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ИНТЕГРАЛЫ И РЯДЫ

ЭЛЕМЕНТАРНЫЕ ФУНКЦИИ

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Книга содержит неопределенные и определенные (в том числе кратные) интегралы, конечные суммы, ряды и произведения с элементарными функциями. Она является наиболее полным справочным руководством, включает результаты, изложенные в аналогичных изданиях, а также в научной и периодической литературе, опубликованной в последние годы. Некоторые результаты публикуются впервые.

Книга предназначена для широкого круга специалистов в различных областях знаний, а также для студентов вузов.

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ОГЛАВЛЕНИЕ

ПРЕДИСЛОВИЕ

Книга содержит неопределенные и определенные (в том числе кратные) интегралы, конечные суммы, ряды и произведения с элементарными функциями Она написана с целью удовлетворения запросов широкого круга специалистов в различных областях знаний и включает результаты, изложенные в аналогичных изданиях, а также в научной и периодической литературе, опубликованной в последнее время. Ряд результатов публикуется впервые. Отметим что, интерпретируя некоторые интегралы и ряды как обобщенные функции, интегральные преобразования обобщенных функций или рассматривая их регуляризации, можно придать им смысл в более широкой области изменения параметров.

Достаточно подробное оглавление позволяет ознакомиться с содержанием книги и отражает ее отличие от ранее выпущенных справочных руководств в этой области математического анализа.

Найти вполне оптимальную систему упорядочения материала книги вряд ли представляется возможным, но мы надеемся, что с помощью оглавления читатель сумеет отыскать нужные формулы.

Применяемые обозначения, как правило, общеприняты в математической литературе и приводятся в указателях в конце книги. При ссылках запись вида 5.1.4.3 обозначает формулу 3 из пункта 5.1.4; k, l, m, $n=0, 1, 2, \ldots$, если не указаны другие условия.

Для компактности изложения в ряде формул используется сокращенная запись. Например, формула

$$\int_0^1 x^{\alpha-1}\begin{Bmatrix}\arcsin x\\ \arccos x\end{Bmatrix}dx=\frac{\pi}{2\alpha}\left(\begin{Bmatrix}1\\0\end{Bmatrix}\mp\frac{1}{\sqrt{\pi}}\Gamma\begin{bmatrix}(\alpha+1)/2\\ 1+\alpha/2\end{bmatrix}\right)\qquad \left[\operatorname{Re}\alpha>-\begin{Bmatrix}1\\0\end{Bmatrix}\right]$$

представляет собой сокращенную запись двух формул:

$$\int_0^1 x^{\alpha-1}\arcsin x\,dx=\frac{\pi}{2\alpha}\left(1-\frac{1}{\sqrt{\pi}}\Gamma\begin{bmatrix}(\alpha+1)/2\\ 1+\alpha/2\end{bmatrix}\right)\qquad [\operatorname{Re}\alpha>-1]$$

(берется только верхний знак и верхнее выражение в фигурных скобках) и

$$\int_0^1 x^{\alpha-1}\arccos x\,dx=\frac{\sqrt{\pi}}{2\alpha}\Gamma\begin{bmatrix}(\alpha+1)/2\\ 1+\alpha/2\end{bmatrix}\qquad [\operatorname{Re} x>0].$$

(берется только нижний знак и нижнее выражение в фигурных скобках).

Следует также отметить, что для некоторых интегралов имеется несколько различных представлений, которые записываются под разными номерами, но без повторения левой части.

Книга содержит три приложения, которые могут быть использованы при вычислении рядов и интегралов, а также при приведении их к виду, содержащемуся в книге.

При составлении настоящего справочного руководства использованы литературные источники, список которых приводится в конце книги.

Мы будем весьма признательны всем читателям, которые обратят наше внимание на неизбежные в работе такого объема недосмотры.

Глава 1. НЕОПРЕДЕЛЕННЫЕ ИНТЕГРАЛЫ

1.1. ВВЕДЕНИЕ

1.1.1. Предварительные сведения.

В этой главе содержатся неопределенные интегралы от элементарных функций; постоянная интегрирования для краткости опущена. Например, вместо

$$\int \sin x\,dx = -\cos x + C$$

пишем

$$\int \sin x\,dx = -\cos x.$$

В ряде формул, где при интегрировании возникает логарифм абсолютной величины, знак абсолютной величины для простоты опущен.

Некоторые формулы при определенных значениях параметров теряют смысл. Если эти значения следуют из структуры формулы, то соответствующие разъяснения опускаются. Выражения для интеграла при этих значениях параметров, как правило, даются в последующих формулах.

Ниже используются следующие обозначения:

x, t, s, u, v — независимые переменные,

f, g — функции,

P, Q, P_i, Q_i — многочлены,

$R = \frac{P}{Q}$ — рациональная функция,

a, b, c, d — действительные числа,

p, q, r, α, λ, μ, ν — комплексные числа,

k, l, m, $n = 0, 1, 2, 3, \ldots$

В случаях, когда имеются ограничения на значения параметров, делаются специальные оговорки.

1.1.2. Основные интегралы.

1. $\int x^{\lambda}\,dx = \frac{x^{\lambda+1}}{\lambda+1}$ $\quad [\lambda \neq -1]$.

2. $\int \frac{dx}{x} = \ln|x|$. $\qquad$ 3. $\int e^{x}\,dx = e^{x}$.

4. $\int a^{x}\,dx = \frac{a^{x}}{\ln a}$ $\quad [a > 0,\ a \neq 1]$.

5. $\int \sin x\,dx = -\cos x$. $\qquad$ 6. $\int \cos x\,dx = \sin x$.

7. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$ 8. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$

9. $\int \frac{\sin x}{\cos^2 x} dx = \sec x.$ 10. $\int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x.$

11. $\int \operatorname{tg} x\, dx = -\ln |\cos x|.$ 12. $\int \operatorname{ctg} x\, dx = \ln |\sin x|.$

13. $\int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right|.$ 14. $\int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$

15. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a}$ $[a \neq 0]$

16. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \operatorname{Arth} \frac{x}{a} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$ $[a \neq 0]$

17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$ $[a \neq 0]$.

18. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{Arsh} \frac{x}{a} = \ln \left(x + \sqrt{x^2 + a^2}\right)$ $[a \neq 0]$.

19. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{Arch} \frac{x}{a} = \ln \left| x + \sqrt{x^2 - a^2} \right|$ $[a \neq 0]$.

20. $\int \operatorname{sh} x\, dx = \operatorname{ch} x.$ 21. $\int \operatorname{ch} x\, dx = \operatorname{sh} x.$

22. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$ 23. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$

24. $\int \operatorname{th} x\, dx = \ln \operatorname{ch} x.$ 25. $\int \operatorname{cth} x\, dx = \ln |\operatorname{sh} x|.$

26. $\int \frac{dx}{\operatorname{sh} x} = \ln \left| \operatorname{th} \frac{x}{2} \right|.$ 27. $\int \frac{dx}{\operatorname{ch} x} = 2 \operatorname{arctg} e^x.$

1.1.3. Общие формулы

28. $\int (af(x) + bg(x))\, dx = a \int f(x)\, dx + b \int g(x)\, dx.$

29. $\int \frac{d}{dx} f(x)\, dx = f(x).$

30. $\int \frac{df(x)}{dx} g(x)\, dx = f(x) g(x) - \int f(x) \frac{dg(x)}{dx} dx$ [интегрирование по частям].

31. $\int f(x)\, dx = \int f(g(y))\, g'(y)\, dy$ [интегрирование подстановкой, $x = g(y)$].

1.2. СТЕПЕННАЯ И АЛГЕБРАИЧЕСКИЕ ФУНКЦИИ

1.2.1. Введение

В этом пункте кратко изложены некоторые сведения общего характера, которые могут быть использованы при вычислении неопределенных интегралов

1. Если $P_1(x)$ и $Q_1(x)$ — произвольные многочлены, то их отношение может быть представлено в виде

$$\frac{P_1(x)}{Q_1(x)} = P_2(x) + \frac{P(x)}{Q(x)},$$

где $P_2(x)$, $P(x)$, $Q(x)$ — многочлены, причем степень $P(x)$ меньше степени $Q(x)$. Пусть a_i, $i=1, \ldots, m$, — корни многочлена $Q(x)$, n_i — соответствующие кратности корней. Тогда

$$\frac{P(x)}{Q(x)}=\sum_{k=1}^{n_1}\frac{A_k^{(1)}}{(x-a_1)^k}+\sum_{k=1}^{n_2}\frac{A_k^{(2)}}{(x-a_2)^k}+\ldots+\sum_{k=1}^{n_m}\frac{A_k^{(m)}}{(x-a_m)^k},$$

где

$$A_k^{(i)}=\frac{1}{(n_i-k)!}\frac{d^{n_i-k}}{dx^{n_i-k}}\left[\frac{P(x)}{Q(x)}(x-a_i)^{n_i}\right]\Bigg|_{x=a_i}.$$

Если $n_i=1$ для всех i, то

$$\frac{P(x)}{Q(x)}=\sum_{i=1}^{m}\frac{A^{(i)}}{x-a_i},$$

где $A^{(i)}=P(a_i)/Q'(a_i)$.

При объединении пар слагаемых, соответствующих комплексно сопряженным корням, получаются дроби вида $\frac{Ax+B}{x^2+bx+c}$, где трехчлен x^2+bx+c имеет действительные коэффициенты. Поэтому вычисление интеграла $\int\frac{P(x)}{Q(x)}dx$ сводится к интегрированию рациональных дробей $\int\frac{dx}{(x-a)^n}$, $\int\frac{Ax+B}{(x^2+bx+c)^n}dx$.

2. Интегралы вида $\int x^p(ax^r+b)^q\,dx$ (p, q, r — рациональные числа) выражаются в виде конечной комбинации элементарных функций только в следующих случаях:

а) q — целое число,

б) $(p+1)/r$ — целое число; тогда

$$\int x^p(ax^r+b)^q\,dx=\frac{1}{r}\int t^{(p+1)/r-1}(at+b)^q\,dt \qquad [t=x^r],$$

в) $\frac{p+1}{r}+q$ — целое число, тогда

$$\int x^p(ax^r+b)^q\,dx=\frac{1}{r}\int\left(\frac{at+b}{t}\right)^q t^{(p+1)/r+q-1}\,dt \qquad [t=x^r].$$

3. Интегралы вида $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^p, \left(\frac{ax+b}{cx+d}\right)^q, \ldots\right)dx$, где $p, q, \ldots$ — рациональные числа, приводятся к интегралам от рациональных функций подстановкой

$$\frac{ax+b}{cx+d}=t^m,$$

где m — общий знаменатель дробей $p, q, \ldots$

4. Интегралы вида

$$I=\int\frac{R_1(x)+R_2(x)\sqrt{P(x)}}{R_3(x)+R_4(x)\sqrt{P(x)}}dx,$$

где R_1, R_2, R_3, R_4 — рациональные функции, а $P(x)$ — многочлен 3-й или 4-й степени с действительными коэффициентами и простыми корнями, приводятся к виду

$$I=\int R_5(x)\,dx+\int\frac{R(x)}{\sqrt{P(x)}}dx,$$

где

$$R_5(x)=\frac{R_1(x)R_3(x)-R_2(x)R_4(x)P(x)}{R_3^2(x)-R_4^2(x)P(x)},$$

$$R(x)=\frac{(R_2(x)R_3(x)-R_1(x)R_4(x))P(x)}{R_3^2(x)-R_4^2(x)P(x)}.$$

Эллиптический интеграл $\int\frac{R(x)}{\sqrt{P(x)}}dx$ при помощи замен переменных которые указаны ниже, может быть приведен к форме, содержащей рациональные функции от эллиптических функций Якоби. Интегралы от эллиптических функций Якоби см. в [24]

1) Пусть $P(x)=a_0(x^2\pm a^2)(x^2\pm b^2)$. Тогда интеграл $\int\frac{R(x)}{\sqrt{P(x)}}dx$ может быть представлен в виде линейной комбинации трех интегралов: $\int\frac{dx}{\sqrt{P(x)}}$, $\int\frac{x^2dx}{\sqrt{P(x)}}$, $\int\frac{dx}{(x^2-p^2)\sqrt{P(x)}}$.

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих шести способов:

$$\sqrt{(a^2-x^2)(x^2-b^2)},\quad \sqrt{(a^2-x^2)(b^2-x^2)},\quad \sqrt{(a^2+x^2)(b^2-x^2)},$$

$$\sqrt{(a^2+x^2)(x^2-b^2)},\quad \sqrt{(x^2+a^2)(x^2+b^2)},\quad \sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)},$$

где a^2, b^2 — действительные числа, ρ^2 и $\bar\rho^2$ — комплексно сопряженные числа.

В первых пяти случаях подстановка

$$x^2=\frac{A_1+A_2\operatorname{sn}^2u}{B_1+B_2\operatorname{sn}^2u}\qquad (0\leqslant u\leqslant K),$$

где $\operatorname{sn}u$ — эллиптический синус Якоби и A_1, A_2, B_1, B_2 — некоторые постоянные, приводит интегралы к виду $\int R_6(\operatorname{sn}u)du$, где R_6 — рациональная функция.

В шестом случае следует использовать замену

$$x^2=\frac{A_1+A_2\operatorname{cn}u}{B_1+B_2\operatorname{cn}u}\qquad (0\leqslant u\leqslant 2K),$$

где $\operatorname{cn}u$ — эллиптический косинус Якоби и A_1, A_2, B_1, B_2 — некоторые постоянные

2) Пусть $P(x)=a_0(x+a_1)(x+a_2)(x+a_3)$, где все a_i различны. Интеграл $\int\frac{R(x)}{\sqrt{P(x)}}dx$ может быть представлен в виде линейной комбинации трех интегралов: $\int\frac{dx}{\sqrt{P(x)}}$, $\int\frac{x\,dx}{\sqrt{P(x)}}$, $\int\frac{dx}{(x-p)\sqrt{P(x)}}$.

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих пяти способов

$$\sqrt{(x-a)(x-b)(x-c)},\quad \sqrt{(a-x)(x-b)(x-c)},\quad \sqrt{(a-x)(b-x)(x-c)},$$

$$\sqrt{(x-a)[(x-b_1)^2+b_2^2]},\quad \sqrt{(a-x)[(x-b_1)^2+b_2^2]},$$

где a, b, c, b_1, b_2 действительны и $a>b>c$. В первых трех случаях к интегралам от эллиптических функций Якоби приводит подстановка

$$x=\frac{A_1+A_2\operatorname{sn}^2u}{B_1+B_2\operatorname{sn}^2u}\qquad (0\leqslant u\leqslant K),\qquad (*)$$

где A_1, A_2, B_1, B_2 — некоторые постоянные. Если два корня многочлена $P(x)$ комплексны, то следует сделать замену

$$x=\frac{A_1+A_2\operatorname{cn}u}{B_1+B_2\operatorname{cn}u}\qquad(0\leqslant u\leqslant 2K).\qquad(**)$$

3) Пусть $P(x)=a_0(x+a_1)(x+a_2)(x+a_3)(x+a_4)$, где все a_i различны. Интеграл $\int\frac{R(x)}{\sqrt{P(x)}}\,dx$ может быть представлен в виде линейной комбинации четырех интегралов: $\int\frac{dx}{\sqrt{P(x)}}$, $\int\frac{x\,dx}{\sqrt{P(x)}}$, $\int\frac{x^2\,dx}{\sqrt{P(x)}}$, $\int\frac{dx}{(x-p)\sqrt{P(x)}}$.

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих шести способов:

$$\sqrt{(x-a)(x-b)(x-c)(x-d)},\quad \sqrt{(a-x)(x-b)(x-c)(x-d)},$$
$$\sqrt{(a-x)(b-x)(x-c)(x-d)},\quad \sqrt{(x-a)(x-b)[(x-b_1)^2+b_2^2]},$$
$$\sqrt{(x-a)(b-x)[(x-b_1)^2+b_2^2]},\quad \sqrt{[(x-b_1)^2+b_2^2][(x-b_3)^2+b_4^2]},$$

где a, b, c, d, b_1, b_2, b_3, b_4 действительны и $a>b>c>d$.

В первых трех случаях следует сделать замену (*), а в остальных — замену (**).

1.2.2. Интегралы вида $\int x^p(ax^r+b)^q\,dx$.

1. $\int x^p(ax^r+b)^q\,dx=\frac{x^{p+1}(ax^r+b)^q}{qr+p+1}+\frac{qrb}{qr+p+1}\int x^p(ax^r+b)^{q-1}\,dx.$

2. $\quad=-\frac{x^{p+1}(ax^r+b)^{q+1}}{(q+1)rb}+\frac{qr+r+p+1}{(q+1)rb}\int x^p(ax^r+b)^{q+1}\,dx.$

3. $\quad=\frac{x^{p+1}(ax^r+b)^q}{p+1}-\frac{qra}{p+1}\int x^{p+r}(ax^r+b)^{q-1}\,dx.$

4. $\quad=\frac{x^{p-r+1}(ax^r+b)^{q+1}}{(q+1)ra}-\frac{p-r+1}{(q+1)ra}\int x^{p-r}(ax^r+b)^{q+1}\,dx.$

5. $\quad=\frac{x^{p-r+1}(ax^r+b)^{q+1}}{(qr+p+1)a}-\frac{(p-r+1)b}{(qr+p+1)a}\int x^{p-r}(ax^r+b)^q\,dx.$

6. $\quad=\frac{x^{p+1}(ax^r+b)^{q+1}}{(p+1)b}-\frac{(qr+r+p+1)a}{(p+1)b}\int x^{p+r}(ax^r+b)^q\,dx.$

7. $\int x^p(ax^r+b)^n\,dx=\sum_{k=0}^{n}\binom{n}{k}\frac{a^kb^{n-k}x^{p+kr+1}}{p+kr+1}.$

8. $\int\frac{x^p\,dx}{(ax^r+b)^q}=\frac{x^{p+1}}{(q-1)rb(ax^r+b)^{q-1}}+\frac{(q-1)r-p-1}{(q-1)rb}\int\frac{x^p\,dx}{(ax^r+b)^{q-1}}.$

9. $\quad=\frac{-x^{p-r+1}}{(q-1)ra(ax^r+b)^{q-1}}+\frac{p-r+1}{(q-1)ra}\int\frac{x^{p-r}\,dx}{(ax^r+b)^{q-1}}.$

10. $\int\frac{x^{r-1}\,dx}{(ax^r+b)^q}=\frac{-1}{(q-1)ra(ax^r+b)^{q-1}}.$

11. $\int\frac{x^{r-1}\,dx}{ax^r+b}=\frac{1}{ra}\ln|ax^r+b|.$

12. $$\int \frac{dx}{x^p (ax^r+b)^q} = \frac{1}{(q-1)\, rbx^{p-1} (ax^r+b)^{q-1}} + \frac{p+(q-1)\, r-1}{(q-1)\, rb} \int \frac{dx}{x^p (ax^r+b)^{q-1}}.$$

13. $$\int \frac{dx}{x^p (ax^r+b)} = -\frac{1}{(p-1)\, bx^{p-1}} - \frac{a}{b} \int \frac{dx}{x^{p-r} (ax^r+b)}.$$

14. $$\int \frac{dx}{x (ax^r+b)} = \frac{1}{rb} \ln \left| \frac{x^r}{ax^r+b} \right|.$$

1.2.3. Интегралы вида $\int \frac{x^m\, dx}{x^n \pm a^n}$.

1. $$\int \frac{dx}{x^{2n}+a^{2n}} = -\frac{1}{2na^{2n-1}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n} \pi + a^2 \right) + \frac{1}{na^{2n-1}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n} \pi}{a \sin \frac{2k+1}{2n} \pi}.$$

2. $$\int \frac{dx}{x^{2n+1}+a^{2n+1}} = \frac{1}{(2n+1)\, a^{2n}} \ln |x+a| - \frac{1}{(2n+1)\, a^{2n}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n+1} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) + \frac{2}{(2n+1)\, a^{2n}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n+1} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}.$$

3. $$\int \frac{dx}{x^{2n}-a^{2n}} = \frac{1}{2na^{2n-1}} \ln \left| \frac{x-a}{x+a} \right| + \frac{1}{2na^{2n-1}} \sum_{k=1}^{n-1} \cos \frac{k\pi}{n} \ln \left(x^2 - 2ax \cos \frac{k\pi}{n} + a^2 \right) - \frac{1}{na^{2n-1}} \sum_{k=1}^{n-1} \sin \frac{k\pi}{n} \operatorname{arctg} \frac{x - a \cos \frac{k\pi}{n}}{a \sin \frac{k\pi}{n}}.$$

4. $$\int \frac{dx}{x^{2n+1}-a^{2n+1}} = \frac{1}{(2n+1)\, a^{2n}} \ln |x-a| - \frac{1}{(2n+1)\, a^{2n}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n+1} \pi \ln \left(x^2 + 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) - \frac{2}{(2n+1)\, a^{2n}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n+1} \pi \operatorname{arctg} \frac{x + a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}.$$

5. $$\int \frac{x^m}{x^{2n}+a^{2n}}\,dx=\frac{1}{na^{2n-m-1}}\sum_{k=0}^{n-1}\sin\frac{(m+1)(2k+1)}{2n}\pi\,\mathrm{arctg}\,\frac{x-a\cos\frac{2k+1}{2n}\pi}{a\sin\frac{2k+1}{2n}\pi}-$$
$$-\frac{1}{2na^{2n-m-1}}\sum_{k=0}^{n-1}\cos\frac{(m+1)(2k+1)\pi}{2n}\ln\left(x^2-2ax\cos\frac{2k+1}{2n}\pi+a^2\right)$$
$[1\leqslant m\leqslant 2(n-1)]$.

6. $$\int \frac{x^m\,dx}{x^{2n+1}+a^{2n+1}}=\frac{(-1)^m}{(2n+1)a^{2n-m}}\ln|x+a|-$$
$$-\frac{1}{(2n+1)a^{2n-m}}\sum_{k=0}^{n-1}\cos\frac{(m+1)(2k+1)}{2n+1}\pi\ln\left(x^2-2ax\cos\frac{2k+1}{2n+1}\pi+a^2\right)+$$
$$+\frac{2}{(2n+1)a^{2n-m}}\sum_{k=0}^{n-1}\sin\frac{(m+1)(2k+1)}{2n+1}\pi\,\mathrm{arctg}\,\frac{x-a\cos\frac{2k+1}{2n+1}\pi}{a\sin\frac{2k+1}{2n+1}\pi}$$
$[1\leqslant m\leqslant 2n-1]$.

7. $$\int \frac{x^m\,dx}{x^{2n}-a^{2n}}=\frac{1}{2na^{2n-m-1}}\left[\ln|x-a|+(-1)^{m+1}\ln|x+a|\right]+$$
$$+\frac{1}{2na^{2n-m-1}}\sum_{k=1}^{n-1}\cos\frac{(m+1)k\pi}{n}\ln\left(x^2-2ax\cos\frac{k\pi}{n}+a^2\right)-$$
$$-\frac{1}{na^{2n-m-1}}\sum_{k=1}^{n-1}\sin\frac{(m+1)k\pi}{n}\,\mathrm{arctg}\,\frac{x-a\cos\frac{k\pi}{n}}{a\sin\frac{k\pi}{n}}$$
$[1\leqslant m\leqslant 2(n-1)]$.

8. $$\int \frac{x^m\,dx}{x^{2n+1}-a^{2n+1}}=\frac{1}{(2n+1)a^{2n-m}}\ln|x-a|+$$
$$+\frac{(-1)^{m+1}}{(2n+1)a^{2n-m}}\sum_{k=0}^{n-1}\cos\frac{(m+1)(2k+1)\pi}{2n+1}\ln\left(x^2+2ax\cos\frac{2k+1}{2n+1}\pi+a^2\right)+$$
$$+\frac{2(-1)^{m+1}}{(2n+1)a^{2n-m}}\sum_{k=0}^{n-1}\sin\frac{(m+1)(2k+1)\pi}{2n+1}\,\mathrm{arctg}\,\frac{x+a\cos\frac{2k+1}{2n+1}\pi}{a\sin\frac{2k+1}{2n+1}\pi}$$
$[1\leqslant m\leqslant 2n-1]$.

1.2.4. Интегралы вида $\int \frac{x^p\,dx}{(x+a)^q}$.

1. $$\int \frac{x^p\,dx}{(x+a)^q}=-\frac{x^p}{(q-1)(x+a)^{q-1}}+\frac{p}{q-1}\int \frac{x^{p-1}\,dx}{(x+a)^{q-1}}.$$

2. $$\int_0^x \frac{x^{\lambda-1}\,dx}{x+a}=\frac{1}{a}x^{\lambda}\Phi\left(-\frac{x}{a},\ 1,\ \lambda\right)$$ $[\mathrm{Re}\,\lambda>0;\ x<a]$.

3. $\int\limits_0^x \frac{x^{\lambda-1}\,dx}{(x+a)^\nu} = \frac{1}{\lambda} a^{-\nu} x^\lambda {}_2F_1\left(\nu, \quad \lambda; \quad 1+\lambda; \ -\frac{x}{a}\right)$ [$\operatorname{Re}\lambda > 0$].

4. $\int\limits_x^\infty \frac{x^{\lambda-1}\,dx}{(x+a)^\nu} = \frac{x^{\lambda-\nu}}{\nu-\lambda} {}_2F_1\left(\nu, \ \nu-\lambda; \ 1+\nu-\lambda; \ -\frac{a}{x}\right)$ [$\operatorname{Re}\lambda < \operatorname{Re}\nu$].

5. $\int \frac{x^m\,dx}{(x+a)^q} = \sum_{k=0}^{m} \binom{m}{k} \frac{(-a)^k (x+a)^{m-q-k+1}}{m-q-k+1}$;

если $k=m-q+1$, то вместо соответствующего члена в сумме следует взять $\binom{m}{q-1}(-a)^{m-q+1}\ln|x+a|$.

