + 1) = q^{-1}(\tau_\sigma + 1)$

whenever $\sigma \in W$ is a reflection.

8.4.5 Proposition/Definition. (Kazhdan and Lusztig [1], Theorem 1.1). For each $w \in W$ there is a unique $C_w \in H$ of the form

$$C_w = \tau_w + \sum_{v \in W-\{w\},\ X_v \subseteq X_w} \tilde{p}_{wv}(q)\, \tau_v,$$

where $\tilde{p}_{wv}(q)$ is a polynomial in q of degree less than

$$\tfrac{1}{2}(\dim X_w - \dim X_v),$$

satisfying

$$DC_w = q^{-\dim X_w} C_w.$$

The polynomials $\tilde{p}_{wv}(q)$ are called *Kazhdan-Lusztig polynomials.*

8.4.6 Theorem. (Kazhdan and Lusztig [2]). The polynomials p_{wv} and $\tilde{p}_{wv}$ defined by 8.3.2 and 8.4.5 coincide.

Sketch proof. Let $\mathbb{C}^{\bullet}_{X_q}$ be the trivial sheaf complex on X_q and if $w \in V$ let $T^{\bullet}_{w,q}$ be the extension by zero of the trivial sheaf complex on $X_{w,q}$ as in §8.3. Consider the set of all formal linear combinations with coefficients in the ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of bounded constructible complexes of ℓ-adic sheaves on X_q with distinguished "Frobenius" endomorphisms, modulo the equivalence relation defined in §8.3. Let H be the submodule generated by the equivalence classes T_w of $T^{\bullet}_{w,q}$ for $w \in W$ and the equivalence class of $\mathbb{C}^{\bullet}_{X_q}$. One can show that H has a natural $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra structure with $\mathbb{C}^{\bullet}_{X_q}$ as multiplicative identity and that there is an isomorphism $\psi: H \to H$ such that $\psi(T_w) = \tau_w$ for all $w \in W$. Verdier duality (see e.g. Borel [1, V §7]) enables one to define an involution Δ of this algebra H such that

$$\Delta(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$$

and

$$\Delta(T_\sigma + 1) = q^{-1}(T_\sigma + 1)$$

if $\sigma \in W$ is a reflection.

It follows from §8.3 that the intersection cohomology sheaf $\underline{\underline{IC}}^{\bullet}_{\bar{X}_{w,q}}$ represents the element

$$T_w + \sum_{v \in W-\{w\}, X_v \subseteq \bar{X}_w} p_{wv}(q)\, T_v$$

of H. Moreover $\underline{\underline{IC}}^{\bullet}_{\bar{X}_{w,q}}$ is self dual with respect to Verdier duality by Theorem 5.4.6 (this is essentially Poincaré duality) but the Frobenius map is multiplied by the scalar factor

$$q^{-\dim X_w}$$

under this duality. It therefore follows from the uniqueness of the Kazhdan-Lusztig polynomials

$$\tilde{p}_{wv}(q)$$

that

$$p_{wv}(q) = \tilde{p}_{wv}(q)$$

for all w and v in W.

Combining 8.4.6 with 8.3.4 and 8.1.5 we obtain the *Kazhdan-Lusztig* *conjecture* (proved by Brylinski and Kashiwara and by Beilinson and Bernstein).

8.4.7 <u>Theorem</u>. (Kazhdan-Lusztig conjecture). The coefficients a_{wv} such that

$$[L_w] = \sum_{v \in W} a_{wv} [M_v]$$

in the Grothendieck group of $\mathfrak{g}$-modules are given by

$$a_{wv} = (-1)^{\dim X_w - \dim X_v} p_{wv}(1)$$

where $p_{wv}(q)$ are the Kazhdan-Lusztig polynomials.

References

M.F. Atiyah et al

1. Representation theory of Lie groups, London Mathematical Society Lecture Note Series 34, Cambridge University Press, Cambridge 1979.

A. Beilinson and J. Bernstein

1. Localisation des G-modules, C.R. Acad. Sci. Paris 292 (1981), 15-18.

A. Beilinson, J. Bernstein and P. Deligne

1. Faisceaux pervers, Analyse et topologie sur les espaces singuliers (I), Astérisque 100 (1982).

J. Bernstein

1. Algebraic theory of $\mathcal{D}$-modules, preprint (course given at C.I.R.M. conference on Differential Systems and Singularities, Luminy, 1983, Proceedings published in Astérisque 130 (1985)).

A. Borel et al

1. Intersection cohomology, Progress in Mathematics 50, Birkhäuser Boston, Basel 1984.
2. Algebraic $\mathcal{D}$-modules, Perspectives in Mathematics 2, Academic Press, Orlando, Florida 1987.

