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Inverse Problems of Mathematical Physics

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Introduction

A feeling for "inverse problems" can be obtained, for example, by noting that not every object in nature is accessible to direct study, and consequently, its properties must be judged indirectly. The bowels of the Earth may serve as an example and problems of this type have been known for some time in geophysics. At the same time the posing of inverse problems is characteristic of scientific investigation and interest in them has been on the increase in many fields of science, physics in particular.

What distinguishes mathematical physics is the fact that in it the study of nature proceeds within the framework of precise mathematical models, formulated on the basis of known regularities. Such models serve as the basis for the solution of inverse problems.

However, due to the specificity of inverse problems—their mathematical "wrongness"—substantial progress has been achieved in the posing and solving of inverse problems only in the last 10–20 years. Responsible for this was the mathematical "regularization theory" developed by Soviet scientists,[71,85,148] as well as the intensive development of computational techniques.

Familiarity with elements of regularization theory and its applications to the solution of inverse problems, as well as the description of possible fields of application, constitute an essential part in the training of scientists and are useful to workers in many branches of knowledge.

This monograph is based on a special topics course taught by the author in the Physics Department of Moscow State University and constitutes a short version of the course "Inverse problems of mathematical physics." It is not meant as a substitute for the excellent monographs on the theory of regularization and some of its applications[19,86,132,139,148] for readers interested in the development of the mathematical theory. Neither does it pretend to provide a full review of known results in the field. However, for the reader not familiar with the subject under discussion, or interested in practical applications of regularization theory, this book should serve as a kind of introduction.

The monograph is addressed to a reader familiar with elements of mathematical analysis, with a course in mathematical physics similar to Ref. 162, with some elements of functional analysis, and with some computational methods. Unfortunately the size of this monograph precludes the replacing of references to various results by their explanation.

The content is based on the fundamental papers [Refs. 71, 85, and 148] including a number of results obtained with the author's participation.

The material is distributed among four chapters. In Chap. 1, we give a broad characterization of the class of inverse problems. In contrast to existing publications and in accordance with the essence of the subject, we consider questions of uniqueness of solutions of inverse problems (Chap. 3), which might be useful from both an instructional and pedagogical point of view. In a slight departure from tradition, we emphasize in Chap. 2 questions relating to the mathematical posing of inverse problems. Regularizing algorithms for their solution are considered in Chap. 4. Concrete examples of inverse problems of mathematical physics are used to illustrate the main assumptions.

Chapter 1
The class of inverse problems

1. The concept and type of inverse problems

1.1. The concept of the inverse problem of mathematical physics

All phenomena in nature have a causal origin and are governed by objective regularities, and the task of scientific investigation is to discover these regularities and determine the cause of the phenomenon in order to master it.

Mathematical physics refers to that branch of science wherein on the basis of known regularities, expressed as (differential or integral) equations, it becomes possible to pose mathematically[162] the problem describing the phenomenon.

Such a problem then constitutes an approximate mathematical model of the physical process, which may be stated in various ways.

As a simple example, consider the mathematical model describing the rectilinear nonuniform motion of a material point of mass m under the action of a force $f(t)$ for given initial conditions

$$\begin{aligned} m\ddot{x}(t) &= f(t) \quad 0<t<T, \\ x(0) &= x_0, \quad \dot{x}(0)=\dot{x}_0. \end{aligned} \tag{1}$$

We may be interested in the following questions:

(1) What is the law of motion of the material point $x = x(t)$, if the "acceleration cause" $f(t)$ and "inertial characteristic" m of the moving object are known? This is a classical problem in mechanics and its solution provides a qualitative idea of the motion; alternatively the process of the motion can be "numerically modelled," for example with the help of an electronic computer.

(2) Suppose the law of motion is known (observed). What then is the required force or (if the force is known) mass of the object—characteristic of its properties? The solution of one of these problems permits the characterization of either the cause of the motion or the properties of the object.

The first of these problems and either of the latter two are invertible. However, in the first case we are to study consequences due to given causes and conditions, whereas in the second case we are to study causes and conditions given the consequences. It is this latter type of problem that will be referred to as "inverse" problems of mathematical physics, provided that a mathematical formulation is possible (as was obviously the case in the example discussed above). It is easily seen that inverse problems are analogous to the task faced by basic scientific research: Given certain characteristics of the phenomenon under study, explain its causes. In Ref. 2 a common signature of inverse problems is noted: they are not realizable physically. Indeed, in the example discussed above, the notion of an experiment which reproduces the force or the mass of the object from the given law of motion is absurd.

At the same time, in contrast to problems involving direct modeling of physical processes (in modern scientific literature such problems are accordingly labeled "direct"), the solving of inverse problems is largely the domain of mathematics.

We shall see later that inverse problems arise in many branches of physics and manufacturing, and their solution often provides an element for the mathematical modeling of the processes, because not all the sources and conditions of the processes are known *a priori*. It could be argued that the theory and methods of solution of inverse problems constitute an important independent direction of research in mathematical physics.

The development of this direction was begun in the fundamental papers of the Soviet scientists A. N. Tikhonov, M. M. Lavrent'ev, and V. K. Ivanov.[71,85,149,160]

1.2. General typification of inverse problems

There is today, probably, no branch of physics in which one or another inverse problem has not been formulated and subjected to mathematical analysis. Accordingly, sufficiently detailed classification has been presented in a number of monographs, based either on the mathematical nature of the sought-for quantities,[116] or on the content of the concrete physical problem (for example, the study of heat exchange processes[2]).

Here we shall consider a sufficiently broad classification of such problems, arising from the development of the fundamental theory[143–148] and its applications.

(a) Scientific studies often deal with objects in nature which are either inaccessible or difficult to access by direct experiment. Glaring examples are cosmic objects, the bowels of the Earth, and objects from the microworld. But even under "earthly" circumstances such a situation is sufficiently typical: many physical (heat, electromagnetic, etc.) properties of, for example, synthetic and porous materials are known poorly, and their experimental study is not always possible or easy.

In all these cases one makes judgments about the properties of the object under study, as being the cause of a phenomenon, on the basis of indirect manifestations which can be directly observed under earthly conditions.

A mathematical model is formulated, reflecting the connection between the characteristic z of the object under study and the characteristic u of the observation, and within the framework of the assumed approximate model the problem of determining z on the basis of u is posed.

It is natural to refer to problems of this type as problems of interpretation of given physical observations.

Any one of numerous examples may serve the problem of the "historical climate,"[162] connected to the study of the history of formation of permafrost zones on the Earth's surface. In this case, in the simplest mathematical description of the phenomenon, the depth dependence of the temperature of the frozen layer at the moment of observation ($t = 0$) may serve as the observed quantity: $u(\zeta) = U(\zeta,0)$. The sought-for quantity, assumed to be the cause of freezing, is the temperature regime of the surface $z(t) = U(0,t)$. Within the framework of the approximate mathematical model for a given $z(t)$, the temperature field $U(\zeta,t)$ [and, correspondingly, $u(\zeta)$] is determined as the solution of a well-known problem of mathematical physics on the effect of the boundary temperature regime[162]:

$$a^2 \frac{\partial^2 U}{\partial \zeta^2} = \frac{\partial U}{\partial t} \quad \zeta \in (0,\infty), \quad t \in (-\infty,0), \qquad U(0,t) = z(t), \quad |U| < +\infty. \tag{2}$$

For the inverse problem of interest to us, the relation between $u(\zeta)$ and $z(t)$ may be given explicitly[162] as

$$u(\zeta) = \frac{1}{2a\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{t^{3/2}} e^{-(\zeta^2/4a^2 t)} z(t)\, dt; \tag{3}$$

consequently the sought-for quantity is a solution of an integral Fredholm equation of the first kind. This equation provides an elementary mathematical formulation of the interpretation problem under discussion.

Let us note that the observed quantity $u(\zeta)$ represents an integral effect of the action of the sought-for boundary regime: its formation is affected by values of $z(t)$ at all times and this is determined by the conditions (2) independently of the explicit expression (3).

(b) A "subset" of problems of interpretation is formed by problems that we shall label "instrumental." The purpose of these type of problems is the reconstruction of the true signal from instrumental readings, and its elementary statement can be given in a sufficiently general form.

Suppose that the apparatus function (impulse characteristic) $K(t)$ of a certain instrument is known. Then, within the framework of the simplest "linear" model, its response $u(t)$ to a continuous influence $z(t)$ ($t > 0$) is given by the formula

$$u(t) = \int_0^t K(t-\tau) z(\tau) d\tau. \tag{4}$$

By using the instrumental markings, the observed readings $u(t)$ can, of course, be taken as characterizing the input signal. This, however, raises questions about the resolution of the instrument.[115] Meanwhile, given the impulse characteristic we may pose the inverse problem of determining $z(\tau)$ from $u(t)$ by means of Eq. (4). It is obvious that if this problem can be solved with sufficient accuracy, it is equivalent to improving the resolution of the instrument.

Sometimes the apparatus function may be defined either by equations describing the process of transformation of the signal by the instrument,[123] or experimentally.[152] In particular, it is conceivable that the experiment could be set up with a directed (controlled by some other means) signal $z(t)$ as input of the instrument, from which the apparatus function can be found by solving the inverse problem resulting from Eq. (4):

$$u(t) = \int_0^t z(t-\xi)K(\xi)d\xi. \tag{5}$$

(c) An important result of scientific research is the creation or "synthesis" of new scientific instruments or technical setups. Although the object in question does not yet exist, we may consider its mathematical model wherein a certain magnitude z characterizes its internal properties on which depends the characteristic u under exploitation. The problem of determining z given u belongs to the class of inverse problems of the synthesis type.

The following mathematical model, connected with the problem of synthesis of optical systems with prescribed "transmission coefficient," may serve as an elementary example of this type of inverse problem.

Under certain simplifying assumptions, the amplitude of an electric field of a light wave of frequency ω, polarized parallel to the surface of a plate of index of refraction $n(\zeta)$, normally incident on the surface, is described by the conditions of the following problem[48]:

$$\begin{aligned}
&\frac{d^2E}{d\zeta^2} + \frac{\omega^2}{c^2}\, n^2(\zeta)E = 0, \quad \zeta\in(0,h),\\
&\frac{dE}{d\zeta}(0) - i\frac{\omega}{c}\, n_0(E(0) - 2E_0) = 0,\\
&\frac{dE}{d\zeta}(h) + i\frac{\omega}{c}\, n_0E(h) = 0,
\end{aligned} \tag{6}$$

where E_0 is the amplitude of the wave and n_0 is the index of refraction of the external medium. By solving this boundary value problem, we can determine for given $n(\zeta)$ the "exploitation" characteristic of the plate—its light transmission coefficient, as a function of the frequency:

$$u \equiv T(\omega) = \frac{n(h)}{n_0}\left|\frac{E(h)}{E_0}\right|^2. \tag{7}$$

The inverse problem of the type considered here consists in the determination of the index of refraction of the plate, $n(\zeta)$, from the given transmission coefficient $T(\omega)$. The solution of this type of problem clearly makes possible the physical synthesis of a plate with *a priori* specified optical properties.

This time the dependence of T on $n(\zeta)$ is given indirectly through conditions (6) and (7); it is, however, clear that the formation of $T(\omega)$ is influenced by values of $n(\zeta)$ for all ζ, i.e., the effect $T(\omega)$ is of integral character.

Examples of papers devoted to the analysis of inverse problems of synthesis type in various branches of physics are to be found in Refs. 126, 127, 150, and 163.

(d) Often in scientific and manufacturing practice it is necessary to solve problems of control of active systems. Insofar as the processes occurring in such systems are subject to known physical laws, it is possible to formulate the corresponding mathematical model. Within the framework of such a model we extract the control characteristic z and the expected effect u. The problem consists in the determination of z from the expected effect.

We shall refer to such problems as inverse problems of control type.

As an example, consider the problem of determining the amplitude of the current $I(t)$ in a solenoidal inductor, governing the heating of a cylindrical sample being inductionally tempered. Since we are dealing here with "case-hardening,"[24] the effect of control will be characterized by the required temperature regime of the sample surface:

$$u(t) = U(R,t) \quad t \in [0,T].$$

In the approximation of the simplest model,[44] for any given $I(t)$ the quantity $u(t)$ is determined by the following nonlinear problem of mathematical physics:

$$\begin{gathered}
\frac{1}{r}\frac{\partial}{\partial r}\left(rk(U)\frac{\partial U}{\partial r}\right) + q(r,t) = c(U)\rho(U)\frac{\partial U}{\partial t}, \\
t \in (0,T), \quad r \in (0,R), \\
q(r,t) = 0.24\left|\frac{\partial H}{\partial r}\right|^2; \\
\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\sigma(U)}\frac{\partial H}{\partial r}\right) = i\omega\mu H + \frac{\partial}{\partial t}[\mu(U)H]; \\
\frac{\partial U}{\partial r}(0,t) = 0, \quad -k\frac{\partial U}{\partial r}(R,t) = \alpha(U(R,t) - u_0), \\
\frac{\partial H}{\partial r}(0,t) = 0, \ H(R,t) = nI(t), \ U(r,0) = u_0, \ H(r,0) = 0,
\end{gathered} \tag{8}$$

where u_0 is the temperature of the surrounding layer of air, n is the number of turns per unit length of the "infinite" solenoid, and the physical parameters of the material are k, c, ρ, μ, and σ in the notation of Ref. 162.

The inverse control problem consists in the determination of $I(t)$ from an *a priori* given $u(t)$, where the dependence of I on u is given implicitly by Eq. (8) and the condition $u(t) = U(R,t)$.

2. Mathematical specifics of inverse problems

2.1. Generalized mathematical statement of the problem

As is clear from the above examples, for any given characteristic of the object (or process) z, a characteristic of the observed phenomenon can be calculated with the help of a set of operations (or an operator) A, given explicitly in cases (3) and (4) and implicitly in cases (6) and (7), such that $u = Az$.

Let us consider a set of such characteristics Z and U, and define on it a measure of closeness of elements (distance): $\rho_Z(z_1, z_2)$ and $\rho_u(u_1, u_2)$. Then the inverse problem—the determination of z from given u—regardless of its physical nature in the elementary statement, can be given in the form of an "operator" equation:

$$Az = u \quad z \in Z, \ u \in U. \tag{9}$$

In what follows we shall lean on this formulation for problems involving interpretation of observations, in particular instrumental problems.

For problems from the synthesis or control classes, at the same elementary formulation level, another approach is *a priori* more natural. In these problems the characteristic of the expected effect $\hat{u}$ is usually "idealized," so that, generally speaking, $\hat{u} \neq Az$. Thus, in example (6), it is required that the transmission coefficient $\hat{T}(\omega) \equiv 1$ for a certain frequency range $[\omega_1, \omega_2]$ and $\hat{T}(\omega) \equiv 0$ for all other frequencies; it is obvious that $T(\omega)$, obtained as a solution of problem (6), being a differentiable function of ω,[151] cannot coincide with $\hat{T}(\omega)$ at every point. In such a case it is natural to introduce an average measure of closeness of Az and $\hat{u}$—a distance in Hilbert space—and seek z from the requirement

$$\inf \rho(Az, \hat{u}) \quad z \in Z \tag{10}$$

[or, in the more general case, inf $\Phi(Az)$, where $\Phi(u)$ is some functional defined on U].

It is easily seen that problems from the first two classes can be viewed analogously. At that if by u is meant, similarly to $\hat{u}$, the exact specification of the characteristic of the phenomenon (as is assumed in the classical analysis), then such a formulation is equivalent to Eq. (6). If, on the other hand, we take into account that the characteristic of the phenomenon $\tilde{u}$ is specified within some error so that, generally speaking, $\tilde{u} \neq Az$, $z \in Z$, then the equivalence no longer holds (the problem $Az = \tilde{u}$ has no solution), and it is more natural to view the problem in the formulation analogous to Eq. (9): $\inf \rho(Az, \tilde{u})$, $z \in Z$.

2.2. The concept of well-posed

Any quantitative problem consists in finding the "solution" z from the "initial data" u: $z = R(u)$. The concept of solution, the meaning of the "operator" A, the character of the sets Z and U from which z and u are chosen (we shall take them to be metrical spaces), are all determined by the statement of the problem. Thus in the above-discussed examples, where the meaning of solution and the character of the sets Z and U are defined, R can be understood to mean the totality of algebraic and analytic operations that solve Eq. (9) or the variational problem (10).

The problem of determining z and u is called well-posed (according to Hadamard) if it satisfies the following conditions: (i) for each $u \in U$ there exists a solution $z \in Z$; (ii) the solution is unique; and (iii) the solution depends continuously on u (it is stable against small variations of u).

If even one of the above conditions is not satisfied, then the problem is called ill-posed.

The concept of being well-posed was formulated by J. Hadamard at the beginning of this century, in application to problems of mathematical physics as a whole. He expressed the opinion, engendered by the success of this discipline in exact description of phenomena, that ill-posed problems have no physical content and are of no interest for mathematical study. This opinion turned out to be erroneous with respect to the subject of inverse problems.

2.3. Peculiarities of the "elementary" mathematical formulation of the inverse problem

In inverse problems of mathematical physics we are usually interested in detailed information about the introduced characteristic property z of the object. Thus, if $z \equiv \mathbf{p} \in E^n$ is a vector, then we are interested in all its components.

Let us consider the case when $z \equiv z(x)$, $x \in X_1$, corresponding to the examples introduced above. It is understood that, having introduced such a characteristic of the object, we are interested in values $z(x)$ at all points. It is natural, therefore, to understand under $\rho_z(z_1, z_2)$ the "uniform" metric:

$$\rho_z(z_1, z_2) \equiv \max_{X_1} |z_1(x) - z_2(x)|; \quad Z \equiv C(X_1).$$

Let the characteristic of the observations (expected effects) also be a certain function $u \equiv u(x)$, $x \in X_2$, as in the discussed examples. Since u is given with uncontrollable errors (or idealized), it is natural to consider on the set U closeness "in the mean"[a]:

$$\rho_U(u_1, u_2) \equiv \left(\int_{X_2} [u_1(x) - u_2(x)]^2 dx \right)^{1/2}; \quad U \equiv L_2(X_2).$$

Let us consider under these conditions the operator equation (9), taking into account that in place of the exact right-hand side we are given the approximation $\tilde{u}$. It is clear that the equation does not have solutions for all $\tilde{u} \in U$, and consequently the problem is ill-posed and the equation $Az = \tilde{u}$ is of purely conditional character. We note that passing to the variational formulation does not, generally speaking, solve the question of existence of solutions: it is not obvious that the existing *a priori* exact lower bound on the distance is reached within the set Z.

Let us, however, extract a subset U, on which Eq. (9) has a solution which, moreover is unique for each u. It is characteristic of inverse problems that the quantity u expresses the integral effect of the influence z; accordingly it should be clarified that the operator A is usually "compact" ("fully continuous").[78] This means, in particular, that for the imaging $u = Az$ an arbitrarily large sufficiently localized variation of z produces an arbitrarily small variation of u. Hence the solution of the operator equation (inverse imaging) is unstable. Consequently, and in this respect problem (10) is in no way different, the introduced measure depends continuously on u and, therefore, to small variations of ρ may correspond arbitrarily large variations in z.

These considerations are readily extended to the case when the operator equation is given explicitly in the form of a Fredholm (3) or Volterra (4) inte-

[a] The measures $\max_i |p_{i1} - p_{i2}|$ and $[\Sigma_i (p_{i1} - p_{i2})^2]^{1/2}$ are equivalent on the set of vectors $z \equiv \mathbf{p} \in E^n$.

gral equation. The compactness of an integral operator with a continuous kernel is well known. Obviously the integral is a continuous function of the parameter (x or t). If the kernel is a smooth function of the parameter, then so is the integral. However, $u(x)$ in Eq. (3) need not be smooth since it carries random perturbations.[b] On the other hand, we may, for example, set in Eq. (4) $\delta z = A_0 \sin \omega\tau$, where A_0 is an arbitrarily large but fixed number. It is obvious that for a continuous kernel ($|K(t)| \leqslant \varkappa$) one can find a sufficiently large value for ω to make δu arbitrarily small[74] on an arbitrary finite interval $t \in [0, T]$ (the integral smooths out the high-frequency oscillations of the integrand function). Moreover, to arbitrarily small perturbations $u(t)$ in $L_2(0, T)$ may correspond arbitrarily large perturbations $z(\tau)$ in $C[0, T]$.

We have discussed the case when z and u are functions. This does not mean that were they elements of a finite-dimensional space (vectors) the elementary formulation of the inverse problem would be *a priori* correct. In that case Eq. (9) turns into a finite system of equations for the vector components $z \equiv \mathbf{p}$. This type of inverse problem was discussed in Refs. 42 and 43. The operator A remains fully continuous, and if it depends weakly on certain components of $\mathbf{p}$, then the corresponding minors of the functional matrix turn out to be close to zero (the system becomes poorly determined). Then, if we are interested in all the components of $\mathbf{p}$, we run into instability of the solution with respect to variation of $\mathbf{u}$. We note that in applied problems a rather small variation in the sought-for vector characteristic $\mathbf{p}$ may lead to physically inconsistent results. Is, for example, the difference between the numbers 1 and 3.5 large? If they stand for the calculated values of density of the Earth's core, then the first one is patently absurd, and the utilized algorithm is unstable in practice. Examples of inverse problems, which give rise in elementary formulation to degenerate systems of linear algebraic equations, are given in Ref. 148. In that case uniqueness is *a priori* absent and variation of the right-hand side which produces, as a rule, the last value from the set of values $\mathbf{u} = A\mathbf{p}$, makes the system conditional. In that case passage to the variational problem of type (10) amounts to the least-squares method. However, for a degenerate operator A the matrix A^*A of the normal system is also degenerate and, consequently, the problem in formulation (10) remains ill-posed.

Being ill-posed in the elementary formulation is a characteristic property of inverse problems of mathematical physics.

2.4. Practical consequences of being ill-posed

The class of ill-posed problems is larger than the class of inverse problems. Many problems in analysis belong here: differentiation of functions given with errors; summation of Fourier series with inexactly given coefficients; solution of systems of linear equations with determinant close to zero in the presence of errors in the matrix elements and in the right-hand sides, and so forth. Corresponding examples can be found in Refs. 86 and 148.

[b] Let us note that its real prototype need not be a continuous function, since chance perturbations refer to any point.

Consequently, the *a priori* possibility of formally utilizing classical methods of solution of ill-posed problems of mathematical physics does not mean that a satisfactory approximation to the exact solution can be obtained in this way.

As an example, consider various realizations of the operator $z = R(u)$ in application to the solution of integral equations, ignoring the fact that the problem is ill-posed.

(a) Within the framework of a simplified model the problem of extending the gravitational field in the direction of its sources gives rise to the equation[154]

$$u(x) = \frac{H}{\pi}\int_a^b \frac{z(\xi)\,d\xi}{H^2 + (x-\xi)^2} \quad x\in(a,b). \tag{11}$$

Let us perform a mathematical experiment for $a = -1$, $b = 1$, $H = 1$, $z(\xi) = \xi^2(1-\xi^2)^2$, having computed values of $u(x)$ from a given mesh $\{x_i\}$ with machine accuracy $\delta = 10^{-9}$. For $R(u)$ we choose the operator composed of the solution of a system of linear equations, obtained by a finite approximation to Eq. (11) with a step h (in ξ and x), and the limit $h\to 0$. In Fig. 1 we show the results of such an experiment for $h = 0.1$. It is seen that the "approximation" has nothing in common with the true solution (even for a relatively large step and machine accuracy of u!).

The effect is explained by noting that as a result of the approximation we arrive at a badly determined system[148] (due to the closeness of columns) and the "degree of determination" decreases with decreasing h.

(b) Since the kernel is symmetric and continuous, one may choose for the analogous problem for $R(u)$ the operation of summation of the Fourier series on the eigenfunctions of the kernel. In that case

$$z(\xi) = \sum_{k=1}^{\infty} (\tilde{u}_k/\lambda_k)\psi_k(\xi).$$

Leaving aside the question of evaluating ψ_k and λ_k, we note that $\tilde{u}_k$ is given with errors if for no other reason than due to errors in $\tilde{u}(x)$. Then, in this case, too, we fail to obtain a satisfactory approximation due to instability in Fourier series summation.

(c) Suppose that an inverse problem is specified in terms of a Volterra equation of the first kind with kernel $K(t,\tau)$, where $K(t,t)\neq 0$ and is differentiable with respect to t. Then it can be reduced to an equation of the second kind, whose solution depends continuously on the right-hand side.[111] For $R(u)$ we choose an operator analogous to that of part (a), but for equations of the second kind. The latter, however, has the form

$$\int_0^t \varkappa(t,\tau)z(\tau)d\tau + z(t) = v'(t),$$

where

$$v'(t) = u_t'(t)K^{-1}(t,t) \quad \varkappa(t,\tau)\equiv K_t'(t,\tau)K^{-1}(t,t).$$

It is to be understood that the stability of the solution of this equation does not mean that to large errors of the right-hand side correspond small errors in the

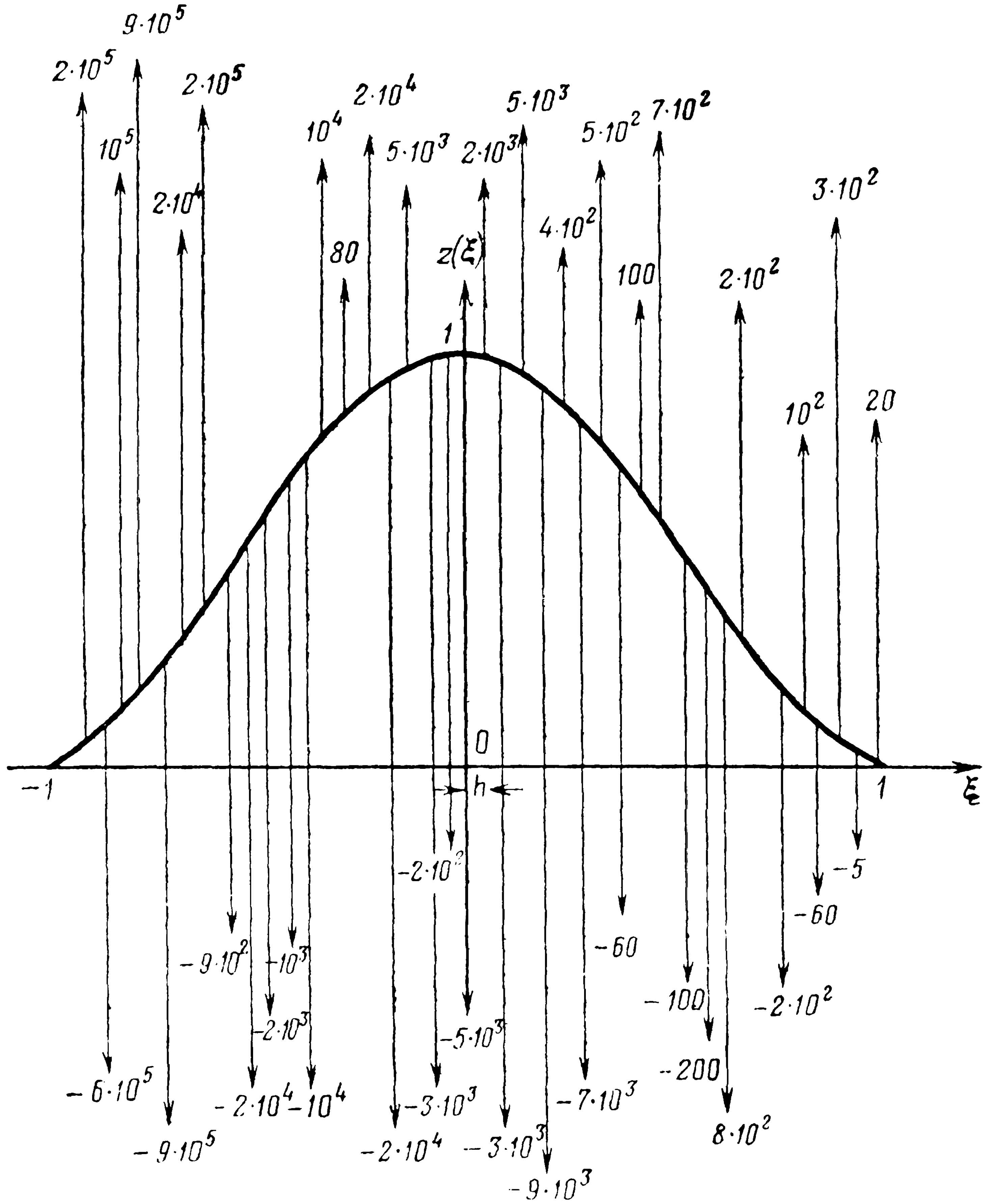

Figure 1

solution. And in this case the right-hand-side errors are uncontrollably large due to the fact that numerical differentiation of the approximate $\tilde{u}(t)$ is not a well-posed operation.

We note that, as a consequence of the just-mentioned circumstances, the inverse problems discussed at the beginning of the first paragraph in connection with Newton's equation are ill-posed. Indeed, to evaluate $f(t)$ or m we must differentiate the function $\tilde{x}(t)$, which is given experimentally with errors. Clearly such a problem is unstable.

(d) Exploiting the fact that in the Volterra equation corresponding to the instrumental problem (4) the kernel is a function of the difference of the arguments, one may choose as the imaging operator the Laplace transform followed by the inverse transform [precisely as the Fourier transform may be used for Eq. (11) in the singular case $a = -\infty$, $b = +\infty$]. However, due to the inexact specification of $\tilde{u}(t)$ [$\tilde{u}(x)$], we run into the fact that the inverse transform is ill-posed.[69,148]

Other examples may be mentioned.

Consequently, the fact that these problems are ill-posed cannot be ignored. There are two paths for overcoming this difficulty: (a) well-posed formulation of the inverse problem of mathematical physics, based on the introduction of additional information about the sought-for solution; and (b) control of classical algorithms for the solution of ill-posed problems.

Both of these paths are analyzed in the regularization theory developed by Soviet scientists, which gives rise to the concept of a regularized operator in numerous realizations.

We study elements of this theory below.

Chapter 2
Well-posing of inverse problems

1. The concept of problem insertion into the correctness class

1.1. Selection method of solution of operator equation

Consider, for example, the conditional equation connected with the instrumental problem (4). Assume that some estimate of the signal duration T is known ($t \in [0, T]$), as well as the instant of maximum intensity $\tau_0 \in (0, T)$, and that the "profile" of the signal can be approximated by the formula

$$z(\tau, \mathbf{p}) = p_1 e^{-p_2(\tau - \tau_0)^2} \quad p_1, p_2 > 0 .$$

Then the signal appearing at the output of the instrument will be a function of t dependent on the vector parameter $\mathbf{p} = \{ p_1, p_2 \}$:

$$\int_0^t K(t - \tau) z(\tau) d\tau \equiv U(t, \mathbf{p}) .$$

Suppose that we have evaluated this function for various values $\mathbf{p} = \mathbf{p}_k$, $k = 1, 2, \ldots, N$, varying the coordinates within the specified intervals:

$$0 < p_{1k} < M_1, \quad 0 < p_{2k} < M_2 .$$

Then, upon comparison of $U(t, p_k)$ with the observed $\tilde{u}(t)$, we may choose that $\mathbf{p}_{k_0}$ which leads to best agreement with observations and accept $z(\tau, \mathbf{p}_{k_0})$ as the approximate solution of the equation.

The just-described search procedure constitutes the content of the "selection" method. This method has been widely used for quite some time in, for example, the study of the bowels of the Earth by means of observations on the Earth's surface of gravitational or electromagnetic fields.

In a sufficiently general way the selection problem consists of the following: Suppose an *a priori* model of the object under study exists, determined with a precision up to the vector parameter $\mathbf{p} = \{ p_1, p_2, \ldots, p_n \}$, where n is given and not too large and $|p_i| \leqslant M$, $i = 1, 2, \ldots, n$; the calculated field is $u = A\mathbf{p} = U(x, \mathbf{p})$, $x \in X$, while the observed field is $\tilde{u} = \tilde{u}(x)$. The vector $\mathbf{p}$ is to be so chosen as to provide, in some well-defined sense, best agreement between the calculated and observed quantities.

This problem may be solved in a variety of ways.

(a) One may prepare ahead of time an album of "templates"—in the present case this would be curves $u = U(x, \mathbf{p})$ for various discrete values of $\mathbf{p}$ in the specified region. The comparison of $\tilde{u}(x)$ then proceeds visually by superimposing $\tilde{u}(x)$ on the "template."

(b) In some cases the process of computing $U(x, \mathbf{p})$ can be modeled on an analog computer equipped with a display. This method is discussed, for example, in Ref. 105. Here it is possible to vary continuously the parameter (from the panel of the analog computer) and visually compare $\tilde{u}$ and $U(x, \mathbf{p})$.

(c) The problem may be reduced to a variational one,[14,18,133] $\inf \rho(A\mathbf{p},\tilde{u})$, and one may delegate to an electronic digital computer the task of finding the minimizing element based on some minimizing algorithm[20,70] on a bounded set $|p_i| \leqslant M$.

It is fairly obvious that the first two methods, in contrast to the last one, are practical only when the number of parameters is small ($n = 1$–3) and, consequently, only in the framework of a rough model.

Moreover, the danger of obtaining as a solution to the problem a quantity significantly different from the "true" one, can be excluded by an "intelligent" choice of *a priori* restrictions on the parameters. But the question now arises of what is meant by an "intelligent" choice of restrictions in application to the class of inverse problems. The answer is particularly important for large n, when the vector parameter p may be an approximation to some continuous function.

A fundamental solution to this question was given by A. N. Tikhonov,[140] where for the first time a direct connection was established between a certain result of functional analysis with the selection practice and inverse problems of mathematical physics. For this reason the corresponding result is commonly referred to as Tikhonov's theorem.

1.2. The theorem of A. N. Tikhonov

Consider an inverse problem of the interpretation-of-observational-data type described by the operator equation (9).

It is readily seen that the mathematical content of the selection process consists of direct representation of the characteristics of the object on the set of characteristics of observations, $u = Az$, $z \in Z$. With this in mind consider Eq. (9), $Az = u$, when the right-hand side is known exactly ($u = \hat{u}$).

Let $\hat{Z}$ be a certain subspace of the metric space Z and U_A its image in the metric space U by the operator A.

THEOREM. Let A be continuous on $\hat{Z}$ and let the equation $Az = \hat{u}$ have for every $\hat{u} \in U_A$ a unique solution $\hat{z} \in \hat{Z}$. Then the inverse image $\hat{z} = R(\hat{u})$ is continuous with respect to the measure of Z if Z is closed compact (with respect to the measure of Z).[a]

In other words, to sufficiently small perturbations $\hat{u}$ in the measure of U correspond arbitrarily small perturbations $\hat{z}$ in the measure of Z.