6. $\int \frac{x^m\,dx}{(x+a)^q} = -\sum_{k=0}^{m} \binom{m}{k} \frac{(-a)^{m-k}}{(q-k-1)(x+a)^{q-k-1}}$;

если $k=q-1$, то вместо соответствующего члена в сумме следует взять $\binom{m}{q-1}(-a)^{m-q-1}\ln|x+a|$.

7. $\int \frac{dx}{(x+a)^n} = -\frac{1}{(n-1)(x+a)^{n-1}}$. 8. $\int \frac{dx}{x+a} = \ln|x+a|$.

9. $\int \frac{x\,dx}{(x+a)^n} = -\frac{1}{(n-2)(x+a)^{n-2}} + \frac{1}{(n-1)(x+a)^{n-1}}$.

10. $\int \frac{x^2\,dx}{(x+a)^n} = -\frac{1}{(n-3)(x+a)^{n-3}} + \frac{2a}{(n-2)(x+a)^{n-2}} - \frac{a^2}{(n-1)(x+a)^{n-1}}$.

11. $\int \frac{x^3\,dx}{(x+a)^n} = -\frac{1}{(n-4)(x+a)^{n-4}} + \frac{3a}{(n-3)(x+a)^{n-3}} -$

$- \frac{3a^2}{(n-2)(x+2)^{n-2}} + \frac{a^3}{(n-1)(x+a)^{n-1}}$.

12. $\int \frac{x^m\,dx}{x+a} = \sum_{k=0}^{m-1} \frac{(-1)^k a^k x^{m-k}}{m-k} + (-a)^m \ln|x+a|$.

13. $\int \frac{x\,dx}{x+a} = x - a\ln|x+a|$. 14. $\int \frac{x^2\,dx}{x+a} = \frac{x^2}{2} - ax + a^2\ln|x+a|$.

15. $\int \frac{x^3\,dx}{x+a} = \frac{x^3}{3} - \frac{ax^2}{2} + a^2x - a^3\ln|x+a|$.

16. $\int \frac{x^m\,dx}{(x+a)^2} = \sum_{k=1}^{m-1} (-1)^k \frac{ka^{k-1}x^{m-k}}{m-k} + \frac{(-1)^{m+1}a^m}{x+a} +$

$+ (-1)^{m-1} m a^{m-1}\ln|x+a|$.

17. $\int \frac{x\,dx}{(x+a)^2} = \frac{a}{x+a} + \ln|x+a|$.

18. $\int \frac{x^2\,dx}{(x+a)^2} = x - \frac{a^2}{x+a} - 2a\ln|x+a|$.

19. $$\int \frac{x^3\,dx}{(x+a)^2} = \frac{x^2}{2} - 2ax + \frac{a^3}{x+a} + 3a^2 \ln|x+a|.$$

20. $$\int \frac{x\,dx}{(x+a)^3} = -\frac{1}{x+a} + \frac{a}{2(x+a)^2}.$$

21. $$\int \frac{x^2\,dx}{(x+a)^3} = \frac{2a}{x+a} - \frac{a^2}{2(x+a)^2} + \ln|x+a|.$$

22. $$\int \frac{x^3\,dx}{(x+a)^3} = x - \frac{3a^2}{x+a} + \frac{a^3}{2(x+a)^2} - 3a\ln|x+a|.$$

1.2.5. Интегралы вида $\int \frac{dx}{x^p(x+a)^q}$.

1. $$\int \frac{dx}{x^p(x+a)^q} = \frac{-1}{(p-1)\,ax^{p-1}(x+a)^{q-1}} - \frac{p+q-2}{(p-1)\,a}\int \frac{dx}{x^{p-1}(x+a)^q}.$$

2. $$= \frac{1}{(q-1)\,ax^{p-1}(x+a)^{q-1}} + \frac{p+q-2}{(q-1)\,a}\int \frac{dx}{x^p(x+a)^{q-1}}.$$

3. $$\int \frac{dx}{x^m(x+a)^n} = \frac{1}{a^{m+n-1}}\sum_{k=0}^{m+n-2}(-1)^{k+1}\binom{m+n-2}{k}\frac{(x+a)^{m-k-1}}{(m-k-1)\,x^{m-k-1}};$$

если $k=m-1$, то вместо соответствующего члена в сумме следует взять $(-1)^m\binom{m+n-2}{m-1}\ln\left|\frac{x+a}{x}\right|$.

4. $$\int \frac{dx}{x(x+a)^n} = \sum_{k=1}^{n-1}\frac{1}{ka^{n-k}(x+a)^k} - \frac{1}{a^n}\ln\left|\frac{x+a}{x}\right|.$$

5. $$= \frac{1}{a^n}\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\binom{n-1}{k}\left(\frac{x}{x+a}\right)^k - \frac{1}{a^n}\ln\left|\frac{x+a}{x}\right|.$$

6. $$\int \frac{dx}{x^2(x+a)^n} = -\frac{1}{ax(x+a)^{n-1}} - \frac{n}{a}\left[\sum_{k=1}^{n-1}\frac{1}{ka^{n-k}(x+a)^k} - \frac{1}{a^n}\ln\left|\frac{x+a}{x}\right|\right].$$

7. $$\int \frac{dx}{x^3(x+a)^n} = \frac{(n+1)\,x-a}{2a^2x^2(x+a)^{n-1}} + \frac{n(n+1)}{2a^2}\int \frac{dx}{x(x+a)^n}.$$

8. $$\int \frac{dx}{x^m(x+a)} = \sum_{k=1}^{m-1}\frac{(-1)^k}{(m-k)\,a^kx^{m-k}} + \frac{(-1)^m}{a^m}\ln\left|\frac{x+a}{x}\right|.$$

9. $$\int \frac{dx}{x(x+a)} = \frac{1}{a}\ln\left|\frac{x}{x+a}\right|.$$

10. $$\int \frac{dx}{x^2(x+a)} = -\frac{1}{ax} + \frac{1}{a^2}\ln\left|\frac{x+a}{x}\right|.$$

11. $$\int \frac{dx}{x^3(x+a)} = \frac{1}{a^2x} - \frac{1}{2ax^2} - \frac{1}{a^3}\ln\left|\frac{x+a}{x}\right|.$$

12. $$\int \frac{dx}{x(x+a)^2} = \frac{1}{a}\left(\frac{1}{x+a} - \frac{1}{a}\ln\left|\frac{x+a}{x}\right|\right).$$

13. $$\int \frac{dx}{x^2(x+a)^2} = -\frac{1}{a^2}\left(\frac{1}{x+a}+\frac{1}{x}-\frac{2}{a}\ln\left|\frac{x+a}{x}\right|\right).$$

14. $$\int \frac{dx}{x^3(x+a)^2} = \frac{1}{a^3}\left(\frac{1}{x+a}+\frac{2}{x}-\frac{a}{2x^2}-\frac{3}{a}\ln\left|\frac{x+a}{x}\right|\right).$$

15. $$\int \frac{dx}{x(x+a)^3} = \frac{1}{a^2(x+a)}+\frac{1}{2a(x+a)^2}-\frac{1}{a^3}\ln\left|\frac{x+a}{x}\right|.$$

16. $$\int \frac{dx}{x^2(x+a)^3} = -\frac{1}{2a^2(x+a)^2}-\frac{2}{a^3(x+a)}-\frac{1}{a^3x}+\frac{3}{a^4}\ln\left|\frac{x+a}{x}\right|.$$

17. $$\int \frac{dx}{x^3(x+a)^3} = \frac{1}{a^4}\left[\frac{3}{x+a}+\frac{a}{2(x+a)^2}+\frac{3}{x}-\frac{a}{2x^2}-\frac{6}{a}\ln\left|\frac{x+a}{x}\right|\right].$$

1.2.6. Интегралы вида $\int (x+c)^p\left(\frac{x+a}{x+b}\right)^q dx$.

1. $$\int x^p\left(\frac{x+a}{x+b}\right)^q dx = \frac{(x+a)(x+b)x^{p-1}}{p+1}\left(\frac{x+a}{x+b}\right)^q -$$
$$-\frac{q(b-a)+p(b+a)}{p+1}\int x^{p-1}\left(\frac{x+a}{x+b}\right)^q dx - \frac{(p-1)ab}{p+1}\int x^{p-2}\left(\frac{x+a}{x+b}\right)^q dx.$$

2. $$\int x\left(\frac{x+a}{x+b}\right)^q dx =$$
$$= \frac{(x+a)^q}{2(x+b)^{q-2}}+\frac{b(x+a)^q}{(q-1)(x+b)^{q-1}}+\frac{q[(q-1)(a-b)-2b]}{2(q-1)}\int\left(\frac{x+a}{x+b}\right)^{q-1}dx.$$

3. $$= \frac{(x+a)^{q+1}}{2(x+b)^{q-1}}-\frac{q(b-a)+a+b}{2}\int\left(\frac{x+a}{x+b}\right)^q dx.$$

4. $$\int \frac{(x+a)^p}{(x+b)^q}dx = -\frac{(x+a)^p}{(q-1)(x+b)^{q-1}}+\frac{p}{q-1}\int\frac{(x+a)^{p-1}}{(x+b)^{q-1}}dx.$$

5. $$\int \frac{(x+a)^p}{(x+b)^{p+2}}dx = \frac{1}{(p+1)(b-a)}\left(\frac{x+a}{x+b}\right)^{p+1}.$$

6. $$\int \frac{(x+a)^p}{(x+b)^{p+3}}dx = \frac{(x+a)^{p+1}[x+(p+1)(b-a)+b]}{(p+1)(p+2)(b-a)^2(x+b)^{p+2}}.$$

7. $$\int \frac{1}{(x+c)^p}\left(\frac{x+a}{x+b}\right)^q dx = \frac{-1}{(p-1)(a-c)(b-c)}\left\{\frac{1}{(x+c)^{p-1}}\cdot\frac{(x+a)^{q+1}}{(x+b)^{q-1}}+\right.$$
$$+[(p-2)(a+b-2c)-q(b-a)]\int\frac{1}{(x+c)^{p-1}}\left(\frac{x+a}{x+b}\right)^q dx+$$
$$\left.+(p-3)\int\frac{1}{(x+c)^{p-2}}\left(\frac{x+a}{x+b}\right)^q dx\right\}.$$

8. $$\int \frac{1}{x+c}\left(\frac{x+a}{x+b}\right)^n dx = \sum_{k=1}^{n-1}\frac{1}{n-k}\left[\left(\frac{a-c}{b-c}\right)^k-1\right]\left(\frac{x+a}{x+b}\right)^{n-k} -$$
$$-\left[\left(\frac{a-c}{b-c}\right)^n-1\right]\ln|x+b|+\left(\frac{a-c}{b-c}\right)^n\ln|x+c|.$$

9. $$\int\left(\frac{x+a}{x+b}\right)^n dx =$$
$$= -\sum_{k=0}^{n-2}\frac{n}{(n-k)(n-k-1)}\,\frac{(x+a)^{n-k}}{(x+b)^{n-k-1}}+n(x+a)-n(b-a)\ln|x+b|.$$

10. $$= -\sum_{k=1}^{n-1}\binom{n}{k+1}\frac{(a-b)^{k+1}}{k(x+b)^k}+x+n(a-b)\ln|x+b|.$$

11. $$\int \frac{x+a}{x+b}\,dx = x+(a-b)\ln|x+b|.$$

12. $$\int x\frac{x+a}{x+b}\,dx = \frac{x^2}{2}+(a-b)(x-b\ln|x+b|).$$

13. $$\int x^2\frac{x+a}{x+b}\,dx = \frac{x^3}{3}+(a-b)\left(\frac{x^2}{2}-bx+b^2\ln|x+b|\right).$$

14. $$\int \frac{x+a}{x(x+b)}\,dx = \frac{a}{b}\ln|x|-\frac{a-b}{b}\ln|x+b|.$$

15. $$\int \left(\frac{x+a}{x+b}\right)^2 dx = x-\frac{(a-b)^2}{x+b}+2(a-b)\ln|x+b|.$$

16. $$\int x\left(\frac{x+a}{x+b}\right)^2 dx = \frac{x^2}{2}+2(a-b)x+(a-b)\left[\frac{b(a-b)}{x+b}+(a-3b)\ln|x+b|\right].$$

17. $$\int x^2\left(\frac{x+a}{x+b}\right)^2 dx = \frac{x^3}{3}+(a-b)x^2+\frac{b(a-b)^2x}{x+b}+$$
$$+(a-b)[(a-3b)(x+b)-2b(a-2b)\ln|x+b|].$$

18. $$\int \frac{1}{x}\left(\frac{x+a}{x+b}\right)^2 dx = \frac{(a-b)^2}{b(x+b)}+\frac{a^2}{b^2}\ln|x|-\left(\frac{a^2}{b^2}-1\right)\ln|x+b|.$$

19. $$\int \frac{(x+a)^2}{x+b}\,dx = \frac{x^2}{2}+(2a-b)x+(a-b)^2\ln|x+b|.$$

20. $$\int x\frac{(x+a)^2}{x+b}\,dx = \frac{x^3}{3}+(2a-b)\frac{x^2}{2}+(a-b)^2(x-b\ln|x+b|).$$

21. $$\int x^2\frac{(x+a)^2}{x+b}\,dx = \frac{x^4}{4}+(2a-b)\frac{x^3}{3}+(a-b)^2\left(\frac{x^2}{2}-bx+b^2\ln|x+b|\right).$$

22. $$\int \frac{(x+a)^2}{x(x+b)}\,dx = x+\frac{1}{b}(a^2\ln|x|-(a-b)^2\ln|x+b|).$$

23. $$\int \frac{x+a}{(x+b)^2}\,dx = -\frac{a-b}{x+b}+\ln|x+b|.$$

24. $$\int x\frac{x+a}{(x+b)^2}\,dx = x+\frac{b(a-b)}{x+b}+(a-2b)\ln|x+b|.$$

25. $$\int x^2\frac{x+a}{(x+b)^2}\,dx = \frac{x^2}{2}+(a-2b)x-\frac{b^2(a-b)}{x+b}-b(2a-3b)\ln|x+b|.$$

26. $$\int \frac{x+a}{x(x+b)^2}\,dx = \frac{a-b}{b(x+b)}-\frac{a}{b^2}\ln\left|\frac{x+b}{x}\right|.$$

1.2.7. Интегралы вида $\int \frac{x^p\,dx}{(x+a)^q(x+b)^r}$.

1. $$\int \frac{x^p\,dx}{(x+a)^q(x+b)^r} = \frac{1}{p-q-r+1}\left\{\frac{x^{p-1}}{(x+a)^{q-1}(x+b)^{r-1}}-\right.$$
$$\left.-[(p-r)a+(p-q)b]\int\frac{x^{p-1}\,dx}{(x+a)^q(x+b)^r}-(p-1)ab\int\frac{x^{p-2}\,dx}{(x+a)^q(x+b)^r}\right\}.$$

2. $$\int \frac{x^k\,dx}{(x+a)^m (x+b)^n} =$$
$$= -\sum_{p=1}^{m-1} \frac{(-1)^{m+k+p+1}}{p\,(x+a)^p} \left[\sum_{q=0}^{m-p-1} \binom{k}{q} \binom{m+n-p-q-2}{n-1} \frac{a^{k-q}}{(b-a)^{m+n-p-q-1}} \right] -$$
$$- \sum_{p=1}^{n-1} \frac{(-1)^m}{p\,(x+b)^p} \left[\sum_{q=0}^{n-p-1} \binom{k}{q} \binom{m+n-p-q-2}{m-1} \frac{(-b)^{k-q}}{(b-a)^{m+n-p-q-1}} \right] +$$
$$+ (-1)^{m+k+1} \sum_{p=0}^{m-1} \binom{k}{p} \binom{m+n-p-2}{n-1} \frac{a^{k-p}}{(b-a)^{m+n-p-1}} \ln|x+a| +$$
$$+ (-1)^m \sum_{p=0}^{n-1} \binom{k}{p} \binom{m+n-p-2}{m-1} \frac{(-b)^{k-p}}{(b-a)^{m+n-p-1}} \ln|x+b|.$$

3. $$\int \frac{dx}{(x+a)^q (x+b)^r} =$$
$$= -\frac{1}{(r-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} - \frac{(q+r-2)}{(r-1)(a-b)} \int \frac{dx}{(x+a)^q (x+b)^{r-1}}.$$

4. $$= \frac{1}{(q-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} +$$
$$+ \frac{q+r-2}{(q-1)(a-b)} \int \frac{dx}{(x+a)^{q-1}(x+b)^r}.$$

5. $$\int \frac{dx}{(x+a)^m (x+b)^n} =$$
$$= \frac{(-1)^{m+n-2}}{(b-a)^{m+n-1}} \sum_{\substack{k=0 \\ k\neq n-1}}^{m+n-2} \frac{(-1)^k}{n-k-1} \binom{m+n-2}{k} \left(\frac{x+a}{x+b}\right)^{n-k-1} +$$
$$+ \binom{m+n-2}{n-1} \frac{(-1)^{m-1}}{(b-a)^{m+n-1}} \ln\left|\frac{x+a}{x+b}\right|.$$

6. $$= \sum_{k=1}^{m-1} \frac{(-1)^{m+k}}{k} \binom{m+n-k-2}{n-1} \frac{1}{(b-a)^{m+n-k-1}(x+a)^k} -$$
$$- (-1)^m \sum_{k=1}^{n-1} \frac{1}{k} \binom{m+n-k-2}{m-1} \frac{1}{(b-a)^{m+n-k-1}(x+b)^k} +$$
$$+ \frac{(-1)^{m+1}}{(b-a)^{m+n-1}} \binom{m+n-1}{n-1} \ln\left|\frac{x+a}{x+b}\right|.$$

7. $$\int \frac{dx}{(x+a)(x+b)} = -\frac{1}{a-b} \ln\left|\frac{x+a}{x+b}\right|.$$

8. $$\int \frac{x\,dx}{(x+a)(x+b)} = \frac{1}{a-b} (a \ln|x+a| - b \ln|x+b|).$$

9. $$\int \frac{x^2\,dx}{(x+a)(x+b)} = x + \frac{1}{a-b} (b^2 \ln|x+b| - a^2 \ln|x+a|).$$

10. $\int \frac{dx}{x(x+a)(x+b)} = \frac{1}{ab} \ln|x| + \frac{1}{a(a-b)} \ln|x+a| - \frac{1}{b(a-b)} \ln|x+b|.$

11. $\int \frac{dx}{x^2(x+a)(x+b)} = -\frac{1}{abx} - \frac{a+b}{a^2b^2} \ln|x| -$

$$- \frac{1}{a^2(a-b)} \ln|x+a| + \frac{1}{b^2(a-b)} \ln|x+b|.$$

12. $\int \frac{dx}{(x+a)(x+b)^2} = \frac{-1}{(a-b)(x+b)} + \frac{1}{(a-b)^2} \ln\left|\frac{x+a}{x+b}\right|.$

13. $\int \frac{x\,dx}{(x+a)(x+b)^2} = \frac{b}{(a-b)(x+b)} - \frac{a}{(a-b)^2} \ln\left|\frac{x+a}{x+b}\right|.$

14. $\int \frac{x^2\,dx}{(x+a)(x+b)^2} = \frac{-b^2}{(a-b)(x+b)} + \frac{a^2}{(a-b)^2} \ln|x+a| + \frac{b^2-2ab}{(a-b)^2} \ln|x+b|.$

15. $\int \frac{dx}{x(x+a)(x+b)^2} = \frac{1}{b(a-b)(x+b)} + \frac{1}{ab^2} \ln|x| -$

$$- \frac{1}{a(a-b)^2} \ln|x+a| + \frac{2b-a}{b^2(a-b)} \ln|x+b|.$$

16. $\int \frac{dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2}\left(\frac{1}{x+a} + \frac{1}{x+b}\right) + \frac{2}{(a-b)^3} \ln\left|\frac{x+a}{x+b}\right|.$

17. $\int \frac{x\,dx}{(x+a)^2(x+b)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{x+a} + \frac{b}{x+b}\right) - \frac{a+b}{(a-b)^3} \ln\left|\frac{x+a}{x+b}\right|.$

18. $\int \frac{x^2\,dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2}\left(\frac{a^2}{x+a} + \frac{b^2}{x+b}\right) + \frac{2ab}{(a-b)^3} \ln\left|\frac{x+a}{x+b}\right|.$

1.2.8. Интегралы вида $\int R(x, ax^2+bx+c)\,dx$.

1. $\int (ax^2+bx+c)^n\,dx = \frac{1}{2(2n+1)a} \Big[(2ax+b)(ax^2+bx+c)^n -$

$$- n(b^2-4ac) \int (ax^2+bx+c)^{n-1}\,dx\Big].$$

2. $$= (-1)^n \frac{(n!)^2}{(2n+1)!} \frac{2ax+b}{2a} \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \left(\frac{b^2-4ac}{a}\right)^{n-k} (ax^2+bx+c)^k.$$

3. $\int \frac{(ax^2+bx+c)^n\,dx}{x^m} = -\frac{(ax^2+bx+c)^{n+1}}{(m-1)cx^{m-1}} +$

$$+ \frac{b(n-m+2)}{c(m-1)} \int \frac{(ax^2+bx+c)^n\,dx}{x^{m-1}} + \frac{a(2n-m+3)}{c(m-1)} \int \frac{(ax^2+bx+c)^n}{x^{m-2}}\,dx.$$

4. $\int \frac{x^m\,dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} +$

$$+ \frac{(m-1)c}{(2n-m-1)a} \int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^n} - \frac{(n-m)b}{(2n-m-1)a} \int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^n}.$$

5. $\int \frac{x^{2n-1}\,dx}{(ax^2+bx+c)^n} = \frac{1}{a} \int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^{n-1}} -$

$$- \frac{c}{a} \int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^n} - \frac{b}{a} \int \frac{x^{2n-2}\,dx}{(ax^2+bx+c)^n}.$$

6. $\int \frac{x\,dx}{(ax^2+bx+c)^n} = -\frac{1}{2(n-1)a(ax^2+bx+c)^{n-1}} - \frac{b}{2a}\int \frac{dx}{(ax^2+bx+c)^n}.$

7. $= \frac{bx+2c}{(n-1)(b^2-4ac)(ax^2+bx+c)^{n-1}} + \frac{(2n-3)b}{(n-1)(b^2-4ac)}\int \frac{dx}{(ax^2+bx+c)^{n-1}}.$

8. $\int \frac{x^2\,dx}{(ax^2+bx+c)^n} = -\frac{(b^2-2ac)(2ax+b)-b(b^2-4ac)}{2(n-1)a^2(b^2-4ac)(ax^2+bx+c)^{n-1}} - \frac{2(2n-3)(b^2-2ac)-2(n-1)(b^2-4ac)}{2(n-1)a(b^2-4ac)}\int \frac{dx}{(ax^2+bx+c)^{n-1}}.$

9. $\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}} - \frac{(m+n-2)b}{(m-1)c}\int \frac{dx}{x^{m-1}(ax^2+bx+c)^n} - \frac{(m+2n-3)a}{(m-1)c}\int \frac{dx}{x^{m-2}(ax^2+bx+c)^n}.$

10. $\int \frac{dx}{x(ax^2+bx+c)^n} = \frac{1}{2(n-1)c(ax^2+bx+c)^{n-1}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^n} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1}}.$

11. $\int \frac{dx}{(ax^2+bx+c)^n} = \frac{-2ax-b}{(n-1)(b^2-4ac)(ax^2+bx+c)^{n-1}} - \frac{2(2n-3)a}{(n-1)(b^2-4ac)}\int \frac{dx}{(ax^2+bx+c)^{n-1}}.$

12. $= \frac{2ax+b}{2n-1} \times \sum_{k=0}^{n-2} \frac{2^k(2n-1)(2n-3)\ldots(2n-2k-1)a^k}{(n-1)(n-2)\ldots(n-k-1)(4ac-b^2)^{k+1}(ax^2+bx+c)^{n-k-1}} + \frac{(2n-3)!!\,(2a)^{n-1}}{(n-1)!\,(4ac-b^2)^{n-1}}\int \frac{dx}{ax^2+bx+c}.$

13. $\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|$ $[b^2-4ac>0].$

14. $= \frac{2}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}}$ $[b^2-4ac<0].$

15. $= -\frac{2}{2ax+b}$ $[b^2=4ac].$

16. $= \frac{1}{a(p-q)} \ln\left|\frac{x-p}{x-q}\right|,$

где p, q — действительные корни многочлена ax^2+bx+c.