A. Borel and J.C. Moore

1. Homology theory for locally compact spaces, Michigan Math. Journal 7 (1960), 137-159.

R. Bott and L.W. Tu

1. Differential forms in algebraic topology, Graduate Texts in Mathematics 82, Springer-Verlag, New York 1982.

N. Bourbaki

1. Groupes et algèbres de Lie, Hermann, Paris 1968.

J.-L. Brylinski

1. (Co)-homologie d'intersection et faisceaux pervers, Séminaire Bourbaki 585, Astérisque 92-93 (1982), 129-158.

J.-L. Brylinski and M. Kashiwara

1. Kazhdan-Lusztig conjecture and holonomic systems, Inventiones mathematicae 64 (1981), 387-410.

H. Cartan and S. Eilenberg

1. Homological algebra, Princeton University Press, Princeton, New Jersey 1956.

J. Cheeger

1. On the spectral geometry of spaces with cone-like singularities, Proceedings National Acad. Sci. U.S.A. 76 (1979), 2103-2106.
2. On the Hodge theory of Riemannian pseudomanifolds, Proceedings of Symposia in Pure Mathematics 36, American Math. Soc. 1980, 91-146.
3. Hodge theory of complex cones, Analyse et topologie sur les espaces singuliers (II-III), Astérisque 101-102, (1983) 118-134.

J. Cheeger, M. Goresky and R. MacPherson

1. L^2-cohomology and intersection homology for singular algebraic varieties, Seminar on Differential Geometry, Annals of Mathematics Studies 102, Princeton University Press, Princeton, New Jersey 1982, 303-340.

C. Chevalley

1. Classification des groupes de Lie algébriques, Séminaire C. Chevalley, 1956-1958, Ecole Normale Sup., Secret. Math., Paris, 1958.

P. Deligne

1. La conjecture de Weil (I), Publications Math. IHES 43 (1974).
2. La conjecture de Weil (II), Publications Math. IHES 52 (1980).
3. Equations différentielles à points singuliers réguliers, Lecture Notes in Mathematics 163, Springer-Verlag, New York 1970.

A. Dold

1. Lectures on algebraic topology, Springer-Verlag, New York 1972.

R. Godement

1. Topologie algébrique et théorie des faisceaux, Hermann, Paris 1958.

M. Goresky

1. Triangulation of stratified objects, Proceedings American Math.Soc. 72 (1978), 193-200.

M. Goresky and R. MacPherson

1. Intersection homology theory, Topology 19 (1980), 135-162.
2. Stratified Morse theory, Singularities, Proceedings of Symposia in Pure Mathematics 49, American Math. Soc. 1983, 517-535 (II-III to appear).
3. On the topology of complex algebraic maps, Proceedings of International Congress of Algebraic Geometry, La Rabida, Springer Lecture Notes in Mathematics 961 (1981), 119-129.
4. Morse theory and intersection homology, Analyse et topologie sur les espaces singuliers (II-III), Astérisque 101-102 (1983), 135-192.
5. Intersection homology II, Inventiones Mathematicae 71 (1983), 77-129.
6. Lefschetz fixed point theorem for intersection homology, Comment. Math. Helvetici 60 (1985), 366-391.

M. Greenberg

1. Lectures on algebraic topology, Benjamin, New York 1966.

P. Griffiths

1. Topics in transcendental algebraic geometry, Annals of Mathematics Studies 106, Princeton University Press, Princeton, New Jersey 1984.

P. Griffiths and J. Harris

1. Principles of algebraic geometry, John Wiley and Sons, New York 1978.

A. Grothendieck

1. Sur quelques points d'algèbre homologique, Tôhoku Math. Journal 9 (1957), 119-221.

R. Hardt

1. Topological properties of subanalytic sets. Transactions American Math. Soc. 211 (1975), 57-70.

R. Hartshorne

1. Algebraic geometry, Graduate Texts in Mathematics 52, Springer-Verlag, New York 1977.

P.J. Hilton and U. Stammbach

1. A course in homological algebra, Graduate Texts in Mathematics 4, Springer-Verlag, Heidelberg 1970.

N. Jacobson

1. Lie algebras, Interscience Publishers, John Wiley and Sons, New York, 1962.

V.G. Kac

1. Infinite dimensional Lie algebras, Cambridge University Press, Cambridge 1985.

M. Kashiwara

1. Faisceaux constructibles et systèmes holonomes d'equations aux derivées partielles linéaires à points singuliers reguliers, Séminaire Goulaouic-Schwartz 19, 1979-80.
2. Systems of microdifferential equations, Progress in Mathematics 34, Birkhauser, Boston 1983.
3. On the holonomic systems of differential equations II, Inventiones mathematicae 49 (1978), 121-135.