Proof. Let $\hat{u}$ be an arbitrarily chosen element of U_A and $A\hat{z} = \hat{u}$. Consider an arbitrary sequence $\{u_n\} \to \hat{u}$ ($u_n \in U_A$, and convergence is with respect to

[a]By closed compact (with respect to some measure) we mean a set such that for every sequence of elements from the set a subsequence can be selected which converges (with respect to the same measure) to an element in the set.[56]

the measure of U). To it corresponds $\{z_n\}\in\hat{Z}$. Since $\hat{Z}$ is closed compact, $\{z_n\}$ has at least one limit point in $\hat{Z}$; label it $z^*\in\hat{Z}$. Then there exists a subsequence $\{z_{n_k}\}\to z^*$ (with respect to the measure of Z). Since the operator A is continuous, it follows that $\{u_{n_k}=Az_{n_k}\}\to Az^*$. However $\{u_{n_k}\}\to\hat{u}$, being a subsequence of a convergent sequence and, consequently, $Az^*=\hat{u}$. Now, by uniqueness of solution, $z^*=\hat{z}$, and therefore $\{z_n\}\to\hat{z}$. This proves the theorem since $\{u_n\}$ was arbitrary.

We observe that convergence of the approximation in the compactum $\hat{Z}$ to the exact solution $\hat{z}$ (with respect to the measure of Z) is also possible if $\hat{Z}$ is not closed but is compact in Z and $\hat{z}\in Z\setminus\hat{Z}$. This case, corresponding to approximating the exact solution by a sequence of elements from a compact set, will not be considered here.

It is obvious that Tikhonov's theorem explains, first of all, the success of the selection method within the framework of the simplest models: an arbitrary bounded set in a finite-dimensional space E^n (n given) is compact with respect to the E^n metric (the Bolzano–Weierstrass theorem). Therefore, if the unique exact solution belongs to this compact set (and that is the meaning of an "intelligent" choice of *a priori* limits on the sought-for vector), the selection method for fixed n is guaranteed to find an approximation to the exact solution for arbitrary errors in the initial data.

On the other hand, the theorem just proved also covers the larger class—when n is large and p approximates some continuous function. Suppose, indeed, that the sought-for characteristic of the object is described by a continuous function $z=z(x)$, $x\in[a,b]$, and Z is the set of continuous functions on $[a,b]$ ($Z\equiv C[a,b]$). Obviously, although this set is bounded [$\max|z(x)|\leqslant M$], it does not follow that it is compact in $C[a,b]$.[b] It is to be expected that, having set $\mathbf{p}=\{z(x_i)\}$ on some mesh $\{x_i\}$ and made use of the introduced limits, a satisfactory approximation will not be obtained by the selection method even if the difficulty posed by large n can be overcome. According to Arzela's theorem,[74] a set compact with respect to the metric of $C[a,b]$ can be obtained, for example, by imposing two conditions: $\max|z(x)|\leqslant M_1$, $\max|z'(x)|\leqslant M_2$, $x\in[a,b]$; if use is made of its analog in the discrete approximation (for arbitrary n) and if the unique exact solution belongs to the chosen compact set, then the efficiency of the selection method (i.e., the ability to come arbitrarily close to the exact solution) is guaranteed.

1.3. The concept of the conditionally well-posed problem

This concept was introduced by M. M. Lavrent'ev[85] for the class of problems of interpretation of observational data, including instrumental problems; it generalizes the conditions of Tikhonov's theorem.

DEFINITION.[86] The problem of solution of the operator equation

$$Az=u,\ z\in\hat{Z},\ u\in U,$$

is called well-posed in the sense of Tikhonov (conditionally well-posed) on the

[b]For example, the sequence of continuous functions $z(x)=\sin\pi nx$, restricted to $[-1,1]$, does not converge to a continuous function.

set $\hat{Z}$ provided (i) it is known *a priori* that its solution exists and belongs to $\hat{Z}$ ($u \in U_A \equiv U$), (ii) the solution is unique, and (iii) to infinitesimally small variations of u, which do not take the solution out of $\hat{Z}$ ($u + \delta u \in U_A$), correspond infinitesimally small variations of z. In this case the set $\hat{Z}$ is called the correctness class (in particular, $\hat{Z}$ is compact).

Here we explicitly emphasize the fact that, in the mathematical formulation of inverse problems of mathematical physics of the type considered here, the existence of a solution (for an exact right-hand side) is a direct consequence of the validity of the physical model (the operator A) and the observability of the phenomenon (u). However, the solution may not be unique and the fulfillment of the second requirement requires special analysis (see Chap. 3). In order that the third requirement be satisfied, it is necessary (i) to choose the correctness class (compactum) such that it contains the exact solution (this also determines U_A) and (ii) to guarantee control over the variations δu such that $u_\delta = u + \delta u \in U_A$.

If the problem is well-posed in the sense of Tikhonov, it is also referred to as being inserted into the correctness class. It is obvious that in this case any of the classical methods may serve to solve the problem for arbitrary δu: $z_\delta = R(u_\delta) = A^{-1} u_\delta$.

In what follows we shall view the concept of "conditionally well-posed" in a narrower sense: we shall require only the conditions of Tikhonov's theorem to be satisfied. Let us note for what follows the formal consequences of this theorem.

CONSEQUENCE (of Tikhonov's theorem). If the problem of solution of the operator equation

$$Az = u, \; z \in \hat{Z} \equiv Z, \; u \in U_A \equiv U,$$

is well-posed in the sense of Tikhonov, then it follows from the equation $\lim_{n \to \infty} \rho_U(Az_n, A\hat{z}) = 0$ that $\lim_{n \to \infty} \rho_Z(z_n, \hat{z}) = 0$.

1.4. Certain practical aspects of being conditionally well posed

As will be seen later, the formulation and algorithms of regularization theory are based on the just-noted results.

The concept of *a priori* problem insertion into the correctness class has certain practical consequences, one of which is the answer to the question of choice of the search region in the selection method. In that case the correctness class is constructed "artificially," taking into account the *a priori* quantitative estimates about the solution, whereas the belonging of u to U_A is ensured "naturally" by discussing the direct effect.

Going beyond selection method problems, another research direction is possible.[85] If the compact set was chosen "felicitously," then U_A—the set of images of the characteristics of the object—was also determined. In practice, what is given is $\tilde{u} \in U_A$ and then one can solve the problem of "projecting" $\tilde{u} \in U$ on $U_A \subset U$: $u_0 = \Pi p_{U_A} \tilde{u}$.[c]

If the characteristic of observation is a function, for example $u = u(x)$, $x \in [c, d]$, then we arrive at the problem of "smoothing" it out, by which is

[c] According to Refs. 9 and 77, u_0 is the projection of $\tilde{u}$ on U_A provided $\rho_U(u_0, \tilde{u}) = \inf_U \rho_U(u, \tilde{u})$.

meant not only ridding $\tilde{u}$ of random perturbations but also the construction of the function from the U_A "closest" to it. That problem can be effectively solved to the extent that properties of U_A, corresponding to the properties of the exact solution $\hat{z}$, are known. For example, if it is known that the exact characteristic of the object cannot produce an effect describable by a not-smooth function, then the search for u_0 should proceed on the set of sufficiently smooth functions (e.g., "splines"[5,23]); if it is known (as in inverse problems of gravimetry) that the effect is a harmonic function, then u_0 should be a function from this class, and so forth. It then often turns out to be possible, having chosen a basis $\{\psi_k(x)\}$ in the appropriate class, to make effective use of the least-squares method[10,81] in searching for the coefficients $\{a_k\}$ in $u_0(x) = \Sigma_k a_k \psi_k(x)$. In that case U_A is also constructed "artificially" to make the formulation of the problem similar to conditionally well posed.

Finally, problems exist for which the compact set is defined "naturally" by the "elementary" formulation of the problem, as a consequence of the physical model. Consider as an example the problem (7) in Chap. 1. This problem is of the control type and usually the expected effect—surface temperature of the sample $U(R, t)$—is given by a not-smooth function (Fig. 2).[44] However, making use of the preceding remark, it is easily "smoothed," and this is sufficient for the existence of the corresponding control function: the current in the inductor, $I(t)$. Something else is of interest. Within the framework of the assumed model, $I(t)$ is proportional to the magnetic field $H(r, t)$ at the sample surface ($r = R$), and the latter satisfies a differential equation which contains the derivative $\partial H/\partial t$. Since the problem is discussed on a finite time segment, it follows from general theory[74] that $\partial H/\partial t$ is bounded and so is (continuous) H. Therefore $H(r, t)$, and with it $I(t)$, belong to some compact class. Taking this into consideration and appealing to Tikhonov's theorem, one may employ for the solution of the inverse problem classical transformation operators (in the present case, finite-difference schemes).[124,125] In the general case, "natural" introduction of the compactum always occurs whenever the sought-for characteristic of the object can be associated with some differential equation.

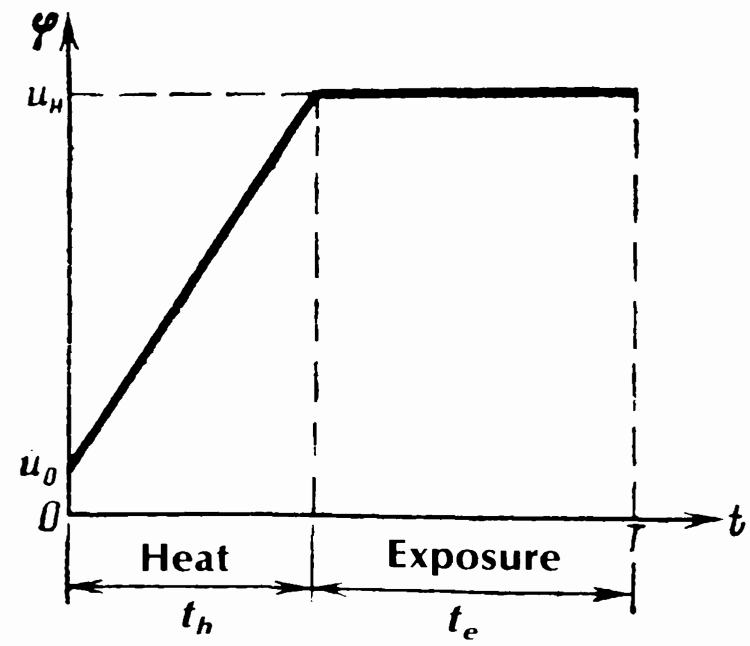

Figure 2

2. The concept of quasi-solution

2.1. The problem of selection automatization and "quasi-solution"

As was already noted (Sec. 1.1), the comparison of the observed and calculated effect can be "automated" with the help of an electronic digital computer. In that case one is solving the problem of minimization of $\rho(A\mathbf{p}, u)$ on some bounded set of values $\mathbf{p}$.

As a result of the analysis, expounded in Sec. 1, a more precise phrasing of the automation of the selection method is possible, as well as its extension to the case when the characteristic of the object is of an arbitrary mathematical nature. This important step was taken by V. K. Ivanov,[71] who introduced the concept of quasi-solution.

DEFINITION. The element z_0, belonging to the compactum $\hat{Z}$, is called the quasi-solution of the conditional operator equation $Az = \tilde{u}$, $z \in Z$, $\tilde{u} \in U$, if it minimizes the functional $\rho_U(Az, \tilde{u})$:

$$\rho_U(Az_0, \tilde{u}) = \inf_{z \in \hat{Z}} \rho_U(Az, \tilde{u}) . \tag{12}$$

This definition states how to search for z_0; the question is what is the relation between the quasi-solution and the sought-for characteristic of the object. It is obvious that for $u = \hat{u}$ and in the case of uniqueness we have $\hat{z} \in \hat{Z}$, $z_0 = \hat{z}$; we shall consider the case when $\tilde{u}$ is given with errors and, generally speaking, does not belong to U_A.

2.2. Correctness of the posing of the quasi-solution problem

THEOREM. If (i) the problem $Az = u$, $z \in \hat{Z}$, $u \in U_A$ is well posed in the sense of Tikhonov on the compactum $\hat{Z}$ and (ii) for arbitrary $\tilde{u} \in U$ $(U \supset U_A)$ $\Pi p_{U_A} \tilde{u}$ is unique,[d] then the quasi-solution problem (12) is well-posed in the sense of Hadamard (from U into $\hat{Z}$).

Proof. Since $\hat{Z}$ is compact the quasi-solution $z_0 \in \hat{Z}$ obviously exists for arbitrary $\tilde{u} \in U$. In view of condition (ii), to each $\tilde{u}$ corresponds a unique $u_0 \in U_A$, and, in view of condition (i), a unique element z_0 such that $Az_0 = u_0$. Consider, lastly, an arbitrary sequence $\{\tilde{u}_n\} \to \hat{u} \in U_A$, and the corresponding sequences $\{u_n = \Pi\, p_{U_A} \tilde{u}_n\}$ and $\{z_n : Az_n = u_n\}$. It is obvious that

$$\rho(Az_n, \tilde{u}_n) \leqslant \rho(A\hat{z}, \tilde{u}_n) \equiv \rho(\hat{u}, \tilde{u}_n) \to 0$$

in the limit as $n \to \infty$. But since $\rho(Az_n, A\hat{z}) \leqslant \rho(Az_n, \tilde{u}_n) + \rho(\hat{u}, \tilde{u}_n)$, hence $\rho(Az_n, A\hat{z}) \to 0$. Then, in view of condition (i) and as a result of Tikhonov's theorem, we have $\lim_{n \to \infty} \rho_Z(z_n, \hat{z}) = 0$ and the proof is complete.

The quasi-solution problem may be viewed as a possible well-posed inverse problem of the interpretation type. Indeed, it is uniquely solvable for any given $\tilde{u}$, and its solution approximates arbitrarily closely (for sufficiently small errors in the initial data) the exact solution.

[d] V. K. Ivanov has also indicated[71,72] sufficient conditions for projection uniqueness, usually fulfilled in interpretation problems.

2.3. Certain practical aspects

For calculational convenience in applications it is natural to formulate the quasi-solution problem in the form

$$\inf_{z\in\hat{Z}} \rho_U^2(Az,\tilde{u}), \tag{13}$$

which is, obviously, equivalent to Eq. (12).

Let us note that here, in contrast to the conditionally well-posed problem, the observational characteristics $\tilde{u}$ serve as the initial data: that means that the operation of preliminary "smoothing" of $\tilde{u}$ loses its meaning if the compact set $\hat{Z}$ includes the exact solution (the "smoothing" proceeds automatically since $\tilde{u}$ is approximated by an element, obtained by the least-squares method, known to belong to U_A). At the same time, as in the conditionally well-posed case, it is necessary to specify *a priori* the compact set, and this is connected with the specification of quantitative restrictions on the sought-for characteristic of the object.[58,63,167]

As an example, we consider the simplest model of the problem of determination of the sources of the gravitational field from its observation on the Earth's surface. The sources will be characterized by the location of their centers of mass $M_k(\xi_k \eta_k, \zeta_k)$ and the masses m_k, $k = 1, 2, \ldots, n$, where n is fixed (and usually not too large). Thus the sought-for quantity is the vector $\mathbf{p} = \{m_1, m_2, \ldots, m_n, \xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n, \zeta_1, \zeta_2, \ldots, \zeta_n\}$. A natural restriction on the components ζ_k could be the requirement that they belong to the sedimentary layer of the Earth's core, $0 < \zeta_k < H$; assuming that the gravitational field, due to the sources being considered, significantly different from zero in the square $K{:}0 < \xi, \eta < L$ has already been extracted from the observations, we impose on ξ_k, η_k the natural restrictions $0 < \xi_k, \eta_k < L$. *A priori* ideas about the size of the masses can be derived from known geological data and available interpretation results: $0 < m_k < M$. In the finite-dimensional space E^{4n}, all these restrictions select the compact set $\mathbf{p}\in P$.

The normal (to the "horizontal" surface of the Earth) component of the gravitational field at the point $N(x, y)$ (consistent with the observations) is given by

$$u = U(N,\mathbf{p}) = \gamma \sum_{k=1}^{n} \frac{m_k \zeta_k}{R^3_{M_k N}},$$

$$R_{M_k N} = [(x - \xi_k)^2 + (y - \eta_k)^2 + \zeta_k^2]^{1/2}.$$

Suppose that its observed size is $\tilde{u}(N)$.

Then the quasi-solution problem is formulated as follows[e]:

$$\mathbf{p}{:}\inf \iint_K (U(N,\mathbf{p}) - \tilde{u}(N))^2 d\sigma, \ \ \mathbf{p}\in P^{5)}. \tag{14}$$

The corresponding gravimetry models in two dimensions, where use can be made of the apparatus of analytic functions of a complex variable,[128] were studied in Refs. 136–171.

[e] Upon discretization on a mesh $\{x_i, y_i\}$, the integral is replaced by a finite sum.

There exist well-developed methods for the solution of the problem of minimization of a functional on a manifold with boundaries.[20,70,75] The complication, however, is due to the fact that in the formulation (13) we are required to find the location of the "absolute" minimum (inf), and there are no economical standard procedures to this end (the economical methods of Ref. 75 are determined in that case by the structure of the functional, which need not be convex). The formal search method of Ref. 75 (or the region sampling method of Ref. 86 might be termed "universal" (although not economical). It can be used in combination with more economical procedures after local minimum regions have been established.

Thus because the inverse problem is ill-posed, one must pay the price of a more involved formulation. For this reason it becomes imperative to choose the compact class "felicitously." The subsequent results of regularization theory reduce the computational difficulties somewhat.

2.4. Construction of the quasi-solution and the inverse problems of synthesis and control

We have previously remarked (Chap. 1, Sec. 2.1) that problems of the indicated classes are *a priori* characterized by a variational formulation. It is often the case that the operating characteristic, or expected effect, $\hat{u}$ is compared with real effects to the same extent as in the previously discussed interpretation problems. The set Z of the characteristics of the object (sythesis or control) is determined first of all by their physical nature. On the other hand, in that type of problem certain additional *a priori* requirements are also relevant: reliability of operation, technical realizability, and so forth. As a result, a subset $\hat{Z}$ of "acceptable" solutions is extracted from Z. If, moreover, $\hat{Z}$ is compact in Z, then formally the posing of the problem is completely analogous to Eq. (14):

$$z_0^* : \inf \rho^2(Az,\hat{u}) \quad z\in\hat{Z} . \tag{15}$$

The solution of such a problem exists for arbitrary $\hat{u}$, and if $\hat{u}$ is single-valuedly projected on U_A then the solution is unique. It coincides with the exact solution of the operator equation if $\hat{u}\in U_A$ (in that case it also coincides with the quasi-solution). Let us note that in applications $\hat{Z}$ is usually specified by stronger restrictions than those needed to specify the compact class and, consequently, is compact.

3. The concept of regularization

Starting from the fundamental concept of a "regularizing operator" (algorithm),[141] we shall mean by "regularization" any method for correctly, in a well-defined sense, posing the inverse problem of mathematical physics or a method for the construction of a stable (regularizing) algorithm for its approximate solution. In this paragraph we shall consider posing problems different from the preceding. Questions relating to the construction of regularizing algorithms (i.e., broader aspects of regularization) will be taken up in Chap. 4.

3.1. The practical equivalence set

The result of a measurement of an arbitrary physical quantity consists of a pair of numbers: its approximate value $\tilde{u}$ and an error estimate δ such that $|\tilde{u} - \hat{u}| \leqslant \delta$. The exact value $\hat{u}$ remains unknown.

The situation is precisely the same in problems of direct calculation of effects of the influence of physical objects. If, for example, one wishes to determine the law of motion of a material point $x(t)$ subject to a given (measured or calculated) force $\tilde{f}(t)$, then one integrates Eq. (1). The error δf is further compounded by calculational errors, and the result is the pair of quantities $[\tilde{x}(t), \delta]$ such that $\rho(\tilde{x},\hat{x}) \leqslant \delta$, where $\hat{x} = \hat{x}(t)$ continues unknown.

In this fashion, the result of our measurements or calculations consists of a certain "indeterminacy" set, and the input data by themselves provide no basis for preferring one point from this set over another for an estimate of the exact value of the sought-for quantity.

In the cases under consideration this "indeterminacy" causes no inconvenience: it is known from the beginning that if we take as an estimate of the exact quantity an arbitrary point from the set, we can obtain an arbitrarily close approximation to its true value by decreasing δ ($\delta \to 0$).

The situation is different for inverse problems of mathematical physics, which are distinguished by instability in their elementary formulation.

Suppose that some problem from the class of interpretation of experimental data is considered within the framework of a given mathematical model A. As in the previous cases the input data consist of a pair of quantities $(\tilde{u}, \delta)$ such that $\rho(\tilde{u}, \hat{u}) \leqslant \delta$, where $\tilde{u}$ is the approximate value of the characteristic of the observation while $\hat{u}$ is unknown. The sought-for quantity is connected to the observational data by the conditional operator equation:

$$Az = u, \quad \tilde{z} \in Z, \quad \tilde{u} \in U, \tag{16}$$
$$\rho(\tilde{u},\hat{u}) \leqslant \delta \,.$$

Suppose that for exact input data the problem has a unique solution $\hat{z}$ where $A\hat{z} = \hat{u}$. As before, the input data provides no guidance on how to choose from the set

$$Z_\delta : \rho_U(Az,\tilde{u}) \leqslant \delta \tag{17}$$

a particular element[f]: all elements are "practically equivalent." However, in contrast to the previous situation, there is now no basis for choosing as an approximation to $\hat{z}$ some arbitrary $z_\delta \in Z_\delta$: the "diameter" Z_δ remains arbitrarily large even when $\delta \to 0$ so that it is not necessarily true that $z_\delta \to \hat{z}$ as $\delta \to 0$.

In the following, the set Z_δ [(17)] will be referred to as the practical equivalence set.

Thus a principal problem of the theory is the question of selection from the practical equivalence set of the approximation to the solution of the interpretation problem.

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[f]The resultant problems of utilizing the statistical information will be discussed in Sec. 4.

3.2. The regularization principle and the regularized approximation

Let us introduce into the basis for making the selection of the approximation $z_\delta \in Z_\delta$ the requirement[141]

$$\lim_{\delta\to 0} \rho_Z(z_\delta,\hat{z}) = 0\,. \tag{18}$$

It is natural to call this requirement the regularization principle since it is automatically satisfied in problems which are *a priori* stable ("regular").

We shall call the approximation z_δ to the solution of the interpretation problem "regularized" if it satisfies the indicated requirement.

Obviously, regularized approximations exist. As an example, consider the "quasi-solution" of the inverse problem (see Sec. 2). However, the quasi-solution is not the sole possible regularized approximation. It could be any element of the practical equivalence set on the corresponding compact class: $z_\delta \in Z_\delta \cap \hat{Z}$, if $\hat{z} \in \hat{Z}$.[56,148,158]

Indeed, if $z_\delta \in Z_\delta$, then $\rho_U(Az_\delta, \tilde{u}) \leqslant \delta$. But then

$$\rho_U(Az_\delta, A\hat{z}) \leqslant \rho_U(Az_\delta,\tilde{u}) + \rho_U(\tilde{u},\hat{u}) \leqslant 2\delta \to 0$$

as $\delta\to 0$. Now $\rho_Z(z_\delta,\hat{z}) \to 0$ as $\delta\to 0$ as a consequence of Tikhonov's theorem since $\hat{z}$, $z_\delta \in \hat{Z}$.

The indicated fact opens up new possibilities for well-posed inverse problems, different from the preceding, and at the same time leaves room for the choice and precision of solution algorithms. In this connection we introduce the following definitions.

DEFINITION. An inverse problem of the interpretation class will be called "generalized-well-posed" for a certain choice of input data $(\tilde{u}, \delta)$ if (i) its solution z_δ exists for arbitrary $\tilde{u} \in U$ and every δ, where $0 \leqslant \delta \leqslant \delta_0$, and (ii) z_δ satisfies the regularization principle.

An arbitrary operator $z = R(u)$ for the solution of a well-posed, conditionally or generalized, problem will be called regularized in the sense of Tikhonov.[g]

It is clear that an example of a generalized-well-posed problem is provided by the problem of "quasi-minimization" of the functional $\rho_U^2(Az, \tilde{u})$ on a compact class containing the unique exact solution of the operator equation, i.e., the problem of selection of an arbitrary element z_δ from the conditions $\rho^2(Az, \tilde{u}_\delta) \leqslant \delta^2$, $z \in \hat{Z}$.

3.3. The "consistency" of the posing of the problem

It is obvious that if we want the approximation z_δ to be regularized it is necessary that it belongs to the equivalence class, at least for sufficiently small δ. If the search proceeds on some compact class $\hat{Z}$, then $z_\delta \in \hat{Z}$.

It is important to remark that the well-posed formulations discussed above are based on the assumption that the model of the phenomenon under study—the operator A—is given, and the compact class $\hat{Z}$ is "consistent" with the model ($\hat{z} \in \hat{Z}$). In practice, this may fail to hold in two cases: (i) the model

[g]The fundamental concept of a regularizing operator (algorithm) introduced in Refs. 141 and 148 is broader than that; we shall discuss it in Chap. 4 in connection with the problem of constructing such operators.

provides a good description of objective regularities of the phenomenon, whereas the compact class contains only "rough" estimates of the characteristics of the object, and (ii) the compact class provides a sufficiently full description of the object characteristics but the model is rough. In these cases it may happen that as the precision δ of observations is increased, $\rho(Az, \tilde{u}) > \delta$ and the approximation $z_\delta \bar{\in} Z_\delta$.

The posing of the problem of the search for z_δ will be called inconsistent[142] if δ is the exact estimate of the error in the input data, $\delta = \inf \rho_U(\tilde{u}, \hat{u})$, $\tilde{u} \in U$, and for the chosen A and $\hat{Z}$ one has $z_\delta \bar{\in} Z_\delta$.

The posing of the quasi-solution problem for z_0 [Eq. (12)] may turn out to be inconsistent. Indeed, suppose that for the chosen A and $\hat{Z}$, one has $\inf \rho_U(Az, \tilde{u}) > \delta$, $z \in \hat{Z}$, at least for sufficiently small δ; it is then obvious that $z_0 \bar{\in} Z_\delta$. We remark that in this case the posing of the quasi-minimization problem is also inconsistent: for arbitrary $z \in \hat{Z}$,

$$\rho_U(Az,\tilde{u}) \geqslant \inf \rho_U(Az,\tilde{u}) > \delta \,.$$

In this fashion, for a specified level of accuracy the quasi-solution problem may serve as a control for consistency of the posing, for example as a test of validity of model choice.

As an example, consider the inverse gravimetry problem on the determination of the form of the boundary between two regions with different densities in the bowels of the Earth from an anomaly in the gravitational field at the surface [$\tilde{u} = \tilde{u}(x)$ in the two-dimensional version]. Let the division boundary on a segment be represented by the "continuous curve" 1 in Fig. 3 and correspondingly by the function $z = z(\xi)$. It is natural to expect that for a bigger error in the "measured" field we could not distinguish it from the step in Fig. 3: although the gravitational field associated with these two surfaces is given by entirely different formulas (operators A_1 and A_2), these formulas (just as in the example in Sec. 3.1) are practically equivalent.

However, as the error is reduced, the regularized approximation corresponds to form (1), and not Eq. (2), and if the interpretation is carried out within the framework of model A_2 [$\inf \rho(A_2\mathbf{p}, \tilde{u})$, $\mathbf{p} \equiv \{ p_1 = c, p_2 = d, p_3 = h \}$, $|p_i| \leqslant M$], the problem is inconsistent.

We note that the concept of "consistency" of the posing should not be confused with the concept of "existence" of solution: the quasi-solution on the compact class exists for arbitrary δ.

The solution of the consistency question goes beyond the boundaries of the mathematical analysis of the problem. In the discussion of the posing of

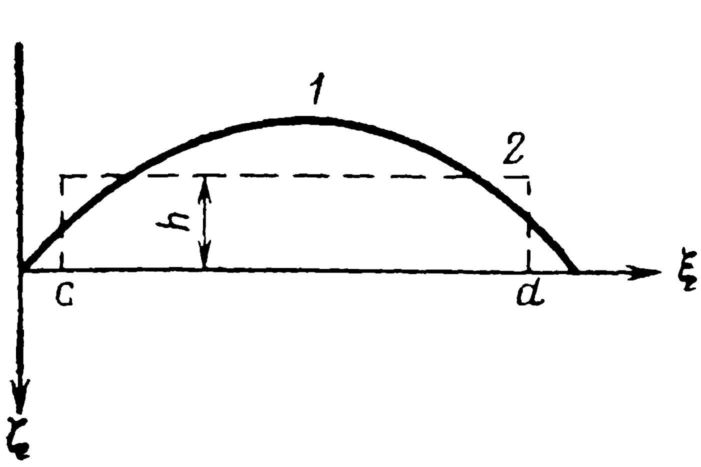

Figure 3

inverse problems that follows, it will be assumed that they are *a priori* consistent in the indicated sense.

To formulate new posings we need one more fundamental concept of regularization theory.

3.4. Stabilizer according to A. N. Tikhonov

Obviously, a compact manifold can be extracted from the finite-dimensional space E^n (in practice for not too large n) by imposing explicit quantitative restrictions on the "Hilbert norm" of the vector $\mathbf{p}\in E^n$:

$$\rho^2(\mathbf{p},0)\equiv\|\mathbf{p}\|^2_{E^n}=\sum_{i=1} p_i^2\leqslant M,$$

where M is some number.[h]

The analogous operation is also possible in the (infinite-dimensional) space of functions $z=z(x)$, defined on $[a, b]$. Here restrictions on the corresponding norm $\|z\|_{L^2}$ do not produce a compact class in $C[a, b]$ since, as we have seen (Sec. 1.2), compactness is not guaranteed even upon restricting the set $\{z(x)\}$. However, as was shown in Ref. 148, the subset of equicontinuous and uniformly bounded on $[a, b]$ functions (compactum) may be defined by restriction to the Sobolev norm:

$$\|z\|^2_{W_2^1}\equiv\int_a^b [\,p_1 z'^2(x)+p_2 z^2(x)]\,dx\leqslant M \quad p_i(x)\geqslant p_{i0}>0\,.$$

We note that the introduction of such a norm already presupposes the isolation in $C[a, b]$ of a certain subset of functions, which possess at the very least square-integrable generalized[129] first derivatives (the Sobolev space W_2^1).

The introduced norms constitute continuous non-negative functionals, and the corresponding method of defining the compactum on the set Z of definition of the operator A admits generalization.

DEFINITION.[148] The non-negative continuous functional $\Omega(z)$, defined on an everywhere dense in Z subset Z_1, is called the stabilizing functional (stabilizer) for the operator equation $Az=\hat{u}$, $z\in Z$, $\hat{u}\in U$, provided (i) the exact solution belongs to Z_1, and (ii) for arbitrary $M>0$ the set Z_M of elements z from Z_1, for which $\Omega(z)\leqslant M$, is compact in Z_1.

The indicated examples show that stabilizers exist. In particular, in the spaces discussed above, it follows from the definition that an arbitrary function $\varphi(\|\mathbf{p}\|)$, respectively $\varphi(\|z\|_{W_2^1})$, could serve as a stabilizer provided that $\varphi(y)$ is continuous, non-negative, and strongly monotonic, with $\varphi(0)=0$.

It could happen that the introduced characteristic of the object is a function of several variables: $z=z(x)$, $x=(x_1,\ldots,x_n)\in E^n$. Such characteristics have to be introduced in connection with the study of the spatial structure of the bowels of the Earth,[132] the solution of inverse problems connected with nonstationary or spatially inhomogeneous processes of heat conductivity,[33]

[h]One may also consider the more general expression $\Sigma_{i=1}^n c_i(p_i-p_{i_0})^2$, where $\mathbf{p}_0$ is a given vector and c_i are given constants.

and so forth. The study of such objects in the framework of ever more "exact" models is the perspective of scientific investigation in many branches of physics.

It is not hard to convince oneself that in these cases the simplest analog to $\|z\|^2_{W_2^1}$, namely $\int_D [(\nabla z)^2 + z^2] d\tau$, no longer provides a stabilizer on $C(D)$.[i] Indeed, upon setting $z(x) = |x - x_0|^{-\lambda}$, where $x_0 \in D \subset E^3$, $0 < \lambda < \frac{1}{2}$, so that $\nabla z \sim |x - x_0|_{-(\lambda+1)}$ in the neighborhood of x_0, we note that the integral remains bounded although $z(x) \overline{\in} C(D)$.

It follows from Ref. 106 and 129 that a stabilizer in $C(D)$, $D \in E^n$, can be taken as

$$\|z\|^2_{W_2^p} \equiv \int_D \sum_{k=0}^{p} \sum_{i_1 + \ldots + i_n = k} \left(\frac{\partial^k z}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \right)^2 dx \qquad (19)$$

provided that $2p \geqslant n + 1$.

In particular, for $n = 2$ $[z = z(x_1, x_2)]$, $p \geqslant 2$; and for $n = 3$ $[z = z(x_1, x_2, x_3)]$, $p \geqslant 2$. It is to be understood that the use of such a stabilizer in formulating the problem presupposes that the true solution of the problem satisfies even more rigid "smoothness" requirements than in the examples discussed above. However, if the true solution of the problem is "close" to some function of the corresponding class, then the introduced assumptions do not give rise to substantial errors in applications.

On the other hand, it could happen that there is available about the solution of the inverse problem, described by some function, *a priori* information of another kind: the presence of "singular points" on the hypersurface $z = z(x)$, the presence of discontinuities, and so forth. In such cases, the corresponding functional that is to serve as stabilizer should be constructed on a space of functions Z larger than continuous. Examples of this can be found in Refs. 44, 64, and 176.

It is not hard to observe that in constructing the stabilizer use is made of available *a priori* qualitative information about the sought-for solution: "similarity" to some known property (e.g., vector), "closeness" to sufficiently smooth functions, existence of discontinuities, and so forth.

Along with this, in the construction of the stabilizer quantitative information about the sought-for solution may turn out to be useful, for example, its value on a certain subset of the region of definition. We consider the following example.[53] Let the boundary dividing two media (such a boundary is called a "contact surface") in the bowels of the near-surface layer of the Earth's core, described by the function $z = z(x)$, $x \in [a, b]$ (Fig. 3), be reproduced from the observed gravitational field near the logging-well slit, so that the value $z_0 = z(c)$ at some point $c \in [a, b]$ is known. Then on the set Z of continuous functions, defined on $[a, b]$ and satisfying the condition $z(c) = z_c$, the stabilizer function satisfies

$$\Omega_1(z) = \int_a^b \left(\frac{dz}{dx} \right)^2 dx .$$

[i]The role of such functionals in the construction of stable algorithms[148] will be discussed in Chap. 4.