17. $\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}.$

18. $$\int \frac{dx}{(ax^2+bx+c)^3} = \frac{2ax+b}{2(4ac-b^2)(ax^2+bx+c)^2} + \frac{3a(2ax+b)}{(4ac-b^2)^2(ax^2+bx+c)} + \frac{6a^2}{(4ac-b^2)^2}\int \frac{dx}{ax^2+bx+c}.$$

19. $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{|b^2-4ac|}}\begin{cases} \dfrac{1}{2}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & [b^2>4ac],\\ \operatorname{arctg}\dfrac{2ax+b}{\sqrt{4ac-b^2}} & [4ac>b^2]. \end{cases}$$

20. $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\ln|ax^2+bx+c| + \frac{b^2-2ac}{2a^2}\int \frac{dx}{ax^2+bx+c}.$$

21. $$\int \frac{x^3\,dx}{ax^2+bx+c} = \frac{ax^2-2bx}{2a^2} + \frac{b^2-ac}{2a^3}\ln|ax^2+bx+c| - \frac{b(b^2-3ac)}{2a^3}\int \frac{dx}{ax^2+bx+c}.$$

22. $$\int \frac{x\,dx}{(ax^2+bx+c)^2} = \frac{bx+2c}{(b^2-4ac)(ax^2+bx+c)} + \frac{b}{b^2-4ac}\int \frac{dx}{ax^2+bx+c}.$$

23. $$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \frac{(2ac-b^2)x-bc}{a(b^2-4ac)(ax^2+bx+c)} - \frac{2c}{b^2-4ac}\int \frac{dx}{ax^2+bx+c}.$$

24. $$\int \frac{x^3\,dx}{(ax^2+bx+c)^2} = \frac{c(2ac-b^2)+b(3ac-b^2)x}{a^2(4ac-b^2)} + \frac{1}{2a^2}\ln|ax^2+bx+c| + \frac{b(6ac-b^2)}{2a^2(b^2-4ac)}\int \frac{dx}{ax^2+bx+c}.$$

25. $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\ln\frac{x^2}{|ax^2+bx+c|} - \frac{b}{c\sqrt{|b^2-4ac|}}\begin{cases} \dfrac{1}{2}\ln\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} & [b^2>4ac],\\ \operatorname{arctg}\dfrac{2ax+b}{\sqrt{4ac-b^2}} & [b^2<4ac]. \end{cases}$$

26. $$\int \frac{dx}{x^2(ax^2+bx+c)} = -\frac{1}{cx} - \frac{b}{2c^2}\ln\frac{x^2}{|ax^2+bx+c|} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{ax^2+bx+c}.$$

27. $$\int \frac{dx}{x^3(ax^2+bx+c)} = \frac{2bx-c}{2c^2x^2} + \frac{b^2-ac}{2c^3}\ln\frac{x^2}{|ax^2+bx+c|} + \frac{b(3ac-b^2)}{2c^3}\int \frac{dx}{ax^2+bx+c}.$$

28. $$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)}\left[1+\frac{b(b+2ax)}{b^2-4ac}\right] + \frac{1}{2c^2}\ln\frac{x^2}{|ax^2+bx+c|} - \frac{b}{2c^2}\left(1-\frac{2ac}{b^2-4ac}\right)\int \frac{dx}{ax^2+bx+c}.$$

29. $$\int \frac{dx}{x^2(ax^2+bx+c)^2} = \frac{(b^2-3ac)(b+2ax)}{c^2(4ac-b^2)(ax^2+bx+c)} - \frac{c+bx}{c^2x(ax^2+bx+c)} - \frac{b}{c^3}\ln\frac{x^2}{|ax^2+bx+c|} + \frac{1}{b^2-4ac}\left(\frac{b^4}{c^3} - \frac{6b^2a}{c^2} + \frac{6a^2}{c}\right)\int \frac{dx}{ax^2+bx+c}.$$

30. $$\int \frac{dx}{x^3(ax^2+bx+c)^2} = \\ = \frac{3bx-c}{2c^2x^2(ax^2+bx+c)} + \frac{3b^2-2ac}{c^2}\int \frac{dx}{x(ax^2+bx+c)^2} + \frac{9ab}{2c^2}\int \frac{dx}{(ax^2+bx+c)^2}.$$

1.2.9. Интегралы вида $\int R(x+d,\ ax^2+bx+c)\,dx$.

1. $$\int (x+d)^m (ax^2+bx+c)^n\,dx = \frac{1}{(m+2n+1)a}\Big[(x+d)^{m-1}(ax^2+bx+c)^{n+1} + (m+n)(2ad-b)\int (x+d)^{m-1}(ax^2+bx+c)^n\,dx - (m-1)(ad^2-bd+c)\int (x+d)^{m-2}(ax^2+bx+c)^n\,dx\Big].$$

2. $$\int \frac{(ax^2+bx+c)^n}{(x+d)^m}\,dx = -\frac{(ax^2+bx+c)^n}{(m-2n-1)(x+d)^{m-1}} - \frac{2n(ad^2-bd+c)}{m-2n-1}\int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^m}\,dx - \frac{n(b-2ad)}{m-2n-1}\int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-1}}\,dx.$$

3. $$= -\frac{1}{(m-1)(ad^2-bd+c)}\Big[\frac{(ax^2+bx+c)^{n+1}}{(x+d)^{m-1}} + (m-n-2)(b-2ad)\int \frac{(ax^2+bx+c)^n}{(x+d)^{m-1}}\,dx + (m-2n-3)a\int \frac{(ax^2+bx+c)^n}{(x+d)^{m-2}}\,dx\Big].$$

4. $$= -\frac{(ax^2+bx+c)^n}{(m-1)(x+d)^{m-1}} + \frac{n(b-2ad)}{m-1}\int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-1}}\,dx + 2na\int \frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-2}}\,dx.$$

5. $$\int \frac{(ax^2+bx+c)^n}{x+d}\,dx = (ad^2-bd+c)^n \ln|x+d| + \int (ax+b-ad)\left[\sum_{k=0}^{n-1}(ad^2-bd+c)^{n-k-1}(ax^2+bx+c)^k\right]dx.$$

6. $$\int \frac{(x+d)^m\,dx}{(ax^2+bx+c)^n} = \frac{(2ad-b)(2ax+b)+(4ac-b^2)}{2(n-1)(4ac-b^2)(ad^2-bd+c)}\,\frac{(x+d)^{m+1}}{(ax^2+bx+c)^{n-1}} + \frac{2(m-2n+4)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)}\,\frac{(x+d)^{m+1}}{(ax^2+bx+c)^{n-2}} + \left[\frac{4(2n-3)a}{(n-1)(4ac-b^2)} - \frac{2(m-n+2)}{(n-1)(ad^2-bd+c)}\right]\int \frac{(x+d)^m\,dx}{(ax^2+bx+c)^{n-1}} - \frac{2(m-2n+4)(m-2n+5)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)}\int \frac{(x+d)^m\,dx}{(ax^2+bx+c)^{n-2}}.$$

7. $$= \frac{1}{(m-2n+1)a}\Big[\frac{(x+d)^{m-1}}{(ax^2+bx+c)^{n-1}} + (m-n)(2ad-b)\int \frac{(x+d)^{m-1}\,dx}{(ax^2+bx+c)^n} - (m-1)(ad^2-bd+c)\int \frac{(x+d)^{m-2}\,dx}{(ax^2+bx+c)^n}\Big].$$

8. $$=\frac{1}{(n-1)(4ac-b^2)}\left[\frac{(2ax+b)(x+d)^m}{(ax^2+bx+c)^{n-1}}-\right.$$
$$\left.-2(m-2n+3)a\int\frac{(x+d)^m\,dx}{(ax^2+bx+c)^{n-1}}-(b-2ad)m\int\frac{(x+d)^{m-1}\,dx}{(ax^2+bx+c)^{n-1}}\right].$$

9. $$\int\frac{dx}{(x+d)^m(ax^2+bx+c)^n}=\frac{-1}{(m-1)(ad^2-bd+c)}\times$$
$$\times\left[\frac{1}{(x+d)^{m-1}(ax^2+bx+c)^{n-1}}+2(m+n-2)(b-2ad)\times\right.$$
$$\left.\times\int\frac{dx}{(x+d)^{m-1}(ax^2+bx+c)^n}+(m+2n-3)a\int\frac{dx}{(x+d)^{m-2}(ax^2+bx+c)^n}\right].$$

10. $$=\frac{1}{2(ad^2-bd+c)}\left[\frac{1}{(n-1)(x+d)^{m-1}(ax^2+bx+c)^{n-1}}+2(ad-b)\times\right.$$
$$\left.\times\int\frac{dx}{(x+d)^{m-1}(ax^2+bx+c)^n}+\frac{m+2n-3}{n-1}\int\frac{dx}{(x+d)^m(ax^2+bx+c)^{n-1}}\right].$$

11. $$=\frac{1}{2(m+n-1)(ad-b)}\left[\frac{1}{(x+d)^m(ax^2+bx+c)^{n-1}}+\right.$$
$$\left.+(m+2n-2)a\int\frac{dx}{(x+d)^{m-1}(ax^2+bx+c)^n}\right]\qquad [ad^2-bd+c=0].$$

12. $$\int\frac{dx}{(x+d)(ax^2+bx+c)^n}=\frac{1}{ad^2-bd+c}\left[\frac{1}{2(n-1)(ax^2+bx+c)^{n-1}}+\right.$$
$$\left.+(ad-b)\int\frac{dx}{(ax^2+bx+c)^n}+\int\frac{dx}{(x+d)(ax^2+bx+c)^{n-1}}\right].$$

13. $$\int\frac{dx}{(x+d)(ax^2+bx+c)}=\frac{1}{2(ad^2-bd+c)}\ln\frac{(x+a)^2}{|ax^2+bx+c|}+$$
$$+\frac{2ad-b}{2(ad^2-bd+c)\sqrt{|b^2-4ac|}}\begin{cases}\operatorname{arctg}\dfrac{2ax+b}{\sqrt{|b^2-4ac|}} & [b^2-4ac<0],\\[2ex] \ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & [b^2-4ac>0].\end{cases}$$

14. $$\int\frac{dx}{(x+d)^2(ax^2+bx+c)}=$$
$$=-\frac{1}{ad^2-bd+c}\left[\frac{1}{x+d}+\frac{2ad-b}{2(ad^2-bd+c)}\ln\frac{(x+a)^2}{|ax^2+bx+c|}+\right.$$
$$\left.+\frac{2c(ad^2-bd+c)-(2ad-b^2)}{2(ad^2-bd+c)^2}\int\frac{dx}{ax^2+bx+c}\right].$$

1.2.10. Интегралы вида $\int\frac{x^m\,dx}{(x^2\pm a^2)^n}$.

1. $$\int\frac{x^m\,dx}{(x^2\pm a^2)^n}=\pm\frac{x^{m+1}}{2(n-1)a^2(x^2\pm a^2)^{n-1}}\mp\frac{(m-2n+3)}{2(n-1)a^2}\int\frac{x^m\,dx}{(x^2\pm a^2)^{n-1}}.$$

2. $$=\frac{x^{m-1}}{(m-2n+1)(x^2\pm a^2)^{n-1}}\mp\frac{(m-1)a^2}{m-2n+1}\int\frac{x^{m-2}\,dx}{(x^2\pm a^2)^n}.$$

3. $$=\frac{-x^{m-1}}{2(n-1)(x^2\pm a^2)^{n-1}}+\frac{m-1}{2(n-1)}\int\frac{x^{m-2}}{(x^2\pm a^2)^{n-1}}.$$

4. $$=\int\frac{x^{m-2}\,dx}{(x^2\pm a^2)^{n-1}}\mp a^2\int\frac{x^{m-2}\,dx}{(x^2\pm a^2)^n}.$$

5. $$\int \frac{x^{2m}\,dx}{(x^2+a^2)^n} = \frac{x^{2m+1}}{2(n-1)a^2}\left[\frac{1}{(x^2+a^2)^{n-1}} + \right.$$
$$+\sum_{k=1}^{n-2} \frac{(2n-2m-3)(2n-2m-5)\ldots(2n-2m-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)}\left.\frac{1}{a^{2k}(x^2+a^2)^{n-k-1}}\right]+$$
$$+(-1)^m a^{2m-2n+1}\frac{(2n-2m-3)(2n-2m-5)\ldots(3-2m)(1-2m)}{2^{n-1}(n-1)!}\times$$
$$\times\left[\sum_{k=1}^{m}\frac{(-1)^k}{2k-1}\left(\frac{x}{a}\right)^{2k-1}+\operatorname{arctg}\frac{x}{a}\right].$$

6. $$\int \frac{x^{2m}\,dx}{(x^2-a^2)^n} = -\frac{x^{2m+1}}{2(n-1)a^2}\left[\frac{1}{(x^2-a^2)^{n-1}} - \right.$$
$$-\sum_{k=1}^{n-2}(-1)^k\frac{(2n-2m-3)(2n-2m-5)\ldots(2n-2m-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)}\times$$
$$\times\left.\frac{1}{a^{2k}(x^2-a^2)^{n-k-1}}\right]+$$
$$+(-1)^{n+1}a^{2m-2n+1}\frac{(2n-2m-3)(2n-2m-5)\ldots(3-2m)(1-2m)}{2^{n-1}(n-1)!}\times$$
$$\times\left[\sum_{k=1}^{m}\frac{1}{2k-1}\left(\frac{x}{a}\right)^{2k-1}-\frac{1}{2}\ln\left|\frac{x+a}{x-a}\right|\right].$$

7. $$\int \frac{x^{2m+1}\,dx}{(x^2\pm a^2)^n} = -\sum_{k=0}^{m}(\mp 1)^k\binom{m}{k}\frac{a^{2k}}{2(n-m+k-1)(x^2\pm a^2)^{n-m+k-1}};$$

8. $$= \frac{1}{2(\pm a^2)^{n-m-1}}\sum_{k=0}^{n-m-2}(-1)^k\binom{n-m-2}{k}\frac{1}{m+k+1}\times$$
$$\times\left(\frac{x^2}{x^2\pm a^2}\right)^{m+k+1} \qquad [n\geqslant m+2]$$

9. $$\int \frac{x^{2m}\,dx}{x^2\pm a^2} = \sum_{k=0}^{m-1}(\mp 1)^k\frac{a^{2k}x^{2m-2k-1}}{2m-2k-1}+(\mp 1)^m a^{2m-1}\left\{\begin{matrix}\operatorname{arctg}\dfrac{x}{a},\\ \dfrac{1}{2}\ln\left|\dfrac{x-a}{x+a}\right|\end{matrix}\right\}.$$

10. $$\int \frac{x^{2m+1}\,dx}{x^2\pm a^2} = \frac{1}{2}\sum_{k=0}^{m-1}(\mp 1)^k\frac{a^{2k}x^{2m-2k}}{m-k}+\frac{1}{2}(\mp a^2)^m\ln|x^2\pm a^2|.$$

11. $$\int \frac{x^{2m}\,dx}{(x^2\pm a^2)^2} = \pm\frac{x^{2m+1}}{2a^2(x^2\pm a^2)}+\frac{1}{2}(1-2m)a^{2m-2}\times$$
$$\times\left[\pm\sum_{k=1}^{m}\frac{(\mp 1)^{m+k}}{2k-1}\left(\frac{x}{a}\right)^{2k-1}+(\mp 1)^m\left\{\begin{matrix}\operatorname{arctg}\dfrac{x}{a},\\ \dfrac{1}{2}\ln\left|\dfrac{x+a}{x-a}\right|\end{matrix}\right\}\right].$$

12. $$\int \frac{dx}{(x^2 \pm a^2)^n} = \pm \frac{x}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \pm \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1}}.$$

13. $$\int \frac{dx}{(x^2 \pm a^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} (\pm 1)^k \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)a^{2k}(x^2 \pm a^2)^{n-k}} +$$
$$+ (\pm 1)^n \frac{(2n-3)!!}{2^{n-1}(n-1)!\,a^{2n-1}} \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{matrix} \right\}.$$

14. $$\int \frac{dx}{x^2 \pm a^2} = \frac{1}{a} \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{matrix} \right\}.$$

15. $$\int \frac{dx}{(x^2 \pm a^2)^2} = \frac{x}{2a^2(a^2 \pm x^2)} + \frac{1}{2a^3} \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{matrix} \right\}.$$

16. $$\int \frac{dx}{(x^2 \pm a^2)^3} = \pm \frac{x}{4a^2(x^2 \pm a^2)^2} + \frac{3x}{8a^4(x^2 \pm a^2)} + \frac{3}{8a^5} \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a}, \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{matrix} \right\}.$$

17. $$\int \frac{x\,dx}{x^2 \pm a^2} = \frac{1}{2} \ln |x^2 \pm a^2|.$$

18. $$\int \frac{x^2\,dx}{x^2 \pm a^2} = x - a \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{matrix} \right\}.$$

19. $$\int \frac{x^3\,dx}{x^2 \pm a^2} = \frac{x^2}{2} \mp \frac{a^2}{2} \ln |x^2 \pm a^2|.$$

20. $$\int \frac{x^4\,dx}{x^2 \pm a^2} = \frac{x^3}{3} \mp a^2 x \pm \left\{ \begin{matrix} a^3 \operatorname{arctg} \frac{x}{a} \\ \frac{a^3}{2} \ln \left| \frac{x+a}{x-a} \right| \end{matrix} \right\}.$$

21. $$\int \frac{x\,dx}{(x^2 \pm a^2)^2} = -\frac{1}{2(x^2 \pm a^2)}.$$

22. $$\int \frac{x^2\,dx}{(x^2 \pm a^2)^2} = -\frac{x}{2(x^2 \pm a^2)} + \left\{ \begin{matrix} \frac{1}{2a} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{4a} \ln \left| \frac{x-a}{x+a} \right| \end{matrix} \right\}.$$

23. $$\int \frac{x^3\,dx}{(x^2 \pm a^2)^2} = \pm \frac{a^2}{2(x^2 \pm a^2)} + \frac{1}{2} \ln |x^2 \pm a^2|.$$

24. $$\int \frac{x^4\,dx}{(x^2 \pm a^2)^2} = x \pm \frac{a^2 x}{2(x^2 \pm a^2)} - \left\{ \begin{matrix} \frac{3a}{2} \operatorname{arctg} \frac{x}{a} \\ \frac{3a}{4} \ln \left| \frac{x+a}{x-a} \right| \end{matrix} \right\}.$$

1.2.11. Интегралы вида $\int \frac{dx}{x^m (x^2 \pm a^2)^n}$.

1. $$\int \frac{dx}{x^{2m} (x^2 \pm a^2)^n} = \mp \frac{1}{(2m-1)\, a^2 x^{2m-1} (x^2 \pm a^2)^{n-1}} \mp$$
$$\mp \frac{2m+2n-3}{(2m-1)\, a^2} \int \frac{dx}{x^{2m-2} (x^2 \pm a^2)^n}.$$

2. $$= \frac{\pm 1}{(2m+2n-1)(x^2 \pm a^2)^{n-1}} \times$$
$$\times \sum_{k=1}^{m} \frac{(\mp 1)^k (2m+2n-1)(2m-2n-3) \ldots (2m+2n-2k+1)}{(2m-1)(2m-3) \ldots (2m-2k+1)\, a^{2k}} \frac{1}{x^{2m-2k+1}} +$$
$$+ (\mp 1)^m \frac{(2n-1)(2n+1) \ldots (2n+2m-3)}{(2m-1)!!\, a^{2m}} \int \frac{dx}{(x^2 \pm a^2)^n}$$ [см. 1.2.10].

3. $$\int \frac{dx}{x^{2m+1} (x^2 \pm a^2)^n} = \mp \frac{1}{2ma^2 x^{2m} (x^2 \pm a^2)^{n-1}} \mp$$
$$\mp \frac{m+n-1}{ma^2} \int \frac{dx}{x^{2m-1} (x^2 \pm a^2)^n}.$$

4. $$= \frac{1}{2(\mp a^2)^{m+n}} \sum_{k=0}^{m+n-1} (-1)^k \binom{m+n-1}{k} \frac{1}{k-n+1} \left(\frac{x^2 \pm a^2}{x^2} \right)^{k-n+1}.$$

5. $$\int \frac{dx}{x (x^2 \pm a^2)^n} = \frac{\pm 1}{2(n-1)\, a^2 (x^2 \pm a^2)^{n-1}} \pm \frac{1}{a^2} \int \frac{dx}{x (x^2 \pm a^2)^{n-1}}.$$

6. $$= \frac{1}{2} \sum_{k=1}^{n-1} \frac{(\pm 1)^k}{(n-k)\, a^{2k} (x^2 \pm a^2)^{n-k}} + \frac{(\pm 1)^n}{2a^{2n}} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

7. $$\int \frac{dx}{x^{2m} (x^2 \pm a^2)} = \sum_{k=0}^{m-1} \frac{(\mp 1)^{m-k}}{(2k+1)\, a^{2(m-k)} x^{2k+1}} + \frac{(\mp 1)^m}{a^{2m+1}} \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \end{matrix} \right\}.$$

8. $$\int \frac{dx}{x^{2m+1} (x^2 \pm a^2)} = \frac{1}{2} \sum_{k=0}^{m-1} \frac{(\mp 1)^{k+1}}{(m-k)\, a^{2(k+1)} x^{2(m-k)}} + \frac{(\mp 1)^{m+1}}{2a^{2(m+1)}} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

9. $$\int \frac{dx}{x^{2m} (x^2 \pm a^2)^2} = \pm \sum_{k=0}^{m-1} \frac{(\mp 1)^{m-k} (m-k)}{(2k+1)\, a^{2m-2k+2} x^{2k+1}} -$$
$$- \frac{(\mp 1)^{m+1} x}{2a^{2m+2} (x^2 \pm a^2)} + \frac{(\mp 1)^m (2m+1)}{2a^{2m+3}} \left\{ \begin{matrix} \operatorname{arctg} \frac{x}{a} \\ \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \end{matrix} \right\}.$$

10. $$\int \frac{dx}{x(x^2 \pm a^2)} = \pm \frac{1}{2a^2} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

11. $$\int \frac{dx}{x^2(x^2 \pm a^2)} = \mp \frac{1}{a^2 x} \mp \frac{1}{a^3} \left\{ \begin{matrix} \operatorname{arctg} \dfrac{x}{a} \\ \dfrac{1}{2} \ln \left| \dfrac{x-a}{x+a} \right| \end{matrix} \right\}.$$

12. $$\int \frac{dx}{x^3(x^2 \pm a^2)} = \mp \frac{1}{2a^2x^2} - \frac{1}{2a^4} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

13. $$\int \frac{dx}{x^4(x^2 \pm a^2)} = \mp \frac{1}{3a^2x^3} + \frac{1}{a^4x} + \frac{1}{a^5} \left\{ \begin{matrix} \operatorname{arctg} \dfrac{x}{a} \\ \dfrac{1}{2} \ln \left| \dfrac{x-a}{x+a} \right| \end{matrix} \right\}.$$

14. $$\int \frac{dx}{x(x^2 \pm a^2)^2} = \frac{1}{2a^2(a^2 \pm x^2)} + \frac{1}{2a^4} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

15. $$\int \frac{dx}{x^2(x^2 \pm a^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(x^2 \pm a^2)} - \frac{3}{2a^5} \left\{ \begin{matrix} \operatorname{arctg} \dfrac{x}{a} \\ \dfrac{1}{2} \ln \left| \dfrac{x-a}{x+a} \right| \end{matrix} \right\}.$$

16. $$\int \frac{dx}{x^3(x^2 \pm a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2 \pm a^2)} \mp \frac{1}{a^6} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

17. $$\int \frac{dx}{x^4(x^2 \pm a^2)^2} = -\frac{1}{3a^4x^3} \pm \frac{2}{a^6x} \pm \frac{x}{2a^6(x^2 \pm a^2)} + \frac{5}{2a^7} \left\{ \begin{matrix} \operatorname{arctg} \dfrac{x}{a} \\ \dfrac{1}{2} \ln \left| \dfrac{x+a}{x-a} \right| \end{matrix} \right\}.$$

18. $$\int \frac{dx}{(x+b)(x^2+a^2)} = \frac{1}{a^2+b^2} \left[\ln|x+b| - \frac{1}{2} \ln(x^2+a^2) + \frac{b}{a} \operatorname{arctg} \frac{x}{a} \right].$$

19. $$\int \frac{dx}{(x+b)(x^2-a^2)} =$$
$$= -\frac{1}{a^2-b^2} \ln|x+b| + \frac{1}{2a(a+b)} \ln|x-a| + \frac{1}{2a(a-b)} \ln|x+a|.$$

1.2.12. Интегралы вида $\int \frac{x^{\pm m}\,dx}{(x^3+a^3)^n}$.

1. $$\int \frac{x^m\,dx}{(x^3+a^3)^n} = \frac{x^{m+1}}{3(n-1)a^3(x^3+a^3)^{n-1}} - \frac{m-3n+4}{3(n-1)a^3} \int \frac{x^m\,dx}{(x^3+a^3)^{n-1}}.$$

2. $$= \frac{x^{m-2}}{(m-3n+1)(x^3+a^3)^{n-1}} - \frac{(m-2)a^3}{m-3n+1} \int \frac{x^{m-3}\,dx}{(x^3+a^3)^n}.$$

3. $$\int \frac{x^m\,dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3 \int \frac{x^{m-3}\,dx}{x^3+a^3}.$$

4. $$\int \frac{dx}{(x^3+a^3)^n} = \frac{x}{3a^3(n-1)(x^3+a^3)^{n-1}} + \frac{3n-4}{3a^3(n-1)} \int \frac{dx}{(x^3+a^3)^{n-1}}.$$

5. $$\int \frac{x\,dx}{(x^3+a^3)^n} = \frac{x^2}{3a^3(n-1)(x^3+a^3)^{n-1}} + \frac{3n-5}{3a^3(n-1)} \int \frac{x\,dx}{(x^3+a^3)^{n-1}}.$$

6. $\int\frac{dx}{x^3+a^3}=\frac{1}{6a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2}+\frac{1}{a^2\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

7. $\int\frac{x\,dx}{x^3+a^3}=\frac{1}{6a}\ln\frac{x^2-ax+a^2}{(x+a)^2}+\frac{1}{a\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

8. $\int\frac{x^2\,dx}{x^3+a^3}=\frac{1}{3}\ln|x^3+a^3|.$

9. $\int\frac{x^3\,dx}{x^3+a^3}=x+\frac{a}{6}\ln\frac{x^2-ax+a^2}{(x+a)^2}-\frac{a}{\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

10. $\int\frac{dx}{x(x^3+a^3)}=\frac{1}{3a^3}\ln\left|\frac{x^3}{x^3+a^3}\right|.$

11. $\int\frac{dx}{x^2(x^3+a^3)}=-\frac{1}{a^3x}+\frac{1}{6a^4}\ln\frac{(x+a)^2}{x^2-ax+a^2}-\frac{1}{a^4\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

12. $\int\frac{dx}{x^3(x^3+a^3)}=-\frac{1}{2a^3x^2}-\frac{1}{6a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2}-\frac{1}{a^5\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

13. $\int\frac{dx}{(x^3+a^3)^2}=\frac{x}{3a^3(x^3+a^3)}+\frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2}+\frac{2}{3a^5\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

14. $\int\frac{x\,dx}{(x^3+a^3)^2}=\frac{x^2}{3a^3(x^3+a^3)}+\frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2}+\frac{1}{3a^4\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

15. $\int\frac{x^2\,dx}{(x^3+a^3)^2}=-\frac{1}{3(x^3+a^3)}.$

16. $\int\frac{dx}{x(x^3+a^3)^2}=\frac{1}{3a^3(x^3+a^3)}+\frac{1}{3a^6}\ln\left|\frac{x^3}{x^3+a^3}\right|.$

17. $\int\frac{dx}{x^2(x^3+a^3)^2}=\frac{1}{3a^3x(x^3+a^3)}-\frac{4}{3a^6x}+$

$$+\frac{2}{9a^7}\ln\frac{(x+a)^2}{x^2-ax+a^2}-\frac{4}{3a^7\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$$

18. $\int\frac{dx}{x^3(x^3+a^3)^2}=-\frac{1}{3a^3x^2(x^3+a^3)}-\frac{5}{6a^6x^2}-$

$$-\frac{5}{18a^8}\ln\frac{(x+a)^2}{x^2-ax+a^2}-\frac{5}{3a^8\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$$

1.2.13. Интегралы вида $\int\frac{x^{\pm m}dx}{(x^4\pm a^4)^n}$, $\int\frac{x^{\pm m}dx}{(ax^4+bx^2+c)^n}$.