M. Kashiwara and T. Kawai

1. On the holonomic systems of linear differential equations (systems with regular singularities) III, Publications of Research Institute for Mathematical Sciences, Kyoto University 17 (1981), 813-979.

M. Kashiwara, T. Kawai and T. Kimura

1. Foundations of algebraic analysis, Princeton Mathematical Series 37, Princeton University Press, Princeton 1986.

N. Katz

1. An overview of Deligne's proof of the Riemann hypothesis for varieties over finite fields (Hilbert's problem 8), Mathematical developments arising from Hilbert's problems, S. Browder (editor), Proceedings of Symposia in Pure Mathematics 28, American Math. Soc. 1976, 275-306.

D. Kazhdan and G. Lusztig

1. Representations of Coxeter groups and Hecke algebras, Inventiones Mathematicae 53 (1979), 165-184.
2. Schubert varieties and Poincaré duality, Proceedings of Symposia in Pure Mathematics 36, American Math. Soc. 1980, 185-203.

Le Dung Trang and Z. Mebkhout

1. Introduction to linear differential equations, Singularities Part 2, Proceedings of Symposia in Pure Mathematics 40, American Math. Soc., Providence, Rhode Island 1983, 31-63.

S. Lojasiewicz

1. Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa (3), 18 (1964), 449-474.
2. Sur les ensembles semi-analytiques, Proceedings of the International Congress of Mathematicians, Nic, 1970, Vol. II, 237-241.

R. MacPherson

1. Global questions in the topology of singular spaces, Proceedings of the International Congress of Mathematicians, Warsaw, 1983, Vol. I, 213-235.

B. Malgrange

1. Sur les points singuliers des equations differentielles, Enseign. Math. XX (1974), 147-176.

Z. Mebkhout

1. Sur le problème de Hilbert-Riemann, C.R. Acad. Sci. 290 (1980), 415-417.
2. Une autre équivalence de catégories, Compositio Math. 51 (1984), 63-88.

J.S. Milne

1. Étale cohomology, Princeton Mathematical Series 33, Princeton University Press, Princeton, New Jersey, 1980.

J. Milnor

1. Morse theory, Princeton University Press, Princeton, New Jersey, 1963.

T. Oda

1. Introduction to algebraic analysis of complex manifolds, Advances in the study of Pure Mathematics 1, North Holland 1983.

F. Pham

1. Singularités des systemes différentiels de Gauss-Manin, Progress in Mathematics 2, Birkhäuser Boston 1979.

H. Poincaré

1. Analysis Situs, Journal de l'École Polytechnique 1 (1895), 1-121. Oeuvres de H. Poincaré VI, Gauthier-Villars, Paris 1953, 193-498.

H. Röhrl

1. Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen, Math. Annalen 133 (1957) 1-25.

M. Saito

1. Modules de Hodge polarisables, Research Institute for Mathematical Sciences preprint 553, Kyoto University, Kyoto, Japan, 1986.
2. Mixed Hodge modules, Research Institute for Mathematical Sciences preprint 585, Kyoto University, Kyoto, Japan 1987.

M. Saito, T. Kawai and M. Kashiwara

1. Microfunctions and pseudo-differential equations, Lecture Notes in Mathematics 287, Springer-Verlag, New York, 1973, 265-529.

J-P. Serre

1. Faisceaux algébrique cohérents, Annals of Mathematics 61 (1955), 197-278.
2. Zeta and L-functions, Arithmetical algebraic geometry, Harper and Row, New York (1965), 82-92.

E.H. Spanier

1. Algebraic topology, McGraw-Hill, New York 1966.

M. Spivak

1. A comprehensive introduction to differential geometry, Second edition, Publish or Perish, Berkeley 1979.

T.A. Springer

1. Linear algebraic groups, Progress in Mathematics 9, Birkhäuser Boston 1981.

S. Sternberg

1. Lectures on differential geometry, Prentice-Hall, Englewood Cliffs, New Jersey 1964.

J-L. Verdier

1. Le théorème de dualité de Poincaré, C.R. Acad. Sci. 256 (1963).
2. Dualité dans la cohomologie des espaces localment compactes, Sém. Bourbaki 300 (1965).

F.W. Warner

1. Foundations of differentiable manifolds and Lie groups, Scott, Foresman and Company, Glenview, Illinois 1971.

H. Whitney

1. Tangents to an analytic variety, Annals of Mathematics 81 (1965), 496-549.

A. Weil

1. Number of solutions of equations over finite fields, Bulletin Amer. Math. Soc. 55 (1949), 497-508.

Index