Indeed, suppose that $\Omega_1(z) \leqslant M$. Then for arbitrary $z(x)$ and an arbitrary pair of points $x_1, x_2 \in [a, b]$,

$$|z(x_1) - z(x_2)| \equiv \left| \int_{x_1}^{x_2} \frac{dz}{dx} dx \right|$$
$$\leqslant \left\{ \left| \int_{x_1}^{x_2} \left(\frac{dz}{dx} \right)^2 dx \right| \times \left| \int_{x_1}^{x_2} dx \right| \right\}^{1/2}$$
$$\leqslant \left\{ \int_a^b \left(\frac{dz}{dx} \right)^2 dx \right\}^{1/2} \sqrt{|x_1 - x_2|} \leqslant \sqrt{M} \sqrt{|x_1 - x_2|} \, ,$$

and since the estimate is independent of z, Z is the set of equicontinuous functions. On the other hand, it follows from this estimate that $|z(x) - z(c)| \leqslant [M(b-a)]^{1/2}$ for an arbitrary point $x \in [a, b]$, but then for arbitrary $z(x) \in Z$ one has $|z(x)| \leqslant |z(c)| + |z(x) - z(c)| \leqslant |z_c| + [M(b-a)]^{1/2}$, and, consequently, Z is the set of uniformly bounded functions. By Arzela's theorem, Z is compact and our assertion is proven.

Similar quantitative information can be particularly helpful in multidimensional problems. Suppose, for example, that the subject being studied is the density of heat sources $z(x, t)$, $t \in [0, T]$, distributed on the segment $0 \leqslant x \leqslant a$, reconstructed from measurements of the temperature field $u(x, t)$ on a certain segment $[c, d]$ inside $[0, a]$ ($[0, a] \cap [c, d] = o$).[33] Assume that the following information is known about the sought-for density: (i) $\hat{z}(x, t)$ is "close" to a twice differentiable function and (ii) $\hat{z}(x, 0) = \hat{z}(0, t) = z_0 = \text{const}$. Then on the set Z of continuous functions, satisfying the above "quantitative" restrictions, the stabilizer turns out to be

$$\Omega_2(z) = \iint_D \left(\frac{\partial^2 z}{\partial x \, \partial t} \right) dx \, dt \, .$$

Indeed, if $\Omega_2(z) \leqslant M$, then

$$\left| \int_{x_1}^{x_2} \int_0^{t_1} \frac{\partial^2 z}{\partial x \, \partial t} \, dx \, dt \right| \leqslant \sqrt{MT} \sqrt{|x_2 - x_1|}$$

and analogously

$$\left| \int_0^{x_2} \int_{t_1}^{t_2} \frac{\partial^2 z}{\partial x \, \partial t} \, dx \, dt \right| \leqslant \sqrt{Ma} \sqrt{|t_2 - t_1|} \, ,$$
$$x_1, x_2 \in [0, a]; \quad t_1, t_2 \in [0, T] \, .$$

However, since $\partial z / \partial x (x, 0) \equiv 0$, the first of these integrals equals

$$\int_{x_1}^{x_2} \left(\frac{\partial z}{\partial x} (x, t_1) - \frac{\partial z}{\partial x} (x, 0) \right) dx = \int_{x_1}^{x_2} \frac{\partial z}{\partial x} (x, t_1) dx \, ,$$

and analogously the second integral equals

$$\int_{t_1}^{t_2} \frac{\partial z}{\partial t} (x_2, t) dt \, .$$

It therefore follows that for an arbitrary pair of points $M_1(x_1, t_1)$, and $M_2(x_2, t_2) \in D$ and an arbitrary $z(x, t)$, one has

$$|z(M_1) - z(M_2)| \leqslant 2d\sqrt{M}\,\rho^{1/2}(M_1,M_2)\,,$$

provided $d = \max(T, a)$. Consequently Z is the set of equicontinuous functions. The uniform boundedness is established in the same fashion as in the example given above. Therefore Z is compact and $\Omega_2(z)$ is the stabilizer.

It is not hard to verify[33] that on the set of continuous functions $z = z(x_1, x_2, \ldots, x_n)$, defined in the region

$$D \equiv \prod_{i=1}^{n} [a_i, b_i] \subset E^n\,,$$

possessing the appropriate derivative and satisfying the conditions $z|_{x_i = a_i} = z_{a_i} = \text{const}$, the stabilizer is given by

$$\Omega_n(z) = \int_D \left(\frac{\partial^n z}{\partial x_1 \partial x_2 \ldots \partial x_n} \right)^2 dx\,.$$

In what follows, stabilizers, obtained with the help of quantitative information about the solution of the inverse problem of the indicated type, will be called conditional. We note that $\Omega_s(z)$ $(s = 1, 2, \ldots, n)$ no longer possesses the properties of the square of a distance.

The concept of the stabilizer provides new ways for well-posing inverse problems.

3.5. Possible well-posed interpretation problems, based on the stabilizer concept

We note that once the stabilizer $\Omega(z)$ is found, the quasi-solution problem (12) can then be formulated as

$$\inf \rho_U(Az, \tilde{u}),\ \ z \in Z_M\colon \Omega(z) \leqslant M\,. \tag{20}$$

Such a formulation of the problem may turn out to be more convenient than the previous formulations, because the quantitative information about the solution is given in terms of a single constant M. It could, however, happen, that even such information is absent and the quantitative restrictions are introduced in the "unfortunate" manner $\Omega(\hat{z}) > M$. For this reason it is of interest to pose interpretation problems in such a way that only "natural" information about the sought-for characteristic of the object is used. The above-discussed concepts of the practical equivalence set and stabilizer make this possible.

Since the stabilizer is constructed with the qualitative peculiarities of the exact solution to the problem taken into account, then for finite accuracy of the input data it is natural to strive for the greatest possible similarity between the approximation and the exact solution with respect to that qualitative characteristic. If, for example, the solution $\hat{z} = \hat{\mathbf{p}} \in E^n$, then one could demand that the approximation deviate minimally from some given $\mathbf{p}_0$, provided that it does not fall outside the bounds of the practical equivalence set Z_δ; if it is known that the solution is "close" to a smooth function, then one could demand maximal smoothness subject to that same condition with respect to the set Z_δ and so forth. This leads to the following posing of the problem on the search of the regularized approximation[148]:

$$\inf \Omega(z),\ z\in Z_1 \cap Z_\delta\ , \tag{21}$$

where $Z_\delta \equiv \{z\in Z: \rho_U(Az, \tilde{u}_\delta) \leqslant \delta\}$. It is clear that we use here along with qualitative considerations only an estimate of the accuracy of the input data; the restriction of $\Omega(z)$ by some, even though unspecified, constant is a natural consequence of the posing of the problem. The operator $z_\delta = R(\tilde{u}_\delta, \delta)$ of the solution of the variational problem (21) on a restricted set can be realized, as also for Eq. (12), by means of known numerical methods.[71,75]

We confine ourselves to "unconditional" stabilizers and introduce the conditions (β), possessing "practical generality" for a broad class of inverse problems:

$$\Omega(z) \equiv \|z\|^2_{Z_1} = (z,z)_{Z_1} = \rho^2_{Z_1}(z,0)$$

(Z_1 is the Hilbert space with the corresponding metric and the stabilizer is the square of the metric). Then the following theorem holds:

THEOREM 1. If, in the posing $Az = \hat{u}$, $\hat{u}\in U_A$, the inverse problem has a unique solution $\hat{z}\in Z_1$, the operator A is continuous, and $\rho_U(\tilde{u}, \hat{u}) < \delta$, then under the conditions (β) the problem (21) is generalized well-posed in the sense of Tikhonov.

To prove this assertion let us first verify that the solution (21) exists for arbitrary δ and $\tilde{u}$. Since $\Omega(z) \geqslant 0$, there exists $\Omega_0 = \inf \Omega(z), z\in Z_1 \cap Z_\delta$, and there exists a minimizing sequence $\{z_n\} \subset Z_1$, where $\lim_{n\to\infty} \Omega(z_n) = \Omega_0$; without loss of generality we may suppose that for arbitrary n

$$\Omega(z_n) \leqslant \Omega(z_{n-1}) \leqslant \Omega_1 \equiv \Omega(z_1)\ .$$

But then, by definition of the stabilizer, $\{z_n\}$ belongs to the compactum and, therefore, one may extract from it a subsequence which converges to $z^*\in Z$. Without change in notation suppose that $\rho_z(z_n, z^*) \to 0$ as $n\to\infty$. We shall prove that $\{z_n\}$ is fundamental in Z_1, i.e., that $z^*\in Z_1$. Assume the opposite. Then there exists $\epsilon_0 > 0$ and a sequence of integers $\{m\}$ and $\{p_m\}$ such that for an arbitrary pair (m, p_m) we have $\|\xi_m\|_{Z_1} \equiv \|z_m - z_{m+p_m}\| \geqslant \epsilon_0$. Set $\gamma_m = 0.5(z_m + z_{m+p_m})$; then $\gamma_m = z_m - 0.5\xi_m = z_{m+p_m} + 0.5\xi_m$. Obviously, $\Omega(\gamma_m) \geqslant \Omega_0$; in view of conditions (β) we have simultaneously

$$\|z_m\|^2_{Z_1} - (z_m, \xi_m)_{Z_1} + 0.25\|\xi_m\|^2_{Z_1} \geqslant \Omega_0$$

and

$$\|z_{m+p_m}\|^2_{Z_1} + (z_{m+p_m}, \xi_m)_{Z_1} + 0.25\|\xi_m\|^2_{Z_1} \geqslant \Omega_0\ .$$

Since $\Omega(z_{m+p_m})$ and $\Omega(z_m)$ converge to Ω_0 as $m\to\infty$ and both these quantities are no smaller than Ω_0, it is obvious that

$$-(z_m, \xi_m)_{Z_1} + 0.25\|\xi_m\|^2_{Z_1} \geqslant -\Delta'_m$$

$$(\Delta'_m > 0, \Delta'_m \to 0 \text{ and } m\to\infty)\ ,$$

$$(z_{m+p_m}, \xi_m)_{Z_1} + 0.25\|\xi_m\|^2_{Z_1} \geqslant -\Delta''_m$$

$$(\Delta''_m \geqslant 0, \Delta''_m \to 0 \text{ and } m\to\infty)\ .$$

Upon combining these inequalities, we obtain

$$-0.5\|\xi_m\|^2_{Z_1} \geqslant -(\Delta'_m + \Delta''_m),$$

and, consequently, $\|\xi_m\|^2_{Z_1}$ is infinitesimally small in the limit as $m \to \infty$. This contradicts the assumption ($\|\xi_m\| \geqslant \epsilon_0$), and, therefore, $z^* \in Z_1$. It is obvious from continuity of the stabilizer that $\Omega(z_n) \to \Omega(z^*)$ as $n \to \infty$, and since $\{z_n\}$ is the minimizing subsequence it follows that $\Omega(z^*) = \Omega_0$. This proves the existence of the solution to problem (21) for arbitrary δ and $\tilde{u}$.

Upon setting $z^* \equiv z_\delta$, let us show that z_δ satisfies the regularization principle. Consider an arbitrary sequence $\{\delta_n\} \to 0$ and the corresponding $\{\tilde{u}_n\}$ (where $\rho_U(\tilde{u}_n, \hat{u}) \to 0$. Obviously $\Omega(z_{\delta_n}) \leqslant \Omega(\hat{z})$ for arbitrary n; consequently z_{δ_n} and $\hat{z}$ belong to the set $\widehat{Z}$, compact in Z. Then the sequence $\{z_{\delta_n}\}$ has a limit (with respect to the measure in Z) point z^{**}. On the other hand, $z_{\delta_n} \in Z_{\delta_n}$, and, therefore, $\rho_U(Az_{\delta_n}, \tilde{u}_n) \to 0$ as $n \to \infty$. Then $\rho_U(Az^{**}, \hat{u}) = 0$ by continuity of A, i.e., $Az^{**} = \hat{u}$. By uniqueness of the solution, $z^{**} = \hat{z}$; consequently the sequence $\{z_{\delta_n}\}$ has a unique limit point and $\lim_{n\to\infty} \rho_Z(z_{\delta_n}, \hat{z}) = 0$. The theorem is proved.

More general conditions under which the assertion is true were established in Ref. 148. As a result the theorem also applies to conditional stabilizers.[j] On the other hand, one can indicate sufficient (although hard to prove) conditions[78] under which the "extremum" of the functional (z_δ) is unique. In that case all the requirements are satisfied for the problem (21) to be well-posed "at the point $\hat{z}$."

The above-discussed formulation of the problem requires minimization on a set with restrictions, which could be difficult. In Ref. 148 the question was studied, in part, on the applicability of the method of Lagrange multipliers and, consequently, on the reduction of the problem to parametrization. Let us introduce the set Z_0 of elements z_0, on which the absolute in Z_1 minimum of the functional $\Omega(z)$ is reached: $\inf \Omega(z)$, $z \in Z_1$. It turns out that if $\rho_U(Az_0, \tilde{u}) > \delta$ (i.e., Z_0 does not intersect Z_δ) and the functional $\Omega(z)$ has no local minima on the set $Z_1 \setminus Z_0$ (is "quasi-monotonic" by definition of Ref. 148, in particular is convex), then the solution to the problem (21) is given by an element satisfying the condition $\rho_U(Az, \tilde{u}) = \delta$ (the conditional minimum is reached on the boundary of Z_δ). In that case one may consider in place of Eq. (21) the problem of absolute minimization of the parametric functional:

$$M^\alpha(z,\tilde{u}_\delta) \equiv \rho^2_U(Az,\tilde{u}_\sigma) + d\Omega(z) \tag{22}$$

on the whole set Z_1, and if z^α is the "extreme point" of this functional then α can be chosen from the condition

$$\rho_U(Az^\alpha,\tilde{u}_\delta) = \delta\,. \tag{23}$$

We note that the functional (22) is known as Tikhonov's smoothing functional and the condition (23) as the discrepancy principle.

Having introduced the concept of a smoothing parametric functional, one may study[148] its minimization problem independently of the formulation (21). In that case the question of well-posing reduces to the choice of the dependence $\alpha = \alpha(\delta)$, such that $z^{\alpha(\delta)}$ satisfies the regularization principle.

[j]This problem was investigated by E. E. Kondorskaya.

THEOREM 2. Under the conditions of Theorem 1, there exists a set of functions $\alpha = \alpha(\delta)$, for which the problem

$$z_0\colon \inf M^{\alpha(\delta)}[z,\tilde{u}_\delta], \quad z\in Z_1, \tag{24}$$

is generalized well-posed in the sense of Tikhonov.

We note first of all that because A and ρ_U are continuous (just as in the proof of the corresponding assertion of Theorem 1) one can verify that $M^\alpha(z, \tilde{u}_\delta)$ reaches the exact lower bound for $z^\alpha\in Z_1$, and at that for arbitrary $\alpha > 0$.[148]

Let $\beta_1(\delta)$ and $\beta_2(\delta)$ be two nondecreasing non-negative continuous functions of δ [$\beta_1(\delta),\beta_2(\delta) \to 0$ as $\delta\to 0$], with $\delta^2/\beta_1(\delta) \to 0$ as $\delta \to 0$ [$\delta^2 = o(\beta_1(\delta))$]. Consider an arbitrary non-negative function $\alpha = \alpha(\delta)$, satisfying the conditions $\delta^2\beta_1^{-1}(\delta)\leqslant\alpha(\delta)\leqslant\beta_2(\delta)$. It follows from the obvious inequality

$$M^\alpha(z^\alpha,u_\delta)\leqslant M^\alpha(\hat{z},\tilde{u}_\delta)\equiv\rho_U^2(\hat{u},\tilde{u}_\delta) + \alpha\Omega(\hat{z})$$

for $\alpha = \alpha(\delta)$ that (i) $\Omega(z^{\alpha(\delta)})\leqslant\delta^2\beta_1^{-1}(\delta) + \Omega(\hat{z})$, and, therefore, $z^{\alpha(\delta)}$ belongs to the same compactum in Z as does $\hat{z}$, and (ii) $\rho_U(Az^{\alpha(\delta)}, \tilde{u}_\delta)\leqslant\delta^2 + \beta_2(\delta)\Omega(\hat{z})$, and, therefore, $\rho_U(Az^{\alpha(\delta)}, \tilde{u}_\delta)\to 0$ as $\delta\to 0$. From this we conclude, just as in the proof of Theorem 1, that $\lim \rho_Z(z^{\alpha(\delta)}, \hat{z}) = 0$ as $\delta\to 0$, which is what was to be proved.

In Ref. 148 were also established sufficient conditions for the uniqueness of the extreme point z^α of the smoothing functional, at least for sufficiently small δ. In that case the problem (24) for α appropriately coordinated with δ satisfies all the conditions to be well-posed with respect to $\hat{z}$. The formulation (22)–(23) constitutes a special case of the above.

We remark that in the last two formulations of the problem the operator $z_\delta = R(\tilde{u}, \delta)$ (regularizing operator) had a simpler realization than in the preceding, since unconditional minimization of a functional was involved. Questions related to the construction of the algorithms will be discussed in Chap. 4, where we also give the fundamental concept of the regularizing operator,[141] which falls outside the framework formulated in this chapter.

3.6. Some examples of posings of inverse problems

In Sec. 2.3 we considered as an example one of the inverse problems of gravimetry, which reduced to the determination of the vector $\mathbf{p} = \{m_k,\xi_k,\eta_k,\zeta_k\}$ $k = 1,2,\ldots,n$, from an approximately specified, in a certain two-dimensional region K function $\tilde{u}(N)$ (the gravitational field on the surface of the Earth).

If no quantitative information about the vector $\mathbf{p}$ is known, then one may introduce as the stabilizer $\Omega(\mathbf{p}) = \Sigma_{i=1}^n p_i^2$, where p_i are the components of the vector $\mathbf{p}$. Then the problem is formulated as follows [compare with Eq. (14)]:

$$\inf\left\{\iint_K (u(N,\mathbf{p}) - \tilde{u}(N))^2 d\sigma + \alpha \sum_{i=1}^n p_i^2\right\}, \tag{25}$$

where *a posteriori* coordination of α with δ is presupposed, either by means of the discrepancy principle (23), or by some other means (see Chap. 4), but such that $\mathbf{p}^{\alpha(\delta)}$ satisfies the regularization principle.

For the problem (mentioned in Sec. 3.4) of determination of the contact surface within the framework of the two-dimensional interpretation model, under the condition that one of the surface points is known, one correlates with the observed gravitational field $\tilde{u}(x)$, $x \in [c, d]$, the quantity $U[x, z] \equiv -\partial v/\partial\zeta|_{\zeta=0}$, where v is the field potential due to the sources of constant (excess) density $\delta\rho$, distributed in the region D: $a \leqslant \xi \leqslant b$, $H \leqslant \zeta \leqslant z(\xi)$, $z(a) = z(b) = H$. Accordingly v is the solution of the Poisson equation $\Delta v = -2\pi\gamma f$, where $f = 0$ outside D and $f = \delta\rho$ inside D. The quantity $U[x, z]$ may be represented explicilty in the form

$$U[x,z] \equiv \int_a^b K(x,\xi,z(\xi))d\xi ,$$

where

$$K(x,\xi,z(\xi)) \equiv \frac{\gamma\delta\rho}{2\pi} \ln \frac{(x-\xi)^2 + H^2}{(x-\xi)^2 + (H - z(\xi))^2} ,$$

with H treated as a known quantity. Since the sought-for surface is assumed to be "close" to some smooth one and since $z(a) = H$, one may introduce the "conditional stabilizer"

$$\Omega_1(z) = \int_a^b z'^2(\xi)d\xi .$$

Then the problem is formulated as follows:

$$\inf \left\{ \int_c^d \left(\int_a^b K(x,\xi,z(\xi))d\xi - \tilde{u}(x) \right)^2 dx + \alpha \int_a^b z'^2(\xi)d\xi \right\} , \tag{26}$$

where again it is presupposed that α and δ are correlated in some manner.

We remark that after a final approximation (of the derivative and the integrals) the problem reduces to the search for a global minimum of a function of many variables, but already with a more "sensitive" to errors finite-dimensional analog of the stabilizer.

3.7. Equivalence sets and stabilizers for synthesis and control problems

In problems of this class, along with the characteristic of the expected effect $\hat{u}$, one often specifies a measure of "tolerance" δ_0, so that one is solving the problem of choosing the characteristic of the object to be synthesized (controlled) from a certain equivalence set $\rho_U(Az, \hat{u}) \leqslant \delta_0$, $z \in Z$, where the metric space Z is defined by the physical nature of the object. The question of uniqueness of this choice is not raised, but certain "technical" requirements, as was already remarked in Sec. 2.5), are imposed on any chosen element. As a result a certain set $\hat{Z} \subset Z$ is *a priori* introduced—the set of admissible solutions.

It is obvious that an arbitrary element z_δ, satisfying the conditions

$$\rho_U(Az,\hat{u}) \leqslant \delta_0, \ z \in \hat{Z} , \tag{27}$$

can be viewed as a solution to the problem.

If $\hat{Z}$ turns out to be compact and in addition

$$d_0 \equiv \inf_{U_A} \rho_U(u,\hat{u}) \equiv \rho(u_0,\hat{u}) \leqslant \delta_0$$

[in the opposite case, the problem (27) is inconsistent], then one may choose as the solution to Eq. (27) the quasi-solution of the conditional operator equation $Az = \hat{u}$.

It is quite obvious that if (for compact $\hat{Z}$) $d_0 \leqslant \rho\delta_0$ $(0 < \rho < 1)$, then the solution can be taken in the form of any element z_μ from the practical equivalence set Z_μ:

$$\rho_U(Az, \tilde{u}_\mu) \leqslant \mu ,$$

where $0 < \mu \leqslant (1-\rho)\delta_0$.

Lastly, if the "technical" requirements can be expressed by the condition $\Omega(z) \leqslant M$, where $\Omega(z)$ posesses the properties of a stabilizer, then the solution to (27) can be chosen in the form of $\tilde{z}_\mu$—the solution of either problem (21) or (24) for the conditional equation.

$$Az = \tilde{u}_\mu, \quad \rho(\tilde{u}_\mu, \hat{u}) \leqslant \mu(\delta = \mu) ,$$

where μ is sufficiently small and $\mu \leqslant (1-\rho)\delta_0$. Indeed, solutions of these problems belong to the indicated compactum, and $\rho(Az_\mu, \hat{u}) \leqslant \delta_0$. We remark that for $\hat{u} = A\hat{z}$, $\hat{z} \in \hat{Z}$, we have $\rho_U(Az_\mu, \hat{u}) \to 0$ as $\delta_0 \to 0$.

In this manner, the above-discussed posing of the interpretation problems can also turn out to be effective for the solution of problems from other classes.

We consider some examples. In the optical covers synthesis problem [Chap. 1, Sec. 1.2(c)], we confine ourselves to layered structures, characterized by the set $\mathbf{p} = \{d_1, \ldots, d_n, n_1, \ldots, n_N\}$, where d_k is the thickness of a layer and n_k is its refraction coefficient.[48] The operator $A\mathbf{p} \equiv A(\omega, \mathbf{p}) = T(\omega)$ is defined by the solution of a system of differential equations corresponding to Eq. (6) for a layered structure, here $T(\omega)$ is the transmission coefficient for a light wave of frequency ω. Let $T(\omega)$ be a given transmission coefficient, equal to unity within some part of the segment $[\omega_1, \omega_2]$ and equal to zero outside; let δ_0 be the tolerance. In this case the following belong to the realm of "technical" requirements: (i) the fixed (by the conditions of preparation) value of N, (ii) natural quantitative restrictions on the components: $d_k \geqslant 0$ and (correlated with the set of utilized materials) $n_{\min} \leqslant n_k \leqslant n_{\max}$; and $\Sigma_{k=1}^{N} d_k \leqslant M$ (this is dictated by stability in operation).

Let us denote by P the obviously compact in E^{2N} set satisfying the indicated restrictions; the distance between $T(\omega)$ and $\hat{T}(\omega)$ will be estimated in $L_2(\omega_1, \omega_2)$. Then the problem is formulated as follows:

$$\rho^2_{L_2}(A(\omega, \mathbf{p}), \hat{T}(\omega)) \leqslant d_0^2, \quad p \in P . \tag{28}$$

We remark that $\Omega(\mathbf{p}) \equiv \Sigma_{i=1}^{n} d_i$ is a conditional stabilizer on the set $\{d_k \geqslant 0\}$: if $\Omega(\mathbf{p}) \leqslant M$, then $0 \leqslant d_k \leqslant M$.

That means that the problem may also be posed as follows:

$$\inf \left\{ \rho^2_{L_2}(A(\omega, \mathbf{p}), \hat{T}(\omega)) + \alpha \sum_{k=1}^{N} d_k \right\} , \tag{29}$$

$$n_{\min} \leqslant n_k \leqslant n_{\max}, \; d_k \geqslant 0, \; k = 1, 2, \ldots N .$$

The formulation of this problem, corresponding to Eq. (21), is also given in Ref. 148. Examples of posing control problems of tempering by induction, based on the regularization concept, can be found in Refs. 39 and 44.

In a number of problems the starting characteristic of the object being synthesized or the result of the control is described by the values of some functional $f(z)$, where the set Z is defined by the physical nature of the object. It is required to determine z from the condition that $f(z)$ be minimum, which we shall call "purposeful." As an example, consider the posing of the problem of controlling a beam of charged particles with the help of an "external" electric field.[127] Consider for definiteness a two-dimensional model, where the particles move in a certain rectangular region from left to right, reaching the right boundary at $x = d$. Let $z = z(x_B, y_B)$ be the controlling potential at the boundary B of the region, and let Θ be the scattering angle of the particles at the boundary $x = d$, defined by the solution of a certain self-consistent system of equations that describe within the framework of some model the motion of the particles; clearly $\Theta = \Theta(z)$ is a functional of z. If the purpose of the control is to focus the beam, then we consider the problem of minimization of this (purposeful) functional.[k] We remark that $\Theta(z)$ is given implicitly and its values are calculated with some error. At the same time the conditions for construction of similar systems clearly impose definite restrictions on z.

We consider the problem of determining $\hat{z}$ from the conditions: $\inf f(z)$, $z \in \hat{Z} \subset Z$, where $\hat{Z}$ is determined by *a priori* technical restrictions, and $f(z)$—the purposeful functional—is continuous and bounded from below $[f(z) \geqslant m]$. (Then there exists $f = \inf f(z)$, $z \in \hat{Z}$, which, as is well known, does not yet imply the existence of $z^{(\min)}$—the "extremum".) Let us suppose that there exists at least one extremum $\hat{z} \in \hat{Z}$; otherwise the problem is *a priori* inconsistent. For problems from the class being discussed, it is immaterial whether the extremum set consists of more than one element; any of its elements is acceptable. For the sake of definiteness we consider the case when $\hat{z}$ is the unique minimizing element. However, it could happen that not every minimizing sequence converges to $\hat{z}$ (corresponding examples are given in Ref. 19), and then the problem is unstable with respect to errors in the specification of $f(z)$ and, consequently, the problem is ill-posed.

Ill-posed variational problems were studied in Refs. 8, 19, 93, 148, and 150.

Suppose that instead of $f(z)$ we are given its approximation $\tilde{f}(z)$: $|\tilde{f}(z) - f(z)| \leqslant \delta(z) \leqslant \delta_0$, where the "estimating" functional $\delta(z)$ or its upper bound δ_0, is specified. Then the variational problem is formulated as

$$z_\delta : \inf \tilde{f}(z), \quad z \in \hat{Z} \subset Z . \tag{30}$$

Generally speaking, $\lim \rho_z(z_\delta, \hat{z}) \neq 0$ as $\delta \to 0$, and the problem in the formulation (30) is ill-posed.[93,163]

In Ref. 148 it is shown that a possible correct posing of the problem of interest to us can be formulated analogously to Eq. (24).

Let the set $\hat{Z}$ be a Hilbert space that admits a stabilizer (in Z): $\Omega(z) = \|z\|^2_{\hat{Z}} = (z, z)_{\hat{Z}}$ and $\delta(z) = \delta\Omega(z)$, where δ is the measure of the error in $\tilde{f}_\delta(z)$ with respect to $\Omega(z)$. We shall call these assumptions conditions (β).

We introduce, analogously to Eq. (22), Tikhonov's smoothing functional

$$F_\alpha(z) = \tilde{f}_\delta(z) + \alpha\Omega(z) . \tag{31}$$

[k]The example under discussion belongs, obviously, to the class of optimal equations problems.[109,148]

THEOREM. Let $\tilde{f}_\delta(z)$ be a functional continuous for every δ. Then under the conditions (β) there exists a set of functions $\alpha = \alpha(\delta)$, for each of which the problem

$$z_\delta:\ \inf F_{\alpha(\delta)}(z) \tag{32}$$

is a generalized well-posed problem for the minimization of $f(z)$ on $\hat{Z}$.

Proof. It follows from conditions (β) that $f(z) - \delta\Omega(z) \leqslant \tilde{f}_\delta(z) \leqslant f(z) + \delta\Omega(z)$. Whence it follows, in particular, that for arbitrary z we have $F_\alpha(z) \geqslant f(z) + (\alpha - \delta)\Omega(z)$; hence for $\alpha > \delta$ we have $F_\alpha(z) \geqslant m$ and there exists an exact lower bound F^α and, therefore, a minimizing sequence $\{z_n\}$. Without loss of generality we have $F_\alpha(z_n) \leqslant F_\alpha(z_{n-1}) \leqslant F_\alpha(z_1)$ for arbitrary $n \geqslant 1$. But then $f(z_n) + (\alpha - \delta)\Omega(z_n) \leqslant F_\alpha(z_1)$and, therefore,

$$(\alpha - \delta)\Omega(z_n) \leqslant F_\alpha(z_1) - m.$$

Since

$$F_\alpha(z_1) \geqslant m + (\alpha - \delta)\Omega(z_1) \geqslant 0, \qquad \alpha > \delta\Omega(z_n) \leqslant M_{\alpha,\delta}(M_{\alpha,\delta} > 0),$$

it follows that the minimizing sequence belongs to the compactum and consequently converges in Z. Because $\tilde{f}_\delta(z)$ is continuous, $\hat{Z}$ is a Hilbert space and $\Omega(z) = \|z\|^2_{\hat{Z}}$ and one verifies, just as in the proof of the theorem of Sec. 3.5 for interpretation problems, that $z_n \to z^\alpha \in \hat{Z}$ as $n \to \infty$. Consequently, for arbitrary δ and arbitrary $\alpha(\delta) > \delta$ the solution to problem (32) exists and belongs to $\hat{Z}$. It remains to discuss the behavior of $z^{\alpha(\delta)}$ as $\delta \to 0$.

Obviously, $F_\alpha(z^\alpha) \leqslant F_\alpha(\hat{z})$, where $\hat{z}$ is the extremum of $f(z)$ on $\hat{Z}$; then from the estimate for $F_\alpha(z)$ for arbitrary z (β), we have

$$(*)\ f(z^\alpha) + (\alpha - \delta)\Omega(z^\alpha) \leqslant F_\alpha(z^\alpha) \leqslant f(\hat{z}) + (\alpha + \delta)\Omega(\hat{z})$$

for arbitrary α and δ. Since $f(z^\alpha) \geqslant f(\hat{z})$, then for $\alpha > \delta$ it follows first of all that $\Omega(z_\alpha) \leqslant [(\alpha + \delta)/(\alpha - \delta)]\Omega(\hat{z})$. Let $\alpha = q\delta$, where $q > 1$. Then $\{z_\delta = z^{\alpha(\delta)}\}$ is a compact set which contains $\hat{z}$. Consider the limit point z^* (in Z) of this set as $\delta \to 0$. From the inequalities (*) it also follows that $0 \leqslant f(z_\delta) - f(\hat{z}) \leqslant (1 + q)\delta\Omega(\hat{z})$, and, therefore, $f(z^\alpha) \to f(\hat{z})$ as $\delta \to 0$. By continuity of the functional $f(z)$ we have $f(z^*) = f(\hat{z})$, and by uniqueness of the extremum we have $z^* = \hat{z}$. Consequently $\rho_Z(z_\delta, \hat{z}) \to 0$ as $\delta \to 0$ and the theorem is proven.

The corresponding assertion is proven in Ref. 148 for a more general case. In Ref. 19 are also considered other correct formulations of inverse problems of this class, analogous to those introduced in Secs. 2 and 3 for interpretation problems.

Suppose now that the problem $\inf f(z)$, $z \in \hat{Z} \subset Z$ has a set of extrema Z_0, as is typical for the class under discussion. In that case the problem will be called generalized well-posed for inexact specification of the functional provided that it leads to the choice of one element from Z_0 as $\delta \to 0$.

In Refs. 19 and 148 sufficient conditions are established under which the solution of problem (32) converges to $\hat{z} \in Z_0$, such that $\Omega(\hat{z}) = \inf \Omega(z)$, $z \in Z_0$ (the so-called Ω-normal solution of the initial problem. Consequently the problem (32) admits approximation from Z_0 and thus becomes well-posed in the indicated sense in that case as well.

4. Regularization possibilities for statistical approaches to the posing of inverse problems

It is not hard to note that for physical problems connected, for example, with the measurement of the quantity $\hat{u}$ one customarily chooses a unique value from the equivalence set $\rho_U(\tilde{u}, \hat{u}) \leqslant \delta$. To this end use is made of measurements statistics, based on the knowledge of the "distribution law" of the random quantity $\tilde{u}$.

The apparatus of probability theory also makes possible this method of choice in problems related to the evaluation of some quantity $\tilde{z}$ from given measurements $\tilde{u}$. In the well-known papers Refs. 55, 95, and 169, it is attempted to extend this approach also to problems of mathematical physics belonging to the inverse class.

It is not hard to verify that, although formal development of the corresponding apparatus is also possible in this field, its effectiveness depends on the introduction of regularization elements, for example into the posing of the problem, or the problem can be considered only in the very restricted region of inverse problems for which real statistical data are available.

Obviously, since inverse problems are being considered, the basis for the "statistical" formulation of the problem is provided by the theorem of Bayes.[38] Let us consider the corresponding model for a certain problem of interpretation of experimental data.

Let us view the characteristic of observations as a random quantity with a known (for example, Gaussian) probability density $p(\tilde{u})$. Correspondingly the sought-for characteristic of the object is also a random quantity with, however, an unknown distribution law; the problem consists in discovering the latter or some moments of the distribution. Let us assume for simplicity that the physical quantities, corresponding to the indicated random ones, are connected deterministically: $Az = u$. Then for any given z we have the conditional probability $p_z(\tilde{u}) = p_z(Az - \tilde{u})$. The conditional probability of the value z for specified $\tilde{u}$ is determined by the formula of Bayes:

$$p_{\tilde{u}}(z) = \hat{p}(z) p_z(Az - \tilde{u}) , \qquad (33)$$

where $\hat{p}(z)$ is the *a priori* unconditional probability of the value z.

From this it already follows that for a probabilistic prognostication of z from given $\tilde{u}$, one must, in essence, have available some statistics about direct observations of this quantity.