1. $\int\frac{x^m\,dx}{(x^4\pm a^4)^n}=\pm\frac{x^{m+1}}{4(n-1)a^4(x^4\pm a^4)^{n-1}}\pm\frac{4n-m-5}{4(n-1)a^4}\int\frac{x^m\,dx}{(x^4\pm a^4)^{n-1}}.$

2. $\int\frac{x^m\,dx}{(x^4\pm a^4)^n}=\frac{x^{m-3}}{(m-4n+1)(x^4\pm a^4)^{n-1}}\mp\frac{(m-3)a^4}{(m-4n+1)}\int\frac{x^{m-4}\,dx}{(x^4\pm a^4)^n}.$

3. $\int\frac{x^m\,dx}{x^4\pm a^4}=\frac{x^{m-3}}{m-3}\mp a^4\int\frac{x^{m-4}\,dx}{x^4\pm a^4}.$

4. $$\int \frac{dx}{x^m (x^4 \pm a^4)^n} = \mp \frac{1}{(m-1)\, a^4 x^{m-1} (x^4 \pm a^4)^{n-1}} \mp$$
$$\mp \frac{4n+m-5}{(m-1)\, a^4} \int \frac{dx}{x^{m-4} (x^4 \pm a^4)^n}.$$

5. $$= \pm \frac{1}{a^4} \int \frac{dx}{x^m (x^4 \pm a^4)^{n-1}} - \frac{1}{a^4} \int \frac{dx}{x^{m-4} (x^4 \pm a^4)^n}.$$

6. $$\int \frac{dx}{x^4 \pm a^4} = \begin{Bmatrix} \dfrac{1}{4a^3\sqrt{2}} \ln \dfrac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \dfrac{1}{2a^3\sqrt{2}} \operatorname{arctg} \dfrac{ax\sqrt{2}}{a^2 - x^2} \\ \dfrac{1}{4a^3} \ln \left| \dfrac{x+a}{x-a} \right| + \dfrac{1}{2a^3} \operatorname{arctg} \dfrac{x}{a} \end{Bmatrix}.$$

7. $$\int \frac{x\, dx}{x^4 \pm a^4} = \begin{Bmatrix} \dfrac{1}{2a^2} \operatorname{arctg} \dfrac{x^2}{a^2} \\ \dfrac{1}{4a^2} \ln \dfrac{|x^2 - a^2|}{x^2 + a^2} \end{Bmatrix}.$$

8. $$\int \frac{x^2\, dx}{x^4 \pm a^4} = \begin{Bmatrix} \dfrac{1}{4a\sqrt{2}} \ln \dfrac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2} + \dfrac{1}{2a\sqrt{2}} \operatorname{arctg} \dfrac{ax\sqrt{2}}{a^2 - x^2} \\ -\dfrac{1}{4a} \ln \left| \dfrac{x+a}{x-a} \right| + \dfrac{1}{2a} \operatorname{arctg} \dfrac{x}{a} \end{Bmatrix}.$$

9. $$\int \frac{x^3\, dx}{x^4 \pm a^4} = \frac{1}{4} \ln |x^4 \pm a^4|.$$

10. $$\int \frac{dx}{x (x^4 \pm a^4)} = \pm \frac{1}{4a^4} \ln \frac{x^4}{|x^4 \pm a^4|}.$$

11. $$\int \frac{dx}{x^2 (x^4 \pm a^4)} = \mp \frac{1}{a^4 x} \mp \frac{1}{a^4} \int \frac{x^2\, dx}{x^4 \pm a^4}.$$

12. $$\int \frac{dx}{x^3 (x^4 \pm a^4)} = \mp \frac{1}{2a^4 x^2} \mp \frac{1}{4a^6} \begin{Bmatrix} 2 \operatorname{arctg} \dfrac{x^2}{a^2} \\ \ln \dfrac{|x^2 - a^2|}{x^2 + a^2} \end{Bmatrix}.$$

13. $$\int \frac{x^m\, dx}{(x^4 \pm a^4)^2} = \frac{x^{m+1}}{4a^4 (a^4 \pm x^4)} \pm \frac{3-m}{4a^4} \int \frac{x^m\, dx}{x^4 \pm a^4}.$$

14. $$\int \frac{dx}{(x^4 \pm a^4)^2} = \frac{x}{4a^4 (a^4 \pm x^4)} \pm$$
$$\pm \begin{Bmatrix} \dfrac{3}{16a^7\sqrt{2}} \ln \dfrac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \dfrac{3}{8a^7\sqrt{2}} \operatorname{arctg} \dfrac{ax\sqrt{2}}{a^2 - x^2} \\ \dfrac{3}{16a^7} \ln \left| \dfrac{x+a}{x-a} \right| + \dfrac{3}{8a^7} \operatorname{arctg} \dfrac{x}{a} \end{Bmatrix}.$$

15. $$\int \frac{x\, dx}{(x^4 \pm a^4)^2} = \frac{x^2}{4a^4 (a^4 \pm x^4)} + \begin{Bmatrix} \dfrac{1}{4a^6} \operatorname{arctg} \dfrac{x^2}{a^2} \\ \dfrac{1}{8a^6} \ln \dfrac{x^2 + a^2}{|x^2 - a^2|} \end{Bmatrix}.$$

16. $$\int \frac{x^2\,dx}{(x^4 \pm a^4)^2} = \frac{x^3}{4a^4(a^4 \pm x^4)} \pm \\ \pm \begin{cases} \dfrac{1}{16a^5\sqrt{2}} \ln \dfrac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2} + \dfrac{1}{8a^5\sqrt{2}} \operatorname{arctg} \dfrac{ax\sqrt{2}}{a^2 - x^2} \\ \dfrac{1}{16a^5} \ln \left| \dfrac{x-a}{x+a} \right| + \dfrac{1}{8a^5} \operatorname{arctg} \dfrac{x}{a} \end{cases}.$$

17. $$\int \frac{x^3\,dx}{(x^4 \pm a^4)^2} = -\frac{1}{4(x^4 \pm a^4)}.$$

18. $$\int \frac{dx}{x(x^4 \pm a^4)^2} = \frac{1}{4a^4(a^4 \pm x^4)} + \frac{1}{4a^8} \ln \frac{x^4}{|x^4 \pm a^4|}.$$

19. $$\int \frac{dx}{x^2(x^4 \pm a^4)^2} = -\frac{1}{a^8 x} - \frac{x^3}{4a^8(x^4 \pm a^4)} - \frac{5}{4a^8} \int \frac{x^2\,dx}{x^4 \pm a^4}.$$

20. $$\int \frac{x^m\,dx}{(ax^4+bx^2+c)^n} = \frac{x^{m-3}}{(m-4n+1)a(ax^4+bx^2+c)^{n-1}} - \\ - \frac{(m-3)c}{(m-4n+1)a} \int \frac{x^{m-4}\,dx}{(ax^4+bx^2+c)^n} - \frac{(m-2n-1)b}{(m-4n+1)a} \int \frac{x^{m-2}\,dx}{(ax^4+bx^2+c)^n}.$$

21. $$= \frac{1}{a} \int \frac{x^{m-4}\,dx}{(ax^4+bx^2+c)^{n-1}} - \frac{c}{a} \int \frac{x^{m-4}\,dx}{(ax^4+bx^2+c)^n} - \\ - \frac{b}{a} \int \frac{x^{m-2}\,dx}{(ax^4+bx^2+c)^n}.$$

22. $$\int \frac{dx}{(ax^4+bx^2+c)^n} = \frac{abx^3 + (b^2-2ac)x}{2(n-1)c(b^2-4ac)(ax^4+bx^2+c)^{n-1}} + \\ + \frac{(4n-7)ab}{2(n-1)c(b^2-4ac)} \int \frac{x^2\,dx}{(ax^4+bx^2+c)^{n-1}} + \\ + \frac{2(n-1)(b^2-4ac)+2ac-b^2}{2(n-1)c(b^2-4ac)} \int \frac{dx}{(ax^4+bx^2+c)^{n-1}}.$$

23. $$\int \frac{dx}{x^m(ax^4+bx^2+c)^n} = \frac{-1}{(m-1)cx^{m-1}(ax^4+bx^2+c)^{n-1}} - \\ - b\frac{m+2n-3}{(m-1)c} \int \frac{dx}{x^{m-2}(ax^4+bx^2+c)^n} - a\frac{m+4n-5}{(m-1)c} \int \frac{dx}{x^{m-4}(ax^4+bx^2+c)^n}.$$

24. $$\int \frac{dx}{ax^4+bx^2+c} = \frac{2a}{\sqrt{b^2-4ac}} \left(\int \frac{dx}{2ax^2+b-\sqrt{b^2-4ac}} - \right. \\ \left. - \int \frac{dx}{2ax^2+b+\sqrt{b^2-4ac}} \right) \qquad [b^2 > 4ac].$$

25. $$= \frac{1}{2a \sin\alpha} \left[\sin\frac{\alpha}{2} \ln \frac{x^2 + 2x\sqrt[4]{a/c}\cos(\alpha/2) + \sqrt{a/c}}{x^2 - 2x\sqrt[4]{a/c}\cos(\alpha/2) + \sqrt{a/c}} + \right. \\ \left. + 2\cos\frac{\alpha}{2} \operatorname{arctg} \frac{x^2 - \sqrt{a/c}}{2x\sqrt{a/c}\sin(\alpha/2)} \right] \qquad \left[b^2 < 4ac;\ \cos\alpha = -\frac{b}{2\sqrt{ac}} \right].$$

26. $$\int \frac{dx}{(ax^4+bx^2+c)^2} = \frac{abx^3+(b^2-2ac)x}{2c(b^2-4ac)(ax^4+bx^2+c)} + \\ + \frac{b^2-6ac}{2a(b^2-4ac)} \int \frac{dx}{ax^4+bx^2+c} + \frac{ab}{2c(b^2-4ac)} \int \frac{x^2\,dx}{ax^4+bx^2+c}.$$

27. $$\int \frac{x\,dx}{ax^4+bx^2+c} = \frac{1}{2\sqrt{b^2-4ac}} \ln \left| \frac{2ax^2+b-\sqrt{b^2-4ac}}{2ax^2+b+\sqrt{b^2-4ac}} \right| \qquad [b^2>4ac].$$

28. $$= \frac{1}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{2ax^2+b}{\sqrt{4ac-b^2}} \qquad [b^2<4ac].$$

29. $$\int \frac{x^2\,dx}{ax^4+bx^2+c} = \frac{b+\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}} \int \frac{dx}{2ax^2+b+\sqrt{b^2-4ac}} - \frac{b-\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}} \int \frac{dx}{2ax^2+b-\sqrt{b^2-4ac}} \qquad [b^2>4ac].$$

30. $$= \frac{1}{4a(\sqrt{c/4a}-b/4a)^{1/2}} \left(\int \frac{x\,dx}{x^2-2(\sqrt{c/4a}-b/4a)^{1/2}x+2\sqrt{c/4a}} - \int \frac{x\,dx}{x^2+2(\sqrt{c/4a}-b/4a)^{1/2}x+2\sqrt{c/4a}} \right) \qquad [b^2<4ac].$$

1.2.14. Интегралы вида $\int R(x,\ ax^{2k}+bx^k+c)\,dx$.

1. $$\int x^m (ax^{2k}+bx^k+c)^n\,dx = \frac{x^{m+1}(ax^{2k}+bx^k+c)^{n+1}}{(m+1)c} - \frac{(m+nk+k+1)b}{(m+1)c} \int x^{m+k}(ax^{2k}+bx^k+c)^n\,dx - \frac{(m+2nk+2k+1)a}{(m+1)c} \int x^{m+2k}(ax^{2k}+bx^k+c)^n\,dx.$$

2. $$= \frac{x^{m+1}(ax^{2k}+bx^k+c)^n}{m+1} - \frac{bnk}{m+1} \int x^{m+k}(ax^{2k}+bx^k+c)^{n-1}\,dx - \frac{2ank}{m+1} \int x^{m+2k}(ax^{2k}+bx^k+c)^{n-1}\,dx.$$

3. $$= \frac{x^{m-2k+1}(ax^{2k}+bx^k+c)^{n+1}}{(m+2nk+1)a} - \frac{(m-2k+1)c}{(m+2nk+1)a} \times \int x^{m-2k}(ax^{2k}+bx^k+c)^n\,dx - \frac{(m+nk-k+1)b}{(m+2nk+1)a} \int x^{m-k}(ax^{2k}+bx^k+c)^n\,dx.$$

4. $$= \frac{x^{m+1}(ax^{2k}+bx^k+c)^n}{m+2nk+1} + \frac{2nkc}{m+2nk+1} \times \int x^m(ax^{2k}+bx^k+c)^{n-1}\,dx + \frac{nkb}{m+2nk+1} \int x^{m+k}(ax^{2k}+bx^k+c)^{n-1}\,dx.$$

5. $$\int \frac{x^m\,dx}{(ax^{2k}+bx^k+c)^n} = \frac{x^{m-2k+1}}{(m-2nk+1)a(ax^{2k}+bx^k+c)^{n-1}} - \frac{(m-2k+1)c}{(m-2nk+1)a} \int \frac{x^{m-2k}\,dx}{(ax^{2k}+bx^k+c)^n} - \frac{(m-nk-k+1)b}{(m-2nk+1)a} \int \frac{x^{m-k}\,dx}{(ax^{2k}+bx^k+c)^n}.$$

6. $$\int \frac{dx}{x^m(ax^{2k}+bx^k+c)^n} = \frac{-1}{(m-1)cx^{m-1}(ax^{2k}+bx^k+c)^{n-1}} - \frac{(m+nk-k-1)b}{(m-1)c} \int \frac{dx}{x^{m-k}(ax^{2k}+bx^k+c)^n} - \frac{m+2nk-2k-1}{(m-1)c} \int \frac{dx}{x^{m-2k}(ax^{2k}+bx^k+c)^n}.$$

1.2.15. Интегралы вида $\int R\left(x^{1/2},\, ax+b\right) dx$.

1. $$\int x^{m+1/2}(ax+b)^n\,dx = 2x^{m+3/2}\sum_{k=0}^{n}\binom{n}{k}\frac{a^k b^{n-k} x^k}{2k+2m+3}.$$

2. $$\int \frac{(ax+b)^n}{x^{m+1/2}}\,dx = \frac{2}{x^{m-1/2}}\sum_{k=0}^{n}\binom{n}{k}\frac{a^k b^{n-k} x^k}{2k-2m+1}.$$

3. $$\int \frac{x^{m+1/2}\,dx}{(ax\pm b)^n} = -\frac{x^{m+1/2}}{(n-1)\,a\,(ax\pm b)^{n-1}} + \frac{2m+1}{2(n-1)\,a}\int \frac{x^{m-1/2}\,dx}{(ax\pm b)^{n-1}}.$$

4. $$= \frac{2}{a^n}\int \frac{t^{2m+2}\,dt}{(t^2\pm b/a)^n} \qquad [t=\sqrt{x};\ \text{см. } 1.2.10].$$

5. $$\int \frac{x^{m+1/2}\,dx}{ax\pm b} = 2\sum_{k=0}^{m}\frac{(\mp 1)^k b^k x^{m-k+1/2}}{(2m-2k+1)\,a^{k+1}} + \left(\mp\frac{b}{a}\right)^{m+1}\int \frac{dx}{x^{1/2}(ax\pm b)}.$$

6. $$\int \frac{dx}{x^{m+1/2}(ax\pm b)^n} = \mp\frac{2}{(2m-1)\,b x^{m-1/2}(ax\pm b)^{n-1}} \mp$$
$$\mp\frac{(2m+2n-3)\,a}{(2m-1)\,b}\int \frac{dx}{x^{m-1/2}(ax\pm b)^n} = \frac{2}{a^n}\int \frac{dt}{t^{2m}(t^2\pm b/a)^n}.$$

7. $$\int \frac{x^{1/2}\,dx}{ax\pm b} = \frac{2x^{1/2}}{a} \mp \left(\frac{b}{a^3}\right)^{1/2}\begin{Bmatrix} 2\,\mathrm{arctg}\left(\dfrac{ax}{b}\right)^{1/2} \\ \ln\left|\dfrac{b^{1/2}-(ax)^{1/2}}{b^{1/2}+(ax)^{1/2}}\right| \end{Bmatrix}.$$

8. $$\int \frac{x^{3/2}\,dx}{ax\pm b} = \frac{2ax^{3/2}\mp 6bx^{1/2}}{3a^2} \pm \left(\frac{b^3}{a^5}\right)^{1/2}\begin{Bmatrix} 2\,\mathrm{arctg}\left(\dfrac{ax}{b}\right)^{1/2} \\ \ln\left|\dfrac{b^{1/2}+(ax)^{1/2}}{b^{1/2}-(ax)^{1/2}}\right| \end{Bmatrix}.$$

9. $$\int \frac{x^{1/2}\,dx}{(ax\pm b)^2} = -\frac{x^{1/2}}{a\,(ax\pm b)} + \frac{1}{(a^3 b)^{1/2}}\begin{Bmatrix} \mathrm{arctg}\left(\dfrac{ax}{b}\right)^{1/2} \\ \dfrac{1}{2}\ln\left|\dfrac{b^{1/2}-(ax)^{1/2}}{b^{1/2}+(ax)^{1/2}}\right| \end{Bmatrix}.$$

10. $$\int \frac{x^{3/2}\,dx}{(ax\pm b)^2} = \frac{2ax^{3/2}\pm 3bx^{1/2}}{a^2\,(ax\pm b)} - 3\left(\frac{b}{a^5}\right)^{1/2}\begin{Bmatrix} \mathrm{arctg}\left(\dfrac{ax}{b}\right)^{1/2} \\ \dfrac{1}{2}\ln\left|\dfrac{b^{1/2}+(ax)^{1/2}}{b^{1/2}-(ax)^{1/2}}\right| \end{Bmatrix}.$$

11. $$\int \frac{dx}{x^{1/2}(ax\pm b)} = \pm\frac{1}{(ab)^{1/2}}\begin{Bmatrix} 2\,\mathrm{arctg}\left(\dfrac{ax}{b}\right)^{1/2} \\ \ln\left|\dfrac{b^{1/2}+(ax)^{1/2}}{b^{1/2}-(ax)^{1/2}}\right| \end{Bmatrix}.$$

12. $\int \frac{dx}{x^{3/2}(ax \pm b)} = \mp \frac{2}{bx^{1/2}} \mp \left(\frac{a}{b^3}\right)^{1/2} \begin{Bmatrix} 2 \operatorname{arctg}\left(\frac{ax}{b}\right)^{1/2} \\ \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \end{Bmatrix}.$

13. $\int \frac{dx}{x^{1/2}(ax \pm b)^2} = \pm \frac{x^{1/2}}{b(ax \pm b)} + \frac{1}{(ab^3)^{1/2}} \begin{Bmatrix} \operatorname{arctg}\left(\frac{ax}{b}\right)^{1/2} \\ \frac{1}{2} \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \end{Bmatrix}.$

14. $\int \frac{dx}{x^{3/2}(ax \pm b)^2} = \frac{-3ax \mp 2b}{b^2 x^{1/2}(ax \pm b)} - 3\left(\frac{a}{b^5}\right)^{1/2} \begin{Bmatrix} \operatorname{arctg}\left(\frac{ax}{b}\right)^{1/2} \\ \frac{1}{2} \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \end{Bmatrix}.$

1.2.16. Интегралы вида $\int R(x^{1/2}, x^2 \pm a^2)\, dx$.

Условие: $a > 0$.

1. $\int \frac{x^{m+1/2}}{(x^2 \pm a^2)^n} dx = \pm \frac{x^{m+3/2}}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \pm \frac{4n-2m-7}{4(n-1)a^2} \int \frac{x^{m+1/2}}{(x^2 \pm a^2)^{n-1}} dx.$

2. $\int \frac{x^{m+1/2}}{x^2 \pm a^2} dx = \frac{2}{2m-1} x^{m-1/2} \mp a^2 \int \frac{x^{m-3/2}}{x^2 \pm a^2} dx.$

3. $\int \frac{x^{2m+1/2}}{x^2 \pm a^2} dx = 2 \sum_{k=1}^{m} \frac{(\mp a^2)^{k-1}}{4m-4k+3} x^{2m-2k+3/2} + (\mp a^2)^m \int \frac{x^{1/2}}{x^2 \pm a^2} dx.$

4. $\int \frac{x^{2m+3/2}}{x^2 \pm a^2} dx = 2 \sum_{k=0}^{m-1} \frac{(\mp a^2)^k}{4m-4k+1} x^{2m-2k+1/2} + (\mp a^2)^m \int \frac{x^{3/2}\, dx}{x^2 \pm a^2}.$

5. $\int \frac{x^{1/2}\, dx}{x^2 + a^2} = \frac{1}{2\sqrt{2a}} \ln \frac{x + a - \sqrt{2ax}}{x + a + \sqrt{2ax}} - \frac{1}{\sqrt{2a}} \operatorname{arctg} \frac{\sqrt{2ax}}{x - a}.$

6. $\int \frac{x^{1/2}\, dx}{x^2 - a^2} = \frac{1}{2\sqrt{a}} \ln \left| \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| - \frac{1}{\sqrt{a}} \operatorname{arctg} \sqrt{\frac{a}{x}}.$

7. $\int \frac{x^{3/2}\, dx}{x^2 + a^2} = 2\sqrt{x} + \frac{\sqrt{2a}}{4} \ln \frac{x + a - \sqrt{2ax}}{x + a + \sqrt{2ax}} + \frac{\sqrt{2a}}{2} \operatorname{arctg} \frac{\sqrt{2ax}}{x - a}.$

8. $\int \frac{x^{3/2}\, dx}{x^2 - a^2} = 2\sqrt{x} + \frac{\sqrt{a}}{2} \ln \left| \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| + \sqrt{a} \operatorname{arctg} \sqrt{\frac{a}{x}}.$

1.2.17. Интегралы вида $\int x^{\pm m}(ax + b)^{n+1/2}\, dx$.

1. $\int x^m (ax + b)^{n+1/2} dx = \frac{2(ax + b)^{n+3/2}}{a^{m+1}} \sum_{k=0}^{m} \frac{(-1)^k b^k}{2m + 2n - 2k + 3} \binom{m}{k} (ax + b)^{m-k}.$

2. $\int (ax + b)^{n+1/2} dx = \frac{2}{(2n+3)a} (ax + b)^{n+3/2}.$

3. $$\int x^m (ax+b)^{1/2}\,dx = \frac{2}{(2m+3)a} x^m (ax+b)^{3/2} - \\ - \frac{2mb}{(2m+3)a}\int x^{m-1}(ax+b)^{1/2}\,dx = \frac{2(ax+b)^{3/2}}{a^{m+1}} \sum_{k=0}^{m} \binom{m}{k} \frac{(-b)^{m-k}}{2k+3} (ax+b)^k.$$

4. $$\int \frac{(ax+b)^{n+1/2}}{x^m}\,dx = -\frac{(ax+b)^{n+3/2}}{(m-1)bx^{m-1}} + \\ + \frac{(2n-2m+5)a}{2(m-1)b}\int \frac{(ax+b)^{n+1/2}}{x^{m-1}}\,dx.$$

5. $$= \frac{2(ax+b)^{n+1/2}}{(2n-2m+3)x^{m-1}} + \frac{(2n+1)b}{2n-2m+3}\int \frac{(ax+b)^{n-1/2}}{x^m}\,dx.$$

6. $$\int \frac{(ax+b)^{n+1/2}}{x}\,dx = \frac{2(ax+b)^{n+1/2}}{2n+1} + b\int \frac{(ax+b)^{n-1/2}}{x}\,dx.$$

7. $$\int \frac{(ax+b)^{1/2}}{x^m}\,dx = -\frac{(ax+b)^{3/2}}{(m-1)bx^{m-1}} + \frac{(5-2m)a}{2(m-1)b}\int \frac{(ax+b)^{1/2}}{x^{m-1}}\,dx.$$

8. $$\int (ax+b)^{1/2}\,dx = \frac{2}{3a}(ax+b)^{3/2}.$$

9. $$\int x(ax+b)^{1/2}\,dx = \frac{2}{a^2}\left[\frac{ax+b}{5} - \frac{b}{3}\right](ax+b)^{3/2}.$$

10. $$\int x^2(ax+b)^{1/2}\,dx = \frac{2}{a^3}\left[\frac{(ax+b)^2}{7} - \frac{2b(ax+b)}{5} + \frac{b^2}{3}\right](ax+b)^{3/2}.$$

11. $$\int x^3(ax+b)^{1/2}\,dx = \\ = \frac{2}{a^4}\left[\frac{(ax+b)^3}{9} - \frac{3b(ax+b)^2}{7} + \frac{3b^2(ax+b)}{5} - \frac{b^3}{3}\right](ax+b)^{3/2}.$$

12. $$\int (ax+b)^{3/2}\,dx = \frac{2}{5a}(ax+b)^{5/2}.$$

13. $$\int x(ax+b)^{3/2}\,dx = \frac{2}{a^2}\left[\frac{(ax+b)}{7} - \frac{b}{5}\right](ax+b)^{5/2}.$$

14. $$\int x^2(ax+b)^{3/2}\,dx = \frac{2}{a^3}\left[\frac{(ax+b)^2}{9} - \frac{2b(ax+b)}{7} + \frac{b^2}{5}\right](ax+b)^{5/2}.$$

15. $$\int x^3(ax+b)^{3/2}\,dx = \frac{2}{a^4}\left[\frac{(ax+b)^3}{11} - \frac{3b(ax+b)^2}{9} + \right. \\ \left. + \frac{3b^2(ax+b)}{7} - \frac{b^3}{5}\right](ax+b)^{5/2}.$$

16. $$\int \frac{(ax+b)^{1/2}}{x} = 2(ax+b)^{1/2} + \begin{cases} b^{1/2}\ln\left|\dfrac{(ax+b)^{1/2}-b^{1/2}}{(ax+b)^{1/2}+b^{1/2}}\right| & [b>0], \\ -2(-b)^{1/2}\operatorname{arctg}\dfrac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b<0]. \end{cases}$$

17. $$\int \frac{(ax+b)^{1/2}}{x^2}\,dx = -\frac{(ax+b)^{1/2}}{x} + \frac{a}{2}\int \frac{dx}{x(ax+b)^{1/2}}.$$

18. $\int \frac{(ax+b)^{1/2}}{x^3} dx = -\frac{(2b+ax)}{4bx^2}(ax+b)^{1/2} - \frac{a^2}{8b}\int \frac{dx}{x(ax+b)^{1/2}}$.

19. $\int \frac{(ax+b)^{3/2}}{x} dx = \frac{2}{3}(ax+b)^{3/2} + 2b(ax+b)^{1/2} + b^2 \int \frac{dx}{x(ax+b)^{1/2}}$.

20. $\int \frac{(ax+b)^{3/2}}{x^2} dx = -\frac{(ax+b)^{5/2}}{bx} + \frac{3a}{2b}\int \frac{(ax+b)^{3/2}}{x} dx$.

21. $\int \frac{(ax+b)^{3/2}}{x^3} dx = -\left(\frac{1}{2bx^2} + \frac{a}{4b^2x}\right)(ax+b)^{5/2} + \frac{3a^2}{8b^2}\int \frac{(ax+b)^{3/2}}{x} dx$.

1.2.18. Интегралы вида $\int \frac{x^m\,dx}{(ax+b)^{n+1/2}}$.

1. $\int \frac{x^m\,dx}{(ax+b)^{n+1/2}} = \frac{2}{a^{m+1}(ax+b)^{n-1/2}} \sum_{k=0}^{m} \frac{(-1)^k b^k}{2m-2n-2k+1}\binom{m}{k}(ax+b)^{m-k}$,

2. $\int \frac{dx}{(ax+b)^{n+1/2}} = -\frac{2}{(2n-1)a(ax+b)^{n-1/2}}$.

3. $\int \frac{x^m\,dx}{(ax+b)^{1/2}} = \frac{2}{(2m+1)a}x^m(ax+b)^{1/2} - \frac{2mb}{(2m+1)a}\int \frac{x^{m-1}\,dx}{(ax+b)^{1/2}}$.

4. $\qquad = \frac{2(ax+b)^{1/2}}{a^{m+1}} \sum_{k=0}^{m} \frac{(-1)^k b^k}{2m-2k+1}\binom{m}{k}(ax+b)^{m-k}$.

5. $\int \frac{dx}{(ax+b)^{1/2}} = \frac{2}{a}(ax+b)^{1/2}$.

6. $\int \frac{x\,dx}{(ax+b)^{1/2}} = \frac{2(ax-2b)}{3a^2}(ax+b)^{1/2}$.

7. $\int \frac{x^2\,dx}{(ax+b)^{1/2}} = \frac{2}{a^3}\left[\frac{(ax+b)^2}{5} - \frac{2b(ax+b)}{3} + b^2\right](ax+b)^{1/2}$.

8. $\int \frac{x^3\,dx}{(ax+b)^{1/2}} = \frac{2}{a^4}\left[\frac{(ax+b)^3}{7} - \frac{3b(ax+b)^2}{5} + b^2(ax+b) - b^3\right](ax+b)^{1/2}$.

9. $\int \frac{dx}{(ax+b)^{3/2}} = -\frac{2}{a(ax+b)^{1/2}}$.

10. $\int \frac{x\,dx}{(ax+b)^{3/2}} = \frac{2(ax+2b)}{a^2(ax+b)^{1/2}}$.

11. $\int \frac{x^2\,dx}{(ax+b)^{3/2}} = \frac{2}{a^3}\left[\frac{(ax+b)^2}{3} - 2b(ax+b) - b^2\right]\frac{1}{(ax+b)^{1/2}}$.

12. $\int \frac{x^3\,dx}{(ax+b)^{3/2}} = \frac{2}{a^4}\left[\frac{(ax+b)^3}{5} - b(ax+b)^2 + 3b^2(ax+b) + b^3\right]\frac{1}{(ax+b)^{1/2}}$.

1.2.19. Интегралы вида $\int \frac{dx}{x^m (ax+b)^{n+1/2}}$.

1. $$\int \frac{dx}{x^m (ax+b)^{n+1/2}} = -\frac{1}{(m-1)x^{m-1}(ax+b)^{n+1/2}} - \frac{(2n+1)a}{2(m-1)} \int \frac{dx}{x^{m-1}(ax+b)^{n+3/2}}.$$

2. $$= \frac{-2}{(2n-1)ax^m (ax+b)^{n-1/2}} - \frac{2m}{(2n-1)a} \int \frac{dx}{x^{m+1}(ax+b)^{n-1/2}}.$$

3. $$= 2a^{m-1} \int \frac{dt}{t^{2n}(t^2-b)^m} \qquad [t=\sqrt{ax+b}, \text{ см. } 1.2\ 10].$$

4. $$\int \frac{dx}{x(ax+b)^{n+1/2}} = \frac{2}{(2n-1)b(ax+b)^{n-1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{n-1/2}}.$$

5. $$\int \frac{dx}{x^m (ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{2(m-1)b} \int \frac{dx}{x^{m-1}(ax+b)^{1/2}}.$$

6. $$\int \frac{dx}{x(ax+b)^{1/2}} = \begin{cases} \frac{1}{b^{1/2}} \ln \left| \frac{(ax+b)^{1/2}-b^{1/2}}{(ax+b)^{1/2}+b^{1/2}} \right| & [b>0], \\ \frac{2}{(-b)^{1/2}} \operatorname{arctg} \frac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b<0]. \end{cases}$$

7. $$\int \frac{dx}{x^2 (ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{bx} - \frac{a}{2b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

8. $$\int \frac{dx}{x^3 (ax+b)^{1/2}} = \frac{3ax-2b}{4b^2x^2}(ax+b)^{1/2} + \frac{3a^2}{8b^2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

9. $$\int \frac{dx}{x(ax+b)^{3/2}} = \frac{2}{b(ax+b)^{1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

10. $$\int \frac{dx}{x^2 (ax+b)^{3/2}} = \frac{-3ax-b}{b^2x(ax+b)^{1/2}} - \frac{3a}{2b^2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

11. $$\int \frac{dx}{x^3 (ax+b)^{3/2}} = \left(-\frac{1}{2bx^2} + \frac{5a}{4b^2x} + \frac{15a^2}{4b^3}\right) \frac{1}{(ax+b)^{1/2}} + \frac{15a^2}{8b^3} \int \frac{dx}{x(ax+b)^{1/2}}.$$

12. $$\int \frac{(ax+b)^{1/2}+c}{(ax+b)^{1/2}+d}\, dx = x + 2\frac{c-d}{a}\left[(ax+b)^{1/2} - d \ln \left| (ax+b)^{1/2}+d \right| \right].$$

1.2.20. Интегралы вида $\int R(x, \sqrt{ax+b}, \sqrt{cx+d})\, dx$.

1. $$\int (ax+b)^{m+1/2}(cx+d)^{n+1/2}\, dx = \frac{(ax+b)^{m+1/2}(cx+d)^{n+3/2}}{(m+n+2)c} - \frac{(2m+1)(ad-bc)}{2(m+n+2)c} \int (ax+b)^{m-1/2}(cx+d)^{n+1/2}\, dx.$$

2. $$= (cx+d)^{n+3/2} \sum_{k=0}^{m} \frac{(-1)^k (2m+1)(2m-1)\ldots(2m-2k+3)(ad-bc)^k}{(m+n+2)(m+n+1)\ldots(m+n-k+2)\,c^{k+1}} \times (ax+b)^{m-k+1/2} + \frac{m!!}{(-2c)^{m+1}}\sqrt{ax+b}\times \sum_{k=0}^{n} \frac{(2n+1)(2n-1)\ldots(2n-2k+3)(ad-bc)^{m+k+1}}{2^k (m+n+2)(m+n+1)\ldots(n-k+1)\,a^{k+1}} \times (cx+d)^{n-k+1/2} + \frac{m!!\,n!!\,(ad-bc)^{m+n+2}}{(m+n+2)!\,(-2c)^{m+1}(2a)^{n+1}} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$

3. $$\int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n+1/2}}dx = \frac{1}{(m-n+1)c}\cdot\frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} - \frac{(2m+1)(ad-bc)}{2(m-n+1)c}\int\frac{(ax+b)^{m-1/2}}{(cx+d)^{n+1/2}}dx.$$

4. $$= \frac{2}{(2n-1)(ad-bc)}\cdot\frac{(ax+b)^{m+3/2}}{(cx+d)^{n-1/2}} + \frac{2(n-m-2)a}{(2n-1)(ad-bc)}\int\frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}}dx.$$

5. $$= (ax+b)^{m+3/2}\sum_{k=0}^{n-m-2}\frac{(n-m-2)(n-m-3)\ldots(n-m-k-1)2^{k+1}a^k}{(2n-1)(2n-3)\ldots(2n-2k-1)(ad-bc)^{k+1}}\times\frac{1}{(cx+d)^{n-k-1/2}}$$ $[n \geqslant m+2]$.

6. $$\int\frac{dx}{(ax+b)^{m+1/2}(cx+d)^{n+1/2}} = \frac{-2}{(2m-1)(ad-bc)(ax+b)^{m-1/2}(cx+d)^{n-1/2}} - \frac{2(m+n-1)}{(2m-1)(ad-bc)}\int\frac{dx}{(ax+b)^{m-1/2}(cx+d)^{n+1/2}}.$$

7. $$= \frac{1}{(cx+d)^{n-1/2}}\sum_{k=0}^{m-1}\frac{(m+n-1)(m+n-2)\ldots(m+n-k)(-2)^{k+1}c^k}{(2m-1)(2m-3)\ldots(2m-2k-1)(ad-bc)^{k+1}}\times\frac{1}{(ax+b)^{m-k-1/2}} + \frac{(-2c)^m}{(m-1)!!}\sqrt{ax+b}\sum_{k=0}^{n-1}\frac{(m+n-1)(m+n-2)\ldots(n-k)2^{k+1}a^k}{(2n-1)(2n-3)\ldots(2n-2k-1)(ad-bc)^{n+k+1}}\times\frac{1}{(cx+d)^{n-k-1/2}}.$$

8. $$\int\frac{dx}{\sqrt{(ax+b)(cx+d)}} = \frac{2}{\sqrt{ac}}\ln\left(\sqrt{c(ax+b)}+\sqrt{a(cx+d)}\right)$$ $[ac>0;\ x>\max(-b/a,\ -d/c)]$.

9. $$= -\frac{2}{\sqrt{ac}} \ln \left(\sqrt{-c(ax+b)} + \sqrt{-a(cx+d)}\right)$$

$[ac > 0;\ x < \max(-b/a,\ -d/c)]$.

10. $$= -\frac{1}{\sqrt{-ac}} \arcsin \frac{c(ax+b)+a(cx+d)}{|ad-bc|} \qquad [ac < 0].$$

11. $$\int \frac{dx}{(x-p)\sqrt{(ax+b)(cx+d)}} =$$

$$= \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{\left[\sqrt{(cp+d)(ax+b)} - \sqrt{(ap+b)(cx+d)}\right]^2}{|x-p|}$$

$[(ap+b)(cp+d) > 0;\ (cp+d)(ax+b) > 0;\ (ap+b)(cx+d) > 0]$.

12. $$= \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{\left[\sqrt{-(cp+d)(ax+b)} - \sqrt{-(ap+b)(cx+d)}\right]^2}{|x-p|}$$

$[(ap+b)(cp+d) > 0,\ (cp+d)(ax+b) < 0,\ (ap+b)(cx+d) < 0]$.

13. $$= \frac{1}{\sqrt{-(ap+b)(cp+d)}} \arcsin \frac{(cp+d)(ax+b)+(ap+b)(cx+d)}{|ad-bc|\,|x-p|}$$

$[(ap+b)(cp+d) < 0]$.

14. $$\int \frac{x^n}{A\sqrt{ax+b}+B\sqrt{cx+d}}\,dx =$$

$$= \frac{A}{aA^2-cB^2}\left(p^n \int \frac{\sqrt{ax+b}}{x-p}\,dx + \sum_{k=1}^{n} p^{n-k} \int x^{k-1}\sqrt{ax+b}\,dx\right) -$$

$$- \frac{B}{aA^2-cB^2}\left(p^n \int \frac{\sqrt{cx+d}}{x-p}\,dx + \sum_{k=1}^{n} p^{n-k} \int x^{k-1}\sqrt{cx+d}\,dx\right)$$

$\left[p = \frac{-bA^2+dB}{aA^2-cB^2}\right]$.

15. $$= \frac{A}{bA^2-dB^2} \int x^n \sqrt{ax+b}\,dx - \frac{B}{bA^2-dB^2} \int x^n \sqrt{cx+d}\,dx$$

$[aA^2-cB^2 = 0]$.

16. $$\int \frac{\sqrt{ax+b}}{x-p}\,dx = 2\sqrt{ax+b} \mp$$

$$\mp \sqrt{|ap+b|} \begin{cases} \ln \left| \dfrac{\sqrt{ax+b}+\sqrt{ap+b}}{\sqrt{ax+b}-\sqrt{ap+b}} \right| & [ap+b > 0], \\ 2 \operatorname{arctg} \sqrt{\dfrac{|ap+b|}{ax+b}} & [ap+b < 0]. \end{cases}$$

17. $$\int \frac{\alpha\sqrt{ax+b}+\beta\sqrt{cx+d}}{\gamma\sqrt{ax+b}+\delta\sqrt{cx+d}}\,dx = \frac{\alpha\gamma a+\beta\delta c}{\gamma^2 a+\delta^2 c}\,x - \frac{\alpha\delta-\beta\gamma}{\gamma^2 a+\delta^2 c}\sqrt{(ax+b)(cx+d)} +$$

$$+ \frac{2\gamma\delta(\alpha\delta-\beta\gamma)(ad-bc)}{(\gamma^2 a-\delta^2 c)^2} \ln \left|\gamma\sqrt{ax+b}+\delta\sqrt{cx+d}\right| -$$

$$- \frac{(\gamma^2 a+\delta^2 c)(\alpha\delta-\beta\gamma)(ad-bc)}{2(\gamma^2 a-\delta^2 c)^2} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$

1.2.21. Интегралы вида $\int R(x, \sqrt[n]{ax+b})\, dx$.

1. $\int R(x, \sqrt[n]{ax+b})\, dx = \frac{n}{a} \int R\left(\frac{t^n - b}{a}, t\right) t^{n-1}\, dt \qquad \left[t = \sqrt[n]{ax+b}\right].$

2. $\int x^p (ax+b)^{1/n}\, dx =$
$$= \frac{n}{(np+n+1)a} x^p (ax+b)^{1+1/n} - \frac{bnp}{(np+n+1)a} \int x^{p-1} (ax+b)^{1/n}\, dx.$$

3. $\int x^m (ax+b)^{1/n}\, dx = \frac{1}{a^{m+1}} (ax+b)^{1+1/n} \sum_{k=0}^{m} \frac{(-b)^k}{m-k+1+1/n} \binom{m}{k} (ax+b)^{m-k}.$

4. $\int (ax+b)^{1/n}\, dx = \frac{n}{(n+1)a} (ax+b)^{1+1/n}.$

5. $\int \frac{(ax+b)^{1/n}}{x^m}\, dx = -\frac{(ax+b)^{1+1/n}}{(m-1)bx^{m-1}} - \frac{(mn-2n-1)a}{(m-1)nb} \int \frac{(ax+b)^{1/n}}{x^{m-1}}\, dx.$

6. $\int \frac{(ax+b)^{1/n}}{x}\, dx = n(ax+b)^{1/n} + nb^{1/n} \int \frac{dt}{t^n - 1} \qquad [t = (ax/b+1)^{1/n}].$

7. $\int \frac{x^m\, dx}{(ax+b)^{1/n}} = \frac{n}{(mn+n-1)a} x^m (ax+b)^{1-1/n} - \frac{mnb}{(mn+n-1)a} \int \frac{x^{m-1}\, dx}{(ax+b)^{1/n}}$

8. $= \frac{1}{a^{m+1}} (ax+b)^{1-1/n} \sum_{k=0}^{m} \frac{(-b)^k}{m-k+1-1/n} \binom{m}{k} (ax+b)^{m-k}.$

9. $\int \frac{dx}{(ax+b)^{1/n}} = \frac{n}{(n-1)a} (ax+b)^{1-1/n}.$

1.2.22. Интегралы вида $\int R\left[x, (ax+b)^{p/n}\right] dx$.

1. $\int x^m (ax+b)^{p/n}\, dx = \frac{n}{(mn+n+p)a} x^m (ax+b)^{1+p/n} -$
$$- \frac{mnb}{(mn+n+p)a} \int x^{m-1} (ax+b)^{p/n}\, dx.$$

2. $= \frac{n}{a} (ax+b)^{1+p/n} \times$
$$\times \sum_{k=0}^{m} \frac{m(m-1)\ldots(m-k+1)}{(mn+p+n)(mn+p)\ldots(mn+n+p-kn)} \left(-\frac{bn}{a}\right)^k x^{m-k}.$$

3. $= \frac{n}{a^{m+1}} (ax+b)^{1+p/n} \sum_{k=0}^{m} \frac{\binom{m}{k}(-b)^k}{(m-k+1)n+p} (ax+b)^{m-k}.$

4. $\int (ax+b)^{p/n}\, dx = \frac{n}{(n+p)a} (ax+b)^{1+p/n}.$

5. $\int \frac{(ax+b)^{p/n}}{(x-\alpha)^m}\, dx = -\frac{1}{(m-1)(a\alpha+b)} \frac{(ax+b)^{1+p/n}}{(x-\alpha)^{m-1}} -$
$$- \frac{[(m-2)n-p]a}{(m-1)n(a\alpha+b)} \int \frac{(ax+b)^{p/n}}{(x-\alpha)^{m-1}}\, dx.$$

6. $$=(ax+b)^{1+p/n}\times$$
$$\times\sum_{k=1}^{n-1}(-1)^k\frac{(m-p-2n)(mn-p-3n)\ldots(mn-p-nk)\,a^{k-1}}{(m-1)(m-2)\ldots(m-k)\,n^{k-1}(a\alpha+b)}\frac{1}{(x-\alpha)^{m-k}}-$$
$$-\frac{p(p+1)\ldots(p+m-1)\,\alpha^{m-1}}{n^{m-1}(m-1)!\,(a\alpha+b)^{m-1}}\int\frac{(ax+b)^{p/n}}{x-\alpha}\,dx.$$

7. $$=\frac{-na^{m-1}}{(m-1)n-p}(ax+b)^{1-m+p/n}\qquad [a\alpha+b=0].$$

8. $$\int\frac{(ax+b)^{p/n}}{x-\alpha}\,dx=n(a\alpha+b)^{p/n}\int\frac{y^{n+p-1}}{y^n-1}\,dy$$
$$\left[y=\left(\frac{ax+b}{a\alpha+b}\right)^{1/n};\quad p=\pm 1,\ \pm 2,\ \ldots;\ \text{см } 1\,2\,3\right].$$

1.2.23. Интегралы вида $\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx.$

1. $$\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx=\frac{n(ad-bc)}{ac}\int R\left[\frac{-adt^n+bc}{ac(t^n-1)},\ \sqrt[n]{\frac{a}{c}}\,t\right]\frac{t^{n-1}}{(t^n-1)^2}\,dt$$
$$\left[t=\sqrt[n]{\frac{c(ax+b)}{a(cx+d)}}\right].$$

2. $$\int\left(\frac{ax+b}{cx+d}\right)^{p/n}dx=\frac{ax+b}{a}\left(\frac{ax+b}{cx+d}\right)^{p/n}+\frac{p(ad-bc)}{ac}\left(\frac{a}{c}\right)^{p/n}\int\frac{t^{n+p-1}}{t^n-1}\,dt$$
$$\left[t=\sqrt[n]{\frac{c(ax+b)}{a(cx+d)}},\quad \text{см } 1\,2\,3\right].$$

3. $$\int x^m\left(\frac{ax+b}{cx+d}\right)^{p/n}dx=\frac{(ax+b)(cx+d)\,x^{m-1}}{(m+1)ac}\left(\frac{ax+b}{cx+d}\right)^{p/n}-$$
$$-\frac{(ad-bc)p+mn(ad+bc)}{(m+1)nac}\int x^{m-1}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx-$$
$$-\frac{(m-1)bd}{(m+1)ac}\int x^{m-2}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx.$$

4. $$\int x\left(\frac{ax+b}{cx+d}\right)^{p/n}dx=\frac{(ax+b)(cx+d)}{2ac}\left(\frac{ax+b}{cx+d}\right)^{p/n}-$$
$$-\frac{(ad-bc)p+(ad+bc)n}{2nac}\int\left(\frac{ax+b}{cx+d}\right)^{p/n}dx.$$

5. $$\int\frac{1}{(x-\alpha)^{m+1}}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx=A\frac{(ax+b)(cx+d)}{(x-\alpha)^m}\left(\frac{ax+b}{cx+d}\right)^{p/n}+$$
$$+B\int\frac{1}{(x-\alpha)^m}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx+D\int\frac{1}{(x-\alpha)^{m-1}}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx,$$
$$A=\frac{-1}{m(a\alpha+b)(c\alpha+d)},\quad B=\frac{-n(m-1)(ad+bc+2ac\alpha)+p(ad-bc)}{mn(a\alpha+b)(c\alpha+d)},$$
$$D=\frac{-(m-2)ac}{m(a\alpha+b)(c\alpha+d)}.$$

6. $$\int\frac{1}{x-\alpha}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx=n\left(\frac{a\alpha+b}{c\alpha+d}\right)^{p/n}\int\frac{t^{p-1}}{t^n-1}\,dt-n\left(\frac{a}{c}\right)^{p/n}\int\frac{s^{p-1}}{s^n-1}\,ds$$
$$\left[t=\sqrt[n]{\frac{(c\alpha+d)(ax+b)}{(a\alpha+b)(cx+d)}};\quad s=\sqrt[n]{\frac{c(ax+b)}{a(cx+d)}};\quad \text{см. } 1.2.3\right].$$

1.2.24. Интегралы вида $\int R(x^{1/n},\ x^2 \pm a^2)\,dx$.

1. $\int \frac{x^{p/n}\,dx}{(x^2 \pm a^2)^m} = \frac{\pm x^{1+p/n}}{2(m-1)a^2(x^2 \pm a^2)^{m-1}} \pm \frac{(2m-3)n-p}{2(m-1)a^2 n} \int \frac{x^{p/n}\,dx}{(x^2 \pm a^2)^{m-1}}$.

2. $$= x^{1+p/n} \sum_{k=1}^{m-1} \frac{(2mn-3n-p)(2mn-5n-p)\dots(2mn-2nk+n-p)}{(m-1)(m-2)\dots(m-k)\,n^{k-1}(\pm 2a^2)^k} \times$$
$$\times \frac{1}{(x^2 \pm a^2)^{m-k}} + \frac{(n-p)(3n-p)\dots(2mn-3n-p)}{(m-1)!\,(\pm 2a^2 n)^{m-1}} \int \frac{x^{p/n}\,dx}{x^2 \pm a^2}.$$

3. $\int \frac{x^{p/n}\,dx}{x^2 \pm a^2} = \frac{n}{p-n} x^{p/n-1} \mp a^2 \int \frac{x^{p/n-2}\,dx}{x^2 \pm a^2}$.

4. $\int \frac{dx}{x^{p/n}(x^2 \pm a^2)} = \frac{\pm n x^{1-p/n}}{(n-p)a^2} \mp \frac{1}{a^2} \int \frac{dx}{x^{p/n-2}(x^2 \pm a^2)}$.