One may indicate inverse problems for which such statistics are available. These are, for example, certain problems of atmospheric physics, where data on the distribution of temperature, water vapor density, etc. are collected, in part, by direct "soundings." It is natural that in this field the direct statistical analysis of the data provides the foundation for the development of subsequent formulations. The possibility of employing aviation and satellites for indirect observations of effects of the sought-for distributions (for example, the heat emission by the Earth's surface) and the successes of regularization theory made possible the formulation of inverse problems of atmospheric physics.[79] At the present time solutions based on algorithms and formulations discussed in Secs. 1 and 3 and Chap. 4 are being tested.[46,47] At the same time the availability of observational statistics permits the development of the probabilistic approach.[95,96]

An opposite example is provided by the physics of the bowels of the Earth, where *a priori* information from direct soundings is practically absent. This is typical of inverse problems of mathematical physics on the whole. Consequently, within the framework of the Bayes model one depends in essence on the hypothesis about the distribution $\hat{p}(z)$, making use, of course, of any quantitative estimates of $\hat{z}$ that might be available. In this case it is not at all obvious, for example, that the formal mean

$$\bar{z} = \int z p_{\tilde{u}}(z) dz$$

converges to the true value as the number of measurements increases without limit.

Let us suppose now, as is natural, that $p_z(Az - \tilde{u})$ corresponds to the normal (error) distribution law; furthermore we shall treat the values $\tilde{u}$ as statistically independent and of equal weight. Then

$$p_z(Az - \tilde{u}) = \beta \exp\left(- \frac{1}{2\sigma^2} \|Az - \tilde{u}\|^2_{\tilde{U}} \right),$$

where σ^2 is the dispersion (assumed known from the experiment) and $\tilde{U}$ is the set of values of the random quantity. We remark that, in accordance with the "least-squares" principle[81] in application to the problems under discussion, $\|Az - \tilde{u}\|_{\tilde{U}} = \rho_{\tilde{U}}(Az, \tilde{u})$ is the previously introduced measure on the observations set.

One possible way of realizing the statistical approach is based on the principle of maximal likelihood.[6,38]

We attempt to determine, following this principle, the most probable value z_0 of the sought-for quantity z. If no additional information is available, then one supposes that $\hat{p}(x) = \text{const}$, and then z_0 is the solution of the problem $\inf \rho_{\tilde{U}}(Az - \tilde{u})$, $z \in Z$ whose ill-posing was noted above. Utilizing quantitative information of the form $z \in \tilde{Z}$, where $\tilde{Z}$ is a given subset of Z, one may set $\hat{p}(z) = \text{const} \neq 0$ for $z \in \tilde{Z}$, and $\hat{p}(z) \equiv 0$ for $z \bar{\in} \tilde{Z}$. Then z_0 is the solution of the same problem on the subset $\tilde{Z}$. It is obvious that the well-posing of such a problem does not follow from the probability-theoretical approach. The results presented in this chapter, as in the case of "selection," give an indication of what *a priori* information is needed for the problem to be well-posed and the sought-for quantity to be effectively obtainable. If $\tilde{Z}$ is compact then z_0 is the quasi-solution. We note that formally one may also arrive at posing the problem as in Eq. (24): it is sufficient to set *a priori*

$$\hat{p}(z) = \exp\left(- \frac{\alpha}{2} \Omega(z) \right),$$

where $\Omega(z)$ is some functional.[1] However, in this case too, the well-posing of the problem becomes possible because of the stabilizer properties of $\Omega(z)$ and is not due to properties of $\tilde{Z}$ that follow from the probability-theoretical approach.

[1]For $\Omega(z) = \|z - \tilde{z}_0\|^2_{\tilde{Z}}$ this corresponds to the normal distribution of the random quantity $z \in \tilde{Z}_0$ about certain mean $\tilde{z}_0$ with dispersion $\sigma_z^2 = \alpha^{-1}$.

In this manner, in the presence of definite sufficient (even though not complete) statistical information about the object under study, the probability-theoretical formulation under consideration can be "regularized" on the basis of the results given in Chap. 1.

When this is done it becomes possible to draw parallels between "deterministic" and statistical estimates of model parameters. For example, the absolute error δ of the input data can be compared with the mean-square error of this quantity treated as random. It is then clear that the selection by discrepancy [problems (24) and (23)] results in an asymptotically "unshifted" parameter estimate for $\sigma \to 0$. Let us suppose now that for the inverse problem under study $\Omega(z) = \|z - \tilde{z}_0\|_Z^2$ and $\Omega(z)$ is the stabilizer. Then the last of the hypotheses about $\hat{p}(z)$ means that the random quantity z is normally distributed about the mean $\bar{z} = \tilde{z}_0$ with dispersion $\sigma_z^2 = \alpha^{-1}$. It is to be understood that the *a priori* specification of the dispersion is usually impossible, and that this equation simply indicates the statistical meaning of the parameter α if it was chosen by some other means, for example by discrepancy.

References 65–67 are devoted to the analysis of other statistical models, corresponding to finite approximation of certain inverse problems (systems of ill-defined linear algebraic equations). In these papers the possibility is studied of obtaining moments of the "true" distribution from the statistics of the observations on the basis of a consistent Bayes approach and under definite hypotheses regarding the *a priori* distribution of the sought-for quantity. It is shown that the "generalized well posing" of the problem within the framework of such models includes the requirement of "multiple" repetition of the observations. In that case the input information may be taken as

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i \ ,$$

and the "regularization parameter" of the problem turns out to be n^{-1}, where n is the number of "measurements."

In this manner, the correct (in the indicated sense) statistical formulation of the problem turns out to be possible, provided sufficiently complete statistical information about the observed quantity is available, as well as additional *a priori* information about the sought-for quantity.

Chapter 3
Uniqueness question in observational data interpretation problems

The meaning of the uniqueness question in problems of mathematical physics is well known.[162] One wants to know whether the mathematical model of the physical process (phenomenon) contains a sufficient number of conditions to isolate one real process (one phenomenon) from among many of the same type and, therefore, to uniquely determine from the input data the sought-for characteristic of the object or the phenomenon created by it. In other words, the question is raised about the uniqueness of the correspondence between the mathematical model and the real process. On the assumption that the basic physical laws governing the process (expressed through differential equations) are already well known, the problem in question becomes that of "identification" (of the mathematical model and the real process).[3] Naturally, the question of uniqueness is studied for an "exact" posing of the problem, free of the influence of any errors. For inverse problems from the interpretation class, studied for "inexact" input data, a positive solution to the uniqueness question is of principal significance: in that case one has the assurance that upon application of stable ("regularizing") algorithms it is possible to obtain, within the framework of the chosen "identified with the object" model, a result arbitrarily close to the real characteristic, provided that the error in the input data is sufficiently small.

Let us consider the interpretation problem, expressed through the operator equation

$$Az = u, \quad z \in Z, \; u \in U_A. \tag{34}$$

It is obvious that the problem has a unique solution in U_A, if the inequality $z_1 \neq z_2 (z_1, z_2 \in Z)$ implies $u_1 \neq u_2 (u_1 = Az_1, u_2 = Az_2)$. Herein is contained a sufficiently general approach to the study of the uniqueness of solutions to inverse problems.

A large literature is devoted to the study of the uniqueness of solutions in various inverse problems of interpretation (for example, Refs. 25–30, 54, 102, 107, 112–114, 119, 130, 137, 144, 146, 166, and others). In this chapter we consider but some examples, focusing attention on questions of formulational or methodological interest.

1. The uniqueness of inverse problems involving a linear correspondence operator

The general theory of equations (including differential and integral equations) involving a linear operator A is well developed and the uniqueness of the solution of Eq. (34) is often a direct consequence of general results.

Many inverse problems of the type under consideration are formulated as Volterra integral equations of the convolution type. As an example may serve instrumental problems (4). Under not too rigid restrictions on the kernel, the question of uniqueness of the solution of such problems is easily analyzed with the help of the Laplace transform and its inverse.[62] In the case of inverse problems expressed in the form of singular Fredholm equations of the first kind, a similar role is played by Fourier transforms, and in other cases by the apparatus of Fourier expansions.

Let us consider, for example, the problem of determining the intensity $f(t)$ of a heat source concentrated at the point $x_0 \in (-\infty, \infty)$ from the temperature at the point $x_1 \neq x_0$:

$$u(x_1, t) \equiv \varphi(t).$$

Such a problem may be viewed as the simplest mathematical model of an experiment on the detection of internal defects in the "walls" of industrial aggregates from observations of the temperature field of the wall.[35] The problem

$$a^2 \frac{\partial^2 u}{\partial x^2} + f(t)\delta(x - x_0) = \frac{\partial u}{\partial t},$$

$|u| \to 0$ as $x \to \infty$, $u(x, 0) = 0$ $[f(0) = 0]$, $u(x_1, t) = \varphi(t)$ $[Af \equiv u(x_1, t)]$ gives rise to the Volterra equation of the first kind

$$\varphi(t) = \int_0^t K(t - \tau) f(\tau) d\tau,$$

where

$$K(t - \tau) = \frac{1}{2a\sqrt{\pi(t - \tau)}} \exp\left(- \frac{(x_1 - x_0)^2}{4a^2(t - \tau)} \right);$$

and, since the kernel has a removable discontinuity at the point $\tau = t$, the question of uniqueness of determination of $f(t)$ is solved in accordance with Ref. 62.

As another example, we consider the problem of determination of the initial temperature $\varphi(t)$ of the finite "segment" $[0, l]$, given the temperature $u(x, T) = u(x)$ for $T > 0$ (for example, Ref. 88). This inverse problem (from $T > 0$ to $t = 0$) is determined by the conditions

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad x \in (0, l), \quad t \in (0, T),$$

$$u(0, t) = u(l, t) = 0, \quad u(x, 0) = \varphi(x), \quad u(x, T) = \hat{u}(x),$$

and in that case the operator is $A\varphi = u(x, t)$, provided that $u(x, t)$ is determined (uniquely, as is known from Ref. 162) by the preceding conditions. One of the possibilities for the solution of the question of uniqueness of $\varphi(x)$ is based on the fact that $u(x, t)$ can be expressed, for any given $\varphi(x)$, in terms of a Fourier series over the closed (in the sense of Ref. 74) orthogonal system of functions $\psi_n(x) = \sin(\pi n/l)x$:

$$\hat{u}(x) = \sum_{n=1}^{\infty} \varphi_n e^{-\omega_n^2 T} \psi_n(x),$$

where $\omega_n = a\pi n/l$, and φ_n are the Fourier coefficients of the function $\varphi(x)$. Let us assume that the problem has two solutions $\varphi_1(x) \not\equiv \varphi_2(x)$ for one and the same $\hat{u}(x)$. Then we have for the difference $\mu(x) \equiv \varphi_1(x) - \varphi_2(x)$

$$\sum_{n=1}^{\infty} \mu_n e^{-\omega_n^2 T} \psi_n(x) \equiv 0;$$

as a consequence of uniqueness of the expansion we have $\mu_n = 0$ for arbitrary n, and since the system $\{\psi_n(x)\}$ is closed it follows that $\mu(t) \equiv 0$, in contradiction with the starting assumption. Thus the solution is unique.

Typical examples are provided by problems of extension of potential fields in the direction of their sources.[17,136,164] In particular, one of the corresponding formulations is equivalent to the Hadamard problem, well known to be ill-posed. Suppose that one is given the potential field and its normal derivative at one boundary ($y = 0$) of a plane layer ($|x| < +\infty, 0 < y < h$):

$$v|_{y=0} = \varphi(x), \quad \left.\frac{\partial v}{\partial y}\right|_{y=0} = \psi(x).$$

Since v is a harmonic function within the layer (containing no sources of the field), then supposing it to be the real part of some analytic function,

$$v(x, y) = \operatorname{Re} f(x),$$

one may assume the value of the latter to be given on the boundary. Indeed, from the given

$$\left.\frac{\partial v}{\partial x}\right|_{y=0} = \varphi'(x), \quad \left.\frac{\partial v}{\partial y}\right|_{y=0} = \psi(x)$$

the imaginary part is determined accurate to within an additive constant C, irrelevant for the determination of v. However, from the known (for some fixed C) values $f(z)$ on the boundary, the function is uniquely determined[128] also within the layer. From this follows the uniqueness of the solution to the problem of extension with respect to v.

Unfortunately, even for equations involving a linear operator "identical" to the inverse problem, the fundamental results are not always helpful in solving the uniqueness question. Thus, for the Fredholm equation of the first kind with a regular symmetric kernel, there exists a criterion (necessary and sufficient condition) for unique solvability (Picard's theorem[111]), which, however, is practically useless in concrete applications since it requires, in addition to the knowledge of the asymptotics with respect to n of the coefficients of the right-hand side of the equation, the asymptotics of the eigenvalues of the kernel.

Consequently, the solution to the uniqueness question for inverse problems is connected with the study of its specifics.[16,27,107] This is even more so in the case of problems involving nonlinear correspondence operators, since the general theory of such equations is developed insufficiently.

In the paragraphs that follow we shall consider problems involving nonlinear operators.

2. The uniqueness of solutions to inverse problems in layered media

The medium in which the physical processes take place is often identified with the one-dimensional plane-layered model.[12] Such are models of the regional structure of the Earth's core, of optical filters, etc. The structure is usually described by the totality of numerical parameters (n **p**): the number of layers n, their thickness, and a set of values for some physical characteristics of the material, constant in each layer (conductivity, density, etc.). In other cases (for example, for large n and for small variation of the material characteristic from layer to layer) the structure is described by a function $z = z(\zeta)$ of a single spatial variable $0 \leqslant \zeta \leqslant h$, where h may also be a parameter.

If a wave of some physical nature is propagating through the layered medium, then the observed wave field depends on the structure and it is possible to pose the inverse problem: determine the parameters of the structure from observations on the wave field. In the present paragraph we shall be concerned with the uniqueness question for the solution of problems of this type.

2.1. Tikhonov's theorem on the uniqueness of the solution of the MTS problem

The Magneto-Telluric Sounding (MTS) method in geophysics (the Tikhonov–Cagniard method[147,173]) consists of the study of the conductivity of the Earth's bowels from observations on the Earth's surface of its natural electromagnetic field.

For the one-dimensional plane model (Fig. 4) the "relative" amplitude

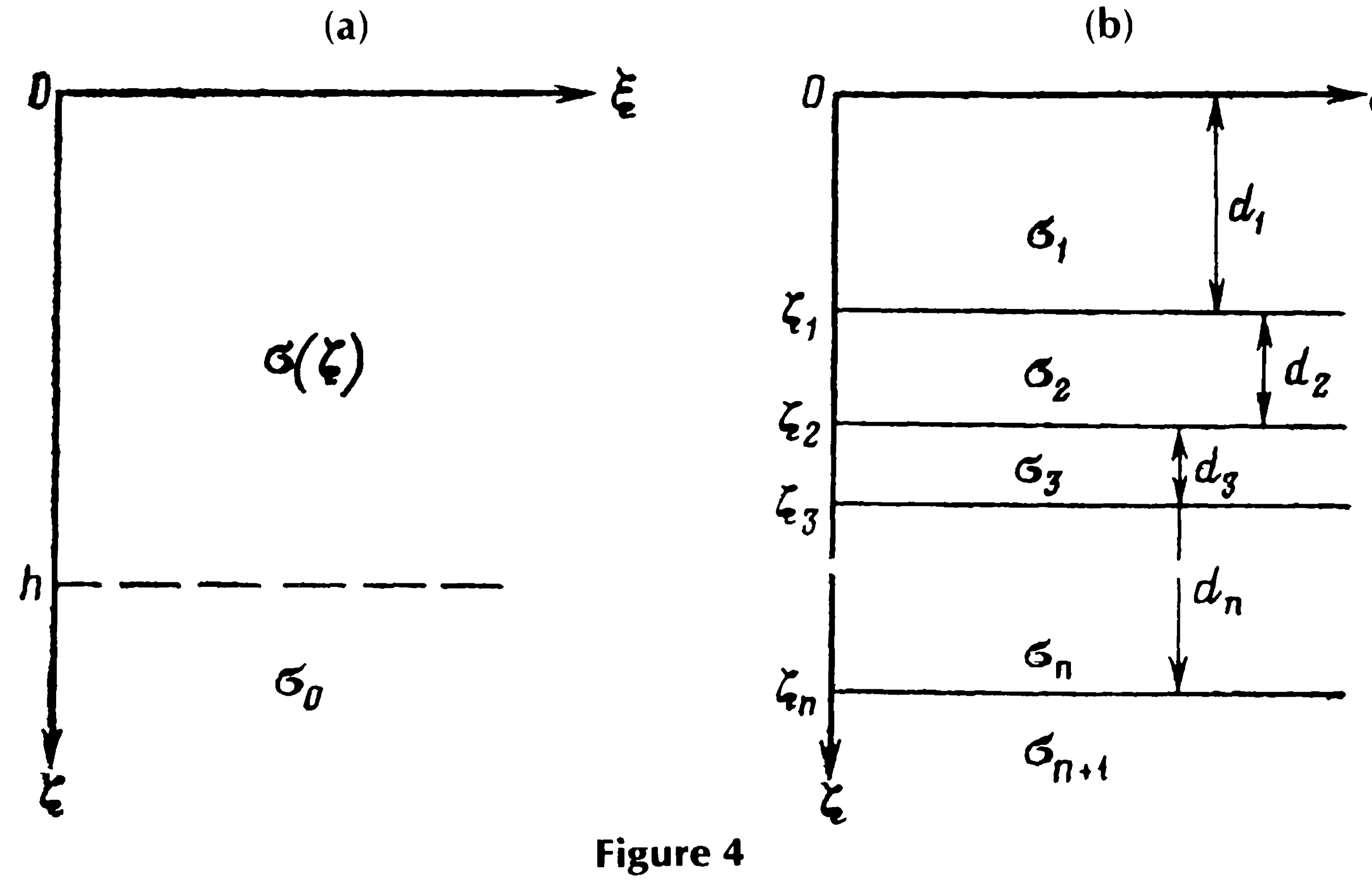

Figure 4

$[v(\zeta)\equiv w(\zeta)/w(0)]$ of the magnetic field H in a plane harmonic wave ($H = We^{i\omega t}$), normally incident on the medium surface, satisfies the conditions

$$v'' + i\omega\sigma(\zeta)v = 0 \quad 0<\zeta<h, \qquad v(0) = 1,\ v(\infty) = 0. \tag{35}$$

We may characterize the observed field by the "impedance" of the medium: $\hat{u}\equiv(E/H)|_{\zeta=0} = v'(0)$. The impedance is a function of ω, and for each ω it is a continuous functional of $\sigma(\zeta)$ (Ref. 144); thus Eq. (35) defines implicitly a continuous nonlinear operator $A\sigma\equiv v'(0)\equiv u(\omega,\sigma)$. The geoelectric section is determined by the equation $A\sigma = \hat{u}(\omega)$, $\omega\in[\omega_1, \omega_2]$, where $\hat{u}(\omega)$ is the exact analog of the "observable" quantity.

Let K now be the set of positive piecewise-analytic functions $\sigma(\zeta)$, defined on $[0, h]$, for which h and $\sigma_0 = \sigma(\zeta)$ for $\zeta > h$ are known quantities.

THEOREM 1.[144] To two different "geoelectric sections" from the class K $[\sigma_1(\zeta)\not\equiv\sigma_2(\zeta)]$ correspond different structure impedances $[\hat{u}_1(\omega)\not\equiv\hat{u}_2(\omega)]$.

Obviously this theorem establishes sufficient conditions for the uniqueness of the solution to the inverse problem, including the case when the geoelectric section is described by a piecewise constant function, i.e., by a choice of parameters **p**.

Following the methodology of Ref. 144, we outline the proof of this theorem for the subset $K_0\subset K$: $\sigma(\zeta) = \sigma_0 = \text{const}$ for $\zeta > h$, where h is known and $\sigma(\zeta)$ is analytic for $0\leqslant\zeta<h$ and continuous at the point h. In that case the condition in (35) as $\zeta\to\infty$ is replaced by

$$v'(h) = -(1-i)\left(\frac{\omega\sigma_0}{2}\right)^{1/2} v(h).$$

Consider two sections: $\sigma_s(\zeta)\in K_0$, $s = 1, 2$, with $\sigma_1(\zeta)\neq\sigma_2(\zeta)$. By making use of the conditions (35) it is not hard to verify that the corresponding impedances are related to $\sigma_s(\zeta)$ by the formula

$$\int_0^h [\sigma_1(\zeta) - \sigma_2(\zeta)]\, v_1(\zeta)v_2(\zeta)d\zeta = I(\omega),$$

where $I(\omega)\equiv(1/i\omega)[\hat{u}_2(\omega) - \hat{u}_1(\omega)]$ and $v_s(\zeta)$ is the amplitude of the corresponding magnetic fields.

Now it is sufficient to verify that $I(\omega)\not\equiv 0$, at least for sufficiently large ω (in the high-frequency asymptotic region). Since $v_s(\zeta)$ satisfies for $\omega\to\infty$ an equation with a small parameter multiplying the highest derivative, $\mu = \omega^{-1}$, analysis analogous to that used in Refs. 21 and 45 gives rise to the asymptotic expression

$$v_s(\zeta) \backsim \exp\left(-\frac{1-i}{\sqrt{2}}\sqrt{\omega}\int_0^{\zeta}\sqrt{\sigma_s(y)}dy\right).$$

Whence follows, for $\omega\to\infty$, that

$$v_1(\zeta)v_2(\zeta) \backsim \exp\left(-\frac{1-i}{\sqrt{2}}\sqrt{\omega}A(\zeta)\right),$$

where

$$A(\zeta)\equiv\int_0^{\zeta}[\sqrt{\sigma_1(y)}+\sqrt{\sigma_2(y)}]\,dy,$$

and $A(\zeta)$ is analytic and $B(\zeta)\equiv A'(\zeta)\neq 0$.

On the other hand, in view of the analyticity of $\sigma_s(\zeta)$, for $\sigma_1(\zeta)\not\equiv\sigma_2(\zeta)$ there exist integers n, $k\geqslant 0$ and c_n and c_{n+k+1} $(c_n, c_{n+k+1}\neq 0)$ such that

$$\sigma_1(\zeta)-\sigma_2(\zeta)=c_n\zeta^n+c_{n+k+1}\zeta^{n+k+1}[1+\zeta\varphi(\zeta)],$$

where $\varphi(\zeta)$ is analytic.

With the help of integration by parts we now obtain for $I(\omega)$

$$I(\omega)=n!c_nB^{-(n+1)}(0)\omega^{-(n+1)/2}+O(\omega^{-(n+2)/2}),$$

and, consequently, $I(\omega)\not\equiv 0$, at least for sufficiently large ω. From here the validity of the assertion follows.

In geophysical practice the impedance model (apparent resistance) is conventionally used as input information for structure prognosis:

$$\rho(\omega)\equiv|\hat{u}(\omega)|.$$

For media with piecewise constant parameters, characterized by the "vector" $\mathbf{p}=\{\sigma_1,\ldots,\sigma_n,d_1,\ldots,d_n\}$ (d_k is the layer thickness, σ_k is the layer conductivity), the additive-asymptotic $(\omega\to\infty)$ representation[28] of the apparent resistance turns out to be valid; it was obtained by N. I. Kulik and has the form[158]

$$\rho^{(n)}(\omega)=\rho^{(k)}(\omega)-2\sqrt{\omega}\sum_{s=k+1}^{n}C_sR_se^{-\sqrt{2\omega}w_s}\left[\cos(\sqrt{2\omega}w_s)+e^{-\sqrt{2\omega}\lambda_s}\varphi_s(\omega)\right],$$

where $\rho^{(k)}(\omega)$ is the apparent resistance of the structure of k layers, lying in the half-space with conductivity σ_{k+1} (for arbitrary k, $0\leqslant k\leqslant n-1$); $w_s=\Sigma_{i=1}^{s}d_i(\sigma_i)^{1/2}$ and, therefore, $w_{s+1}>w_s$;

$$R_s=\frac{1-\varkappa_s}{1+\varkappa_s},\quad \varkappa_s=\sqrt{\frac{\sigma_{s+1}}{\sigma_s}},$$

furthermore, without loss of generality we may assume that $R_s\neq 0$ (in the opposite case one relabels d_s);

$$C_s=4^{s-1}(\sigma_s)^{1/2}\left(\prod_{i=1}^{s-1}(1+\varkappa_s)^2\right)^{-1}$$

is independent of d_s and σ_{s+1}; φ_s is a bounded function as $\omega\to\infty$; $\lambda_s=\min_{1\leqslant k\leqslant s}\{d_k(\sigma_k)^{1/2}\}$. In turn, $\rho^{(0)}(\omega)=(\omega\sigma_1)^{1/2}$.

This representation makes possible for "properly layered" structures application of the method of analysis, developed in the proof of the preceding theorem (see also Sec. 4). The following theorem is valid.

THEOREM 1*. To two structures, differing for known σ_1 (or σ_{n+1}) in even one of the parameters n, $\sigma_2,\ldots,\sigma_{n+1}$ (or $\sigma_1,\ldots,\sigma_n$), d_1, $d_2,\ldots,d_n$, correspond different apparent resistance "curves" $\rho^{(n)}(\omega)$.

In this manner, the apparent resistance minimally augmented by some information about the structure uniquely determines the geoelectric section.

A similar representation is also valid for the phase of the impedance. It follows from this analysis that (a) the phase of the impedance may also be used to obtain a unique prognosis of the structure, but (b) it provides no additional information about the structure in comparison with $\rho(\omega)$.

2.2 Equivalence of structures and effective parameters

The indicated results testify to the fact that in MTS problems given the impedance, it is possible to obtain sufficiently detailed information about the structure. We note, however, that a certain quantitative restriction was imposed here on the sought-for characteristic; the value of σ at a certain point was assumed known.

We consider next a certain class of inverse problems for which the "observed" quantity is the impedance, or the reflection coefficient. As will be seen below, the formulation of the latter is identical to the former.

Let the plane structure consist of n layers (horizontal, for the sake of definiteness) $(0<\zeta<h)$, confined between two media. To the upper medium corresponds the wave number k_0, and to the lower (filling half-space), k_{n+1}; the wave numbers of the layers will be denoted by k_s $(s=1, 2, \ldots, n)$. The amplitude of the harmonic wave of arbitrary physical nature satisfies, for normal incidence on the (without loss of generality) upper boundary of the structure, the conditions[a]

$$v_s''+k_s^2 v_s=0,\ \zeta_{s-1}<\zeta<\zeta_s,\ s=1,2,\ldots,n;$$
$$[v]_{\zeta=\zeta_s}=0,\ [pv']_{\zeta=\zeta_s}=0,\ s=1,2,\ldots,n,$$

and the radiation conditions for $\zeta\to\infty$ are transferred to the point $h=\zeta_n$:

$$(v_{n+1}'-ik_{n+1}v_{n+1})_{\zeta=h}=0.$$

Let r_s denote the amplitude reflection coefficient at the lower boundary of each layer, and $I_s=(v_s'/ik_s v_s)|\zeta=\zeta_s$ be the impedance at the same point. Then the following relations (for arbitrary s, where $0\leqslant s\leqslant n$) can be readily obtained from the above-indicated conditions:

$$I_s=\frac{1-r_s}{1+r_s};$$
$$Vr_s=\frac{R_s+r_{s+1}\exp(-2ik_{s+1}d_{s+1})}{1+R_s r_{s+1}\exp(-2ik_{s+1}d_{s+1})},\quad r_{n+1}=0(I_{n+1}=1), \tag{37}$$

where $R_s=(1-\theta_s)/(1+\theta_s)$ is the Fresnel reflection coefficient and $\theta_s=(p_{s+1}/p_s)(k_{s+1}/k_s)$ is the wave contact resistance,[12] and d_s is the layer thickness.

It is seen from Eq. (37) that the reflection coefficient r_0, and therefore the structure impedance I_0, are determined by certain combinations of physical parameters, entering into the formulas for $k_s d_s$ and θ_s; if upon varying the physical parameters the indicated combinations remain unchanged, then the

[a]Here $[v]_{\zeta=\zeta_s}$ denotes the "jump" in v at the corresponding point: $v_{s+1}(\zeta_s+0)-v_s(\zeta_s-0)$; $p=p_s$ has its value in each layer (it is some parameter, determined by the physical nature of the problem).

field characteristic $r_0(\omega)$ $[I_0(\omega)]$ under consideration remains unchanged as well. To these characteristics corresponds a class of equivalent structures.

The indicated combinations of physical parameters will be called the effective parameters of the layered structure.[28,54] We include among effective parameters also the number of layers of the structure. It is clear that the question of uniqueness in problems $r_0(\omega)\to(n, \mathbf{p})$ can be discussed, generally speaking, only with respect to the effective parameters.

Naturally, given certain additional information about a part of the physical parameters of the structure, one may expect unique determination of the remaining ones from the effective ones. We shall be interested in the minimal additional information of this kind.

Let us consider examples of concrete physical fields.

(a) In the MTS problem,

$$k_s^2 = i\omega\sigma_s \quad s = 0,1,2,\ldots,n+1; \quad \rho_s = \sigma_s^{-1} \quad 1\leqslant s\leqslant n+1;$$

correspondingly $\theta_s = (\sigma_s/\sigma_{s+1})^{1/2}$ and $k_s d_s = (i\omega)^{1/2}(\sigma_s)^{1/2}d_s$. Consequently the effective parameters consist of θ_s and $\nu_s = (\sigma_s)^{1/2}d_s$.

Obviously, the minimal information needed within the framework of this model, for the totality of the effective parameters to determine uniquely all the physical parameters, consists of the specification of the value of one of the σ_s, for example (as is natural) σ_0.

(b) Let us consider a (e.g., optical) system of dielectric layers. In that case $k_s^2 = \omega^2 n_s^2$, where $n_s = (\epsilon_s \mu_s)^{1/2}$ is the index of refraction[b] of the layer ($s = 0, 1, \ldots, n+1$); $p_s = (i\omega\epsilon_s)^{-1}$. Correspondingly $\theta_s = (\epsilon_s \mu_s + 1/\epsilon_{s+1} \mu_s)^{1/2}$ $k_s d_s = (\epsilon_s \mu_s)^{1/2} d_s \omega$, and the effective parameters consist of θ_s and $\nu_s = n_s d_s$ (optical thickness).

Since μ_0 and ϵ_0 are known, then from the totality $\{\theta_s, \nu_s\}$ the quantities $\theta_s = (\mu_s/\epsilon_s)^{1/2}$ for $s = 1, 2, \ldots, n-1$ and ν_s for $s = 1, 2, \ldots, n$ are uniquely determined.

However, in a number of cases it may be assumed that μ_s is known for every layer: $\mu_s \equiv \mu_0$ (or ϵ_s). Then the totality of effective parameters uniquely determines the physical ones: ϵ_s (and n_s) for $s = 1, 2, \ldots, n+1$ and d_s for $s = 1, 2, \ldots, n$.

(c) An analogous state of affairs may be noted in the case of absorption-free propagation of elastic compression or shear waves. In that case, for shear waves for example, it is not hard to see that the effective parameters consist of $\theta_s = \mu_{s+1} b_{s+1}/\mu_s b_s$ and $\nu_s = d_s/b_s$, where μ_s is the shear modulus and $b_s = (\mu_s/\rho_s)^{1/2}$ with ρ_s the layer density. Thus, for problems involving elastic ("seismic") oscillations it is not possible to determine uniquely from a given impedance such physical parameters as $\{d_s, b_s\}$ ("velocity section of the medium"), for example. If, however, the material of the layers is known, then their thickness is determined uniquely.

In Ref. 174 it is shown that the pair of functions $\mu(\zeta)$ and $\rho = \rho(\zeta)$, and therefore the complete characteristic of the structure, are uniquely determined by, for example, independent measurements of displacements and stresses on the surface of the elastic medium as a function of the surface point.

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[b] μ_s and ϵ_s are the magnetic and electric permeability.

Moreover, it is sufficient to obtain values of the harmonic components of these quantities for two values of the frequency.

2.3. Additive representations of the reflection coefficient for a layered structure

Special additive representations[28] of r_0 are useful in the analysis of uniqueness in the problem of determination of effective parameters for a layered medium.

Setting $n = k\,(1 \leqslant k < n)$ in formula (37) results in a structure with k layers on a half-space with parameters of the $(k+1)$ th. Let us denote the corresponding reflection coefficient by $r_0^{(k)}$ $(r_0 \equiv r_0^{(n)})$. The following identities are obvious:

$$r_0^{(n)} \equiv r_0^{(s-1)} + \sum_{m=s}^{n} \Delta_m \equiv r_0^{(s-1)} + r_0^{(n)} - r_0^{(s-1)}$$

for arbitrary s, provided $\Delta_m \equiv r_0^{(m)} - r_0^{(m-1)}$.

We introduce the notation $r_0^{(n)} - r_0^{(s-1)} \equiv D_s(r_s)$. If $n = s$ then $r_{s+1} = 0$ and $r_s = R_{s+1}$; that is, $\Delta_s \equiv r_0^{(s)} - r_0^{(s-1)} = D_s(R_{s+1})$. At the same time, be definition $D_{s+1}(r_{s+1}) = r_0^{(n)} - r_0^{(s-1)} = D_s(r_s) - \Delta_s$. Whence by induction we arrive at the expressions

$$r_0^{(n)} = r_0^{(0)} + \sum_{m=1}^{n} \Delta_m \equiv r_0^{(s-1)} + \sum_{m=s}^{n} \Delta_m,$$
$$r_0^{(0)} = R_1, \tag{38}$$
$$\Delta_m = W_m \frac{R_{m+1}}{q_{m+1}} \exp\left(2i \sum_{j=1}^{m} k_j d_j\right),$$

where[c]

$$W_m = \prod_{j=1}^{m} \frac{1-R_j^2}{q_j^2}, \quad q_s = 1 + \rho_s R_s \exp(2ik_s d_s),$$
$$\rho_s = \frac{1}{q_{s-1}} [R_{s-1} + \rho_{s-1} \exp(2ik_{s-1} d_{s-1})],$$
$$s = 1, 2, \ldots, n, \ \rho_0 = 0, \ q_0 = 1.$$

The formulas (38) provide an "additive" representation of the quantity $r_0^{(n)}$ in the sense that the reflection coefficient of the structure is given additively in terms of the corresponding quantity of any of the "partial" structures $(s = 1, 2, \ldots, n-1)$. Also, $r_0^{(s)}$ is, apparently, independent of the parameters of the lower-lying layers of the structure [the $(s+1)$th taken to be as the half-space]. Moreover, each term in this representation $(m \geqslant 1)$ contains an exponent whose modulus increases with increasing m. The indicated peculiarities of Eq. (38) make the additive representation of $r_0^{(n)}$ useful in the uniqueness analysis of the problems under consideration.

[c] It is not hard to note that ρ_s corresponds to the amplitude coefficient of the "moving in the opposite direction" (fictitious) wave.