5. $$\int \frac{x^{p/n}}{x^2+a^2}\,dx = \frac{1}{a^{1-p/n}} \sum_{k=0}^{n-1} \left(\sin\frac{4k+1}{2n} p\pi \ln\left|x^{1/n} - a^{1/n}\right| + \right.$$
$$\left. + \cos\frac{4k+1}{2n} p\pi \operatorname{arctg} \frac{x^{1/n} - a^{1/n}\cos\frac{4k+1}{2n}\pi}{a^{1/n}\sin\frac{4k+1}{2n}\pi} \right) \qquad [p = 1,\ 2,\ \dots,\ n-1;\ a>0].$$

6. $$\int \frac{x^{p/n}}{x^2-a^2}\,dx = \frac{na^{p/n-1}}{2} \int \frac{t^{n+p-1}}{t^n-1}\,dt - \frac{na^{p/n-1}}{2} \int \frac{t^{n+p-1}}{t^n+1}\,dt$$
$[t = (x/a)^{1/n};\ p = \pm 1,\ \pm 2\ \dots;\ a>0;$ см. 1.2 3$]$.

1.2.25. Интегралы вида $\int_a^x R(\sqrt{x-a},\ \sqrt{x-b},\ \sqrt{x-c})\,dx$.

Условие: $x > a > b > c$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{x-a}{x-b}},\ k = \sqrt{\frac{b-c}{a-c}}$.

1. $\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi,\ k)$.

2. $\int_a^x \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} [a(a-b)\Pi(\varphi,\ 1,\ k) + b^2 F(\varphi,\ k)]$.

3. $\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi,\ k) - \frac{2}{(b-c)\sqrt{a-c}} F(\varphi,\ k)$.

4. $$\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi,\ k) - E(\varphi,\ k)] +$$
$$+ \frac{2}{a-c} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

5. $$\int_a^x \frac{dx}{(x-p)\sqrt{(x-a)(x-b)(x-c)}} =$$
$$= \frac{2}{(b-p)(a-p)\sqrt{a-c}} \left[(b-a)\,\Pi\left(\varphi, \frac{b-p}{a-b}, k\right) + (a-p)\,F(\varphi, k)\right] \qquad [p \neq a].$$

6. $$\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} =$$
$$= \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(2a-b-c)\,E(\varphi, k) - 2(a-b)\,F(\varphi, k)] + \frac{2}{(a-c)(b-c)}\sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

7. $$\int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)}}\,dx = -2\sqrt{a-c}\,E(\varphi, k) + 2\sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

8. $$\int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)}}\,dx = \frac{2(a-b)}{\sqrt{a-c}}F(\varphi, k) - 2\sqrt{a-c}\,E(\varphi, k) + 2\sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

9. $$\int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)}}\,dx = 2\sqrt{a-c}\,[F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

10. $$\int_a^x \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}[F(\varphi, k) - E(\varphi, k)].$$

11. $$\int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\,dx =$$
$$= \frac{2\sqrt{a-c}}{b-c}E(\varphi, k) - \frac{2(a-b)}{(b-c)\sqrt{a-c}}F(\varphi, k) - 2\sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

12. $$\int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}E(\varphi, k) - 2\frac{b-c}{a-c}\sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

13. $$\int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\varphi, k).$$

14. $$\int_a^x \sqrt{\frac{(x-a)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\varphi, k) - (a-b)\,F(\varphi, k)] + \frac{2}{3}(x+2c-a-2b)\sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

15. $\int\limits_a^x \sqrt{\frac{(x-a)(x-c)}{x-b}}\,dx = \frac{2}{3}\frac{\sqrt{(a-c)^3}}{b-c}\,[(a+c-2b)\,E(\varphi, k) - (a-b)\,F(\varphi, k)] +$
$$+\frac{2(a-c)}{3(b-c)}(x+b-a-c)\sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

16. $\int\limits_a^x \sqrt{\frac{(x-b)(x-c)}{x-a}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(a-b)\,F(\varphi, k) + (b+c-2a)\,E(\varphi, k)] +$
$$+\frac{2}{3}(x+2a-2b-c)\sqrt{\frac{(x-a)(x-b)}{x-c}}.$$

1.2.26. Интегралы вида $\int\limits_x^\infty R\left(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}\right)dx$.

Условие: $x \geqslant a > b > c$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{a-c}{x-c}}$, $k = \sqrt{\frac{b-c}{a-c}}$.

1. $\int\limits_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}\,F(\varphi, k).$

2. $\int\limits_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}}\,E(\varphi, k) +$
$$+\frac{2}{a-b}\sqrt{\frac{x-b}{(x-a)(x-c)}} \qquad [x>a].$$

3. $\int\limits_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\varphi, k) -$
$$-\frac{2}{(b-c)\sqrt{a-c}}\,F(\varphi, k) - \frac{2}{a-b}\sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

4. $\int\limits_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\varphi, k) - E(\varphi, k)].$

5. $\int\limits_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} =$
$$= \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\,[(a-b)\,F(\varphi, k) - (a+b-2c)\,E(\varphi, k)] +$$
$$+\frac{2x-a-b}{(a-b)^2\sqrt{(x-a)(x-b)(x-c)}} \qquad [x>a].$$

6. $$\int_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}[(a+c-2b)E(\varphi, k)-(a-b)F(\varphi, k)] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x>a].$$

7. $$\int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}[(2a-b-c)E(\varphi, k)-2(a-b)F(\varphi, k)] - \frac{2}{(a-b)(b-c)}\sqrt{\frac{x-a}{(x-b)(x-c)}} \quad [x>a].$$

8. $$\int_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}}\times \times[(a-b)(2a-b-c)F(\varphi, k)-2(a^2+b^2+c^2-ab-ac-bc)E(\varphi, k)] + \frac{2[x(a+b-2c)-a(a-c)-b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(x-a)(x-b)(x-c)}} \quad [x>a].$$

9. $$\int_x^\infty \frac{dx}{(x-p)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(p-c)\sqrt{a-c}}\left[\Pi\left(\varphi, \frac{p-c}{a-c}, k\right)-F(\varphi, k)\right].$$

10. $$\int_x^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}}\,dx = \frac{2}{\sqrt{a-c}}F(\varphi, k)-\frac{2\sqrt{a-c}}{a-b}E(\varphi, k)+\frac{2(a-c)}{a-b}\sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x>a].$$

11. $$\int_x^\infty \sqrt{\frac{x-b}{(x-a)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}[F(\varphi, k)-E(\varphi, k)]+2\sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x>a].$$

12. $$\int_x^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\varphi, k)-2\frac{b-c}{a-b}\sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

13. $$\int_x^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}[F(\varphi, k)-E(\varphi, k)]+2\sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

14. $\int\limits_x^\infty \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx=\frac{2}{\sqrt{a-c}}E(\varphi,k).$

15. $\int\limits_x^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\,dx=\frac{2\sqrt{a-c}}{b-c}E(\varphi,k)-\frac{2(a-b)}{(b-c)\sqrt{a-c}}F(\varphi,k)$

1.2.27. Интегралы вида $\int\limits_x^a R\left(\sqrt{a-x},\ \sqrt{x-b},\ \sqrt{x-c}\right)dx.$

Условие: $a>x\geqslant b>c$.

Обозначения: $\varphi=\arcsin\sqrt{\frac{a-x}{a-b}},\quad k=\sqrt{\frac{a-b}{a-c}}.$

1. $\int\limits_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}}=\frac{2}{\sqrt{a-c}}F(\varphi,k).$

2. $\int\limits_x^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}}=\frac{2c}{\sqrt{a-c}}F(\varphi,k)+2\frac{a}{b}\sqrt{a-c}\,E(\varphi,k).$

3. $\int\limits_x^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}}=\frac{2}{(a-b)\sqrt{a-c}}F(\varphi,k)-$

$-\frac{2\sqrt{a-c}}{(a-b)(b-c)}E(\varphi,k)+\frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-x)(x-c)}{x-b}}$ $[x>b]$.

4. $\int\limits_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}}=\frac{2}{(b-c)\sqrt{a-c}}E(\varphi,k)-$

$-\frac{2}{(b-c)(a-c)}\sqrt{\frac{(a-x)(x-b)}{x-c}}.$

5. $\int\limits_x^a \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)}}=\frac{2}{(a-p)\sqrt{a-c}}\Pi\left(\varphi,\frac{a-b}{a-p},k\right)$

$[p\neq a]$.

6. $\int\limits_x^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}}=$

$=\frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(b-c)F(\varphi,k)-2(2a-b-c)E(\varphi,k)\right]+$

$+\frac{2(a-b-c+x)}{(a-b)(b-c)(a-c)}\sqrt{\frac{a-x}{(x-b)(x-c)}}$ $[x>b]$.

7. $\int\limits_x^a \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx=2\sqrt{a-c}\,[F(\varphi,k)-E(\varphi,k)].$

8. $$\int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}\,E(\varphi, k).$$

9. $$\int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\varphi, k) - \frac{2(b-c)}{\sqrt{a-c}}F(\varphi, k).$$

10. $$\int_x^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{c-b}E(\varphi, k) + \frac{2}{b-c}\sqrt{\frac{(a-x)(x-c)}{x-b}} \qquad [x>b].$$

11. $$\int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}E(\varphi, k) - \frac{2}{\sqrt{a-c}}F(\varphi, k) - \frac{2}{b-c}\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

12. $$\int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c}\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

13. $$\int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-b}\sqrt{\frac{(a-x)(x-c)}{x-b}} \qquad [x>b].$$

14. $$\int_x^a \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] - \frac{2}{3}\sqrt{(a-x)(x-b)(x-c)}.$$

15. $$\int_x^a \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+c-2b)E(\varphi, k) + (b-c)F(\varphi, k)] - \frac{2}{3}\sqrt{(a-x)(x-b)(x-c)}.$$

16. $$\int_x^a \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \frac{2}{3}\sqrt{(a-x)(x-b)(x-c)}.$$

1.2.28. Интегралы вида $\int_b^x R\left(\sqrt{a-x},\ \sqrt{x-b},\ \sqrt{x-c}\right) dx$.

Условие: $a \geqslant x > b > c$.

Обозначения: $\varphi = \arcsin\sqrt{\dfrac{(a-c)(x-b)}{(a-b)(x-c)}}$, $k = \sqrt{\dfrac{a-b}{a-c}}$.

1. $$\int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

2. $$\int_b^x \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c)\,\Pi(\varphi, k^2, k) + cF(\varphi, k)].$$

3. $$\int_b^x \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-b}\sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

4. $$\int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\varphi, k).$$

5. $$\int_b^x \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} =$$
$$= \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(b-c)F(\varphi, k) - (2b-a-c)E(\varphi, k)] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

6. $$\int_b^x \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)}} =$$
$$= \frac{2}{(c-p)(b-p)\sqrt{a-c}} \left[(c-b)\,\Pi\left(\varphi, k^2\frac{c-p}{b-p}, k\right) + (b-p)F(\varphi, k)\right] \quad [p \neq b].$$

7. $$\int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}\,[F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

8. $$\int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\varphi, k) - \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k) - 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

9. $$\int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}\,E(\varphi, k) - 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

10. $$\int_b^x \sqrt{\frac{x-b}{(a-x)^3(x-c)}}\,dx = -\frac{2}{\sqrt{a-c}}E(\varphi,\ k)+2\sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x<a].$$

11. $$\int_b^x \sqrt{\frac{x-c}{(a-x)^3(x-b)}}\,dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}}F(\varphi,\ k)-$$
$$-\frac{2\sqrt{a-c}}{a-b}E(\varphi,\ k)+2\frac{a-c}{a-b}\sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x<a].$$

12. $$\int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}[F(\varphi,\ k)-E(\varphi,\ k)].$$

13. $$\int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}E(\varphi,\ k)-\frac{2}{\sqrt{a-c}}F(\varphi,\ k).$$

14. $$\int_b^x \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)E(\varphi,\ k)-2(b-c)F(\varphi,\ k)]+$$
$$+\frac{2}{3}(x+c-a-b)\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

15. $$\int_b^x \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(b-c)F(\varphi,\ k)+(a+c-2b)E(\varphi,\ k)]+$$
$$+\frac{2}{3}(2b-a-2c+x)\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

16. $$\int_b^x \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)E(\varphi,\ k)-(b-c)F(\varphi,\ k)]+$$
$$+\frac{2}{3}(b+2c-2a-x)\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

1.2.29. Интегралы вида $\int_x^b R\left(\sqrt{a-x},\ \sqrt{b-x},\ \sqrt{x-c}\right)dx$.

Условие: $a>b>x\geqslant c$.

Обозначения: $\varphi=\arcsin\sqrt{\frac{(a-c)(b-x)}{(b-c)(a-x)}},\ k=\sqrt{\frac{b-c}{a-c}}$.

1. $$\int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\varphi,\ k).$$

2. $$\int_x^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}[(b-c)\Pi(\varphi,\ k^2,\ k)+aF(\varphi,\ k)].$$

3. $$\int_x^b \frac{dx}{\sqrt{(a-x)^3 (b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\varphi, k).$$

4. $$\int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-c}\sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x>c].$$

5. $$\int_x^b \frac{dx}{\sqrt{(a-x)^3 (b-x)(x-c)^3}} = \frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}} [(a-b)F(\varphi, k) + (2b-a-c)E(\varphi, k)] + \frac{2}{(b-c)(a-c)}\sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x>c].$$

6. $$\int_x^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(p-a)(p-b)\sqrt{a-c}}\left[(b-a)\,\Pi\left(\varphi, k^2\frac{p-a}{p-b}, k\right) + (p-b)F(\varphi, k)\right] \quad [p \neq b].$$

7. $$\int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\varphi, k) - 2\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

8. $$\int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\varphi, k) - \frac{2(a-b)}{\sqrt{a-c}}F(\varphi, k) - 2\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

9. $$\int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}\,[F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

10. $$\int_x^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\varphi, k) - \frac{2}{\sqrt{a-c}}F(\varphi, k).$$

11. $$\int_x^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}[F(\varphi, k) - E(\varphi, k)].$$

12. $\int\limits_x^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}}\,dx =$

$$= 2\frac{a-b}{(b-c)\sqrt{a-c}} F(\varphi, k) - \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) + 2\frac{a-c}{b-c}\sqrt{\frac{b-x}{(a-x)(x-c)}}.$$

13. $\int\limits_x^b \sqrt{\frac{b-x}{(a-x)(x-c)^3}}\,dx = -\frac{2}{\sqrt{a-c}} E(\varphi, k) + 2\sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x>c].$

14. $\int\limits_x^b \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)E(\varphi, k) - (a-b)F(\varphi, k)] +$

$$+ \frac{2}{3}(2c-2a-b+x)\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

15. $\int\limits_x^b \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(c-b)F(\varphi, k) + (2b-a-c)E(\varphi, k)] +$

$$+ \frac{2}{3}(2a+c-2b-x)\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

16. $\int\limits_x^b \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(b-a)F(\varphi, k) + (2a-b-c)E(\varphi, k)] +$

$$+ \frac{2}{3}(2c-b-x)\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

1.2.30. Интегралы вида $\int\limits_c^x R\left(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}\right) dx$.

Условие: $a > b > x > c$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{x-c}{b-c}}$, $k = \sqrt{\frac{b-c}{a-c}}$.

1. $\int\limits_c^x \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$

2. $\int\limits_c^x \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\varphi, k) - 2\sqrt{a-c}\,E(\varphi, k).$

3. $\int\limits_c^x \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\varphi, k) -$

$$- \frac{2}{(a-b)(a-c)}\sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

4. $$\int_c^x \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad [x < b].$$

5. $$\int_c^x \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\varphi, k) - (a+b-2c)E(\varphi, k)] + \frac{2[a^2+b^2-ac-bc-x(a+b-2c)]}{(a-b)^2(b-c)(a-c)} \sqrt{\frac{x-c}{(a-x)(b-x)}} \quad [x < b].$$

6. $$\int_c^x \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(p-c)\sqrt{a-c}} \Pi\left(\varphi, \frac{b-c}{p-c}, k\right) \quad [p \neq c].$$

7. $$\int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)}}\, dx = 2\sqrt{a-c}\, E(\varphi, k).$$

8. $$\int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)}}\, dx = 2\sqrt{a-c}\, E(\varphi, k) - \frac{2(a-b)}{\sqrt{a-c}} F(\varphi, k).$$

9. $$\int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)}}\, dx = 2\sqrt{a-c}\, [F(\varphi, k) - E(\varphi, k)].$$

10. $$\int_c^x \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\, dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

11. $$\int_c^x \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\, dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

12. $$\int_c^x \sqrt{\frac{a-x}{(b-x)^3(x-c)}}\, dx = \frac{2\sqrt{a-c}}{b-c} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad [x < b].$$

13. $$\int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)^3}}\,dx = -\frac{2\sqrt{a-c}}{a-b}E(\varphi, k) + \frac{2}{a-b}\sqrt{\frac{(a-x)(x-c)}{b-x}} \qquad [x<b].$$

14. $$\int_c^x \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)E(\varphi, k) - (a-b)F(\varphi, k)] + \frac{2}{3}\sqrt{(a-x)(b-x)(x-c)}.$$

15. $$\int_c^x \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2b-a-c)E(\varphi, k) + (a-b)F(\varphi, k)] - \frac{2}{3}\sqrt{(a-x)(b-x)(x-c)}.$$

16. $$\int_c^x \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)E(\varphi, k) - 2(a-b)F(\varphi, k)] - \frac{2}{3}\sqrt{(a-x)(b-x)(x-c)}.$$

1.2.31. Интегралы вида $\int_x^c R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x})\,dx$.

Условие. $a > b > c > x$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{c-x}{b-x}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

1. $$\int_x^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\varphi, k).$$

2. $$\int_x^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}[cF(\varphi, k) + (a-c)E(\varphi, k)] - 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

3. $$\int_x^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}}[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c}\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

4. $$\int_x^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k).$$

5. $$\int_x^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}[(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

6. $$\int_x^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(p-b)(p-c)\sqrt{a-c}} \times \Pi\left(\varphi, \frac{p-b}{p-c}, k\right) + \frac{2}{(p-b)\sqrt{a-c}} F(\varphi, k) \quad [p \neq c].$$

7. $$\int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)}}\, dx = 2\sqrt{a-c}[F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

8. $$\int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)}}\, dx = \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k) - 2\sqrt{a-c}\, E(\varphi, k) + 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

9. $$\int^c \sqrt{\frac{c-x}{(a-x)(b-x)}}\, dx = -2\sqrt{a-c}\, E(\varphi, k) + 2\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

10. $$\int_x^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\, dx = \frac{2}{\sqrt{a-c}} E(\varphi, k) - \frac{2(a-b)}{a-c}\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

11. $$\int_x^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\, dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k) - 2\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

12. $$\int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\, dx = \frac{2\sqrt{a-c}}{a-b}[F(\varphi, k) - E(\varphi, k)].$$

13. $$\int_x^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\, dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k).$$

14. $$\int_x^c \sqrt{\frac{(a-x)(b-x)}{c-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(b-c)F(\varphi, k) + (2c-a-b)E(\varphi, k)] + \frac{2}{3}(a+2b-2c-x)\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

15. $$\int_x^c \sqrt{\frac{(a-x)(c-x)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2b-a-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \frac{2}{3}(a+c-b-x)\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

16. $$\int_x^c \sqrt{\frac{(b-x)(c-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \frac{2}{3}(2b-2a+c-x)\sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

1.2.32. Интегралы вида $\int_{-\infty}^{x} R\left(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}\right)dx$.

Условие: $a > b > c \geq x$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{a-c}{a-x}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

1. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\varphi, k).$$

2. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}}[F(\varphi, k) - E(\varphi, k)].$$

3. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}E(\varphi, k) - \frac{2}{(a-b)\sqrt{a-c}}F(\varphi, k) - \frac{2}{b-c}\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

4. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}}E(\varphi, k) + \frac{2}{b-c}\sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].$$

5. $$\int_{-\infty}^{x} \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{(a-p)\sqrt{a-c}}\left[\Pi\left(\varphi, \frac{a-p}{a-c}, k\right) - F(\varphi, k)\right].$$

6. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} =$$
$$= \frac{2}{(b-c)(a-b)^2\sqrt{a-c}}\left[(a+b-2c)E(\varphi, k)-2(b-c)F(\varphi, k)\right]-$$
$$-\frac{2}{(a-b)(b-c)}\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

7. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}} =$$
$$= \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(b-c)F(\varphi, k)-(a+b-2c)E(\varphi, k)\right]+$$
$$+\frac{2(b+c-2x)}{(b-c)^2\sqrt{(a-x)(b-x)(c-x)}} \qquad [x<c].$$

8. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} =$$
$$= \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(2b-a-c)E(\varphi, k)-(b-c)F(\varphi, k)\right]+$$
$$+\frac{2}{(b-c)(a-c)}\sqrt{\frac{b-x}{(a-x)(c-x)}} \qquad [x<c].$$

9. $$\int_{-\infty}^{x} \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}}\times$$
$$\times\left[(b-c)(a+b-2c)F(\varphi, k)-2(c^2+a^2+b^2-ab-ac-bc)E(\varphi, k)\right]+$$
$$+\frac{2\left[c(a-c)+b(a-b)-x(2a-c-b)\right]}{(a-b)(a-c)(b-c)^2\sqrt{(a-x)(b-x)(c-x)}} \qquad [x<c].$$

10. $$\int_{-\infty}^{x} \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}}E(\varphi, k).$$

11. $$\int_{-\infty}^{x} \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}E(\varphi, k)-\frac{2(b-c)}{(a-b)\sqrt{a-c}}F(\varphi, k).$$

12. $$\int_{-\infty}^{x} \sqrt{\frac{a-x}{(b-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}F(\varphi, k)-\frac{2\sqrt{a-c}}{b-c}E(\varphi, k)+$$
$$+\frac{2(a-c)}{b-c}\sqrt{\frac{b-x}{(a-x)(c-x)}} \qquad [x<c].$$

13. $$\int_{-\infty}^{x} \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{a-c}}{b-c}E(\varphi, k)-2\frac{a-b}{b-c}\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

14. $$\int\limits_{-\infty}^{x} \sqrt{\frac{b-x}{(a-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}\left[F(\varphi, k) - E(\varphi, k)\right] + 2\sqrt{\frac{b-x}{(a-x)(c-x)}} \qquad [x<c].$$

15. $$\int\limits_{-\infty}^{x} \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\left[F(\varphi, k) - E(\varphi, k)\right] + 2\sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

1.2.33. Интегралы вида $\int\limits_{a}^{x} R\left(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d}\right) dx$.

Условие: $x > a > b > c > d$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{(b-d)(x-a)}{(a-d)(x-b)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

1. $$\int\limits_{a}^{x} \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

2. $$\int\limits_{a}^{x} \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-b)\,\Pi\left(\varphi, \frac{a-d}{b-d}, k\right) + bF(\varphi, k)\right].$$

3. $$\int\limits_{a}^{x} \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{ab\sqrt{(a-c)(b-d)}}\left\{(b-a)\,\Pi\left[\varphi, \frac{b(a-d)}{a(b-d)}, k\right] + aF(\varphi, k)\right\}.$$

4. $$\int\limits_{a}^{x} \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \left\{(a-b)\,\Pi\left[\varphi, \frac{(a-d)(p-b)}{(b-d)(p-a)}, k\right] + (p-a)F(\varphi, k)\right\}.$$

5. $$\int\limits_{a}^{x} \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi, \frac{a-d}{b-d}, k\right) - F(\varphi, k)\right].$$

6. $$\int\limits_{a}^{x} \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi, \frac{a-d}{b-d}, k\right).$$

7. $$\int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-b)\Pi\left(\varphi, \frac{a-d}{b-d}, k\right) + (b-c)F(\varphi, k)\right].$$

8. $$\int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-b)\Pi\left(\varphi, \frac{a-d}{b-d}, k\right) + (b-d)F(\varphi, k)\right].$$

9. $$\int_a^x \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)].$$

10. $$\int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}E(\varphi, k) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}}F(\varphi, k) - \frac{2}{c-d}\sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

11. $$\int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{-2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k) + \frac{2}{c-d}\sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}}.$$

12. $$\int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}}\,dx = \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}} + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}}F(\varphi, k) + 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}E(\varphi, k).$$

13. $$\int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k) - \frac{2(b-c)}{(a-c)(c-d)}\sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

14. $$\int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k).$$

15. $$\int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-d}\sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}}.$$

16. $$\int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{(a-c)(b-a)}}{(a-b)(b-c)}E(\varphi, k) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}F(\varphi, k).$$

17. $$\int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k)-E(\varphi, k)]+\frac{2}{a-c}\sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

1.2.34. Интегралы вида $\int_x^a R\left(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d}\right)dx$.

Условие $a>x>b>c>d$.

Обозначения $\varphi=\arcsin\sqrt{\frac{(b-d)(a-x)}{(a-b)(x-d)}}$, $k=\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

1. $$\int_x^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\varphi, \frac{b-a}{b-d}, k\right)+dF(\varphi, k)\right].$$

2. $$\int_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\varphi, k).$$

3. $$\int_x^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{ad\sqrt{(a-c)(b-d)}}\left\{(d-a)\,\Pi\left[\varphi, \frac{d(b-a)}{a(b-d)}, k\right]+aF(\varphi, k)\right\}.$$

4. $$\int_x^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}}\times$$
$$\times\left\{(a-d)\,\Pi\left[\varphi, \frac{(b-a)(p-d)}{(b-d)(p-a)}, k\right]+(p-a)F(\varphi, k)\right\} \qquad [p\neq a].$$

5. $$\int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi, \frac{b-a}{b-d}, k\right)-F(\varphi, k)\right].$$

6. $$\int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-a)\,\Pi\left(\varphi, \frac{b-a}{b-d}, k\right)+(b-d)F(\varphi, k)\right].$$

7. $$\int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\varphi, \frac{b-a}{b-d}, k\right) + (d-c)F(\varphi, k)\right].$$

8. $$\int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi, \frac{b-a}{b-d}, k\right).$$

9. $$\int_x^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{c-b}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k) + \frac{2}{b-c}\sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}} \quad [x>b].$$

10. $$\int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\varphi, k) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\varphi, k) - \frac{2}{b-c}\sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}$$

11. $$\int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)].$$

12. $$\int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c}\sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

13. $$\int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}\,E(\varphi, k) - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\varphi, k).$$

14. $$\int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-b}\sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}}.$$

15. $$\int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k).$$

16. $$\int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}}\,dx=\frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}F(\varphi,k)-$$
$$-\frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}E(\varphi,k)+\frac{2(b-d)}{(a-b)(b-c)}\sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}}\qquad [x>b].$$

17. $$\int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx=$$
$$=\frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\varphi,k)-\frac{2(c-d)}{(a-c)(b-c)}\sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

1.235. Интегралы вида $\int_b^x R\left(\sqrt{a-x},\ \sqrt{x-b},\ \sqrt{x-c},\ \sqrt{x-d}\right)dx$.

Условие: $a \geqslant x > b > c > d$.

Обозначения: $\varphi=\arcsin\sqrt{\frac{(a-c)(x-b)}{(a-b)(x-c)}}$, $k=\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

1. $$\int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}}=\frac{2}{\sqrt{(a-c)(b-d)}}F(\varphi,k).$$

2. $$\int_b^x \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}}=$$
$$=\frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-c)\,\Pi\left(\varphi,\frac{a-b}{a-c},k\right)+cF(\varphi,k)\right].$$

3. $$\int_b^x \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}}=$$
$$=\frac{2}{bc\sqrt{(a-c)(b-d)}}\left\{(c-b)\,\Pi\left[\varphi,\frac{c(a-b)}{b(a-c)},k\right]+bF(\varphi,k)\right\}.$$

4. $$\int_b^x \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}}=\frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}}\times$$
$$\times\left\{(b-c)\,\Pi\left[\varphi,\frac{(a-b)(p-c)}{(a-c)(p-b)},k\right]+(p-b)F(\varphi,k)\right\}\qquad [p\neq b].$$

5. $$\int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx=$$
$$=\frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-b)\,\Pi\left(\varphi,\frac{a-b}{a-c},k\right)+(a-c)F(\varphi,k)\right].$$

6. $$\int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx=\frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi,\frac{a-b}{a-c},k\right)-F(\varphi,k)\right].$$

7. $$\int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\,dx=\frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi,\ \frac{a-b}{a-c},\ k\right).$$

8. $$\int_b^x \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx=$$
$$=\frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-c)\,\Pi\left(\varphi,\ \frac{a-b}{a-c},\ k\right)+(c-d)\,F(\varphi,k)\right].$$

9. $$\int_b^x \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}}\,dx=\frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi,k)-E(\varphi,k)]+$$
$$+\frac{2}{a-b}\sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}}\qquad [x<a].$$

10. $$\int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx=$$
$$=\frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\varphi,k)-\frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\varphi,k).$$

11. $$\int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx=$$
$$=\frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi,k)-E(\varphi,k)]+\frac{2}{b-d}\sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

12. $$\int_b^x \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}}\,dx=$$
$$=\frac{2}{d-a}\sqrt{\frac{b-d}{a-c}}\,E(\varphi,k)+\frac{2}{a-d}\sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}}.$$

13. $$\int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx=\frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi,k)-E(\varphi,k)].$$

14. $$\int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx=\frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}}\times$$
$$\times[(a-c)(b-d)\,E(\varphi,k)-(a-d)(b-c)\,F(\varphi,k)]-\frac{2}{a-d}\sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

15. $$\int_b^x \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}}\,dx=\frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\varphi,k)-$$
$$-\frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\varphi,k)+\frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}}\qquad [x<a].$$

16. $\int\limits_b^x \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx =$

$$= \frac{2}{a-d}\left[\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k) - \frac{c-d}{b-d}\sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}\right].$$

17. $\int\limits_b^x \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k).$

1.2.36. Интегралы вида $\int\limits_x^b R\left(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}, \sqrt{x-d}\right) dx$

Условие: $a > b > x \geq c > d$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{(a-c)(b-x)}{(b-c)(a-x)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

1. $\int\limits_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\varphi, k).$

2. $\int\limits_x^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$

$$= \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-a)\,\Pi\left(\varphi, \frac{b-c}{a-c}, k\right) + aF(\varphi, k)\right].$$

3. $\int\limits_x^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$

$$= \frac{2}{ab\sqrt{(a-c)(b-d)}}\left\{(a-b)\,\Pi\left[\varphi, \frac{a(b-c)}{b(a-c)}, k\right] + bF(\varphi, k)\right\}.$$

4. $\int\limits_x^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}}\times$

$$\times\left\{(b-a)\,\Pi\left[\varphi, \frac{(b-c)(p-a)}{(a-c)(p-b)}, k\right] + (p-b)\,F(\varphi, k)\right\} \qquad [p \neq b].$$

5. $\int\limits_x^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi, \frac{b-c}{a-c}, k\right).$

6. $\int\limits_x^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi, \frac{b-c}{a-c}, k\right) - F(\varphi, k)\right].$

7. $\int\limits_x^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\,dx =$

$$= \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-a)\,\Pi\left(\varphi, \frac{b-c}{a-c}, k\right) + (a-c)\,F(\varphi, k)\right].$$

8. $$\int_x^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-a)\,\Pi\left(\varphi, \frac{b-c}{a-c}, k\right)+(a-d)\,F(\varphi, k)\right].$$

9. $$\int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}}\,dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}}F(\varphi, k) - 2\frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}E(\varphi, k)+\frac{2(a-c)}{(b-c)(c-d)}\sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x>c].$$

10. $$\int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}E(\varphi, k)-\frac{2(a-d)}{(b-d)(c-d)}\sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

11. $$\int_x^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k)-E(\varphi, k)].$$

12. $$\int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{d-c}\sqrt{\frac{b-d}{a-c}}E(\varphi, k)+\frac{2}{c-d}\sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x>c].$$

13. $$\int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}}[(a-c)(b-d)E(\varphi, k)-(a-b)(c-d)F(\varphi, k)] - \frac{2}{c-d}\sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

14. $$\int_x^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}E(\varphi, k)-\frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}F(\varphi, k).$$

15. $$\int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k)-E(\varphi, k)]+\frac{2}{b-d}\sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

16. $$\int_x^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k).$$

17. $$\int_x^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-c}\sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x > c].$$

2.37. Интегралы вида $\int_c^x R\left(\sqrt{a-x},\ \sqrt{b-x},\ \sqrt{x-c},\ \sqrt{x-d}\right)dx$.

Условие: $a > b \geq x > c > d$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

1. $$\int_c^x \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\varphi, k).$$

2. $$\int_c^x \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\,\Pi\left(\varphi, \frac{b-c}{b-d}, k\right) + dF(\varphi, k)\right].$$

3. $$\int_c^x \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{cd\sqrt{(a-c)(b-d)}}\left\{(d-c)\,\Pi\left[\varphi, \frac{d(b-c)}{c(b-d)}, k\right] + cF(\varphi, k)\right\}.$$

4. $$\int_c^x \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \left\{(c-d)\,\Pi\left[\varphi, \frac{(b-c)(p-d)}{(b-d)(p-c)}, k\right] + (p-c)F(\varphi, k)\right\} \quad [p \neq c].$$

5. $$\int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\,\Pi\left(\varphi, \frac{b-c}{b-d}, k\right) + (a-d)F(\varphi, k)\right].$$

6. $$\int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\,\Pi\left(\varphi, \frac{b-c}{b-d}, k\right) + (b-d)F(\varphi, k)\right].$$

7. $$\int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi, \frac{b-c}{b-d}, k\right) - F(\varphi, k)\right].$$

8. $$\int_c^x \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi, \frac{b-c}{b-d}, k\right).$$

9. $$\int_c^x \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-c}\sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}} \quad [x < b].$$

10. $$\int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k).$$

11. $$\int_c^x \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c}\sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

12. $$\int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}}\times [(a-c)(b-d)\,E(\varphi, k) - (a-b)(c-d)\,F(\varphi, k)].$$

13. $$\int_c^x \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\varphi, k) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\varphi, k) - \frac{2}{a-b}\sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

14. $$\int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{b-a}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k) + \frac{2}{a-b}\sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}}.$$

15. $$\int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)].$$

16. $$\int_c^x \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k) - \frac{2(a-d)}{(a-b)(a-c)}\sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

17. $\int\limits_c^x \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}}\,dx = \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\varphi, k) -$

$- \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}}$ $[x < b]$.

1.2.38. Интегралы вида $\int\limits_x^c R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d})\,dx$.

Условие: $a > b > c > x \geqslant d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(c-x)}{(c-d)(b-x)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

1. $\int\limits_x^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k)$.

2. $\int\limits_x^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \times$

$\times \left[(c-b)\,\Pi\left(\varphi, \frac{c-d}{b-d}, k\right) + bF(\varphi, k)\right]$.

3. $\int\limits_x^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$

$= \frac{2}{bc\sqrt{(a-c)(b-d)}} \left\{(b-c)\,\Pi\left[\varphi, \frac{b(c-d)}{c(b-d)}, k\right] + cF(\varphi, k)\right\}$.

4. $\int\limits_x^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \times$

$\times \left\{(c-b)\,\Pi\left[\varphi, \frac{(c-d)(p-b)}{(b-d)(p-c)}, k\right] + (p-c)F(\varphi, k)\right\}$ $[p \neq c]$.

5. $\int\limits_x^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx =$

$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c)\,\Pi\left(\varphi, \frac{c-d}{b-d}, k\right) + (a-b)F(\varphi, k)\right]$.

6. $\int\limits_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{c-d}{b-d}, k\right)$.

7. $\int\limits_x^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{c-d}{b-d}, k\right) - F(\varphi, k)\right]$.

8. $$\int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-b)\,\Pi\left(\varphi, \frac{c-d}{b-d}, k\right)+(b-d)\,F(\varphi, k)\right].$$

9. $$\int_x^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k).$$

10. $$\int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k)-E(\varphi, k)]+ \frac{2}{c-d}\sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \qquad [x>d].$$

11. $$\int_x^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k)-\frac{2(a-b)}{(a-c)(a-d)}\sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

12. $$\int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}}\,[(b-c)(a-d)\,F(\varphi, k)-(a-c)(b-d)\,E(\varphi, k)]+ \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \qquad [x>d].$$

13. $$\int_x^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,F(\varphi, k)- \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\varphi, k)-\frac{2}{a-d}\sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

14. $$\int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k)-E(\varphi, k)].$$

15. $$\int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{d-a}\left[\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k)-\sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}}\right] \qquad [x>d].$$

16. $$\int_x^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k)-E(\varphi, k)]+\frac{2}{a-c}\sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

17. $$\int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\varphi, k)-\frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\varphi, k).$$

1.2.39. Интегралы вида $\int_d^x R\left(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d}\right)dx$.

Условия: $a > b > c \geqslant x > d$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{(a-c)(x-d)}{(c-d)(a-x)}}$, $k=\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

1. $$\int_d^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\varphi, k).$$

2. $$\int_d^x \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-a)\,\Pi\left(\varphi, \frac{d-c}{a-c}, k\right)+aF(\varphi, k)\right].$$

3. $$\int_d^x \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{ad\sqrt{(a-c)(b-d)}}\left\{(a-d)\,\Pi\left[\varphi, \frac{a(d-c)}{d(a-c)}, k\right]+dF(\varphi, k)\right\}.$$

4. $$\int_d^x \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}}\times$$
$$\times\left\{(d-a)\,\Pi\left[\varphi, \frac{(d-c)(p-a)}{(a-c)(p-d)}, k\right]+(p-d)\,F(\varphi, k)\right\} \qquad [p \neq d].$$

5. $$\int_d^x \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi, \frac{d-c}{a-c}, k\right).$$

6. $$\int_d^x \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\varphi, \frac{d-c}{a-c}, k\right)-(a-b)\,F(\varphi, k)\right].$$

7. $$\int\limits_d^x \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\varphi, \frac{d-c}{a-c}, k\right)-(a-c)\,F(\varphi, k)\right].$$

8. $$\int\limits_d^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi, \frac{d-c}{a-c}, k\right)-F(\varphi, k)\right].$$

9. $$\int\limits_d^x \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k)-\frac{2(a-b)}{(b-c)(b-d)}\sqrt{\frac{(x-d)(c-x)}{(a-x)(b-x)}}.$$

10. $$\int\limits_d^x \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}}\,dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\varphi, k)-$$
$$-2\frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\varphi, k)+2\,\frac{a-c}{(b-c)(c-d)}\sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x<c].$$

11. $$\int\limits_d^x \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k).$$

12. $$\int\limits_d^x \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k)-E(\varphi, k)]+$$
$$+\frac{2}{c-d}\sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x<c].$$

13. $$\int\limits_d^x \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\varphi, k)-\frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\varphi, k).$$

14. $$\int\limits_d^x \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k)-E(\varphi, k)]+\frac{2}{b-d}\sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

15. $$\int\limits_d^x \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k)-E(\varphi, k)].$$

16. $$\int_d^x \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}E(\varphi, k) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}F(\varphi, k) + \frac{2}{b-c}\sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

17. $$\int_d^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{-2}{b-c}\sqrt{\frac{b-d}{a-c}}E(\varphi, k) + \frac{2}{b-c}\sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}}.$$

1.2.40. Интегралы вида $\int_x^d R\left(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{d-x}\right) dx.$

Условие. $a > b > c > d > x$

Обозначения $\varphi = \arcsin\sqrt{\frac{(a-c)(d-x)}{(a-d)(c-x)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

1. $$\int_x^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\varphi, k)$$

2. $$\int_x^d \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[c\Pi\left(\varphi, \frac{a-d}{a-c}, k\right) - (c-d)F(\varphi, k)\right].$$

3. $$\int_x^d \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{cd\sqrt{(a-c)(b-d)}}\left\{(c-d)\Pi\left[\varphi, \frac{c(a-d)}{d(a-c)}, k\right] + dF(\varphi, k)\right\}.$$

4. $$\int_x^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \left\{(d-c)\Pi\left[\varphi, \frac{(a-d)(p-c)}{(a-c)(p-d)}, k\right] + (p-d)F(\varphi, k)\right\}$$ $[p \neq d]$.

5. $$\int_x^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\Pi\left(\varphi, \frac{a-d}{a-c}, k\right) + (a-c)F(\varphi, k)\right].$$

6. $$\int\limits_x^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\,\Pi\left(\varphi, \frac{a-d}{a-c}, k\right) + (b-c)\,F(\varphi, k)\right].$$

7. $$\int\limits_x^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\varphi, \frac{a-d}{a-c}, k\right).$$

8. $$\int\limits_x^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\varphi, \frac{a-d}{a-c}, k\right) - F(\varphi, k)\right].$$

9. $$\int\limits_x^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d}\sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

10. $$\int\limits_x^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\varphi, k) - \frac{a-b}{b-c}\,\frac{2}{\sqrt{(a-c)(b-d)}}\,F(\varphi, k)$$

11. $$\int\limits_x^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-d}\sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

12. $$\int\limits_x^d \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k).$$

13. $$\int\limits_x^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}}\,dx = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\varphi, k) -$$
$$- \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\varphi, k) + \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

14. $$\int\limits_x^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,E(\varphi, k) - \frac{2(b-c)}{(a-b)(b-d)}\sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

15. $$\int_x^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{b-a}\sqrt{\frac{b-d}{a-c}}\,E(\varphi, k) + \frac{2}{a-b}\sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

16. $$\int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\varphi, k) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\varphi, k) - \frac{2}{a-b}\sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

17. $$\int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\varphi, k) - E(\varphi, k)].$$

2.41. Интегралы вида $\int x^m (x^2 \pm a^2)^{n+1/2}\,dx$.

1. $$\int x^m (x^2 \pm a^2)^{n+1/2}\,dx = \frac{x^{m-1}(x^2 \pm a^2)^{n+3/2}}{m+2n+2} \mp \frac{(m-1)a^2}{m+2n+2}\int x^{m-2}(x^2 \pm a^2)^{n+1/2}\,dx.$$

2. $$\int x^{2m}(x^2 \pm a^2)^{n+1/2}\,dx = \frac{(x^2 \pm a^2)^{n+3/2}}{2n+2m+2} \times$$
$$\times \left[x^{2m-1} + \sum_{k=1}^{m-1} (\mp 1)^k \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k (m+n)(m+n-1)\ldots(m+n-k+1)}\, a^{2k} x^{2m-2k-1}\right] +$$
$$+ (\mp 1)^m \frac{a^{2m}(2m-1)!!}{2^m (n+m+1)(n+m)\ldots(n+2)}\int (x^2 \pm a^2)^{n+1/2}\,dx.$$

3. $$\int x^{2m+1}(x^2 \pm a^2)^{n+1/2}\,dx = \sum_{k=0}^{m} \binom{m}{k} \frac{(\mp a^2)^{m-k}(x^2 \pm a^2)^{n+k+3/2}}{2n+2k+3}.$$

4. $$\int x(x^2 \pm a^2)^{n+1/2}\,dx = \frac{(x^2 \pm a^2)^{n+3/2}}{2n+3}.$$

5. $$\int (x^2 \pm a^2)^{n+1/2}\,dx = \frac{x(x^2 \pm a^2)^{n+1/2}}{2(n+1)} \pm \frac{(2n+1)a^2}{2(n+1)}\int (x^2 \pm a^2)^{n-1/2}\,dx.$$

6. $$= \frac{x(x^2 \pm a^2)^{1/2}}{2n+2} \times$$
$$\times \left[(x^2 \pm a^2)^n + \sum_{k=1}^{n} (\pm 1)^k \frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k n(n-1)\ldots(n-k+1)}\, a^{2k}(x^2 \pm a^2)^{n-k}\right] +$$
$$+ \frac{(\pm 1)^{n+1} a^{2n+2}(2n+1)!!}{2^{n+1}(n+1)!}\ln\left|x + (x^2 \pm a^2)^{1/2}\right|.$$

7. $$\int x^{2m}(x^2\pm a^2)^{1/2}\,dx=\frac{(x^2\pm a^2)^{3/2}}{2(m+1)}\times$$
$$\times\left[x^{2m-1}+\sum_{k=1}^{m-1}(\mp 1)^k\frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k m(m-1)\ldots(m-k+1)}a^{2k}x^{2m-2k-1}\right]+$$
$$+(\mp 1)^m\frac{a^{2m}(2m-1)!!}{2^m(m+1)!}\int(x^2\pm a^2)^{1/2}\,dx$$

8. $$\int(x^2\pm a^2)^{1/2}\,dx=\frac{1}{2}x(x^2\pm a^2)^{1/2}\pm\frac{a^2}{2}\ln\left|x+(x^2\pm a^2)^{1/2}\right|.$$

9. $$\int x(x^2\pm a^2)^{1/2}\,dx=\frac{1}{3}(x^2\pm a^2)^{3/2}.$$

10. $$\int x^2(x^2\pm a^2)^{1/2}\,dx=$$
$$=\frac{1}{4}x(x^2\pm a^2)^{3/2}\mp\frac{a^2}{8}x(x^2\pm a^2)^{1/2}-\frac{a^4}{8}\ln\left|x+(x^2\pm a^2)^{1/2}\right|.$$

11. $$\int x^3(x^2\pm a^2)^{1/2}\,dx=\frac{1}{5}(x^2\pm a^2)^{5/2}\mp\frac{a^2}{3}(x^2\pm a^2)^{3/2}.$$

12. $$\int(x^2\pm a^2)^{3/2}\,dx=$$
$$=\frac{1}{4}x(x^2\pm a^2)^{3/2}\pm\frac{3}{8}a^2x(x^2\pm a^2)^{1/2}+\frac{3}{8}a^4\ln\left|x+(x^2\pm a^2)^{1/2}\right|.$$

13. $$\int x(x^2\pm a^2)^{3/2}\,dx=\frac{1}{5}(x^2\pm a^2)^{5/2}.$$

14. $$\int x^2(x^2\pm a^2)^{3/2}\,dx=$$
$$=\frac{1}{6}x(x^2\pm a^2)^{5/2}\mp\frac{a^2}{24}x(x^2\pm a^2)^{3/2}-\frac{a^4}{16}x(x^2\pm a^2)^{1/2}\mp$$
$$\mp\frac{a^6}{16}\ln\left|x+(x^2\pm a^2)^{1/2}\right|.$$

15. $$\int x^3(x^2\pm a^2)^{3/2}\,dx=\frac{1}{7}(x^2\pm a^2)^{7/2}\mp\frac{a^2}{5}(x^2\pm a^2)^{5/2}.$$

1.2.42. Интегралы вида $\int\frac{(x^2\pm a^2)^{n+1/2}}{x^m}dx$.

1. $$\int\frac{(x^2\pm a^2)^{n+1/2}}{x^m}dx=\mp\frac{(x^2\pm a^2)^{n+3/2}}{(m-1)a^2x^{m-1}}\mp\frac{2n-m+4}{(m-1)a^2}\int\frac{(x^2\pm a^2)^{n+1/2}}{x^{m-2}}dx.$$

2. $$=-\frac{(x^2\pm a^2)^{n+1/2}}{(m-1)x^{m-1}}+\frac{2n-1}{m-1}\int\frac{(x^2\pm a^2)^{n-1/2}}{x^{m-2}}dx.$$

3. $$\int\frac{(x^2\pm a^2)^{n+1/2}}{x^{2m}}dx=\mp\frac{(x^2\pm a^2)^{n+3/2}}{(2m-1)a^2}\times$$
$$\times\left[\frac{1}{x^{2m-1}}+\sum_{k=1}^{m-1}\frac{(\mp 1)^k 2^k(m-n-2)(m-n-3)\ldots(m-n-k-1)}{(2m-3)(2m-5)\ldots(2m-2k-1)a^{2k}}\frac{1}{x^{2m-2k-1}}\right]+$$
$$+\frac{(\mp 1)^m 2^m(m-n-2)(m-n-3)\ldots(-n)(-n-1)}{a^{2m}(2m-1)!!}\int(x^2\pm a^2)^{n+1/2}\,dx.$$

4. $$= \frac{(x^2 \pm a^2)^{n+3/2}}{(\pm a^2)^{m-n-1} x^{2n+3}} \sum_{k=0}^{m-n-2} (-1)^k \binom{m-n-2}{k} \frac{1}{2n+2k+3} \left(\frac{x^2 \pm a^2}{x^2}\right)^k$$

$[m \geqslant n+2]$.

5. $$\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{2n}}\,dx =$$

$$= -\sum_{k=0}^{n-1} \frac{(2n+1)(x^2 \pm a^2)^{n-k-1/2}}{[(2n-2k)^2-1]\,x^{2n-2k-1}} + (2n+1)\int (x^2 \pm a^2)^{1/2}\,dx.$$

6. $$\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{2m+1}}\,dx =$$

$$= -\frac{(x^2 \pm a^2)^{n+3/2}}{2n-2m+1} \sum_{k=1}^{m} (\pm 1)^k \frac{(2n-2m+1)(2n-2m+3)\ldots(2n-2m+2k-1)}{2^k m(m-1)\ldots(m-k+1)\,a^{2k} x^{2m-2k+2}} +$$

$$+ (\pm 1)^m \frac{(2n-2m+1)(2n-2m+3)\ldots(2n-1)}{2^m a^{2m} m!} \int \frac{(x^2 \pm a^2)^{n+1/2}}{x}\,dx.$$

7. $$\int \frac{(x^2 \pm a^2)^{n+1/2}}{x}\,dx =$$

$$= \sum_{k=0}^{n} \frac{(\pm a^2)^k (x^2 \pm a^2)^{n-k+1/2}}{2n-2k+1} + (\pm 1)^{n+1} a^{2n+1} \begin{Bmatrix} \frac{1}{2}\ln\left|\dfrac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+a^2}+a}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

8. $$\int \frac{(x^2 \pm a^2)^{1/2}}{x}\,dx = (x^2 \pm a^2)^{1/2} - a \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

9. $$\int \frac{(x^2 \pm a^2)^{1/2}}{x^2}\,dx = -\frac{(x^2 \pm a^2)^{1/2}}{x} + \ln\left|x + (x^2 \pm a^2)^{1/2}\right|.$$

10. $$\int \frac{(x^2 \pm a^2)^{1/2}}{x^3}\,dx = -\frac{(x^2 \pm a^2)^{1/2}}{2x^2} + \frac{1}{2a} \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

11. $$\int \frac{(x^2 \pm a^2)^{3/2}}{x}\,dx =$$

$$= \frac{1}{3}(x^2 \pm a^2)^{3/2} \pm a^2 (x^2 \pm a^2)^{1/2} \mp a^3 \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

12. $$\int \frac{(x^2 \pm a^2)^{3/2}}{x^2}\,dx =$$

$$= -\frac{(x^2 \pm a^2)^{3/2}}{x} + \frac{3}{2} x (x^2 \pm a^2)^{1/2} \pm \frac{3}{2} a^2 \ln\left|x + (x^2 \pm a^2)^{1/2}\right|.$$

13. $$\int \frac{(x^2 \pm a^2)^{3/2}}{x^4}\,dx = -\frac{(x^2 \pm a^2)^{3/2}}{2x^3} + \frac{3}{2}(x^2 \pm a^2)^{1/2} - \frac{3a}{2}\left\{\begin{matrix} \ln\left|\frac{a + (x^2 \pm a^2)^{1/2}}{x}\right| \\ \arccos\left|\frac{a}{x}\right| \end{matrix}\right\}.$$

14. $$\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{2n+2}}\,dx = -\sum_{k=0}^{n} \frac{1}{2n-2k+1}\left(\frac{x^2 \pm a^2}{x^2}\right)^{n-k+1/2} + \ln\left|x + \sqrt{x^2 \pm a^2}\right|.$$

15. $$\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{2n+4}}\,dx = \pm \frac{(x^2 \pm a^2)^{n+3/2}}{(2n+3)\,a^2 x^{2n+3}}.$$

1.2.43. Интегралы вида $\int \frac{x^m\,dx}{(x^2 \pm a^2)^{n+1/2}}$.