2.4. Solution uniqueness for impedance interpretation

Let us introduce for MTS problems a class K of structures of the above-discussed type, for which the quantity σ_0 is known. We shall refer to these structures as "different" provided they are distinguished by at least one value of the physical parameters $(n, \sigma_1, \ldots, \sigma_n, d_1, \ldots, d_n)$.

THEOREM 2. To two different structures from the class K correspond two different functions $r_0(\omega)$ (correspondingly impedances).

Let the structure K_t $(t = 1, 2)$ correspond to $r_0,t(\omega)$. We verify, just as in the proof of Theorem 1 (Sec. 2.1), that $r_{0,1}(\omega) \neq r_{0,2}(\omega)$ at least for sufficiently large ω $(\omega \to \infty)$.

We note that, since $k_j = -(1-i)(\omega\sigma_j/2)^{1/2}$, the quantity Δ_m is of higher asymptotic order than Δ_{m-1}, and consequently for sufficiently large ω subsequent terms cannot "compensate" for preceding ones. On the other hand, for a structure containing more than m layers, the argument of the exponential in Δ_m is different from zero. Finally, in the asymptotic representation for Δ_m,

$$\Delta_m \sim A_m R_{m+1} \exp\left[-(1-i)\sqrt{2\omega}(a_m + \sqrt{\sigma_m} d_m)\right]$$

the quantities A_m and a_m depend only on the preceding layers of the structure $(1 \leqslant s \leqslant m-1)$.[d]

We now suppose that the structures K_t are distinguished by the values of $R_1 = r_0^{(0)}$; then obviously $r_{0,1}(\omega) \neq r_{0,2}(\omega)$, since the "corrections" are exponentially small as $\omega \to \infty$. Let us isolate a subset of structures for which R_1 has the same value; if one of them is the half-space $(n = 0)$, and for the other $n \geqslant 1$, then

$$r_{0,1}(\omega) - r_{0,2}(\omega) = \Delta_1^{(2)}(\omega) + \cdots \neq 0,$$

i.e., the reflection coefficients are different as $\omega \to \infty$. Let us consider the subset of structures for which values of R_1 coincide and at the same time $n \geqslant 1$; in that case A_1 and a_1 $(a_1 = 0)$ also have the same values in the asymptotic representation of Δ_1, and therefore $r_{0,1}(\omega) \neq r_{0,2}(\omega)$ provided only either $R_2^{(1)} \neq R_2^{(2)}$ or $(\sigma_1^{(1)})^{1/2} d_1^{(1)} \neq (\sigma_1^{(2)})^{1/2} d_1^{(2)}$. Clearly induction is possible, as a result of which we verify that a difference in the values of any of the effective parameters results in different reflection coefficients $r_0(\omega)$ (consequently, in different impedances). But in the class K the set of effective parameters determines the indicated physical ones in a one-to-one fashion. This proves the theorem.

Theorem 1* of Sec. 2.1, involving the use of apparent resistance in structure prognosis, is proven analogously.

We consider now an optical or elastic system. In that case it is not hard to see that in the expression (38):

$$\Delta_m = A_m R_{m+1} \exp[2i(a_m + n_m d_m)\omega][1 + \alpha_m(\omega)],$$

where n_m is the index of refraction, A_m and a_m are constants independent of

[d]For $\omega \to \infty$ the representation (38) becomes "additive-asymptotic."[28]

R_{m+1}, n_m, and d_m, and $a_m(\omega)$ is an "almost-periodic" function[89,e] with lowest Fourier exponent different from zero. Correspondingly Δ_m is an almost-periodic function with lowest exponent $\nu_0 \equiv 2(a_m + n_m d_m)$ and lowest generalized Fourier coefficient $f_0 \equiv A_m R_{m+1}$. As is known,[39] to such a function corresponds uniquely a set (ν_k, f_k) and, therefore, values of ν_0 and f_0. This means that for two structures K_1 and K_2, with coincident parameters of the first $m-1$ layers and, correspondingly,

$$r_{0,1}^{(m-1)}(\omega) \equiv r_{0,2}^{(m-1)}(\omega), \quad a_m^{(1)} = a_m^{(2)}, \quad A_m^{(1)} = A_m^{(2)},$$

a difference in the values of R_{m+1}, or $n_m d_m$, or in the number of layers $(n \geqslant m+1)$ results in $r_{0,1}^{(m)}(\omega) \not\equiv r_{0,2}^{(m)}(\omega)$. By means of induction, as in the previous case, we conclude that the following theorem is valid.

THEOREM 3. For the interpretation problem in the case of the reflection coefficient (impedance) of an optical or elastic layered medium, the input information uniquely determines the totality of the effective parameters of the structure.

2.5. The uniqueness of the solution of problems of interpretation of data on dispersion of surface seismic waves

An elastic impulse in a layered medium such as, in particular, the Earth's core, creates a wave field which contains waves propagating along the Earth's surface ("surface" waves, whose amplitude is damped with depth). The dependence of the phase (or group) velocity of such waves on the spectral frequency may serve as input information for the problem of determination of the core "elastic-density" section: $\mu = \mu(\zeta)$, $\rho = \rho(\zeta)$ for displacement waves ("Lyav's waves").[122]

Let $c(\omega)$ be the phase velocity of the corresponding ω-"harmonic" of Lyav's wave, propagating along the boundary $\zeta = 0$ in a horizontally uniform plane-layered medium, and let $v = v(\zeta)$ be the wave amplitude. Then (for each ω) $c(\omega)$ can be found as the solution of the eigenvalue problem[40]

$$\begin{gathered} v_s'' + \omega^2(b_s^{-2} - c^{-2})v_s = 0, \quad \zeta_{s-1} < \zeta < \zeta_s; \\ [v]_{\zeta=\zeta_s} = 0, \quad [\mu v']_{\zeta=\zeta_s} = 0, \quad s = 1,2,\ldots,n; \\ (v_n' + \omega\sqrt{c^{-2} - b_{n+1}^{-2}}\, v_n)_{\zeta=\zeta_n} = 0, \quad v'|_{\zeta=0} = 0. \end{gathered} \tag{39}$$

It can be shown that for $c < b_{n+1}$ there exists a discrete spectrum of eigenvalues $\{c_k\}$.[28,92] In practice it is customary to make use of one of them, $c_0 = c_0(\omega)$, and in what follows we shall mean by c precisely this quantity.

It is fairly obvious that in terms of this information the effective parameters $(n, \mathbf{p})$ of the structure are the quantities n, b_s $(s = 1, 2, \ldots, n+1)$, $\nu_s = \mu_s/\mu_{s+1}$, and d_s (the layer thickness) for $s = 1, 2, \ldots, n$.[28]

[e] The function f will be called "almost-periodic" if it is expressible in the form of a generalized trigonometric series

$$f(\omega) = \sum_{k=-\infty}^{+\infty} f_k \exp(i\nu_k \omega),$$

where ν_k and f_k are numbers. The ratio of two generalized trigonometric polynomials is an almost-periodic function.[89]

To study the inverse problem $c(\omega)\to(n, \mathbf{p})$, one may use, for example, recursion relations analogous to Eq. (37), and then the condition $v'|_{\zeta=0}=0$ results in the equation $I_0(\omega, c)=0$ with a recursively determined left-hand side. In the same manner as above, one obtains for the latter an additive representation.[28] Assuming that $b_1=\min_s b_s$ and $b_1<c<b_{n+1}$ (as is the case in real problems), one finds that $c(\omega)\to b_1$ as $\omega\to\infty$ and, consequently, $c(\omega)<b_s$ $(s=2,3,\ldots,n+1)$ for sufficiently large ω. Therefore in the indicated region the additive representation for $I_0(\omega, c)$ displays the influence of lower-lying layers, similarly to the MTS problem (skin effect), and one may apply asymptotic analysis $(\omega\to\infty)$, as in the proof of Theorem 2. To this end an additive-asymptotic formula for $c(\omega)$ is "extracted" from the equation $I_0(\omega, c)=0$ (in the additive representation). Its main (nonexponential) term—the eigenvalue for one layer on a half-space with parameters of another—has the form

$$c_1^{-2}(\omega)=b_1^{-2}-\frac{\pi^2}{4}\frac{1}{\omega^2 d_1^2}\left\{1-\frac{2\rho_1}{\omega}+\frac{3\rho_1^2}{\omega^2}-\frac{4\rho_1^3}{\omega^3}\right.$$

$$\left.\times\left[1+\frac{\pi^2}{16}\left(\frac{1}{\nu_1^2}-\frac{2}{3}\right)\right]\right\}+\frac{1}{\omega^6}\varphi(\omega),$$

where $\varphi(\omega)=O(1)$ is a function determined by the parameters of the first element of the structure (b_1, d_1, ν_1, b_2), and $\rho_1=(\nu_1/d_1)(b_1^{-2}-b_2^{-2})$. It is clear that $c_1^{-2}(\omega)$, and therefore $c(\omega)$, uniquely determines the indicated parameters. Performing for the additive-asymptotic representation of $c(\omega)$ the previous induction, we arrive at the conclusion that the following theorem is valid.

THEOREM 4. In the class of structures for which $b_1=\min_s b_s$, the dispersion of the fundamental tone of the Lyav wave uniquely determines all effective parameters of the structure.

It can also be shown that "overtones" of the Lyav wave $\{c_k(\omega), k>0\}$ have completely analogous representations and carry no additional information about the structure.

The same assertions are valid for Rayleigh waves—elliptically polarized compression-shear waves.[29]

In Refs. 26 and 30 the more general case of not necessarily discrete structures is discussed.

We note that in the problem under discussion the sought-for quantity consists of the totality of the coefficients of a certain type of equation, with a certain element of the eigenvalue "spectrum" given. The fundamental result for this type of problem (inverse spectral) was obtained in Ref. 25, where one-to-one correspondence was established in Schrödinger-type problems between the equation coefficient [compared with $b^{-2}(\zeta)$] and the full spectrum characteristic (the spectral function). In the above-mentioned papers the formulation of the problem "approximates" the experimental situation.

3. The uniqueness of the solution in certain inverse problems of heat conductivity

3.1. Different aspects of the study of uniqueness of the solution (on the example of heat conductivity problems)

In discussing inverse problems for layered media we called attention to the fact that to an exactly given, correlated with experiment, field characteristic may correspond a multitude of equivalent structures, determined by the totality **p** of effective parameters.

An even broader statement can be made: in problem (39), for example, to the set of such structures for a given harmonic and specified normalization of the eigenfunction corresponds one and the same wave field and, therefore, the same pair $[c(\omega), v(x, \zeta)]$. The analogous situation is possible for any mathematical model formulated as a (mixed) boundary-value problem for a differential equation. It is obvious, for example, that the temperature field, determined by the boundary-value problem for the equation $k\,\partial^2 u/\partial x^2 = c\rho\,\partial u/\partial t$ for constant k, c, and ρ, remains unchanged under a change in the parameters that preserves the ratio $a^2 = k/c\rho$.

This means that it is not possible to determine uniquely all the parameters, which may be of interest in the inverse problem, even if complete information on the solution of the differential equation (the complete field) is available, and therefore there exists no field characteristic that could provide additional information for the determination of the sought-for parameters.

In this vein one of the directions of research on inverse problems, connected with differential equations of mathematical physics, consists at the present time of the extraction of a subset of object characteristics (in particular, the totality of coefficients), on which the solution of the boundary-value problem is invariant, and also the study of such solutions.

If by "identifiability" of the model of the inverse problem with the object under study we mean, as before, the existence of a one-to-one correspondence between characteristics of object and effect, then problems for which such correspondence is violated, even for a full characterization of the effect, may be called "nonidentifiable in full."[117,118]

Consider as an example[119] the boundary-value problem

$$a_1 \frac{\partial u}{\partial t} = a_2 \frac{\partial^2 u}{\partial x^2} + 1, \quad u|_{t=0} = x(x-1), \quad u|_{x=0} = u|_{x=1} = t.$$

It is well known[162] that its solution is unique for given values of a_1 and a_2. Set $u^*(t) \equiv t + x(x-1)$; $u^*(t)$ satisfies additional conditions. Upon substitution of u^* into the equation we find $a_1 = 2a_2 + 1$. Suppose now that the coefficients a_1 and a_2 are the sought-for quantities. Then if the field has the form $u^*(x, t)$, there corresponds to it a multitude of coefficient pairs and there exists no additional information about the field that could be used to extract a unique pair.

The indicated direction is reflected, in part, in Refs. 119, 166, and 170. The analogous problem has been studied long since in problems of gravimetry[131,179] (see Sec. 4).

The main direction of research on the uniqueness question for solutions of inverse problems, connected with differential equations, consists in clarification of what constitutes minimal additional information about the solution of the equation and, possibly (as we have seen in the examples), the sought-for characteristic, under which the solution of the inverse problem becomes unique.

While searching for such a posing of the problem, qualitative considerations are useful. For example, if the sought-for quantity is the coefficient of heat conductivity as a function of temperature [$k = k(u)$], then in addition to the conditions of the boundary-value problem on $u(x, t)$, $x \in E^1$, it makes sense to specify the independent characteristic of the field as a function of one variable. If instead $k = k(u, x)$, then additionally one should specify a function of two variables. However, the subsequent proof of the uniqueness of the solution is necessary, since the preliminary considerations may turn out to be erroneous.

As an example, consider the problem of determination of the location of a "plane" heat source of given intensity from observations on the temperature field. In the simplest model the temperature field is determined from the conditions

$$a^2 \Delta u + f(t)\delta(x,x_0) = \frac{\partial u}{\partial t}, \quad 0 < x_1^2 + x_2^2 < +\infty, \quad u|_{t=0} = 0(f(0) = 0),$$

where $x = (x_1, x_2) \in E^2$.

It would follow from purely qualitative considerations that for the determination of the pair (x_1, x_2) it is sufficient to state additionally the temperature at two points (at some fixed instant of time). In Refs. 35 and 80 it is shown, in particular, that this is not so; the minimal information turns out to be the temperature at three not colinear points: $u(x_s, t_1) = u_s$ $(t_1 > 0, s = 1, 2, 3)$. The fact that, in general, two observation points are insufficient becomes clear upon a more careful consideration of the problem: the integral connection between $u(x_s, t_1)$ and x_0 determines the distance $\Delta_s \equiv |x_s - x_0|$. If $s = 1, 2$ then (in view of the *a priori* solvability of the problem) the pair Δ_s determines the intersection of two circles, i.e., two values for x_0.

3.2. The uniqueness in the determination of the coefficient of heat conductivity in high-temperature processes

As is well known, information on the behavior of the physical characteristics of many technical materials in high-temperature, rapidly proceeding processes is far from complete.[22,187] Their experimental study is often difficult if not possible.

For this reason the "inverse problems method,"[2] replacing the physical experiment by a mathematical one, has become more and more popular in the heat technology field. It is based on the fact that the sought-for characteristics are (as also in the above-discussed problems) coefficients in differential equations, describing a certain technological process.

In many cases it is sufficient to consider the simplest model of such a process.

Let us consider the spatially uniform model of heat conductivity:

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)=c\rho\frac{\partial u}{\partial t}\quad 0<x<l,\ 0<t<T,$$

$$u|_{t=0}=u_0=\text{const},\quad k\frac{\partial u}{\partial x}\bigg|_{x=0}=0,\quad k\frac{\partial u}{\partial x}\bigg|_{x=l}=\Phi(t)>0,\qquad (40)$$

where $k(u)$ is also an unknown function. To determine it we specify further $u|_{x=1}=f(t), f(0)=u_0, f'(t)>0$. Correspondingly, if $v(x, t, k(u))$ is the solution of the problem (40) for some $k=k(u)$, then the latter is a solution of the operator equation $Ak\equiv u(l, t, k)=f(t)$, with A defined implicitly. Such a problem corresponds to, for example, the modeling of casehardening[44,102] with other parameters of the material known.

Let us note certain peculiarities of the problem and facts, helpful in the study of uniqueness of the solution: (i) at an arbitrary instant $t\in(0, T)$ in a certain finite interval in x, we have $u'_x>0$; (ii) the solution to problem (40) satisfies the maximum principle: $u_0\leqslant u\leqslant f(t)$; and (iii) if $\varphi(x\ t)$ is determined as the solution of the problem with "inverse time" $[\varphi_t+p\varphi_{xx}=0,\ 0<t<\tau,$ $0<x<l,\quad \varphi_x(0, t)=0,\quad \varphi_x(1, t)=\chi(t)\quad (\chi>0,\quad \chi'>0),\quad u(x, \tau)=0,$ where $p=p(x, t)>0$ and satisfies the Lipschitz condition[128]], then $\varphi_x>0$ almost everywhere. The first two facts are understandable from the physics of the process, and all three can be proven rigorously[102] on the basis of the results in Refs. 73 and 87.

Let us introduce the class K of functions $k=k(u)$, analytic in the neighborhood of the segment $[u_0, f(t)]$ or piece wise-polynomial on the segment.

THEOREM 5. If $f(t)$ is continuous and strictly monotonic on $[0, T]$ and $f(0)=u_0$, then to it corresponds a unique $k(u)\in K$.

We sketch the proof of this assertion.[102] The problem (40) is reduced by the substitution

$$b(u)=\int_{u_0}^{u}k(\zeta)d\zeta$$

to the form

$$u_t=b_{xx},\ u|_{t=0}=u_0,\ b_x|_{x=0}=0,\ b_x|_{x=l}=\Phi(t),$$

and $b_x=ku_x$. Multiplying the equation $u_t=b_{xx}$ by an arbitrary function $\varphi(x, t)\in C^{2,1}$ and integrating over the region $Q_\tau=\{0<x<l, 0<t<\tau\}$ for arbitrary τ, we arrive at the integral identity for (u, b):

$$\int_0^l[u(x,\tau)\varphi(x,\tau)-u_0\varphi(x,0)]dx-\int_0^\tau[\Phi(t)\varphi(l,t)$$
$$-b(f(t))\varphi_x(l,t)+b(u(0,t))\varphi_x(0,t)]dt$$
$$-\iint_{Q_\tau}[u\varphi_t+b(u)\varphi_{xx}]dx\,dt\equiv 0.\qquad (41)$$

Suppose now that there exist two solutions to the problem and correspondingly two pairs of functions (u, b). Since each of them satisfies the identity (41), by performing a subtraction and choosing $\varphi(x, t)$ such that $\varphi(x, \tau)=0$, $\varphi_x(0, t)=0$ (τ is a parameter), we arrive at the equation

$$\iint_{Q_\tau} [(u_1 - u_2)\varphi_t + [b_1(u_1) - b_1(u_2)]\varphi_{xx}]dx\,dt$$

$$= \int_0^\tau [b_1(f(t)) - b_2(f(t))]\varphi_x(l, t)dt.$$

Let us isolate under the first integral sign the difference

$$b_1(u_1) - b_2(u_2) \equiv [b_1(u_1) - b_1(u_2)] + r(u_2)$$

and carry out the integration involving $r(u_2)$ by parts. We then obtain, as can be easily shown,

$$\iint_{Q_\tau} [(u_1 - u)\varphi_1 + [b_1(u_1) - b_2(u_2)]\varphi_{xx}]dx\,dt$$

$$= \iint_{Q_\tau} [k_1(u_2) - k_2(u_2)]u'_{2x}\varphi_x\,dx\,dt.$$

Finally, let us choose $\varphi(x, t)$ such that $\varphi t + p(x, t)\varphi_{xx} = 0$, where $p = [b_1(u_1) - b_1(u_2)]/(u_1 - u_2)$ *for* $u_1 \neq u_2$ and $p = k_1(u_1)$ for $u_2 = u_1$ (these p's satisfy, as can be verified, the Lipschitz condition); and let us require $\varphi_x(l, t) = \chi(t)$ $(\chi'(t) > 0)$ to be positive. Then, summarizing the restrictions on $\varphi(x, t)$, we conclude in view of condition (iii) that $\varphi_x > 0$ almost everywhere. As a result we have, for arbitrary $\tau > 0$,

$$I_\tau \equiv \iint_{Q_\tau} [k_1(u_2) - k_2(u_2)]u'_{2x}\varphi_x\,dx\,dt = 0,$$

and furthermore $u'_{2x} > 0$ (2), $\varphi_x > 0$ almost everywhere, and, by assumption, $k_1(u_2) \not\equiv k_2(u_2)$.

Suppose now (without loss of generality) that $k_1(u_0) > k_2(u_0)$. Since $k_s(u) \in K$, there exist u_1 and ϵ such that $k_1(u) - k_2(u) > \epsilon$ for $u_0 \leqslant u \leqslant u_1$. Let us choose τ from the condition $f(t) = u_1$. This is possible in view of the conditions of the theorem. Then $u(l, t) = u_1$, and in view of the maximum principle (ii), $u_0 \leqslant u(x, t) \leqslant u_1$ for $0 \leqslant t \leqslant \tau$. But on the indicated set of variation of u, we have $k_1(u) - k_2(u) > \epsilon$. Consequently, $I_\tau > 0$, which contradicts the resultant identity.

Let now $k_1(u_0) = k_2(u_0)$. Then there exist u_1 and ϵ such that $k_1(u) = k_2(u)$ for $u_0 \leqslant u \leqslant u_1$ and (without loss of generality) $k_1(u) > k_2(u)$ for $u_1 < u \leqslant u_1 + \epsilon$. Choosing τ from the equation $f(t) = u_1 + \epsilon$, we verify that there exists a finite interval $\Delta \subset [0, \tau]$ on which $u_1 < u < u_1 + \epsilon$, and, consequently, $k_1(u) - k_2(u) > \epsilon$. Thus in this case, too, $I_\tau \neq 0$ for some τ.

Therefore our assumption that $k_1(u) \not\equiv k_2(u)$ is false and the theorem is proven.

In Ref. 102 a more general result is obtained: for the equation $c(u)u_t = (k(u)u_x)_x$ with the previous initial and boundary conditions and with the additional conditions $u(0, t) = f_1(t)$, $u(l, t) = f_2(t)$, $t \in [0, T]$, $f_s(t) \in C^1[0, T]$ and are monotonic, and $f_s(0) = u_0$, there results a unique determination of a pair of functions $[c(u), k(u)] \in K$.

Similar results make it apparently possible to plan a physical experiment (control of the heat flow and measurement of the temperature) sufficient, in conjunction with calculations on an electronic computer with the help of regularizing algorithms, to determine both characteristics of the material.

4. Uniqueness in a certain inverse problem in gravimetry

The gravimetric method in geophysics consists of the study of the structure of local sources of the gravitational field, located in the Earth's bowels, from field observations carried out on the Earth's surface. In the mining papers (Refs. 132 and 137), the objects under study were isolated bodies and the sought-for characteristics were their form and density. We shall consider as the characteristic of the observed field its potential. The connection between the indicated quantities is easily established explicitly in accordance with the universal gravitation law.

It is well known[131] that the form of the "gravitating" body and the density distribution within it cannot be uniquely determined from the "external" (with respect to the source distribution region) potential. Indeed, suppose that the field is due to a density distribution $\rho(M')$ within the volume of the body T. Then the external potential is given by

$$V(M) = \int_T \rho(M') R_{MM'}^{-1} \, d\tau_{M'}.$$

Let us add to $\rho(M')$ the function $\sigma(M') = -(1/4\pi)\Delta W$, where $W(M')$ is an arbitrary twice differentiable function, satisfying the conditions $W|_\Sigma = (\partial W/\partial n)|_\Sigma = 0$, with Σ the surface of the body T. It is easily shown, by making use of the second Green formula for W and R^{-1} (for observations outside T),[162] that

$$\int_T \frac{\sigma(M')}{R_{MM'}} \, d\tau_{M'} = 0.$$

Consequently, for a given form of the surface Σ, there corresponds to the potential $V(M)$, generally speaking, an infinite set of density distributions $\rho'(M') = \rho(M') + \sigma(M')$. In contemporary terminology the gravimetry problem under consideration is nonidentifiable in full."

Let us note[131] that the above-introduced density $\sigma(M)$ exhausts for arbitrary W the set of densities for which the external potential of the body T vanishes.[f] It turns out that $\sigma(M)$ possesses a remarkable property.

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[f]For arbitrary planar case,

$$\sigma^*(M) = -\Delta W/2\pi, \quad M \in D, \quad W|_C = \partial W/\partial n|_C = 0,$$

where C is the boundary of D, and W is arbitrary satisfying the indicated conditions.

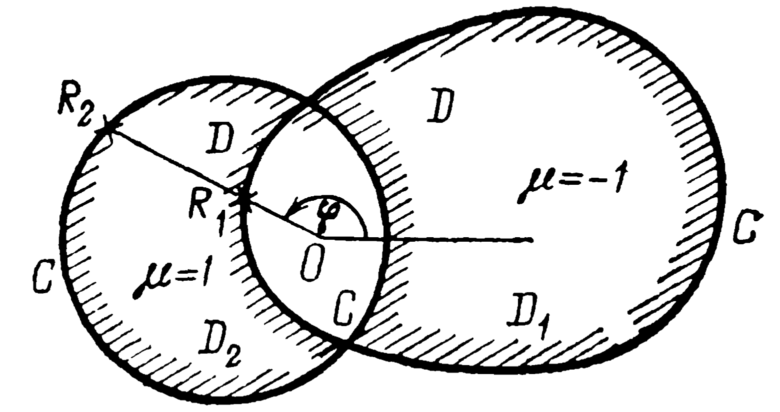

Figure 5

LEMMA. $\sigma(M)$ is orthogonal within T to an arbitrary harmonic function: $\int_T U\sigma\, d\tau = 0$, provided $\Delta U = 0$ within T (in the two-dimensional case, $\int_D U\sigma^*\, dS = 0$, $\Delta U = 0$ within D).

The validity of the assertion follows from the second Green formula for U and W.[131,162]

Properties of harmonic functions will also be helpful down the line in further studies of the uniqueness question.

We consider next the question of the uniqueness of determination of the shape of the body given its density distribution from an external potential. We confine ourselves to the case of constant density.

Let us isolate the class K of convex bodies T having in common a certain inner point, whose mass density $\rho = \text{const}$ is known.

THEOREM (S. P. Novikov–L. N. Sretenskii).[131] To two different bodies from the class K correspond different external potentials:

$$T_1 \neq T_2 \Rightarrow V_1(M) \neq V_2(M), \quad M \in E \setminus T_1 \cup T_2.$$

We sketch the proof of this assertion for two-dimensional bodies (regions) D_1 and D_2 (Fig. 5).

Let O be the common point of D_1 and D_2, which consequently intersect, and $\rho_0 = \text{const}$ be the density of each. Let us suppose that $D_1 \neq D_2$, but for arbitrary $M \bar{\in} D_1 \cup D_2$ we have $V_1(M) = V_2(M)$, where $V_s(M)$ are the corresponding external potentials.

Consider the region $D \equiv (D_1 \cup D_2) \setminus (D_1 \cap D_2)$. By assumption

$$V_1(M) = -\frac{\rho_0}{2\pi} \int_{D_1} \ln R_{MM'}\, d\sigma_{M'} = -\frac{\rho_0}{2\pi} \int_{D_2} \ln R_{MM'}\, d\sigma_{M'} = V_2(M).$$

Consequently $\int_D \mu(M') \ln R_{MM'}\, d\sigma_{M'} = 0$, where $\mu(M') = -1$ if $M' \in D_1 \setminus D_2$ and $\mu(M') = 1$ if $M' \in D_2 \setminus D_1$ (Fig. 5). But then $\mu(M') \equiv \sigma(M')$, as defined above, for some $W(M')$ and consequently (by the lemma) $I = \int_D U\mu\, d\sigma = 0$ for arbitrary harmonic in D function U.

Let us verify that there exists a function, harmonic in D, for which the last equality is false.[g] Let us introduce polar coordinates, with the point O as origin, and let $r = R_s(\varphi)$ $(s = 1, 2)$ be the polar equation of the boundary nearest to and farthest from the point O, respectively. Then

$$I = \int_0^{2\pi} \mu\, d\varphi \int_{R_1(\varphi)}^{R_2(\varphi)} u(r, \varphi) r\, dr.$$

[g]The proof of the theorem would follow from the fact that the resultant $\mu(M')$ cannot equal $\sigma(M')$ for any W.

Let us introduce the function

$$V(r, \varphi) = \int_0^r U(r, \varphi) r\, dr.$$

It is not hard to show that if $U(r, \phi)$ is a harmonic function then so is $V(r, \varphi)$ and vice versa, so that the problem reduces to the construction of a harmonic in D function V. For it,

$$I = \int_0^{2\pi} \mu(\varphi)[R_2^2(\varphi) V(R_2, \varphi) - R_1^2(\varphi) V(R_1, \varphi)]\, d\varphi.$$

Let us isolate a finite interval $\Delta \in (0, 2\pi)$, corresponding for the sake of definiteness to the region wherein $\mu = 1$. We construct a sequence of functions $\Phi_n(\varphi)$, defined on the boundary C as follows:

$$\Phi_0(\varphi) = \begin{cases} \dfrac{q}{R_2^2(\varphi) \text{mes } \Delta} & \varphi \in \Delta, \\ 0 & \varphi \bar{\in} \Delta; \end{cases}$$

$$\Phi_n(\varphi) = \begin{cases} \int_\Delta R_1^2 \Phi_{n-1}\, d\varphi \int_\Delta R_2^2\, d\varphi & \varphi \in \Delta, \\ 0 & \varphi \bar{\in} \Delta; \end{cases} \qquad (42)$$

where $q > 0$. It is not hard to show that then

$$\int_\Delta (R_2^2 \Phi_n - R_1^2 \Phi_{n-1}) d\varphi = 0,$$

and the integral has zero value upon completion of Δ to $[0, 2\pi]$. Consider next a sequence of solutions to the Dirichlet problem $\Delta V_n(M) = 0$, $M \in D$, $V_n|_C = \Phi_n(\varphi)$, where C is the boundary of D, and construct $V(r, \varphi) = \Sigma_{n=0}^{+\infty} V_n(r, \varphi)$. This series converges uniformly in an arbitrary closed subregion of D. Indeed,

$$|V_n(r, \varphi)| \leqslant \max_\Delta |\Phi_n(\varphi)| \leqslant \eta \max_\Delta |\Phi_{n-1}(\varphi)|,$$

where

$$\eta = \int_\Delta R_1^2(\varphi) d\varphi \quad \int_\Delta R_2^2(\varphi) d\varphi < 1;$$

since $|\Phi_0| \leqslant q/[\min_\Delta R_2^2(\phi)(\phi_2 - \phi_1)] \leqslant C_0 = \text{const}$, it follows that $|V_n(r, \varphi)| \leqslant C_0 \eta^n$. Whence we conclude by the first theorem of Harnack[131] that $V(r, \varphi)$ is harmonic in D. Moreover, $V_n(R_2, \varphi) = V_n(R_1, \varphi) = \Phi_n$ in view of the boundary conditions. Consequently

$$R_2^2 V_n(R_2, \varphi) - R_1^2 V(R_1, \varphi) = R_2^2 \Phi_0 + \sum_{n=1}^{\infty} [R_2^2 \Phi_n - R_1^2 \Phi_{n-1}],$$

and therefore, in view of the remarks made above,

$$I = \int_\Delta R_2^2 \Phi_0(\varphi) d\varphi = q > 0.$$

Thus there exists a function U, harmonic in D, not orthogonal to the body density, and giving rise to a vanishing external potential. That means the external potential for the region $D_1 \cup D_2$ is not identically zero, i.e., $V_1(M) \neq V_2(M)$. The assertion is proven.

More general theorems are proven in Refs. 107 and 113. The question of uniqueness in gravimetry is also studied in Refs. 112, 124, and 137 and others.

It is obvious that the theorem establishes one-to-one correspondence between the external potential field and the shape of the body. It is presumed that sufficiently complete input information for the corresponding inverse problem is known. It is known, however, that the potential as a harmonic function is uniquely determined by specifying its value and the value of its normal derivative[17,84] on some interval of the region boundary, for example (for the model under consideration) for $\zeta = 0$. Such data may serve as minimal input information for the problem of determination of the body shape. It is usual in such problems to make indirect use of more complete data—an explicit expression for the potential everywhere in a source-free region. It is sufficient[105,162] to specify its value on the boundary ($\zeta = 0$).

Chapter 4
The stability problem and regularizing operators

1. Regularizing operator

1.1. The concept of a regularizing operator

The fundamental concept of a regularizing operator (algorithm) for the solution of ill-posed problems was introduced in Ref. 141 and may be applied to the entire class of problems of interpretation of physical observations. Suppose that to such a problem corresponds, with measurement inaccuracies taken into account, the equation

$$Az = \tilde{u}_\delta, \ z \in Z, \ \tilde{u}_\delta \in U, \ \rho(\hat{u}, \tilde{u}_\delta) \leqslant \delta , \tag{43}$$

where $\hat{u}$ is the hypothetical exact value of the right-hand side ($\hat{u} \in U_A$), for which the existence of a unique exact solution ($A\hat{z} = \hat{u}$) has already been established. Then the question of stability of the approximation in the neighborhood of the exact solution of the problem comes to the forefront, i.e., the question whether the principle of regularization expounded in Chap. 2 (Sec. 3.2) is satisfied. It follows from Chap. 1 that $z = A^{-1}\tilde{u}_\delta$, where A^{-1} does not satisfy the regularization principle; here A^{-1} (whose existence is assumed) is some operator of the classical (exact) inversion of Eq. (43) with approximate data. We have seen, however, that the reduction of Eq. (43) to being well-posed or generalized well-posed results in the possibility of obtaining an operator whose values on $\tilde{u}_\delta$ are stable in the neighborhood $\hat{z}$. It is any classical operator of the inversion problem in the new posing.

DEFINITION 1. The operator $z_\delta = R_\delta(\tilde{u}_\delta, A)$ is called regularizing (in the sense of Tikhonov) for the problem (43) in the metric space Z (with respect to the element $\hat{u}$) if (i) it is defined for some δ_0 for arbitrary δ ($0 < \delta < \delta_0$), and for arbitrary $\tilde{u}_\delta$ for which $\rho_U(\tilde{u}_\delta, \hat{u}) \leqslant \delta$, and (ii) z_δ satisfies in the metric Z the principle of regularization: $\rho_Z(z_\delta, \hat{z}) \to 0$ as $\delta \to 0$.

We have seen above that regularizing operators exist. However, the just-introduced concept is not connected with the manner in which the operator is constructed, and we shall see in the following that for each concrete problem more than one operator of this type may exist.

The choice of the regularizing operator (RO) is determined mainly by considerations of "economy" in the solution of inverse problems of mathematical physics. (Here the concept of economy is analogous to that of Ref. 125 and has to do with the use of computer time.)

1.2. Types of regularizing operators

As already noted (Chap. 1), there are two possibilities for overcoming the instability of problem (43): either the corresponding correct posing, or a "regularized" correction of the classical algorithm for its solution. The fundamental RO concept embraces both these possibilities.

Let us agree, first of all, to give the name "general regularizing operators" to those whose construction involves the element of well-posing (generalized well-posing) of the problem corresponding to Eq. (43). As is clear from the preceding, such posings include broad classes of inverse problems. An arbitrary algorithm for the solution of such a problem is regularizing (by definition, given in Chap. 2).

Regularizing operators obtained by correcting the classical ones, without preliminary analysis of the posing of the problem, will be given the name "adaptive" (ARO).[a] As a rule, ARO are connected with the specifics of the concrete problem and, as we shall see, are constructed in general for problems with linear, explicitly given operators A. The latter is natural, since it is precisely for such problems that classical inversion procedures have been developed. This type of operator will be considered in Sec. 3.

One of the simplest examples is provided by the problem on the extension of the potential in the direction of the sources (Chap. 1, Sec. 2.4), if the solution is sought in the form of a Fourier series. Due to inaccuracies in $\tilde{u}$ such a series diverges. However, as is shown in Refs. 13 and 148, for a given δ the number $n = n(\delta)$ of the partial sum S_n can be chosen so that $S_{n(\delta)} \to \hat{z}$ as $\delta \to 0$; obviously $S_{n(\delta)}$ exists for arbitrary δ. Consequently, the indicated algorithm, including the method of correlating n and δ, is regularizing by definition. Moreover it is adaptive because (a) it corrects the classical (ineffective) algorithm with the help of the choice of $n(\delta)$ and (b) it makes use of the specifics of the problem (not for every kernel can a system of eigenfunctions be explicitly constructed).

It follows from the results in Chap. 2 that in the well posing of inverse problems it is helpful to introduce the auxiliary parameter α. Then also the corresponding resolving operator turns out to be dependent on this parameter. Auxiliary parameters may also be introduced in the construction of ARO. Thus in the example discussed above, instead of the controlled replacement of the Fourier series by its partial sum one could introduce a multiplier $\gamma_n(\alpha)$ for each term of the series,[148] ensuring convergence of the series in the region $\alpha > 0$ (existence of the operator). In such cases construction of the ARO reduces to correlating α with δ, so that the resultant approximation satisfies the regularization principle.

Definition 2. The operator $z^\alpha = R(\tilde{u}_\delta, A)$, dependent on the parameter α, is called regularizing in the sense of Tikhonov in Z for the problem (43) (with respect to $\hat{u}$) (i) provided it is defined for all α $(0 < \alpha \leqslant \alpha_0$ for some $\alpha_0)$ and arbitrary $\tilde{u}_\delta$ for which $\rho_U(\tilde{u}_\delta, \hat{u}) \leqslant \delta$, and (ii) there exists a dependence $\alpha = \alpha(\delta)$ such that $z^{\alpha(\delta)}$ satisfies the regularization principle in the metric Z.

In what follows such RO will be called α-parametric. Let us note that it is also conventional to call α the "regularization parameter." However, it is not

[a]This term was introduced by V. N. Strakhov[138] in application to certain algorithms, in the sense of "adaptation" to the specifics of the initial data of the problem.

hard to see that such a term has a broader meaning: any RO depends on some parameter, since it depends on the degree of accuracy; in the example above, such a parameter is the number n of terms in the series.

A sufficient sign of an α-parametric regularizing operator $z = R(u, \alpha)$, useful for the construction of RO, was established in Ref. 149.

THEOREM. Let A be an operator from Z into U and $R(u, \alpha)$ be an operator from U into Z, defined for every $u \in U$ and arbitrary $\alpha > 0$. Then $R(u, \alpha)$ is a regularizing operator for the equation $Az = u$ provided (a) $R(u, \alpha)$ is continuous in u for each $\alpha > 0$ and (b) for arbitrary $z \in Z$ we have $\lim_{\alpha \to 0} R(Az, \alpha) = z$.

From the conditions of the theorem it is enough to prove that $z_\alpha = R(\tilde{u}_\delta, \alpha)$ satisfies the regularization principle for some $\alpha = \alpha(\delta)$. Let

$$\hat{u} = A\hat{z}, \quad \rho_U(\tilde{u}_\delta, \hat{u}) \leqslant \delta \quad \text{and} \quad \bar{z}_\alpha = R(\hat{u}, \alpha).$$

Then

$$\rho_Z(\bar{z}_\alpha, \hat{z}) \equiv \rho_Z(R(A\hat{z}, \alpha), \hat{z}) \to 0$$

as $\alpha \to 0$ independently of the choice of δ by condition (b).

This means that for arbitrary $\epsilon > 0$ we can find α_ϵ such that $\rho_Z(\bar{z}_{\alpha_\epsilon}, \hat{z}) < \delta/2$. On the other hand,

$$\rho_Z(z_{\alpha_\epsilon}, \bar{z}_{\alpha_\epsilon}) \equiv \rho_Z(R(\tilde{u}_\delta, \alpha_\epsilon) R(\hat{u}, \alpha_\epsilon)) \leqslant \omega_{\alpha_\epsilon}(\delta) \to 0$$

as $\delta \to 0$ by condition (a) of the theorem. That means there exists δ_ϵ such that $\rho_Z(z_{\alpha_\epsilon}, \bar{z}_{\alpha_\epsilon}) < \epsilon/2$, provided only that $\rho_U(\tilde{u}_{\delta_\epsilon}, \hat{u}) \leqslant \delta_\epsilon$. Hence, provided the last inequality is satisfied, we have

$$\rho_Z(z_{\alpha_\epsilon}, \hat{z}) \leqslant \rho_Z(\bar{z}_{\alpha_\epsilon}, \hat{z}) + \rho_Z(\bar{z}_{\alpha_\epsilon}, z_{\alpha_\epsilon}) < \epsilon .$$

This proves the theorem since ϵ was arbitrary.

Recently, along with the study of operators regularizing with respect to the metric of some space in the sense of Definitions 1 and 2, consideration has been given to operators for which $\rho(z_\delta, \hat{z})$ converges to zero "probabilistically"[b] on a certain statistical ensemble of approximations as the error measure tends to zero (see also Sec. 3.1).[37,156] Such operators will be called "stochastically regularized" in the sense of Tikhonov.

As was noted in Chap. 2, Sec. 2.5, generalized well-posings of problems based on the regularization concept also turn out to be effective for problems of the synthesis or control type. The mathematical formulation of such a problem reduces typically to the selection of some element from a set of acceptable ones ($z \in \hat{Z}$), satisfying the inequality $\rho_U(Az, \hat{u}) \leqslant \delta_0$, where $\hat{u}$ is the expected effect and δ_0 is the given tolerance.

Any operator that makes possible a unique choice of "solution" of this problem will be called "conditionally regularizing" (for problems of synthesis or control type).[36] The construction of conditionally regularizing operators may also depend on either the general framework (Chap. 2) or the specifics of the problem (Sec. 3).

We turn next to the study of certain concrete regularizing operators.

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[b]A sequence $\rho_n \equiv \rho_Z(z_n, \hat{z})$ is called convergent to zero probabilistically in P, or stochastically ($\rho_n \to 0$), if for arbitrary $\epsilon > 0$ one has $\lim_{n \to \infty} P(\rho_n < \epsilon) = 1$.

2. The general regularizing operator of A. N. Tikhonov

2.1. Basic construction

Under Tikhonov's regularizing operator we shall understand the totality of operations $z^\alpha = R(\tilde{u}_\delta, \alpha)$, $\alpha = \alpha(\delta)$ which solve the problem of minimization of the smoothing functional (24).

It is obvious that it refers to parametric operators of a general type and indeed encompasses a rather broad class since the basic element of the construction—Tikhonov's smoothing functional—is restricted by neither the dimension of the spaces Z and U, nor the character of the operator A, which can be both nonlinear and specified only implicitly.

Examples of use of such an operator in all of the above situations in problems of interpretation of data on physical observations can be found in Refs. 34, 43, 47, 49, 50–53, 60, 101, 121, 155, 159, and 164, devoted to problems of geophysics. In other branches of physics, the basic construction was used, in particular, in Refs. 17, 32, 117, 152, 176, and 178 for the solution of problems of interpretation or control.

In constructing $R(\tilde{u}_\delta, \alpha)$ for concrete inverse problems one needs first of all (i) to formulate the mathematical model for the direct correspondence (43) (i.e., display the operator A); the solution of the resultant problem with respect to z is not necessary (!); and (ii) choose a stabilizer in conformity with the character of the *a priori* information about the solution (Chap. 2).

It then remains to complete the construction by (a) indicating the method of selection of $\alpha = \alpha(\delta)$, corresponding to the regularization principle (this need not be explicit; an algorithm is sufficient, and (b) choosing the method of minimization of the smoothing functional.

Let us discuss in order some general possibilities for the solution of these last two questions.

2.2. The choice of regularizing parameter by the discrepancy principle

Departing somewhat from generally accepted terminology, we shall understand "discrepancy principle" to mean the totality of methods for selecting the parameter, based on the natural (for finite δ) practical criterion of validity of the approximations: coincidence of the observed effect ($\tilde{u}$) with the calculated one (Az) within the specified observations accuracy.

As was established in Chap. 2, the well-posing of the inverse problem with this criterion taken into account is possible [see Eqs. (22) and (23)].

Let us verify that the corresponding choice of $\alpha = \alpha(\delta)$ results in a regularized approximation within the framework (24).

Let z^α be the extremum of Tikhonov's smoothing functional (taken, for the sake of definiteness, to be unique[c]) and let δ be the given measure of error. Consider for the given δ the function $\varphi(\alpha) = \rho_U^2(Az^\alpha, \tilde{u}_\delta)$ to be the value of

[c] Sufficient conditions for the extremal to be unique are established in Ref. 148, where a more general case is discussed.

square of the discrepancy $[\rho_U(Az, \tilde{u})]$ on the extremum (z^α). It is calculated for each α, when the extremum is already known.

The selection of the regularizing parameter "by discrepancy" is determined, generally speaking, algorithmically by the equation

$$\varphi(\alpha) = \delta^2 . \tag{44}$$

Algorithms for numerical solution of this equation for concrete models are indicated in Refs. 57 and 100.

It is not hard to show that $\varphi(\alpha)$ is a monotonically nondecreasing function, bounded from below and (under certain conditions) above. Indeed,[99,148] with the notation $\psi(\alpha) \equiv \Omega(z^\alpha)$, we have for arbitrary $\alpha_1 < \alpha_2$ and corresponding extrema $z^{\alpha_1}, z^{\alpha_2}$ that

$$\begin{aligned}\varphi(\alpha_1) + \alpha_1\psi(\alpha_1) &\leqslant \varphi(\alpha_2) + \alpha_1\psi(\alpha_2)\\ &= [\varphi(\alpha_2) + \alpha_2\psi(\alpha_2)] + (\alpha_1 - \alpha_2)\psi(\alpha_2)\\ &\leqslant \varphi(\alpha_1) + \alpha_2\psi(\alpha_1) + (\alpha_1 - \alpha_2)\psi(\alpha_2) .\end{aligned}$$

Upon comparison of the first and last expression in this chain we conclude that $(\alpha_1 - \alpha_2)\psi(\alpha_1) \leqslant (\alpha_1 - \alpha_2)\psi(\alpha_2)$, i.e., $\psi(\alpha_1) \geqslant \psi(\alpha_2)$; it then follows from the inequality $\varphi(\alpha_1) + \alpha_1\psi(\alpha_1) \leqslant \varphi(\alpha_2) + \alpha_1\psi(\alpha_2)$ that $\varphi(\alpha_1) \leqslant \varphi(\alpha_2)$, which establishes the monotonicity. Further, it is obvious that $\varphi(\alpha) \geqslant 0$ is bounded from below.

Suppose, further, that $\hat{Z}$ and U are Hilbert spaces and $\Omega(z) \equiv \|z - z_0\|^2_{\hat{Z}}$, where z_0—a certain given element—is an extreme point of $\Omega(z)$. Then for arbitrary α we have

$$M^\alpha(z^\alpha, \tilde{u}_\delta) = \varphi(\alpha) + \alpha\psi(\alpha) \leqslant M^\alpha(z_0, \tilde{u}_\delta) = \rho^2_U(Az_0, \tilde{u}_\delta) . \tag{45}$$

From here it follows, in particular, that $\varphi(\alpha) \geqslant \rho^2_U(Az_0, \tilde{u}_\delta)$ is bounded from above.

From the established properties of the discrepancy follows the existence of its limits for arbitrary infinitely small or infinitely large sequences of values α. Indeed, it follows from Eq. (45) that for an infinitely large sequence $\{\alpha_p\}$, we have

$$\psi(\alpha_p) \leqslant \frac{1}{\alpha_p} \rho^2_U(Az_0, \tilde{u}_\delta) \to 0 ;$$

then $z^\alpha_p \to z_0$ in the metric of Z, since it is an element of the compactum containing z_0. Hence, by continuity, $\lim_{\alpha_p \to \infty} \varphi(\alpha_p) = \rho^2_U(Az_0, \tilde{u}_\delta)$. On the other hand, it was shown in Ref. 148 that if U_A is everywhere dense in U, then $\lim_{\alpha \to 0} \varphi(\alpha) = 0$ for arbitrary δ and $\tilde{u}_\delta$.

We have assumed that to each α corresponds a unique value of z^α—the extreme point of the functional. If the inverse representation $z^\alpha \Rightarrow \alpha$ is also single valued, then it is not hard to see that $\varphi(\alpha)$ is strictly monotonic. Sufficient conditions for strict monotonicity, as well as for continuity, of this function are established in Refs. 99 and 148. (Under the above-introduced assumptions, linearity of the operator is sufficient to this end.)

THEOREM. If U and $\hat{Z}$ are Hilbert spaces, $\Omega(z) = \|z - z_0\|^2_{\hat{Z}}$, $\varphi(\alpha)$ $(0 < \alpha < +\infty)$ is strictly monotonic and continuous (semicontinuous at $\alpha = 0$), and $0 < \delta < \rho_U^2(Az_0, \tilde{u}_\delta)$, then for every δ and arbitrary $\tilde{u}_\delta$ there exists a unique regularized approximation, selected by discrepancy.

Proof. Existence and uniqueness follow, obviously, from the above-established properties on the conditions of the theorem (Fig. 6). Let us verify that z_δ satisfies the regularization principle. Indeed, suppose that α_δ is a value selected by discrepancy; then $\varphi(\alpha_\delta) = \delta^2 \to 0$ as $\delta \to 0$ and, consequently, $\rho_U(Az_\delta, A\hat{z}) \to 0$ as $\delta \to 0$. On the other hand,

$$\varphi(\alpha_\delta) + \alpha_\delta \psi(\alpha_\delta) = \delta^2 + \alpha_\delta \Omega(z_\delta) \leqslant M^{\alpha(\delta)}(A\hat{z}, \tilde{u}_\delta) \leqslant \delta^2 + \alpha_\delta \Omega(\hat{z})\,;$$

this means that $\Omega(z_\delta) \leqslant \Omega(\hat{z})$, i.e., that z_δ and $\hat{z}$ belong to the same compactum. From this, it follows by Tikhonov's theorem (Chap. 2) that $\lim \rho_Z(z_\delta, \hat{z}) = 0$ as $\delta \to 0$, as was to be proved.

We remark that the upper bound on δ in the conditions of the theorem is quite natural. Suppose, for example, that $z_0 = 0$; then $\rho_U^2(Az_0, \tilde{u}_\delta) = \|\tilde{u}_\delta\|^2$. Should we now have $\delta > \|\tilde{u}_\delta\|$, this would mean that the error exceeds the useful information about the object, contained in $\tilde{u}_\delta$, and the problem becomes meaningless.

For problems for which the operator A is nonlinear, $\varphi(\alpha)$ need be neither strictly monotonic nor continuous (Fig. 6b). To deal with this case there is described in Ref. 42 a "generalized discrepancy principle," when α_δ is selected by the condition

$$\alpha_\delta = \inf_{\varphi(\alpha) \geqslant \delta} \alpha\,. \tag{46}$$

We note that in practice both methods are equivalent, since for finite δ the selection of the approximation proceeds on some sequence of values $\{\alpha_s\}$.

For numerical realization of the algorithm the operator A is itself given with some error, and if its effect is comparable to the error in $\tilde{u}$ then this should be taken into account in selecting the regularizing parameter. A method to this end is described in Refs. 57 and 148, applicable to problems with a linear operator A.

Let Z and U be real Hilbert spaces and suppose that an estimate of the quantity

$$h = \sup_Z \frac{\|\tilde{A}_{hz} - Az\|}{\|z\|_Z} U$$

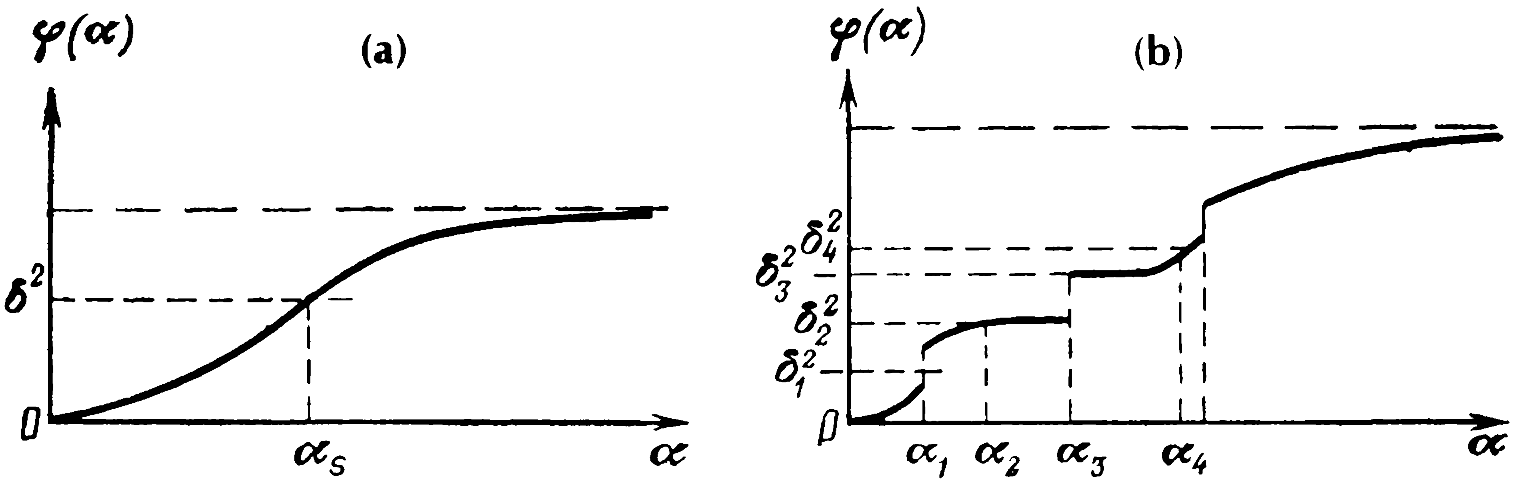


Figure 6

is known for the approximate operator $\tilde{A}_h$. Then the relative error of the operator is given by the formula $\|\tilde{A}_h z - Az\| \leqslant h\|z\|$ for arbitrary $z \in Z$. In that case for the smoothing functional, corresponding to the problem $\tilde{A}_h z = \tilde{u}_\delta$, $z \in Z$, with previous estimate of the error in $\tilde{u}_\delta$, the correlation of α with (h, δ) proceeds via the equation

$$\varphi(\alpha) - (\delta + h\|z\|)^2 = \mu \,, \tag{47}$$

where $\mu = \inf_Z \rho^2(\tilde{A}_h z, \tilde{u}_\delta)$. The left-hand side of Eq. (47) is called[57] the generalized discrepancy. Iterative processes for the calculation of μ are discussed in Ref. 56.

In this case, by regularized approximation is understood z^σ, which satisfies the condition $\rho_z(z^\sigma, \hat{z}) \to 0$ as $\sigma \to 0$ [translator's note: the original has δ everywhere instead of σ], provided $\sigma = \max(h, \delta)$. A more general formulation of the problem of selection of α in the presence of errors in the operator and corresponding methods are discussed in Ref. 148.

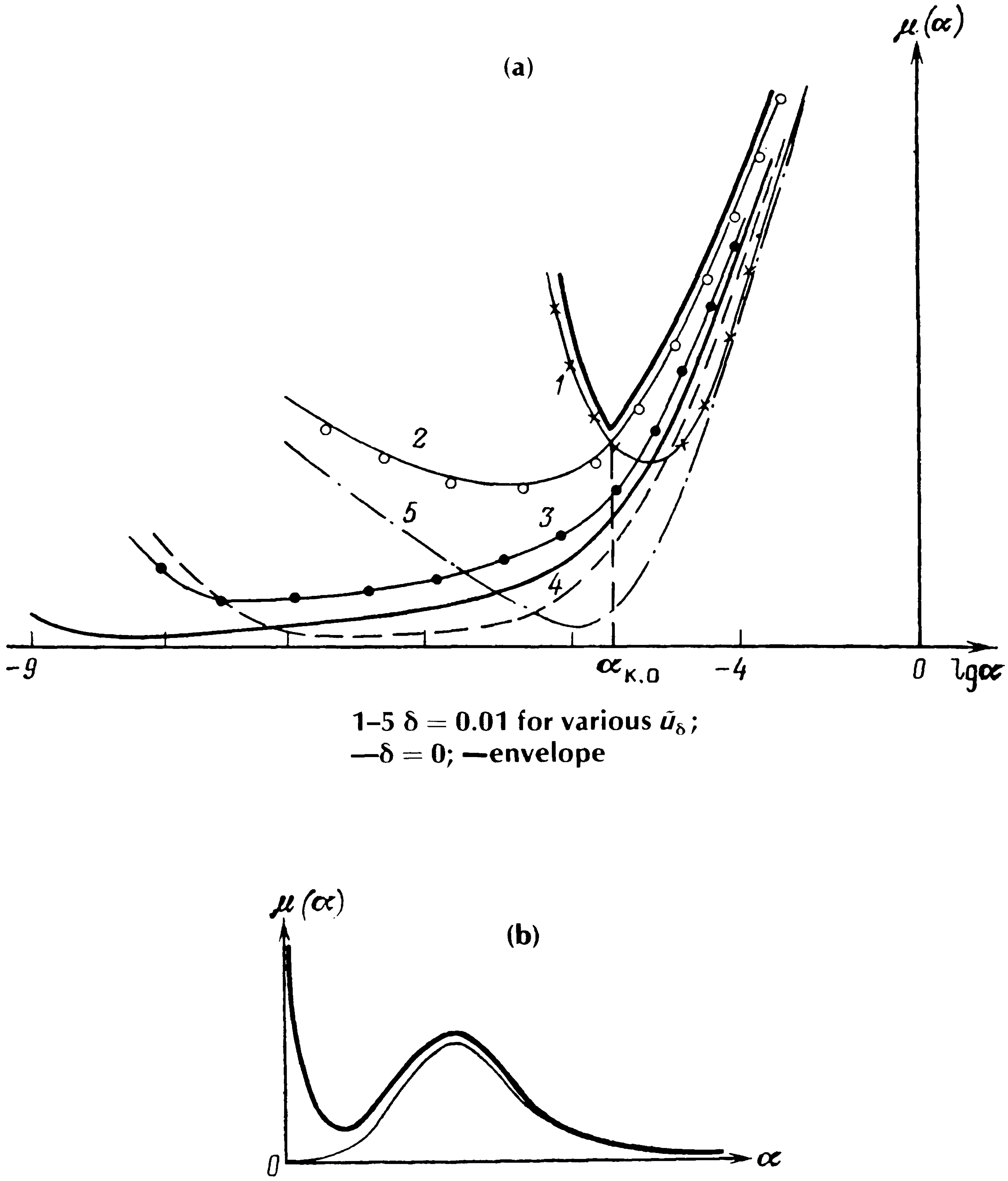
We remark that the discrepancy principle of Eqs. (44) and (46) can also be extended to adaptive regularizing operators.

2.3. The quasi-optimal method of parameter selection

Another approach to the question of selection of the regularizing parameter is based more on the mathematical nature of inverse problems as ill-posed and a mental comparison of the exact solution of the problem with its "random" approximation. It can already be noted in the elementary posing of the inverse problem (Chap. 1) that if the input characteristic $\tilde{u}$ carries a random error, then the probability that it belongs to the set U_A is arbitrarily close to zero and, therefore, there is a gap between the exact solution of the problem and the mental random approximation to it; the latter is simply absent with a probability arbitrarily close to unity. In this respect the situation is completely analogous to what happens when attempting to obtain a rational number by the process of division by a rational of an arbitrarily chosen real.

This nature of the ill-posed problem suggests that things will be the same with respect to the approximation to $\hat{z}$, obtained by means of some α-parametric algorithm, if the question of correlating α with the random error of the problem is ignored: $\rho_Z(z^\alpha, \hat{z}) \to \infty$ as $\alpha \to 0$ with probability close to unity. At the same time, in the absence of any errors the corresponding limit should equal zero. Indeed we have, for example,[148] for the extreme point of Tikhonov's smoothing functional for $\delta = 0$ ($z^\alpha = \bar{z}^\alpha$) that $\rho^2(A\bar{z}^\alpha, \hat{u}) + \alpha\Omega(\bar{z}^\alpha) \leqslant \alpha\Omega(\hat{z})$, which shows that $\bar{z}^\alpha$ belongs to the same compactum as $\hat{z}$ and $\rho^2(A\bar{z}^\alpha, A\hat{z}) \to 0$ as $\alpha \to 0$; this is sufficient for the convergence of $\bar{z}^\alpha$ to $\hat{z}$.

If, in turn, α were correlated with the error in accordance with the regularization principle, it would follow for sufficiently small δ that $\rho^2(Az^{\alpha(\delta)}, \hat{z})$ is close to zero also for arbitrarily small α_δ. Thus one may assert with "high probability" the existence of a "critical value" $\alpha = \alpha(\delta)$, corresponding to the exact lower bound of values of $\nu(\alpha) \equiv \rho_Z^2(z^\alpha, \hat{z})$ as $\alpha \to 0$, provided z^α is the result of the action of the regularizing parametric operator in the presence of errors, but without the errors being correlated with α. The approximation corresponding to the indicated minimum, is optimal in the sense of the problem. It



1–5 δ = 0.01 for various $\tilde{u}_\delta$;
—δ = 0; —envelope

characteristic behavior of μ(α)—δ ≠ 0;
—δ = 0

Figure 7

remains to indicate a certain indirect characteristic of the quantity $\nu(\alpha)$, since $\hat{z}$ is unknown, and, once that is done, one may speak of a "quasi-optimal" approximation. In Ref. 153 such a characteristic was taken to be a certain norm of the quantity $\alpha\, \partial z^\alpha/\partial\alpha$, correlated with the first term in the Taylor-series expansion of the difference $\bar{z}^\alpha - \hat{z}$ in the absence of errors.

DEFINITION 1. (a) The quantity $\hat{\mu}(\alpha) = \sup_{U_\delta} |\alpha \partial z^\alpha/\partial\alpha|_Z$ will be called the quasi-optimal measure on the equivalence set $U_\delta : \rho_U(\tilde{u}_\delta, \hat{u}) \leqslant \delta$; (b) the quantity $\mu(\alpha) \equiv |\alpha \partial z^\alpha/\partial\alpha|_Z$ will be called the weakly quasi-optimal measure for the concrete given $\tilde{u}_\delta$ [translator's note: original has here $\tilde{u}_s$ instead.][d]

In Fig. 7(a) are shown the results of a numerical experiment,[153] giving an idea about the behavior of $\mu(\alpha)$ and $\hat{\mu}(\alpha)$ [the "envelope" curve is for the family

[d]An elementary modification of these concepts is admissible such as, for example, the replacement of Z by any of its subsets[148,152,156]; in the following we do not restrict these concepts to concrete forms of subsets.

$\mu(\alpha)$, corresponding to various specifications of $\tilde{u}_\delta$]. Characteristic behavior of $\mu(\alpha)$ is shown in Fig. 7(b). The experiment was performed on the problem of determination of the boundary form separating two media in the Earth's bowels, which gives rise to a nonlinear integral equation. In this case $\hat{u} = \hat{u}(x)$, $x \in [a, b]$, and its various values were imitated by random numbers generated by an electronic computer: $\tilde{u}_\delta(x) = \hat{u}(x) + \delta\xi(x)$, where $\xi(x)$ is a random function such that $\rho(u_\delta, \hat{u}) \leqslant \delta$. Analogous behavior of $\mu(\alpha)$ in a certain region $0 < \delta \leqslant \delta_0$ was established in a broad class of problems using the quasi-optimal selection method.[34,45,49,51,152,164] This fact is confirmed by a rigorous mathematical analysis, based either on the expounded-above "statistical" considerations[37,156] or (for a particular problem, Sec. 3.2) on the equivalent functional apparatus.[91]

It has been established that for a broad class of problems of the interpretation type, an α_0 exists such that for $\alpha > \alpha_0$ one has $\mu(\alpha)$ decreasing monotonically [$\mu(\alpha) \to 0$ as $\alpha \to \infty$].

DEFINITION 2. We call $\alpha = \alpha_{qo}$ the quasi-optimal value for concrete specification of $\tilde{u}$ provided $\mu(\alpha_{qo}) = \inf \mu(\alpha)$, $0 < \alpha \leqslant \alpha_0$. The corresponding selection α is called the weakened quasi-optimal criterion.

The concept of the "quasi-optimal criterion" in the proper sense, i.e., on the ensemble $\tilde{u}$, is introduced analogously. However, the weakened criterion is more convenient in practical applications. One should also keep in mind that the indicated behavior of $\mu(\alpha)$ refers to a region of values of δ bounded from above. If δ is "excessively large," $\mu(\alpha)$ may turn out to be a monotonically nonincreasing function [$\mu(\alpha) \to 0$ as $\alpha \to \infty$, $\mu(\alpha) \to \infty$ as $\alpha \to 0$]. In this respect the situation is the same as for the "discrepancy principle" for $\delta > \|\tilde{u}\|$.

We now call attention to the fact that the weakened quasi-optimal criterion involves no *a priori* information on the quantity δ and, therefore, may be used in those cases where such an estimate is unknown.[153]

On the other hand, in the practical realization of the regularizing algorithm one unavoidably includes errors of the operator A, of the calculation of the minimizing element, and of the roundings-off on the electronic computer. The quasi-optimal criterion "summarizes" the effect of all these errors.

It was established in Refs. 37, 91, 148, and 156 that the "quasi-optimal approximation" is regularized. In particular, the following theorem, based on a statistical modeling of the aggregate error, is valid:

THEOREM. Let A be a linear operator and $\Omega(z) = \|z\|^2_{\hat{Z}}$. Then one can find a class of models of aggregate error in $\tilde{u}$ and calculations, characterized by dispersion σ, such that (a) as $\sigma \to 0$ we have with unit probability that $\alpha_{qo} \to 0$ and $\mu(\alpha_{qo}) \to 0$ and (b) $\rho_Z(z^{\alpha_{qo}}, \hat{z})$ converges stochastically to zero as $\sigma \to 0$. (The concept of stochastic convergence was described in Sec. 1.)

This means that under the indicated conditions Tikhonov's generalized RO for quasi-optimal choice of the approximation is, at the very least, stochastically regularizing. The analogous fact is also established for problems involving nonlinear operators. This assertion also extends to a series of "adaptive" RO[156]; in the latter case the operator is defined as regularizing in the sense of the basic definition (Sec. 1).[37] In practice the approximation is taken on some sequence of values $\{\alpha_s\}$. If this is a geometric progression [$\alpha_s = \lambda\alpha_{s-1}$ $(0 < \lambda < 1)$], then, as is easily noted, the quasi-optimal measure is "approximated" by the simple formula $\tilde{\mu}(\alpha_s) = \|z^{\alpha_s} - z^{\alpha_{s-1}}\|$, which is convenient for calculations. Algorithms are also given for the calculation of the quasi-optimal measure in Ref. 68, based on the reduction of the minimization problem to the Euler equation (see Sec. 6).

The "ratio criterion," proposed in Ref. 68, is similar in its mathematical nature to the quasi-optimal one. In that case $\alpha = \alpha_{\text{ratio}}$ is defined by the condition sup $[\mu(\alpha)/\varphi(\alpha)]$ in the region $0 < \alpha < \alpha_0$ for some α_0.[68] The experimental and analytical definition of this criterion may be found in Refs. 37, 68, and 148.

2.4. The parameter-descent method for Tikhonov's smoothing functional

One of the questions in the construction of general-type RO, connected with the variational formulation of the problem, is the method of minimization of the functional, and A. N. Tikhonov's operator is no exception in this regard. The point is that, as is also the case in the "quasi-solution" concept, one wishes to reach the exact lower bound of the functional, and not one of its local minima. These concepts usually coincide for inverse problems with a linear operator A, and in that case the smoothing functional plays a decisive role, being "strictly convex" for $\alpha > 0$ due to the choice of the stabilizer.[56] This assertion cannot be made in the case of a nonlinear operator.

It turns out that Tikhonov's α-parametric smoothing functional may fulfill one more function, related to the construction of a stable "absolutely minimizing" sequence $\{z^{\alpha_s}\}$.

We call attention to the fact that the convergence of any of the well-elaborated iteration processes, as well as the rate of convergence, depend, generally speaking, on the choice of the initial approximation.[75,76] For the smoothing functional, viewed on the sequence $\{\alpha_s\}$, it is possible to make a rational choice.

Indeed, as was noted in Sec. 2.2, for the case when $\Omega(z) = \|z - z_0\|_Z^2$, the extreme point $z^\alpha \to z_0$ as $\alpha \to \infty$. This makes possible the realization of the following procedure.[43,132,153,159] Let $\{\alpha_s\}$ be an infinitely small sequence for a sufficiently "large" value of α_0 $(s = 0)$. Choose as the initial approximation to z^{α_0} the quantity z_0; it belongs to the neighborhood of the extreme point and therefore any classical iteration process will converge to z^{α_0} for such an initial approximation. For $\alpha = \alpha_1 < \alpha_0$ choose as the initial approximation z^{α_0} and use the same iteration process; if α_1 is not "too different" from α_0, we remain in a situation favorable to the convergence of the process to z^{α_1}. Clearly induction is possible. We shall refer to this procedure as "parameter descent" for the smoothing potential.[e]

The parameter descent is interrupted by one of the criteria of selection of α from the sequence $\{\alpha_s\}$, thus closing the construction of the regularizing algorithm.

Practice shows that even for a "not too painstaking" choice of $\{\alpha_s\}$ the described procedure is effective. On the other hand, the result is independent

[e] This terminology was used previously[172] in a different context: the calculation of solutions of "correct" operator equations from a known solution of another equation, taken as the initial approximation.

of the choice z_0 (in the stabilizer) in a sufficiently large range of values, validated by the available *a priori* information.[31,42,50,51,60,135]

This last fact is easily illustrated in the example of interpretation of gravitational field data to determine the depth and shape of the boundary dividing two media of differing density in the Earth's bowels (contact surface).[51] The result of the mathematical experiment is given in Fig. 8. It is seen that for a given initial depth and shape, substantially different from the true ones, the latter are reproduced with an accuracy sufficient for practical purposes. In this experiment (as in many other calculations) we have chosen for $\{\alpha_s\}$ a geometric progression with ratio $\lambda = 0.1$ and $\alpha_0 = 1$; for $\delta = 0.01 = 5\%$ max $\tilde{u}$, there results $\alpha_{qo} = 10^{-3}$.

It turns out further that the parameter-descent procedure admits "economization" (by one iteration for each α_s) in the case when the operator is "weakly nonlinear." This possibility—the so-called diagonal process—was studied in Refs. 42 and 132.

2.5. The Gauss–Newton procedure for minimization of the smoothing functional

The choice of the minimization method $M^\alpha(z, \tilde{u})$, with the parameter-descent procedure taken into account, is not the principal problem. Nonetheless it is worth calling attention to the fact that the specifics of the main element of the smoothing functional in interpretation problems with nonlinear A [discrepancy $\rho^2(Az,\tilde{u})$] make convenient the "linearization" of A in the neighborhood

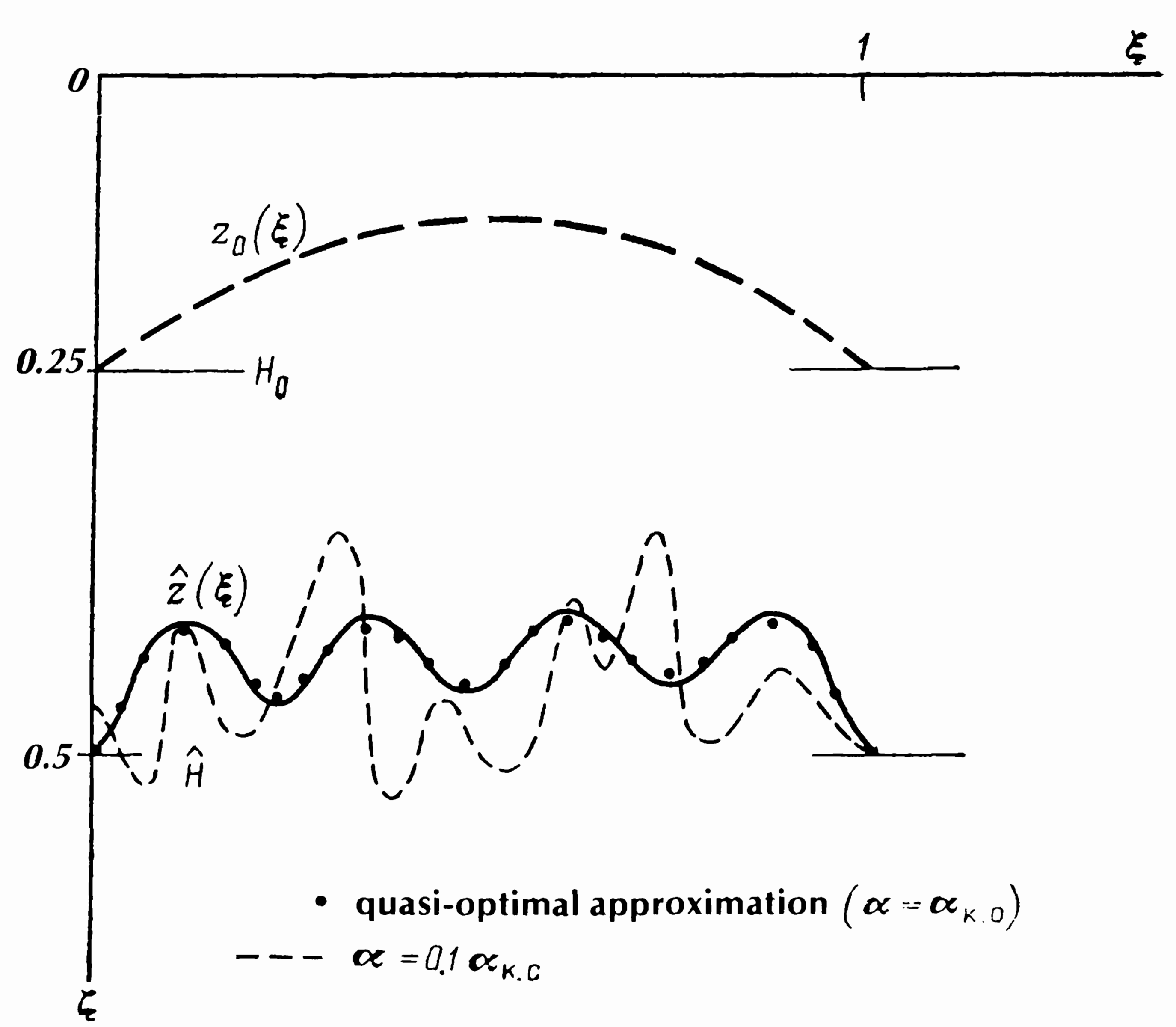


Figure 8

of the sequential (for given α) approximation z_n. In that case one may use for the determination of z_{n+1} minimization methods for quadratic functions.[20,75,98]

Indeed, suppose that the nonlinear operator A is represented for arbitrary $z, z_n \in \hat{Z}$ in the form

$$A(z) = A(z_n) + A'_z(z_n)(z - z_n) + o(\|z - z_n\|), \tag{48}$$

where $A'_z(z_n)$ is a certain operator called the Frechet derivative,[76,77] and $A'_z(z_n)(z - z_n)$ is the "Frechet differential" at the point z_n.

Introducing the notation $B \equiv A'_z(z_n)$ and $\tilde{v}_n \equiv \tilde{u}_n - A(z_n) + A'_z(z_n)z_n$, we arrive for a predetermined z_n at the problem of minimization of the functional $\rho_U^2(Bz, \tilde{v}_n) + \alpha\Omega(z)$, which is quadratic provided

$$\Omega(z) = \|z - z_0\|_Z^2 .$$

In a concrete problem there is usually no doubt as to the linearizability of A since the Frechet differential can be found by the "perturbation method" and, moreover, is given explicitly even when the operator A is known only implicitly. Let us consider as an example the operator for the direct correspondence in the MTS problem (Chap. 3, Sec. 2). According to Eq. (5) we have $Az = v'_\zeta(0, \omega)$, where $v'(0, \omega)$ is determined (implicitly) by the boundary-value problem $v'' + i\omega\sigma(\zeta)v = 0$, $v(0) = 1$, $v'(h) = -\varkappa v(h)$.

Upon setting

$$v' = pv \quad \left(v = \exp \int_0^\zeta p(y)dy \right),$$

we obtain $Az = p(0)$, $p' + p^2 + i\omega\sigma(\zeta) = 0$, $p(h) = \varkappa$. Suppose that $\sigma_n(\zeta)$ and $p_n(\zeta)$ $[v_n(\zeta)]$ are already known. Setting $\sigma = \sigma_n + \eta, p = p_n + w$ and viewing η and w as small, we find that $Az = p_n(0) + w(0)$, where $w(0)$—the linear part of the operator increment—is the Frechet differential. The latter, obviously, is determined from the conditions $w' + 2p_n w = -i\omega\eta$, $w(h) = 0$ (with terms $\sim w^2$ being ignored). Then, with the help of the variation-of-constants method and taking into account the relation between p and w, we obtain

$$w(0) = i\omega \int_0^h v_n^2(\zeta)\eta(\zeta)d\zeta .$$

Correspondingly

$$B\sigma \equiv i\omega \int_0^h v_n^2(\zeta)\sigma(\zeta)d\zeta ,$$

where $v_n(\zeta)$ is determined algorithmically, once $\sigma_n(\zeta)$ has been found, by the above-indicated boundary-value problem.

Linearization of the operator in the discrepancy functional (10) is referred to in the contemporary scientific literature as the Gauss–Newton procedure.[20,132] Questions of realization of the parameter-descent method and the Gauss–Newton procedure are discussed in detail in Ref. 132. *A priori* linearization of A allows the utilization of the Euler operator[98] in the search for z^α for each α.

2.6. The gradient of Tikhonov's smoothing functional and use of the Euler equation for minimization

The formal search method[75] is most universal for the minimization $M^\alpha(z, \tilde{u})$ for fixed α, applicable as an element of the parameter-descent procedure. In a number of cases[158,183] it may also turn out to be sufficiently "economical." However, "gradient" methods are more economical.[75]

Under the gradient of the functional $\Phi(z)$ is understood[20,77] the operator $Cz \equiv \Phi'(z)$, provided the following representation is valid: $\Phi(z + \Delta z) = \Phi(z) + [\Phi'(z), \Delta z] + o(\|\Delta z\|)$. In the case that such a representation is valid for arbitrary z, $\Delta z \in Z$, $\Phi(z)$ is called differentiable in Z, and $[\Phi'(z), \Delta z]$ is called the differential of Φ. It is obvious upon setting $\Delta z = t\zeta$, where t is a numerical parameter and ζ an arbitrary fixed element, that the differential of Φ can be expressed by the formula $[\Phi'(z), \Delta z] = (d/dt)[\Phi'(z + t\zeta)]|_{t=0}$. In practice the differential is calculated by "perturbation methods," similarly to the differential of the nonlinear operator A.

From here it is not hard to obtain the general expression for the gradient of the smoothing functional with nonlinear operator A in the neighborhood of the approximation z_n:

$$\tfrac{1}{2} M'_\alpha(z) = A'^{*}_{z}(z_n)(A(z_n) - \tilde{u}_\delta) + \alpha L z_n \; ; \tag{49}$$

$2Lz_n$ is the gradient of the stabilizer, and $A'^{*}_{z}(z_n)$ is the operator conjugate to the Frechet derivative.[76,77] In the case that Z and U are Hilbert spaces and $\Omega(z) = \|z\|^2_{\hat{Z}} = (z, z)_{\hat{Z}}$, the operator A'^{*}_{z} is determined by the formula

$$(A'^{*}_{z} v, z)_Z = (v, A'_z z)_U$$

for arbitrary $v \in U$ and $z \in Z$. Analogously L is determined by the formula $(z, \zeta)_{\hat{Z}} = (Lz, \zeta)_Z$ for an arbitrary pair $z, \zeta \in \hat{Z}$.

We note that if $\hat{Z}$ is a function manifold [for example, $z = z(x, y)$ $(x, y) \in D$ with boundary Γ], then L is a differential operator containing derivatives of higher order than $\Omega(z)$. Consequently the use of gradient methods is connected in this case with the extraction from the introduced compactum of a subset of functions, possessing the corresponding (classical) derivatives. Thus, for example, for the "simplified" conditional stabilizer (Chap. 2, Sec. 3.4)

$$\Omega(z) = \iint_D \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 dx\, dy, \quad z|_\Gamma = 0 \, ,$$

it is easily verified that with a scalar product conventionally defined for a function manifold we have $Lz = \partial^4 z/\partial x^2\, \partial y^2$ with the same boundary condition.

The indicated "narrowing" of the compactum is guaranteed not to violate the regularizing properties of Tikhonov's operator only in the case when the sought-for exact solution belongs to the corresponding subset of $\hat{Z}$ and, in particular, satisfies the boundary conditions, which are unavoidably used in the construction of L; otherwise the RO guarantees convergence to $\hat{z}$ "almost everywhere" (for example, with the exception of points where the higher derivatives are discontinuous and points on the region boundary), which usually is acceptable in concrete problems.

If the operator A is linear, or has been linearized by the Gauss–Newton procedure (Sec. 2.5), then a natural method for searching out the extremum for fixed α consists of going over to the Euler equation. As is well known,[148] we obtain such an equation by equating to zero the gradient of the functional. Consequently, for the smoothing functional with the operator linearized at the point z_n it has the form

$$(B^*(z_n)B(z_n) + \alpha L)z = B^*(z_n)\tilde{v}_n \ . \qquad (50)$$

If $\hat{Z}$ is a function manifold, then Eq. (50) explicitly includes boundary conditions for z, which must also be obeyed by the exact solution $\hat{z}$ in order that the RO guarantee uniform convergence of z^α to $\hat{z}$.

As an example let us consider the "Euler equation" for the search for the next approximation to the extremum for fixed α in the contact surface problem, treating as unknown also the depth of the "base" layer. To simplify the description, the problem will be considered within the framework of a two-dimensional model with known excess density $\Delta\rho$ (Fig. 8). In that case the space Z consists of the manifold of pairs $z \equiv \{w(\zeta), H\}$ with the scalar product

$$(z_1,z_2) = \int_a^b w_1(\xi)w_2(\xi)d\xi + H_2H_2 \ .$$

The quantity under observation—the ζ component of the gravitational field at the Earth's surface—is explicitly given in terms of z:

$$Az \equiv u(x,w,H) = \frac{\gamma\Delta\rho}{4\pi}\int_a^b \ln\frac{(x-\xi)^2 + H^2}{(x-\xi)^2 + [H - w(\xi)]^2}\,d\xi \ .$$

The result of linearizing u in the neighborhood of $z_n = (w_n, H_n)$ may be written in the form

$$B_n z \equiv B(z_n)z = \int_a^b K'_{nw}(x,\xi)w(\xi)d\xi + H\int_a^b K'_{nH}(x,\xi)d\xi \ ,$$

where K'_w and K'_H are derivatives of the "kernel" of the integral operator Az with respect to the corresponding arguments. Correspondingly $\tilde{v}_n(x) = \tilde{u}(x) - u(x, w_n, H_n) + B(z_n)z_n$. Since U is a function manifold, $u = u(z)$, $x \in [a, b]$, with a standard scalar product, we have for arbitrary $v(x)$

$$(v,B_n z)_U = \int_a^b \varkappa_{n1}(\eta,v)w(\eta)d\eta + H\varkappa_{n2}(v) \equiv (B_n^* v,z)_Z \ ,$$

where

$$\varkappa_{n1} = \int_c^d K_{nw}(x,\eta)v(x)dx, \quad \varkappa_{nz} = \int_a^b\int_c^d K_{nH}(x,\eta)v(x)dx\,d\eta$$

is determined by the expression for $B_n z$. Consequently, $B_n{}^*v = \{\varkappa_{n1}(\eta, v), \varkappa_{n2}(v)\} \in Z$. From this we easily determine $B_n{}^*(B_n z)$ and $B_n{}^*\tilde{v}_n$ for Eq. (50). Suppose it is known that $w(\xi)$ is a smooth function satisfying the conditons $w(a) = w(b) = 0$. Then we may introduce as the stabilizer

$$\Omega(z) = \int_a^b \left(\frac{dw}{d\xi}\right)^2 d\xi + H_2 \ .$$

$\hat{Z}$ is the manifold of pairs $\{w(\xi), H\}$ with the scalar product

$$(z,\zeta)_{\hat{Z}} = \int_a^b w_1'(\xi)w_2'(\xi)\delta\xi + H_1H_2, \quad w_s(a) = w_s(b) = 0$$

for $s = 1,2$. It is not hard to see that

$$(z_1,z_2)_{\hat{Z}} = w_1' w_2|_a^b - \int_a^b w_1'' w_2 \, d\xi + H_2H_2 = (Lz_1,z_2)_Z ,$$

provided

$$Lz_1 = \{ -w_1''(\eta),H_1\}, \; z_1 = w_1(\xi), \; z_2 = w_2(\xi) .$$

Comparing in accordance with Eq. (50) the components of the pair, we arrive at the following expression for the "Euler equation" for the problem under discussion:

$$\int_a^b K_{n11}(\eta,\xi)w(\xi)d\xi + K_{n12}(\eta)H - \alpha w''(\eta) = b_{n1}(\eta), \; w(z) = w(b) = 0 ,$$

$$\int_a^b K_{n21}(\xi)w(\xi)d\xi + K_{n22}H - \alpha H = b_{n2} ,$$

where

$$K_{n11}(\eta,\xi) = \int_c^d K'_{nw}(x,\eta)K'_{nw}(x,\xi)dx ,$$

$$K_{n12}(\eta) = \int_c^d \int_a^b K_{nw}(x,\eta)K_{nH}(x,\xi)dx \, d\xi ,$$

$$b_{n1}(\eta) = \int_c^d K_{nw}(x,\eta)\tilde{v}_n(x)dx ,$$

$$K_{n21}(\xi) = \int_c^d \int_a^b K_{nw}(x,\xi)K_{nH}(x,\eta)dx \, d\eta,$$

$$K_{n22} = \int_c^d \int_a^b \int_a^b K_{nH}(x,\eta)K_{nH}(x,\xi)dx \, d\xi \, d\eta ,$$

$$b_{n2} = \int_c^d \int_a^b K_{nH}(x,\eta)\tilde{v}_n(x)dx \, d\eta .$$

In this manner there corresponds to Eq. (50), in the example under discussion, a system of integro-differential (with specified boundary conditions) and integro-algebraic equations. Obviously, as a result of finite-difference approximation, it reduces to a system of algebraic equations with respect to the $(n+1)$th variable $\{w(\xi_1), \ldots ,w(\xi_n), H\}$, and inversion of the latter reduces to a standard electronic computer procedure.

3. Certain adaptive regularizing operators

3.1. Fourier regularizing operators

One of the methods for locating sources of anomalous gravitational fields in geoprospecting is based on extending the field from the Earth's surface towards the sources. Let the normal component of the gravitational field $v(x)$, $x = (\xi, \eta, \zeta)$, be measured on the surface $[v|_{\zeta=0} = \tilde{u}(x)]$ in a certain rectangle Π of sufficiently large dimensions so that on its boundary Γ we may set $(\partial\tilde{u}/\partial n)|_\Gamma = 0$. Then it is obvious that in a certain source-free layer $0 < \zeta < h$ beneath the surface, $v(x)$ satisfies the same condition and is there harmonic. Consequently a formal representation of $v(x)$ is possible in terms of a Fourier series over an orthogonal system: $\{\psi_n(y)\}$, $y = (\xi, \eta)$, where the ψ are eigenfunctions of the problem $\Delta_{\xi,\eta}\psi + \lambda\psi = 0$, $(\partial\psi/\partial n)|_\Gamma = 0$. Use of the classical method results in the following expression for $v|_{\zeta=h} \equiv w(y)$:

$$w(y) = \sum_{n=1}^{\infty} \tilde{u}_n e^{\sqrt{\lambda_n}h}\psi_n(y),$$

where λ_n are the eigenvalues of the boundary-value problem. It is obvious that for exact values of the Fourier coefficients $\hat{u}_n$ this series converges: $\hat{u}(x)$ is produced by sources located on the level $H > h$, and therefore $\hat{u}_n = O(e^{-(\lambda_n)^{1/2}H})$. However, in the presence of errors $u_n = \tilde{u}_n$ the series diverges and, consequently, is useless in the search for an approximation.

Let us introduce multipliers for the terms in the series, $\gamma_n(\alpha) = (1 + \alpha\lambda_n e^{2(\lambda_n)^{1/2}h})^{-1}$, obtained "heuristically," for example with the help of a construction analogous to Tikhonov's functional:

$$M^\alpha(w, \tilde{u}) = \rho^2_{L_2}(Aw, \tilde{u}) + \alpha\Omega_0(w),$$

where

$$\Omega_0(w) = \int_\Pi (\nabla w)^2 \, d\sigma$$

(it is not the stabilizer on the function manifold under consideration[f]). Indeed, taking into account Green's formula

$$\Omega_0(w) = -\int_\Pi w\Delta w \, d\sigma$$

and the fact that Aw is the result of recalculating the field from the level $\zeta = h$ to the level $\zeta = 0$, we obtain the representation

$$M^\alpha(w,\tilde{u}) = \sum_{n=1}^{\infty} (w_n e^{-\sqrt{\lambda_n}h} - \tilde{u}_n)^2 + \alpha \sum_{n=1}^{\infty} \lambda_n w_n^2.$$

The indicated expressions then follow from the condition $\partial M^\alpha/\partial w_n = 0$.

Let us consider the "regularized" Fourier series:

$$w_\alpha(y) = \sum_{n=1}^{\infty} \tilde{u}_n \gamma_n(\alpha) e^{\sqrt{\lambda_n}h}\psi_n(y) \equiv R_\Phi(\alpha, \tilde{u}_\delta). \tag{51}$$

THEOREM. The operator $R_\Phi(\alpha, \tilde{u}_\delta)$ is regularizing in the sense of Tikhonov on the manifold of functions continuous in Π.

[f] $\Omega_0(w)$ may be the conditional stabilizer, for example, on the manifold of sufficiently smooth functions satisfying the conditions $w|_\Gamma = 0$ (Chap. 2, Sec. 3).

Proof. For any $\tilde{u}\in L_2(\Pi)$ and any $\alpha > 0$, the series (51) obviously converges and, therefore, the operator is defined. It remains to verify that for any $\tilde{u}_\delta$ such that $\rho_{L_2}(\tilde{u}_\delta, \hat{u}) \leqslant \delta$, there exists a function $\alpha = \alpha(\delta)$ such that

$$\lim_{\delta\to 0} \max_{\Pi} |w_{\alpha(\delta)}(y) - \hat{w}(y)| = 0.$$

To this end we replace $\hat{w}(y)$ by the set of its Fourier coefficients $\{\hat{w}_n\}$ and take into account that $\hat{w}_n = O(e^{-(\lambda_n)^{1/2}H})$, where $H > h$ and $|\psi_n(y)| \leqslant 1$. Then for arbitrary α and δ we have

$$|w_\alpha(y) - \hat{w}(y)| = \left| \sum_{n=1}^{\infty} (\tilde{w}_n \gamma_n(\alpha) - \hat{w}_n) e^{\sqrt{\lambda_n} h} \psi_n(y) \right|$$

$$\leqslant \sum_{n=1}^{\infty} |\tilde{w}_n - \hat{w}_n| \gamma_n(\alpha) e^{\sqrt{\lambda_n} h} + \sum_{n=1}^{\infty} |\hat{w}_n| |\gamma_n(\alpha) - 1| e^{\sqrt{\lambda_n} h}$$

$$\equiv \sigma_1 + \sigma_2 .$$

Since

$$\gamma_n(\alpha) e^{\sqrt{\lambda_n} h} \leqslant e^{-\sqrt{\lambda_n} h} (\alpha \lambda_n)^{-1} ,$$

it follows from the Cauchy–Bunyakovskii inequality that

$$\sigma_1 \leqslant \frac{1}{\alpha} \left(\sum_{n=1}^{\infty} (\Delta \tilde{w}_n)^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} e^{-\sqrt{\lambda_n} h} \right)^{1/2} \leqslant C_1 \frac{\delta}{\alpha} ,$$

where $C_1 = \text{const}$.
On the other hand,

$$|\gamma_n(\alpha) - 1| = \alpha \lambda_n e^{2\sqrt{\lambda_n} h} (1 + \alpha \lambda_n e^{2\sqrt{\lambda_n} h})^{-1} \equiv \lambda_n \varkappa_n(\alpha) ,$$

and the series

$$\sum_{n=1}^{\infty} |\hat{w}_n| \lambda_n e^{(\lambda_n)^{1/2} h}$$

converges. Therefore $\sigma_2 \leqslant \max_n \varkappa_n(\alpha) C$, where $C = \text{const}$. Next we take into account the fact that $\lambda_n \equiv \lambda_{pq} = (\pi p / l_1)^2 + (\pi q / l_2)^2$, where l_1 and l_2 are the lengths of the sides of Π, and p and q are natural numbers. Then $\max_n \varkappa_n(\alpha) \leqslant \max \tilde{\varkappa}(\tilde{\xi}, \tilde{\eta})$, $c_0^2 \leqslant \tilde{\xi}^2 + \tilde{\eta}^2 < +\infty$, $c_0 = \text{const}$, and

$$\tilde{\varkappa}(\tilde{\xi}, \tilde{\eta}) = \alpha e^{2\sqrt{\tilde{\xi}^2 + \tilde{\eta}^2} h} [1 + \alpha (\tilde{\xi}^2 + \tilde{\eta}^2) e^{2\sqrt{\tilde{\xi}^2 + \tilde{\eta}^2} h}]^{-1} .$$

Let us put

$$2\sqrt{\tilde{\xi}^2 + \tilde{\eta}^2} h = \tau .$$

Then $\sigma_2 \leqslant C \max \mu_\alpha(\tau)$, $\tau_0 \leqslant \tau < +\infty$, where

$$\mu_\alpha(\tau) \equiv \alpha e^\tau \left(1 + \alpha \frac{\tau^2}{4h^2} e^\tau \right)^{-1} \to 0$$

is continuous for $\tau \to \infty$ (for arbitrary α). From this it follows, provided $d\mu_\alpha / d\tau = 0$ for sufficiently small α, that the extremum point (τ_m) corresponds to a maximum of $\mu_\alpha(\tau)$. The quantity τ_m is determined by the equation αe^τ

$= 2h^2\tau^{-1}$, and for $\alpha \to 0$ it equals asymptotically $|\ln \alpha|$.[61] Therefore $\max \mu_\alpha(\tau) = \mu_\alpha(\tau_m) \leqslant C^*(2h^2/\ln^2 \alpha)$; hence $\sigma_2 \leqslant C_2 \ln^{-2} \alpha$, where $C_2 = \text{const}$. It is now obvious that for $\alpha = \sqrt{\delta}$

$$|w_{\alpha(s)}(y) - \hat{w}(y)| \leqslant C_1\sqrt{\delta} + C_2 \ln^{-2}\sqrt{\delta} \to 0$$

for $\delta \to 0$, uniformly with respect to y. The theorem is proved.

In this fashion the operator $R_\Phi(\alpha, \tilde{u}_\delta)$ introduced above is regularizing in the sense of Tikhonov for the potential extension problem.

One example of utilizing this operator for the extension of gravitational fields is shown in Fig. 9. Here one was able to determine the depth of the sources to a 10% accuracy.

In Ref. 17 it is also shown that an operator similar to Eq. (51), applied to the Hadamard problem for a circular region (the problem of extension of a static magnetic field in a plasma trap) with a "normal" stabilizer $\Omega(z)$, guarantees regularization in arbitrarily "high order",[148] i.e., provides convergence also for $D^k w$ for arbitrary finite k.

The generalization of problem (43) to the case when A is an arbitrary self-conjugate linear operator in L_2, possessing a discrete spectrum $A\psi_n = \mu_n \psi_n$, is discussed in Ref. 56. In particular the following theorem is established.

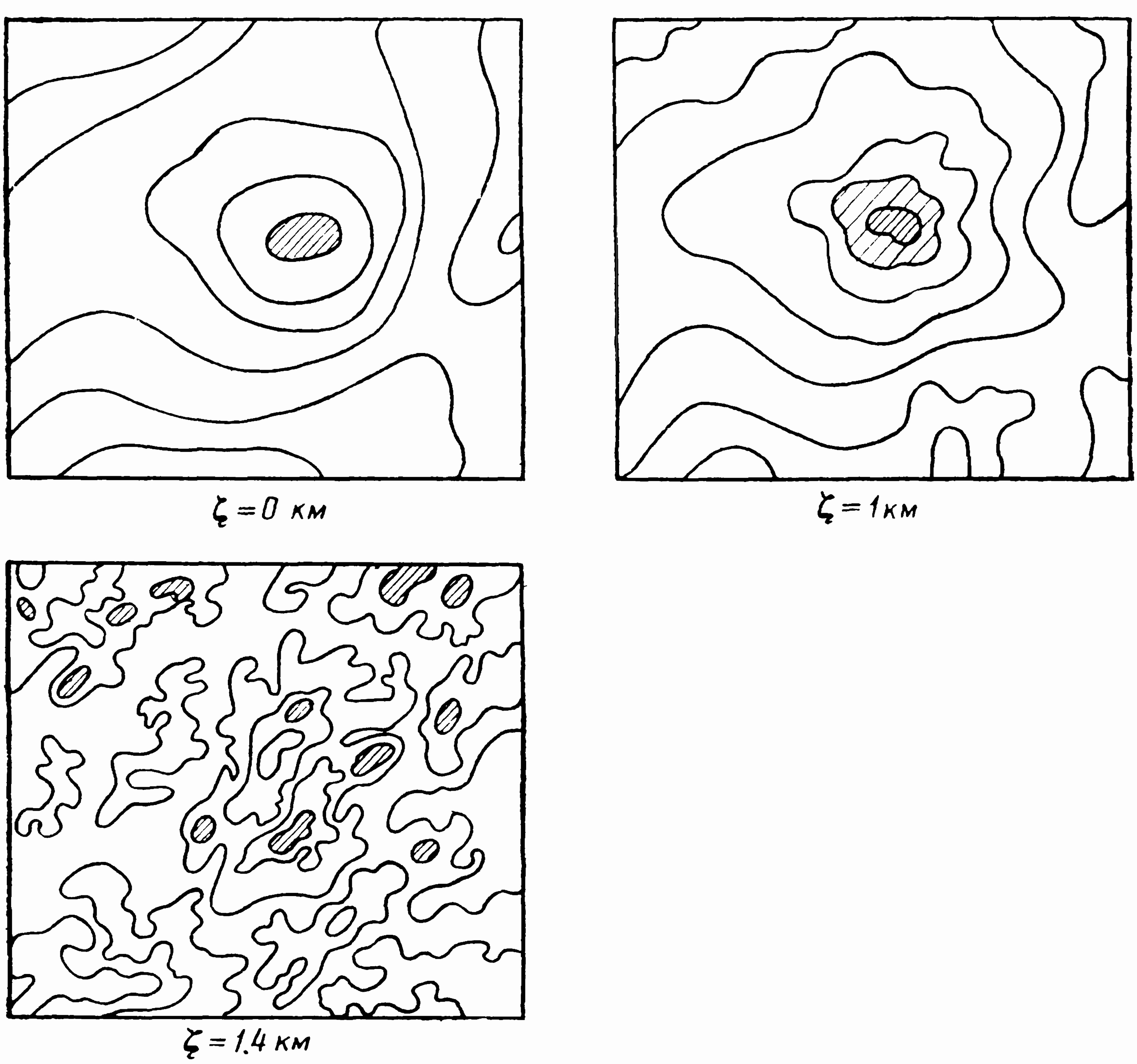


Figure 9

THEOREM. Suppose that (i) the asymptotic order of μ_n as $n \to \infty$ is one of the following: either $\mu_n = O(n^{-\gamma})$, $\gamma > 1$, or $\mu_n = O(e^{-bn})$, or $\mu_n = O(e^{-bn^2})$; (ii) the random errors in the Fourier coefficients $\Delta u_n = \tilde{u}_n - \hat{u}_n$ are independent in the aggregate and distributed uniformly on the segment $\Delta_n = [-C\delta/n^\lambda, C\delta/n^\lambda]$, $0 < \lambda \leqslant 1$, $C = \text{const}$; and (iii)

$$z^\alpha = \sum_{n=1}^{\infty} \mu_n \tilde{u}_n/(\mu_n^2 + \alpha)\psi_n \equiv R_S(\alpha, \tilde{u}).$$

Then for a quasi-optimal choice of α in the norm of the L_2 space, we have $z^\alpha_{\text{qo}} \to \hat{z}$ (see Sec. 1), provided that $\hat{z}$ is the unique solution of the equation $Az = \hat{u}$.

In this fashion the regularized Fourier series becomes, for a quasi-optimal choice of α, a "stochastic RO."

It is not hard to see that in the concrete problem discussed above one has $\mu_n = O(e^{-hn})$; on the other hand the restrictions on the errors in the statement of the theorem are natural if the Fourier coefficients $\tilde{u}_n$ decrease with increasing n according to a law analogous to that of the error. The model includes the case when $\{u_n\} \overline{\in} l_2$: $0 < \lambda < \frac{1}{2}$.

Questions arising in regularization of Fourier transformations are also discussed in Refs. 148 and 156.

3.2. Simplified regularization

In Ref. 141 attention was called to the fact that for specified requirements on the operator A, naturally arising in certain inverse problems in potential theory and heat conductivity, which are expressed as linear integral equations, it is possible to obtain the regularizing (in the required metric) operator by direct insertion of the parameter-dependent "differential increment" into the equation. Similar constructions were studied in Ref. 85.

Consider the operator equation (43) under the conditions of uniqueness of the exact solution, where A is a linear operator defined in $W_2^1[a, b]$. Let $L(z) = -p_1 d^2z/dx^2 + p_0 z$, where p_1 and p_0 are positive functions of $x \in [a, b]$.

Introduce the operator $z^\alpha = R(\alpha, \tilde{u}_\delta)$, which solves the equation

$$Az(x) + \alpha L(z) = \tilde{u}_\delta(x), \quad z(a) = z(b) = 0\,, \tag{52}$$

where $\rho_{L_2}(\tilde{u}, \hat{u}) \leqslant \delta$.

THEOREM.[103] If the linear operator A is self-conjugate and positive definite, then the inversion operator for Eq. (52) is regularizing in the sense of Tikhonov.

By making use of the variational method it is not hard to verify that problem (52) corresponds to the Euler equation (see Sec. 2) for the smoothing "energetic" functional

$$P_\alpha(z) \equiv (Az - 2\tilde{u}_\delta, z)_{L_2} + \alpha\Omega(z)\,, \tag{53}$$

where $\Omega(z) = \|z\|^2_{W_2^1}$ is Tikhonov's stabilizer, viewed on the manifold of functions twice differentiable on $[a, b]$ satisfying the conditions $z(a) = z(b) = 0$. It is clear that for a self-conjugate positive definite A the functional (53) is strongly convex: $P_\alpha(z + \zeta) > P_\alpha(z) + [P'_\alpha(z), \zeta]$ for $\zeta \neq 0$; consequently it has on the indicated manifold a unique extremum for arbitrary $\alpha > 0$, coinciding with the solution of Eq. (53). This then establishes the existence of $z^\alpha = R(\alpha, \tilde{u}_\delta)$ for arbitrary $\delta > 0$. Then the uniqueness of the solution of Eq. (52) under the conditions of the theorem follows from the Fredholm alternative[111]; as a result the problem (52) is equivalent to the problem of minimization of Eq. (53). Let us verify that an $\alpha = \alpha(\delta)$ exists, such that $z_{\alpha(\delta)}$ satisfies the regularization principle.

We note that the variational equation for Eq. (53) has the form

$$(Az - \tilde{u}_\delta, \zeta)_{L_2} + \alpha(z,\zeta)_{W_2^1} = 0 .$$

Putting here $\zeta = z = z^\alpha$ we find

$$(Az,z)_{L_2} + \alpha\|z\|^2_{W_2^1} - (\tilde{u}_\delta, z)_{L_2} = 0 ,$$

and since $(Az, z) \geqslant 0$, it follows that

$$\|z\|^2_{W_2^1} \leqslant \frac{1}{\alpha} (\tilde{u}_\delta z)_{L_2} \leqslant \frac{1}{\alpha} \|\tilde{u}_\delta\|_{L_2} \|z\|_{L_2} \leqslant \frac{1}{\alpha} \|\tilde{u}_\delta\|_{L_2} \|z\|_{W_2^1} .$$

Consequently, for any $\tilde{u}_\delta \in L_2$ (in particular also for $\hat{u}$) $\|z^\alpha\|_{W_2^1} \leqslant (1/\alpha)\|\tilde{u}_\delta\|_{L_2}$. Next we estimate $\mu_\alpha = \|z^\alpha - \hat{z}\|_{C[a,b]}$. Obviously we have

$$\mu_\alpha \leqslant \|z^\alpha - \bar{z}^\alpha\|_C + \|\bar{z}^\alpha - \hat{z}\|_C \leqslant \|z^\alpha - \bar{z}^\alpha\|_{W_2^1} + \|z^\alpha - \hat{z}\|_C ,$$

where $\bar{z}^\alpha$ is the solution of Eq. (52) for $u = \hat{u}$. From the previous estimate we have in view of the linearity of the operator that

$$\|z^\alpha - \bar{z}^\alpha\|_{W_2^1} \leqslant \frac{\|\tilde{u}_\delta - \hat{u}\|_{L_2}}{\alpha} \leqslant \frac{\delta}{\alpha} .$$

To estimate the second component of μ_α, we make use of the obvious inequality

$$(A\bar{z}^\alpha - 2\hat{u}, \bar{z}^\alpha)_{L_2} + \alpha\|\bar{z}^\alpha\|^2_{W_2^1} \leqslant (A\hat{z} - 2\hat{u}, \hat{z})_{L_2} + \alpha\|\hat{z}\|^2_{W_2^1} .$$

From here it is not hard to obtain, under the conditions of the theorem with respect to A,

$$(A(\bar{z}^\alpha - \hat{z}), \bar{z}^\alpha - \hat{z})_{L_2} + \alpha\|\bar{z}^\alpha\|^2_{W_2^1} \leqslant \alpha\|\bar{z}\|^2_{W_2^1} .$$

Consequently, $\|\bar{z}^\alpha\|^2_{W_2^1} \leqslant \|\hat{z}\|^2_{W_2^1}$, and therefore $\bar{z}^\alpha$ and $\hat{z}$ belong for arbitrary $\alpha > 0$ to the same compactum in the metric $C[a, b]$; at the same time we have as $\alpha \to 0$ that $(A(\bar{z}^\alpha - \hat{z}), \bar{z}^\alpha - \hat{z})_{L_2} \leqslant \alpha\|\hat{z}\|^2_{W_2^1} \to 0$. Next we use the fact that, under the conditions of the theorem, A has a "square root": $A = B \times B$, where B is an operator with the same properties as A.[76] Then

$$\begin{aligned}\rho^2_{L_2}(A\bar{z}^\alpha, A\hat{z}) &= \|A\bar{z}^\alpha - A\hat{z}\|^2_{L_2} = \|A(\bar{z}^\alpha - \hat{z})\|^2_{L_2} \\ &= \|BB(\bar{z}^\alpha - \hat{z})\|^2_{L_2} \leqslant \|B\|^2 \|B(\bar{z}^\alpha - \hat{z})\|^2_{L_2} \\ &\leqslant \|B\|^2 (B(\bar{z}^\alpha - \hat{z}), B(\bar{z}^\alpha - \hat{z}))_{L_2} \leqslant (A(\bar{z}^\alpha - \hat{z}), \bar{z}^\alpha - \hat{z})_{L_2} \|B\|^2 \to 0\end{aligned}$$

as $\alpha \to 0$. Since $\rho_{L_2}(Ajz^\alpha, Axz) \to 0$ as $\alpha \to 0$ and jz^α and xz belong to the same compactum (in the metric $C[a, b]$), it follows from Tikhonov's theorem (Chap. 2) that $\|jz^\alpha - xz\|_C = \rho_C(jz^\alpha, xz) \leqslant \epsilon(\alpha) \to 0$ as $\alpha \to 0$. Setting for example, $\alpha = \sqrt{\delta}$, we find that $\mu_{\alpha(\delta)} \leqslant \sqrt{\delta} + \epsilon(\sqrt{\delta}) \to 0$ as $\delta \to 0$, as was to be proven.

Consequently the operator introduced above turns out to be regularizing in the sense of Tikhonov. In Refs. 103 and 141 it is referred to as "simplified" RO.

It is interesting to note that in place of Eq. (52) one could solve the variational problem, although only for the indicated restricted class of operators.

3.3. The quasi-inversion operator

The quasi-inversion method was developed in a monograph [Ref. 88] in application to inverse problems of control type, modeled by bounds to evolution equations in partial derivatives. It amounts to the introduction into the equation of a differential "increment" of higher order, which depends on some parameter. This permits the construction of a "stable" time inversion operator. The "stability" question is framed and analyzed with respect to the control "effect," although the numerical examples given in Ref. 88 show that also the controlling characteristic z is reconstructed in a stable fashion.

It turns out that the "quasi-inversion" operator may be viewed as regularizing in the sense of Tikhonov for interpretation problems as well, where uniqueness of solution is presumed.

Let us consider a model of the heat conductivity process, described by the equation $u_{xx} = u_t$, $-\infty < x < +\infty$, $0 < t < T$. Suppose that the temperature at the instant T is known, $u(x, T) = \tilde{u}(x)$, and one wishes to determine the temperature at the initial instant of time, $u(x, 0) = z(x)$. The problem is equivalent to a conditional integral equation of the convolution type

$$\tilde{u}(x) = \int_{-\infty}^{+\infty} G(x-\xi, T) z(\xi) d\xi \, .$$

The uniqueness of the solution $\hat{z}(x)$ for an exact $\hat{u}(x)$ can be proved by means of Fourier analysis, and for such an equation one may construct the general (Sec. 2) or the "simplified" regularizing operator.

Following the quasi-inversion method of Ref. 88, we consider the parametric problem

$$u_t = u_{xx} + \alpha u_{xxxx}, \quad -\infty < x < +\infty, \quad t \in (O,T), \quad u|_{t=T} = \tilde{u}(x) \, . \tag{54}$$

Let us denote by $z^\alpha = R(\alpha, \tilde{u})$ any algorithm for its inversion, and let $Az^\alpha \equiv v(x, T)$, where $v(x, t)$ is determined by the Cauchy condition $v(x, 0) = z^\alpha(x)$ for Eq. (54). Let $\rho_{L_2}(\hat{u}, \tilde{u}_\delta) \leqslant \delta$. Then, according to Ref. 88, we can choose $\alpha = \alpha(\delta)$ such that $\rho_{L_2}(Az^{\alpha(\delta)}, \tilde{u}_\delta) \leqslant \delta$ for arbitrary predetermined δ. It turns out that a stronger assertion is possible.

THEOREM.[104] Let $\hat{z}(x)$ be the exact solution of the problem $u_{xx} = u_t$, $-\infty < x < +\infty$, $t \in (0, T)$, $u(x, T) = \hat{u}(x)$, and $\rho_{L_2}(\hat{u}, \tilde{u}_\delta) \leqslant \delta$. Then there exists a manifold of functions $\alpha = \alpha(\delta)$ such that $\lim_{\delta \to 0} \max |z^{\alpha(\delta)} - \hat{z}(x)| = 0$, where z^α is the quasi-inversion operator; the manifold $\alpha = \alpha(\delta)$ includes selection by discrepancy.

This assertion can be proven in complete analogy to the proof of the theorem in Sec. 3.1 after Fourier-transforming the problem. In Ref. 104 it is also shown that upon selection of α by discrepancy the algorithm converges in W_2^k for arbitrary fixed k.

This, then, establishes that the "quasi-inversion" operator for the problem under consideration is regularizing in the sense of Tikhonov.

3.4. Iteration operators

Under certain restrictions on the operator A, analogous to the ones assumed in Sec. 3.2, various iteration processes[11,15,82,85168] may be used to construct a stable approximation to the solution of Eq. (43). Suppose that the input data are given with some error: $\rho_U(\hat{u}, \tilde{u}_\delta) \leqslant \delta$. Then an iteration operator, with "discontinuities" in accordance with a certain criterion corresponding to the regularization principle,

$$z_{n+1} = R(z_n, \tilde{u}_\delta), \quad n = 0,1,...,N; \quad N = N(\delta) \tag{55}$$

turns out to be regularizing. In this case the number N serves as the regularization parameter and the selection criterion may be one of those described in Sec. 2.

The analog to the quasi-optimal measure for "one-step"[10] iteration processes was introduced in Ref. 37:

$$\mu_n = \|n(z_n - z_{n-1})\| \,. \tag{56}$$

Correspondingly the quasi-optimal value of N is that for which $\mu_N = \inf \mu_n$ on some set $n_0 \leqslant n < +\infty$. In turn, approximation by "discrepancy" may be determined by the formula

$$|\delta - \rho_U(Az_N, \tilde{u}_\delta)| = \min_n |\rho_U(Az_n, \tilde{u}_\delta) - \delta| \,.$$

Suppose next that the operator equation is reduced to the variational problem $\inf \tilde{\Phi}(z), z \in Z$, where $\tilde{\Phi}(z) \equiv \rho_U^2(Az, \tilde{u}_\delta)$. It is obvious that the iteration operators[10] can be used to obtain the solution, provided that a choice $N = N(\delta)$, satisfying the regularization principle, is possible.

Suppose that $\tilde{z}_{n+1} = R_A(\tilde{z}_n, \tilde{u}_\delta) \equiv R(\hat{z}_n, \tilde{u}_\delta)$ is some iteration process for the variational problem posed above, possessing (for the sake of definiteness) a unique solution $\hat{z}$ for $\delta = 0$. We shall call the operator $R(z, u)$ continuous at the point (z^*, u^*) provided that for arbitrary $\epsilon > 0$ there exists an $\eta = \eta(\epsilon) > 0$ and a $\delta = \delta(\epsilon) > 0$ such that $\rho_Z(R(z, u), R(z^*, u^*)) < \epsilon$, provided only that $\rho_Z(z, z^*) < \eta$ and $\rho_U(u, u^*) < \delta$.

THEOREM.[4] Let (i) the iteration process $z_{n+1} = R(z_n, \hat{u})$ converge to $\hat{z}$ for exact input data $u = \hat{u}$ and for an arbitrary choice of the initial approximation $z_0 \in Z$ and (ii) the operator $R(z, u)$ be continuous at every point except possibly $(\hat{z}, \hat{u})$. Then there exists a function $N = N(\delta)$ such that $\lim_{\delta \to 0} \rho_Z (\tilde{z}_{N(\delta)}, \hat{z}) = 0$.

Indeed, according to condition (i), for any $\epsilon > 0$ we can find z_0 and N such that (a) $\rho_Z(z_N, \hat{z}) < \epsilon/2$ and (b) $z_n \neq \hat{z}$ for $n = 0,1,\ldots,N-1$, where $\{z_n\}$ is an iteration sequence obtained for exact data. On the other hand, for the indicated N, as a consequence of condition (ii) for the chosen ϵ there exist for the sequence $\tilde{z}_{n+1} = R(\tilde{z}_n, \tilde{u}_\delta)$ such η_{N-1} and δ_{N-1} that $\rho_Z(\tilde{z}_N, z_N) < \epsilon/2$, provided only that $\rho_Z(\tilde{z}_{N-1}, z_{N-1}) < \eta_{N-1}$ and $\rho_U(\tilde{u}_\delta, \hat{u}) < \delta_{N-1}$. By induction, for any

$k = N-1, \ldots, 1$, for given η_{N-k} we have $\rho_Z(\tilde{z}_{N-k}, z_{N-k}) < \eta_{N-k}$, provided only that $\rho_Z(\tilde{z}_{N-k-1}, z_{N-k-1}) < \eta_{N-k-1}$ and $\rho_U(\tilde{u}_\delta, \hat{u}) < \delta_{N-k-1}$. Since $\rho_Z(\tilde{z}_0, z_0) = 0$ (by the conditions of the theorem), upon making the choice $\delta = \min \delta_s$, $0 \leqslant s \leqslant N-1$, we find $\rho_Z(\tilde{z}_N, \hat{z}) \leqslant \rho_Z(\hat{z}_N, z_N) + \rho_Z(z_N, \hat{z}) < \epsilon$. Clearly one can find a monotonic infinitesimal sequence $\{\delta_N\}$ for which $\lim_{N\to\infty} \rho_Z(\tilde{z}_N, \hat{z}) = 0$, since ϵ is arbitrary. But then for arbitrary $\delta > 0$ and arbitrary $\tilde{u}_\delta$ [$\rho_U(\tilde{u}_\delta, \hat{u}) < \delta$] one can find $N(\delta) = \max N$, for which $\delta \leqslant \delta_N$, such that $\lim_{\delta\to 0} \rho_Z(\tilde{z}_{N(\delta)}, \hat{z}) = 0$. The theorem is proved.

It is not hard to note that in the case of a linear operator A properties of the discrepancy functional ensure that the first condition of the theorem is fulfilled for well-developed iteration processes on the search for a local minimum (coinciding with a global one.[20,75,77] In Refs. 4 and 120 it was also established that under the same circumstances a number of gradient methods (in particular, the method of conjugate gradients[75]) give rise to iteration operators that satisfy also the second condition of the theorem. More general results have also been obtained, in particular when the solution of the exact equation is not unique. Thus if the operator A in Eq. (43) is linear, then the classical iteration procedures with discontinuity according to any criterion satisfying the regularization principle are regularizing operators. In Ref. 120 are stated, in particular, certain modifications of the above-mentioned consistency criteria $n = n(\delta)$.

The situation is different for equations where the operator A is nonlinear. In that case, as remarked in Sec. 2, the functional need not be convex, and condition (i) of the theorem is violated so that the "adaptive" iteration operator loses, generally speaking, its regularizing properties.

In Sec. 2 it was shown, in particular, in what way the iteration operators can be used in the framework of the general regularizing algorithm of A. N. Tikhonov (for nonlinear A). The fullest study of the construction of regularizing iteration operators for "nonlinear" inverse problems was carried out in Refs. 108, 120, and 132.

4. The solution of a control problem with the help of regularizing operators

4.1. The control problem of induction tempering of a steel sample

As is well known,[24,59] the process of induction tempering of a steel part includes two basic stages: (a) heating of the part by induction Foucault currents and (b) rapid cooling by a "cleansing" liquid stream. The thickness Δ of the near-surface "tempered" layer (where the material underwent "martensite" transformation[59] may be viewed as the effect of the tempering. The controlling factor consists of the pair of quantities $z = \{I, v\}$, where $I = I(t)$ is the amplitude of the current in the inductor during the tempering heating as a function of time, v being some characteristic of the speed of cooling of the sample layers.

The solution of the problem of "purposeful" control of max $\Delta(z)$, where Δ is an implicitly specified functional, naturally divides into three steps, particularly since in the framework of existing mathematical-physics models v also is not given explicitly and, moreover, depends on the law of heat exchange on the sample surface during the cooling, which is also unknown.[83,110]

These three steps will be discussed concisely in the example of the axisymmetric model of tempering long cylindrical samples $(0<r<R, |z|<+\infty)$ by heating in a solenoidal inductor.[44,157]

(1) The determination of the heat exchange law on the surface under realistic conditions is the subject of the problem of interpretation of observations of the temperature field on the sample surface during cooling. The heat exchange condition can be written in the form

$$-k(u)\frac{\partial u}{\partial r}\bigg|_{r=R} = H(u)(u-u^*)|_{r=R},$$

where $H(u)$ is the sought-for function of temperature and u^* is the temperature of the cleansing stream. The solution of the nonlinear heat conductivity equation with this condition and for a given initial temperature, obtained as a result of heating, determines (implicitly) the nonlinear operator $A(H) \equiv u|_{r=R}$, whose values are compared with the measured surface temperature $\tilde{u}$. The solution of the operator equation $A(H)=\tilde{u}$ by means of some regularizing algorithm (see Sec. 3.2) permits the *a priori* determination of the heat exchange law. It was shown in Ref. 157 that in spite of the influence of certain fine physical effects ("bubbling," skin effect), one may view the heat exchange as obeying the "classical" law $H(u)=H_0=\text{const}$. As a result one may view the quantity H_0, which depends on the properties of the cooling stream and its speed, as one of the parameters that control the speed of cooling of the sample layers.

(2) In the control problem of heat tempering one usually specifies the desirable surface temperatures regime: $u|_{r=R}=\hat{u}(t)$. Since in this case the heat exchange law on the surface is known,

$$-k\frac{\partial u}{\partial r}\bigg|_{r=R} = h(u-u_0)|_{r=R},$$

where h is a given constant and u_0 is the air temperature, then for an arbitrary given current $I(t)e^{i\omega t}$ in the inductor the temperature field in the sample may be determined by solving a system of nonlinear Maxwell and heat conductivity equations, analogous to Eq. (7).[39,44] That determines the operator $A(I) \equiv u|_{r=R}$, whose values are compared with the desired $\hat{u}$, and for the solution of the control problem one may also utilize the regularizing algorithm.[g]

(3) If $I(t)$ is determined then a series of mathematical experiments on "electronic computer tempering" can be realized. (a) The temperature field in the sample is calculated during both tempering stages as a function of the parameters of the heating ($\hat{u}$) and cooling (H_0) regimes.[h] (b) "Temperature

[g] It was shown in Chap. 2 (Sec. 1) that for the simplest model (7) this problem turns out to be automatically inserted into the correctness class.

[h] We note that for each given ($\hat{u}$, H_0) step (3) is carried out automatically following step (2).

curves" for cooling in various layers $u(r_i, t)$ are compared (automatically by the electronic computer)[157] with "thermokinetic diagrams" for phase transitions[110] and, as a result, the percent content of martensite after cooling in each (ith) layer is established. (c) The thickness of the near-surface layer containing the *a priori* specified percent of martensite, i.e., Δ, is determined by elementary means. The nomogram $\Delta = \Delta(\hat{u}, H_0)$ obtained on the electronic computer permits the selection of optimal parameters.

4.2. Consecutively stabilizing Tikhonov operator

One and the same regularizing operator, based on the evolution character of the process, turned out to be effective for problems of the first and second step.

Suppose that the sought-for characteristic $[I(t)$ or $H(u(t))]$ is $z(t)$, $t\in[t_0,\hat{t}]$; suppose further that $\tilde{v}(t)$ is the measured (or prescribed) temperature of the sample surface for which the allowed deviation (or measure of error) is known: $\rho_{L_2}(\tilde{v}, v)\leqslant\delta$, where v is either the exact (unique) value of the temperature, or that possible in the given physical process (allowable) effect.

We introduce on the segment $[t_0, \hat{t}]$ the grid $t_0 < t_1 < \cdots < t_n = \hat{t}$ ($\Delta_s = [t_{s-1}, t_s]$, n fixed) and the grid function $\{z_s\}$. We set $\delta_s = \delta/\sqrt{n}$ and

$$\rho_s^2(v,v) \equiv \int_{t_{s-1}}^{t_s} [\tilde{v}(t) - v(t)]^2 dt.$$

It is obvious that for given $\{z_s\}$ this defines on each of the segments Δ_s the operator $A_s(z_s)$; its values are obtained by solving either the mixed problem for the heat conductivity equation with initial condition $u_s(t_{s-1}, r) = u_{s-1}(t_{s-1}, r)$, or the analogous problem for the system of heat conductivity and Maxwell equations.

Let us consider the sequence of problems with respect to $\zeta = z_s$:

$$\min(\zeta - z_{s-1})^2, \quad \zeta\in Z_s \equiv \{\zeta\in E^1 : \rho_s^2(A_s(\zeta),\tilde{v}) \leqslant \delta^2\}, \quad s = 1,2,\ldots,n. \tag{57}$$

It is obvious that for each s this posing is analogous to Eq. (21) (Chap. 2). It was established in Refs. 44 and 161 that the inversion operator for Eq. (57) is conditionally regularizing for control problems. Such a consecutively stabilizing operator[i] can be realized on the sequence of Gauss–Newton iterations (Sec. 2), minimizing the discrepancy $\rho_s^2(A_s(\zeta), \tilde{v})$ with natural discontinuity in the quantity δ_s^2.

4.3. Some results on solutions of tempering control problems

One of the tempering heating regimes requires that the sample surface temperature be rapidly raised to u_H (above the temperature of austenite transitions), and then held at a constant level for a longer or shorter time interval (Fig. 10); the latter is called the "isothermal holding time" of the surface. In this case u_H may be viewed as the controlling parameter.

[i]In Refs. 39, 44, and 157 it is called "stepwise."

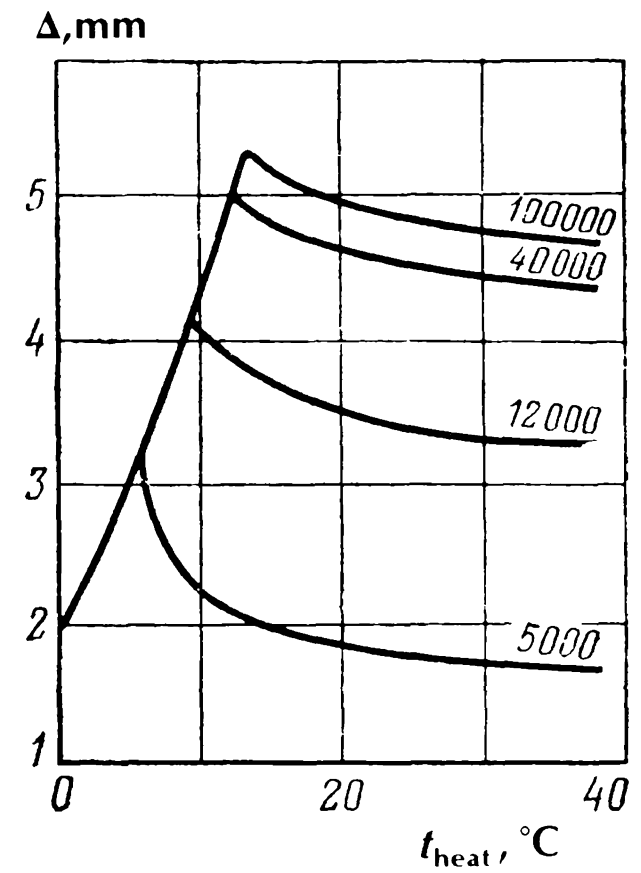


Figure 10. $R = 20$ mm, $t_{heat} = 4$ sec, $u_{heat} = 850$ °C (the numbers labeling the curves refer to H_0 in kcal/m^2 hour deg).

In Fig. 10 is shown a nomogram of the dependence of the layer thickness Δ, containing 100% of martensite, on the duration of the isothermal holding time, for various values of H_0.[39,157]

It turns out to be possible to select uniquely an optimal (for given other parameters) heating regime and correspondingly reduce the tempering time. As was noted in Ref. 157, an increase in excess of 10% was made possible in equipment productivity by incorporation of these results in manufacturing.

We note that the behavior of tempering characteristic indicated in Fig. 10 has a natural physical explanation. It is the result of interaction between the heating currents during the cooling stage: the "cooling" (from the surface) and the heating (from the deep layers of the sample, previously heated to higher temperatures for sufficiently long holding time). Such an effect was first discovered as a result of mathematical modeling of the process carried out in Refs. 39, 44, and 157.

The characteristic behavior of the current amplitude, obtained in solving second-stage problems, is shown in Fig. 11. This information allows one to control the heating stages appropriately.

The results quoted in the present paragraph testify to the fact that use of concepts of regularization theory in the posing and solving of inverse problems permits the solution of actual current problems: the problem of automatization of the analysis of data from a physical experiment and the automatization of the control of technological processes.

5. The solution of certain seismic data interpretation problems with the help of regularizing operators

In the solution of every concrete interpretation problem, from the multitude of RO one is chosen or constructed anew, corresponding to the actually exist-

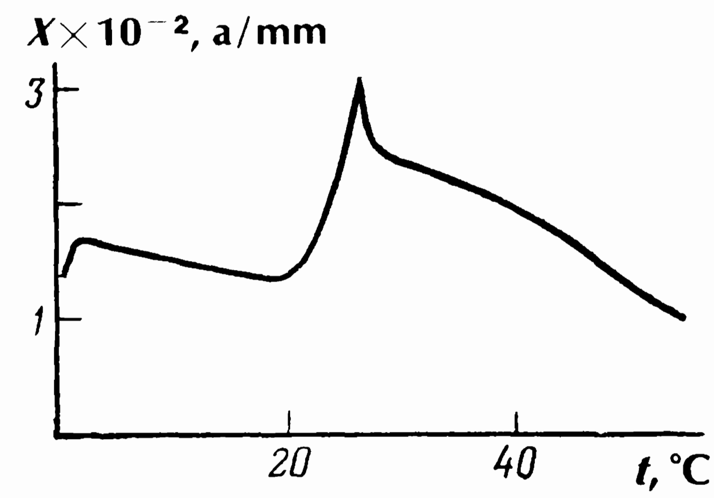


Figure 11

ing information about the solution. We consider the various possibilities involved in this choice for the example of seismic problems.

In seismology one studies the structure of the bowels of the Earth from data resulting from observations on the Earth's surface of elastic waves caused by earthquakes or by artificial sources (seismic prospecting). Seismic waves of one type or another,[122] whose formation was influenced by deep-lying structures, carry information about them.

5.1. Interpretation of observations on surface seismic waves

Of the two types of surface waves (Rayleigh and Lyav), distinguished by the character of vibration and polarization, we shall consider for clarity the latter, confining ourselves—as in Chap. 3—to the plane-layered model of the medium. The interpretation problem consists of the following: to determine the effective parameters of the elastic-density section $(n, \mathbf{p})$ (see Chap. 3, Sec. 2.5) from that extracted from observations "dispersion curve" $u \equiv \tilde{c}(\omega)$ of the fundamental tone of the Lyav wave.

It is obvious that the conditions (39) uniquely determine the operator of direct correspondence $Az \equiv c(\omega, n, \mathbf{p})$—the principal eigenvalue of the boundary-value problem (39), which can be calculated for each pair $(n, \mathbf{p})$ by standard algorithms on electronic computers.[46] Consequently, the fact that this operator is only specified implicitly does not create an obstacle to the solution of the inverse problem corresponding to the conditional operator equation $c(\omega, n, \mathbf{p}) = \tilde{c}(\omega), \quad 0 < \omega < \infty$.

Upon taking into account the character of the dispersed data (the random scatter of points about some mean with noticeable dispersion) and, on the other hand, the continuous dependence on ω of the "left-hand side" of the equation, it becomes obvious that the posed problem has no solution. However, it is known that the phase velocity depends only weakly on certain "passive"[176] parameters, and for an unknown number of layers with *a priori* specified sufficiently large n the phase velocity for the true structure (with a smaller number of layers) is altogether independent of any of the values of the layer thickness d_s. Consequently, the operator equation is "poorly conditioned," and to small variations $\tilde{c}(\omega)$ may correspond quite large variations of the sought-for parameters. Upon considering the fact that physically reasonable values for medium parameters lie in a rather narrow interval, it becomes clear that the problem is unstable in practice.

For some time a "visual" method of selection has been used for the interpretation of dispersion data, which consists of correlating "experimental" dispersion curves $\tilde{c}(\omega)$ with calculated "templates," based on peculiarities of the behavior of the dispersion curves depending on the structure character. To automatize the selection with the help of an electronic computer the interpretation problem must be well-posed, for which it is necessary to choose a "regularization strategy" from a number of possible variants. Taking into account the discrete character of the input information $\tilde{c}(\omega) \equiv \{\tilde{c}_i \equiv \tilde{c}(\omega_i), i = 1, 2, \ldots, N\}$, we introduce in the role of $\rho(c, \tau)$ the mean-square deviation on the specified grid of values $\{\omega_i\}$. It is clear that the problem of searching for the minimum of this function (the least-squares method) does not ensure the automatization of the selection in view of the instability noted above.

The following aspects of regularization are possible (Chap. 2).

(1) Insertion of the problem into the correctness class by means of exclusion of the "passive" parameters for a known number of layers n. Useful information may be lost if the accuracy of the input data is sufficiently high.

(2) Posing the quasi-solution problem by extracting the compactum $\mathscr{P}$ by explicit restrictions on the parameters: $|p_i| \leqslant M, \quad i = 1, 2, \ldots, m$ for known n. As usual the compactum should be chosen felicitously, in the sense of consistency with the accuracy δ of the input data. Since the operator $c(\mathbf{p})$ is defined algorithmically, the solution of such a problem may be obtained, for example, by direct tabulation of the function $\rho^2(c, \tilde{c})$ on the specified grid in $\mathscr{P}$ with automatic extraction of regions where $\rho^2 \leqslant \delta^2$, and exact location of the minimum in each of these regions.[18]

(3) Use of the general Tikhonov RO by introducing the stabilizer $\Omega(\mathbf{p})$, based on considerations of "similarity" between the sought-for structure and some specified one (see footnote h of Chap. 2). n plays a role of a parameter and may be quite arbitrary. In that case we arrive at the problem of minimization of the smoothing functional (22), solvable by standard methods for each α, with α and δ correlated either by discrepancy, $\rho^2(c(\omega, \mathbf{p}^\alpha), \tilde{c}) \leqslant \delta^2$, $\mathbf{p}^\alpha$ being the minimizing element, or by the quasi-optimal method.

The solution of the interpretation problem of data on Rayleigh waves is in principle the same as above. As an example we consider some data for the New York–Pennsylvania region. The values of $\mathbf{p}$, obtained with the help of RO,[41] are as follows: $\mathbf{p} = \{d_1 = 36.03$ km, $b_1 = 3.61$ km/s, $\rho^1 = 3.07$ g/cm^3, $b_2 = 4.68$ km/s$\}$. This result is in good agreement with data obtained by other geophysical methods for these parameters, when known. On the other hand, the least-squares method without regularization gives the following "section fragment"[176]: $\mathbf{p} = \{d_1 = 46.5, b_1 = 3.70, \rho_1 = 1.34, b_2 = 4.29\}$. The erroneous value of ρ_1 stands out.

The mathematical experiment described in Ref. 41 shows that use of the same algorithm, for input data of sufficiently high accuracy, makes possible the unique determination, along with the physical parameters, of the number of layers n; by choosing n unquestionably larger than the true value, we arrive at a structure with "coinciding" parameter values in certain neighboring layers.

5.2. The inverse kinematic seismic problem

Conclusions about the structure of the Earth's bowels can be drawn from the arrival time at various points on the Earth's surface S of seismic waves, produced by sources whose location and instant of activity are known: $t = t(M)$, $M \in S$. Such a dependence is called a "hodograph." The sought-for quantities could be the location and form of the boundary dividing two media with differing "speed" (see Chap. 3) characteristics and the values of the latter. This interpretation problem, called "kinematic," is discussed in the framework of the "ray" approximation neglecting diffraction of the seismic waves by "small" (~ 300 m) inhomogeneities, and a large number of papers have been devoted to it (see Refs. 27, 86, and 122).

We consider for clarity a plane model of the medium, characterized by the function $v(\zeta)$ which determines the depth dependence of the speed of volume waves (for example, shear: $v \equiv b$). We shall assume that in the observation field signals have been isolated corresponding to identically formed rays, for example refracted, not containing segments parallel to the Earth's surface. Then for arbitrary specified $v = v(\zeta)$, the seismic ray originating at the point $\mathscr{P}(0, 0)$ and ending at the point $\mathscr{Q}(x, 0)$ can be described by the solution of the following boundary-value problem, resulting from the eikonal equation for the front of the seismic wave. Let $s = L\tau$ be the length of the arc along the ray $\mathbf{q} = (\zeta(r), \zeta(\tau))$ and L its full length, let $t = t(\tau)$ be the time of propagation of the seismic wave along the ray, $u = v^{-1}$. Then $(u\mathbf{q}')' = |\mathbf{q}'|^2 \nabla u$, $t' = |\mathbf{q}'| u$ $(0 < \tau < 1)$, $\mathbf{q}(0) = \{0, 0\}$, $\mathbf{q}(1) = \{x, 0\}$, $t(0) = 0$. Methods for solution of boundary-value problems are well developed, and consequently for each point x on the surface we have defined an operator for the direct correspondence $Av \equiv t(1) \equiv T[x, v]$, where T is the time of propagation of the signal from the source to the chosen point on the surface.

It is known[27] that the existence of deep-down waveguides—layers with lowered speed—results in an "exact" hodograph $T(x)$ having a nonunique speed section corresponding to it. However, in problems of regional seismic prospecting it can be assumed that the speed $v(\zeta)$ is a monotonically increasing function of depth, and then the solution of the operator equation $Av = T$ is unique. It is natural, therefore, to solve the inverse kinematic problem for seismic prospecting purposes on the manifold of monotonic functions.

This already defines a possible regularization strategy since the manifold of monotonic and bounded functions is compact on the manifold of continuous ones. Taking into account the fact that the observations are carried out on a discrete grid of values x, we introduce the mean-square measure of deviation $\rho(T, \tilde{T})$ on this grid. Suppose that an upper estimate R for $v(\zeta)$ and the ray penetration depth H are known in the given region. Then the speed section of interest can be determined[7] as the quasi-solution

$$v(\zeta): \inf \rho^2(Av, \tilde{T}), \quad v \in V,$$

where V is the manifold of functions monotonically increasing with depth, defined on $[0, H]$, with values in $[0, R]$. Furthermore it is immaterial whether $v(\zeta)$ is a continuous or piecewide continuous function of the depth. It was shown in Ref. 7, in particular, that for sufficiently high accuracy of the input data it is possible to reconstruct quite precisely by this method speed sections of various types also in the case when the coordinates x_i are specified with some error.

It is understood that if we are interested in some averaged characteristic, *a priori* expressible in the form of some smooth function, then the corresponding stabilizer may be introduced and use can be made, for example, of the general parametric regularizing operator (Sec. 1). In both cases the choice of the initial approximation for the iterational minimization process, unavoidable in view of the nonlinearity of the problem, is immaterial. The corresponding formulations are also easily generalized to the case when v depends on a larger number of variables.

The special *a priori* parametrization of the speed section, for example representing the dependence $v(M)$ in the form of a polynomial, requires either the imposition of quantitative restrictions on each of the parameters (coefficients of the polynomial), and then we arrive again at the problem with quasi-solution, or the development of an adaptive iterational regularizing operator. Naturally, in the absence of information on monotonicity, for even the "average" section, for the problem to be well-posed, in view of nonuniqueness of the solution, the introduction of additional well-founded conditions on the choice of approximation is unavoidable.

The examples discussed in this paragraph and above testify to the wide range of possibilities for the utilization of the regularization concept at the level of data interpretation. It is understood that this does not exhaust the problem of constructing automated systems for data analysis by electronic calculators, since, as is easily noted on these examples, there exist the problems of (a) primary analysis of the observational data, giving rise to the interpreted characteristic of the observation field, and (b) supply of *a priori* information about the object under study, which can be done by means of complex interpretation of observational data of various nature.

Nevertheless, the use of RO is an indispensable element of any automated analysis system.

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