1. $$\int \frac{x^m\,dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{-x^{m-1}}{(2n-1)(x^2 \pm a^2)^{n-1/2}} + \frac{m-1}{2n-1}\int \frac{x^{m-2}\,dx}{(x^2 \pm a^2)^{n-1/2}}.$$

2. $$= \pm \frac{x^{m+1}}{(2n-1)\,a^2 (x^2 \pm a^2)^{n-1/2}} \pm \frac{2n-m-2}{(2n-1)\,a^2}\int \frac{x^m\,dx}{(x^2 \pm a^2)^{n-1/2}}.$$

3. $$= \frac{x^{m-1}}{(m-2n)(x^2 \pm a^2)^{n-1/2}} \mp \frac{(m-1)\,a^2}{m-2n}\int \frac{x^{m-2}\,dx}{(x^2 \pm a^2)^{n+1/2}}.$$

4. $$\int \frac{x^{2m}\,dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{x^{2m+1}\sqrt{x^2 \pm a^2}}{(2n-1)\,a^2} \times$$
$$\times\left[\frac{1}{(x^2 \pm a^2)^n} + \sum_{k=1}^{n-1} (\pm 1)^k \frac{2^k (n-m-1)(n-m-2)\ldots(n-m-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \frac{a^{-2k}}{(x^2 \pm a^2)^{n-k}}\right] +$$
$$+ \frac{(\pm 1)^n 2^n (n-m-1)(n-m-2)\ldots(-m+1)(-m)}{a^{2n}(2n-1)!!}\int \frac{x^{2m}\,dx}{\sqrt{x^2 \pm a^2}}.$$

5. $$= (\pm 1)^{n-m} a^{2m-2n} \sum_{k=0}^{n-m-1} \frac{(-1)^k}{2m+2k+1}\binom{n-m-1}{k}\left(\frac{x^2}{x^2 \pm a^2}\right)^{m+k+1/2}$$
$[n \geqslant m+1]$

6. $$\int \frac{x^{2m}\,dx}{\sqrt{x^2 \pm a^2}} = \frac{\sqrt{x^2 \pm a^2}}{2m} \times$$
$$\times\left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(\mp 1)^k (2m-1)(2m-3)\ldots(2m-2k+1)}{2^k (m-1)(m-2)\ldots(m-k)} a^{2k} x^{2m-2k-1}\right] +$$
$$+ (\mp 1)^m a^{2m} \frac{(2m-1)!!}{2^m m!} \ln\left|x + \sqrt{x^2 \pm a^2}\right|.$$

7. $$\int \frac{x^{2n-2}\,dx}{(x^2 \pm a^2)^{n+1/2}} = \pm \frac{x^{2n-1}}{(2n-1)\,a^2 (x^2 \pm a^2)^{n-1/2}}.$$

8. $$\int \frac{x^{2m+1}\,dx}{(x^2 \pm a^2)^{n+1/2}} = \sum_{k=0}^{m} (\mp 1)^{m+k} \binom{m}{k} \frac{a^{2m-2k}}{(2n-2k-1)(x^2 \pm a^2)^{n-k-1/2}}.$$

9. $$\int \frac{dx}{(x^2 \pm a^2)^{n+1/2}} = \pm \frac{x}{(2n-1)a^2(x^2 \pm a^2)^{n-1/2}} \pm \frac{2n-2}{(2n-1)a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1/2}}.$$

10. $$\int \frac{dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{\pm x(x^2 \pm a^2)^{1/2}}{(2n-1)a^2} \times$$
$$\times \left[\frac{1}{(x^2 \pm a^2)^n} + \sum_{k=1}^{n-1} (\pm 1)^k \frac{2^k (n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \cdot \frac{1}{a^{2k}(x^2 \pm a^2)^{n-k}} \right].$$

11. $$= \frac{(\pm 1)^n}{a^{2n}} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \left(\frac{x^2}{x^2 \pm a^2} \right)^{k+1/2}.$$

12. $$\int \frac{x\,dx}{(x^2 \pm a^2)^{n+1/2}} = -\frac{1}{(2n-1)(x^2 \pm a^2)^{n-1/2}}.$$

13. $$\int \frac{dx}{(x^2 \pm a^2)^{1/2}} = \ln|x + (x^2 \pm a^2)^{1/2}|.$$

14. $$\int \frac{x\,dx}{(x^2 \pm a^2)^{1/2}} = (x^2 \pm a^2)^{1/2}.$$

15. $$\int \frac{x^2\,dx}{(x^2 \pm a^2)^{1/2}} = \frac{x(x^2 \pm a^2)^{1/2}}{2} \mp \frac{a^2}{2} \ln|x + (x^2 \pm a^2)^{1/2}|.$$

16. $$\int \frac{x^3\,dx}{(x^2 \pm a^2)^{1/2}} = \frac{1}{3}(x^2 \pm a^2)^{3/2} \mp a^2 (x^2 \pm a^2)^{1/2}.$$

17. $$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \pm \frac{x}{a^2(x^2 \pm a^2)^{1/2}}.$$

18. $$\int \frac{x\,dx}{(x^2 \pm a^2)^{3/2}} = -\frac{1}{(x^2 \pm a^2)^{1/2}}.$$

19. $$\int \frac{x^2\,dx}{(x^2 \pm a^2)^{3/2}} = -\frac{x}{(x^2 \pm a^2)^{1/2}} + \ln|x + (x^2 \pm a^2)^{1/2}|.$$

20. $$\int \frac{x^3\,dx}{(x^2 \pm a^2)^{3/2}} = \frac{x^2 \pm 2a^2}{(x^2 \pm a^2)^{1/2}}.$$

1.2.44. Интегралы вида $\int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}}$.

1. $$\int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}} =$$
$$= \pm \frac{1}{(2n-1)a^2 x^{m-1}(x^2 \pm a^2)^{n-1/2}} \mp \frac{m+2n-2}{(2n-1)a^2} \int \frac{dx}{x^m (x^2 \pm a^2)^{n-1/2}}.$$

2. $$= \frac{-1}{(2n-1)x^{m+1}(x^2 \pm a^2)^{n-1/2}} - \frac{m+1}{2n-1} \int \frac{dx}{x^{m+2}(x^2 \pm a^2)^{n-1/2}}.$$

3. $$= \frac{-1}{(m-1)x^{m-1}(x^2\pm a^2)^{n-1/2}} - \frac{2n-1}{m-1}\int \frac{dx}{x^{m-2}(x^2\pm a^2)^{n+1/2}}.$$

4. $$\int \frac{dx}{x^{2m}(x^2\pm a^2)^{n+1/2}} =$$
$$= \frac{(\mp 1)^{m+n}}{a^{2m+2n}} \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2k-2m+1}\binom{m+n-1}{k}\left(\frac{x^2}{x^2\pm a^2}\right)^{k-m+1/2}.$$

5. $$\int \frac{dx}{x(x^2\pm a^2)^{n+1/2}} = \sum_{k=0}^{n-1} \frac{(\pm 1)^{k+1}}{(2n-2k-1)a^{2k+2}(x^2\pm a^2)^{n-k-1/2}} -$$
$$- \frac{(\pm 1)^{n+1}}{a^{2n+1}} \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

6. $$\int \frac{dx}{x(x^2\pm a^2)^{1/2}} = \frac{1}{a} \begin{Bmatrix} \ln\left|\dfrac{x}{a+(x^2+a^2)^{1/2}}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

7. $$\int \frac{dx}{x^2(x^2\pm a^2)^{1/2}} = \mp \frac{(x^2\pm a^2)^{1/2}}{a^2 x}.$$

8. $$\int \frac{dx}{x^3(x^2\pm a^2)^{1/2}} = \mp \frac{(x^2\pm a^2)^{1/2}}{2a^2x^2} + \frac{1}{2a^3} \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

9. $$\int \frac{dx}{x(x^2\pm a^2)^{3/2}} = \pm \frac{1}{a^2(x^2\pm a^2)^{1/2}} - \frac{1}{a^3} \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

10. $$\int \frac{dx}{x^2(x^2\pm a^2)^{3/2}} = -\frac{2x^2\pm a^2}{a^4 x(x^2\pm a^2)^{1/2}}.$$

11. $$\int \frac{dx}{x^3(x^2\pm a^2)^{3/2}} =$$
$$= \mp \frac{1}{2a^2x^2(x^2\pm a^2)^{1/2}} - \frac{3}{2a^4(x^2\pm a^2)^{1/2}} \pm \frac{3}{2a^5} \begin{Bmatrix} \ln\left|\dfrac{a+(x^2+a^2)^{1/2}}{x}\right| \\ \arccos\left|\dfrac{a}{x}\right| \end{Bmatrix}.$$

1.2.45. Интегралы вида $\int \frac{dx}{(x+b)^n\sqrt{x^2\pm a^2}}$, $\int \frac{dx}{(x^2\pm b^2)\sqrt{x^2\pm a^2}}$.

1. $$\int \frac{dx}{(x+b)^n\sqrt{x^2\pm a^2}} = \frac{-1}{(n-1)(b^2\pm a^2)}\left[\frac{\sqrt{x^2\pm a^2}}{(x+b)^{n-1}} -\right.$$
$$\left. -(2n-3)b\int \frac{dx}{(x+b)^{n-1}\sqrt{x^2\pm a^2}} + (n-2)\int \frac{dx}{(x+b)^{n-2}\sqrt{x^2\pm a^2}}\right].$$

2. $$\int \frac{dx}{(x+b)\sqrt{x^2+a^2}} = \frac{1}{\sqrt{a^2+b^2}} \ln \left| \frac{a^2-bx-\sqrt{(a^2+b^2)(x^2+a^2)}}{x+b} \right|.$$

3. $$\int \frac{dx}{(x+b)\sqrt{x^2-a^2}} = \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{a^2+bx+\sqrt{(b^2-a^2)(x^2-a^2)}}{x+b} \right|$$ $[b^2 > a^2]$.

4. $$= \frac{1}{\sqrt{a^2-b^2}} \operatorname{arctg} \frac{\sqrt{(a^2-b^2)(x^2-a^2)}}{a^2+bx}$$ $[a^2 > b^2]$.

5. $$= \frac{1}{\sqrt{a^2-b^2}} \arccos \frac{a^2+bx}{a(x+b)}$$ $[a^2 > b^2]$.

6. $$\int \frac{dx}{(x+a)\sqrt{x^2-a^2}} = \frac{1}{a}\sqrt{\frac{x-a}{x+a}}.$$

7. $$\int \frac{dx}{(x-a)\sqrt{x^2-a^2}} = -\frac{1}{a}\sqrt{\frac{x+a}{x-a}}.$$

8. $$\int \frac{dx}{(x^2+b^2)^m (x^2+a^2)^{n+1/2}} =$$
$$= \frac{x}{2(m-1)(a^2-b^2)b^2(x^2+b^2)^{m-1}(x^2+a^2)^{n-1/2}} +$$
$$+ \frac{(2m-3)a^2-(2n+4m-6)b^2}{2(m-1)(a^2-b^2)b^2} \int \frac{dx}{(x^2+b^2)^{m-1}(x^2+a^2)^{n+1/2}} +$$
$$+ \frac{m+n-2}{(m-1)(a^2-b^2)b^2} \int \frac{dx}{(x^2+b^2)^{m-2}(x^2+a^2)^{n+1/2}}.$$

9. $$\int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n+1/2}} = \frac{-x}{(2n-1)(a^2-b^2)a^2(x^2+a^2)^{n-1/2}} +$$
$$+ \frac{(4n-3)a^2-(2n-2)b^2}{(2n-1)(a^2-b^2)a^2} \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n-1/2}} -$$
$$- \frac{2n-2}{(2n-1)(a^2-b^2)a^2} \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n-3/2}}.$$

10. $$\int \frac{dx}{(x^2+b^2)\sqrt{x^2+a^2}} = \frac{1}{b\sqrt{b^2-a^2}} \ln \frac{|x\sqrt{b^2-a^2}+b\sqrt{x^2+a^2}|}{\sqrt{x^2+b^2}}$$ $[b^2 > a^2]$.

11. $$= \frac{1}{b\sqrt{a^2-b^2}} \operatorname{arctg} \frac{x\sqrt{a^2-b^2}}{b\sqrt{x^2+a^2}}$$ $[a^2 > b^2]$.

12. $$\int \frac{dx}{(x^2-b^2)\sqrt{x^2+a^2}} = \frac{1}{2b\sqrt{a^2+b^2}} \ln \left| \frac{x\sqrt{a^2+b^2}-b\sqrt{x^2+a^2}}{x\sqrt{a^2+b^2}+b\sqrt{x^2+a^2}} \right|.$$

13. $$\int \frac{dx}{(x^2+b^2)\sqrt{x^2-a^2}} = \frac{1}{b\sqrt{a^2+b^2}} \ln \frac{|x\sqrt{a^2+b^2}+b\sqrt{x^2-a^2}|}{\sqrt{x^2+b^2}}.$$

14. $$\int \frac{dx}{(x^2-b^2)\sqrt{x^2-a^2}} = \frac{1}{b\sqrt{b^2-a^2}} \ln \frac{|x\sqrt{b^2-a^2}-b\sqrt{x^2-a^2}|}{\sqrt{x^2-b^2}}$$ $[b^2 > a^2]$.

15. $$= \frac{1}{b\sqrt{a^2-b^2}} \arcsin \frac{b\sqrt{x^2-a^2}}{a\sqrt{x^2-b^2}} \qquad [a^2 > b^2].$$

16. $$= \frac{1}{b\sqrt{a^2-b^2}} \operatorname{arctg} \frac{b\sqrt{x^2-a^2}}{x\sqrt{a^2-b^2}} \qquad [a^2 > b^2].$$

1.2.46. Интегралы вида $\int x^m (a^2-x^2)^{n+1/2}\,dx$.

1. $$\int x^m (a^2-x^2)^{n+1/2}\,dx = -\frac{x^{m-1}(a^2-x^2)^{n+3/2}}{m+2n+2} + \frac{(m-1)a^2}{m+2n+2}\int x^{m-2}(a^2-x^2)^{n+1/2}\,dx.$$

2. $$= \frac{x^{m+1}(a^2-x^2)^{n+1/2}}{m+2n+2} + \frac{(2n+1)a^2}{m+2n+2}\int x^m (a^2-x^2)^{n-1/2}\,dx.$$

3. $$\int x^{2m}(a^2-x^2)^{n+1/2}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{2n+2m+2} \times \left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m+n)(m+n-1)\ldots(m+n-k+1)} a^{2k}x^{2m-2k-1}\right] + \frac{a^{2m}(2m-1)!!}{2^m(m+n+1)(m+n)\ldots(n+2)}\int (a^2-x^2)^{n+1/2}\,dx.$$

4. $$\int x^{2m+1}(a^2-x^2)^{n+1/2}\,dx = \sum_{k=0}^{m} \frac{(-1)^{k+1}a^{2m-2k}}{2n+2k+3}\binom{m}{k}(a^2-x^2)^{n+k+3/2}.$$

5. $$\int x(a^2-x^2)^{n+1/2}\,dx = -\frac{1}{2n+3}(a^2-x^2)^{n+3/2}.$$

6. $$\int (a^2-x^2)^{n+1/2}\,dx = \frac{x}{2n+2}(a^2-x^2)^{n+1/2} + \frac{(2n+1)a^2}{2n+2}\int (a^2-x^2)^{n-1/2}\,dx.$$

7. $$\int (a^2-x^2)^{n+1/2}\,dx = \frac{x(a^2-x^2)^{1/2}}{2n+2} \times \left[(a^2-x^2)^n + \sum_{k=1}^{n} \frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k n(n-1)\ldots(n-k+1)} a^{2k}(a^2-x^2)^{n-k}\right] + \frac{(2n+1)!!\,a^{2n+2}}{2^{n+1}(n+1)!}\arcsin\frac{x}{|a|}.$$

8. $$\int (a^2-x^2)^{1/2}\,dx = \frac{x}{2}(a^2-x^2)^{1/2} + \frac{a^2}{2}\arcsin\frac{x}{|a|}.$$

9. $$\int x(a^2-x^2)^{1/2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2}.$$

10. $$\int x^2(a^2-x^2)^{1/2}\,dx = \frac{x}{8}(2x^2-a^2)(a^2-x^2)^{1/2} + \frac{a^4}{8}\arcsin\frac{x}{|a|}.$$

11. $$\int x^3(a^2-x^2)^{1/2}\,dx = \frac{1}{5}(a^2-x^2)^{5/2} - \frac{a^2}{3}(a^2-x^2)^{3/2}.$$

12. $\int (a^2-x^2)^{3/2}\,dx = \frac{x}{4}(a^2-x^2)^{3/2}+\frac{3a^2x}{8}(a^2-x^2)^{1/2}+\frac{3a^4}{8}\arcsin\frac{x}{|a|}.$

13. $\int x(a^2-x^2)^{3/2}\,dx = -\frac{1}{5}(a^2-x^2)^{5/2}.$

14. $\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x}{48}(8x^4-14a^2x^2+3a^4)(a^2-x^2)^{1/2}+\frac{a^6}{16}\arcsin\frac{x}{|a|}.$

15. $\int x^3(a^2-x^2)^{3/2}\,dx = \frac{1}{7}(a^2-x^2)^{7/2}-\frac{a^2}{5}(a^2-x^2)^{5/2}.$

1.2.47. Интегралы вида $\int \frac{(a^2-x^2)^{n+1/2}}{x^m}\,dx.$

1. $\int \frac{(a^2-x^2)^{n+1/2}}{x^m}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{(m-1)a^2x^{m-1}}-\frac{2n-m+4}{(m-1)a^2}\int \frac{(a^2-x^2)^{n+1/2}}{x^{m-2}}\,dx.$

2. $= -\frac{(a^2-x^2)^{n+1/2}}{(m-1)x^{m-1}}-\frac{2n+1}{m-1}\int \frac{(a^2-x^2)^{n-1/2}}{x^{m-2}}\,dx.$

3. $\int \frac{(a^2-x^2)^{n+1/2}}{x^{2m}}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{(2m-1)a^2}\times$

$$\times\left[\frac{1}{x^{2m-1}}+\sum_{k=1}^{m-1}\frac{2^k(m-n-2)(m-n-3)\dots(m-n-k-1)}{(2m-3)(2m-5)\dots(2m-2k-1)}\,\frac{a^{-2k}}{x^{2m-2k-1}}\right]+$$

$$+\frac{2^m(m-n-2)(m-n-3)\dots(-n)(-n-1)}{a^{2m}(2m-1)!!}\int (a^2-x^2)^{n+1/2}\,dx.$$

4. $\int \frac{(a^2-x^2)^{n+1/2}}{x}\,dx =$

$$= \sum_{k=0}^{n}\frac{a^{2k}}{2n-2k+1}(a^2-x^2)^{n-k+1/2}+\frac{a^{2n+1}}{2}\ln\left|\frac{a-(a^2-x^2)^{1/2}}{a+(a^2-x^2)^{1/2}}\right|.$$

5. $\int \frac{(a^2-x^2)^{n+1/2}}{x^{2n+2}}\,dx = \sum_{k=0}^{n}\frac{(-1)^{k+1}}{2n-2k+1}\left(\frac{a^2-x^2}{x^2}\right)^{n-k+1/2}+(-1)^{n+1}\arcsin\frac{x}{|a|}.$

6. $\int \frac{(a^2-x^2)^{n+1/2}}{x^{2n+4}}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{(2n+3)a^2x^{2n+3}}.$

7. $\int \frac{(a^2-x^2)^{1/2}}{x}\,dx = (a^2-x^2)^{1/2}-a\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$

8. $\int \frac{(a^2-x^2)^{1/2}\,dx}{x^2} = -\frac{(a^2-x^2)^{1/2}}{x}-\arcsin\frac{x}{a}.$

9. $\int \frac{(a^2-x^2)^{1/2}}{x^3}\,dx = -\frac{1}{2x^2}(a^2-x^2)^{1/2}+\frac{1}{2a}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$

10. $\int \frac{(a^2-x^2)^{3/2}}{x}\,dx = \frac{1}{3}(a^2-x^2)^{3/2}+a^2(a^2-x^2)^{1/2}-a^3\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$

11. $\int \frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{1}{x}(a^2-x^2)^{3/2} - \frac{3x}{2}(a^2-x^2)^{1/2} - \frac{3a^2}{2}\arcsin\frac{x}{|a|}.$

12. $\int \frac{(a^2-x^2)^{3/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3}{2}(a^2-x^2)^{1/2} + \frac{3a}{2}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$

1.2.48. Интегралы вида $\int \frac{x^m\,dx}{(a^2-x^2)^{n+1/2}}$.

1. $\int \frac{x^m\,dx}{(a^2-x^2)^{n+1/2}} = \frac{x^{m-1}}{(2n-1)(a^2-x^2)^{n-1/2}} - \frac{m-1}{2n-1}\int \frac{x^{m-2}\,dx}{(a^2-x^2)^{n-1/2}}.$

2. $= \frac{x^{m+1}}{(2n-1)a^2(a^2-x^2)^{n+1/2}} + \frac{2n-m-2}{(2n-1)a^2}\int \frac{x^m\,dx}{(a^2-x^2)^{n-1/2}}.$

3. $= -\frac{x^{m-1}}{(m-2n)(a^2-x^2)^{n-1/2}} + \frac{(m-1)a^2}{m-2n}\int \frac{x^{m-2}\,dx}{(a^2-x^2)^{n+1/2}}.$

4. $\int \frac{x^{2m+1}\,dx}{(a^2-x^2)^{n+1/2}} = \sum_{k=0}^{m} \frac{(-1)^k a^{2m-2k}}{2n-2k-1}\binom{m}{k}\frac{1}{(a^2-x^2)^{n-k-1/2}}.$

5. $\int \frac{x^{2m}\,dx}{(a^2-x^2)^{n+1/2}} = \frac{x^{2m+1}\sqrt{a^2-x^2}}{(2n-1)a^2}\times$

$\times\left[\frac{1}{(a^2-x^2)^n} + \sum_{k=1}^{n-1}\frac{2^k(n-m-1)(n-m-2)\ldots(n-m-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\frac{a^{-2k}}{(a^2-x^2)^{n-k}}\right]+$

$+\frac{2^n(n-m-1)(n-m-2)\ldots(-m+1)(-m)}{a^{2n}(2n-1)!!}\int \frac{x^{2m}\,dx}{\sqrt{a^2-x^2}}.$

6. $\int \frac{x^{2m}\,dx}{(a^2-x^2)^{n+1/2}} = \frac{1}{a^{2n-2m}}\sum_{k=0}^{n-m-1}\frac{1}{2m+2k+1}\binom{n-m-1}{k}\left(\frac{x^2}{a^2-x^2}\right)^{m+k+1/2}$

[$n\geqslant m+1$].

7. $\int \frac{x^{2n-2}\,dx}{(a^2-x^2)^{n+1/2}} = \frac{x^{2n-1}}{(2n-1)a^2(a^2-x^2)^{n-1/2}}.$

8. $\int \frac{x^{2m}\,dx}{(a^2-x^2)^{1/2}} = -\frac{(a^2-x^2)^{1/2}}{2m}\times$

$\times\left[x^{2m-1} + \sum_{k=1}^{m-1}\frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)}a^{2k}x^{2m-2k-1}\right]+$

$+a^{2m}\frac{(2m-1)!!}{2^m m!}\arcsin\frac{x}{|a|}.$

9. $\int \frac{dx}{(a^2-x^2)^{n+1/2}} = \frac{x(a^2-x^2)^{1/2}}{2n-1}\times$

$\times\left[\frac{1}{(a^2-x^2)^n a^2} + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\frac{a^{-2k-2}}{(a^2-x^2)^{n-k}}\right]$

[$n\geqslant 1$].

10. $$= \frac{1}{a^{2n}} \sum_{k=0}^{n-1} \frac{1}{2k+1} \binom{n-1}{k} \left(\frac{x^2}{a^2-x^2} \right)^{k+1/2}.$$

11. $$\int \frac{x\,dx}{(a^2-x^2)^{n+1/2}} = \frac{1}{(2n-1)(a^2-x^2)^{n-1/2}}.$$

12. $$\int \frac{dx}{(a^2-x^2)^{1/2}} = \arcsin \frac{x}{|a|}.$$ 13. $$\int \frac{x\,dx}{(a^2-x^2)^{1/2}} = -(a^2-x^2)^{1/2}.$$

14. $$\int \frac{x^2\,dx}{(a^2-x^2)^{1/2}} = -\frac{x}{2}(a^2-x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x}{|a|}.$$

15. $$\int \frac{x^3\,dx}{(a^2-x^2)^{1/2}} = \frac{1}{3}(a^2-x^2)^{3/2} - a^2(a^2-x^2)^{1/2}.$$

16. $$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2(a^2-x^2)^{1/2}}.$$ 17. $$\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{(a^2-x^2)^{1/2}}.$$

18. $$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{(a^2-x^2)^{1/2}} - \arcsin \frac{x}{|a|}.$$

19. $$\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \frac{2a^2-x^2}{(a^2-x^2)^{1/2}}.$$

1.2.49. Интегралы вида $\int \frac{dx}{x^m (a^2-x^2)^{n+1/2}}$.

1. $$\int \frac{dx}{x^m (a^2-x^2)^{n+1/2}} =$$
$$= \frac{1}{(2n-1)a^2 x^{m-1}(a^2-x^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1)a^2} \int \frac{dx}{x^m (a^2-x^2)^{n-1/2}}.$$

2. $$= \frac{-1}{(m-1)a^2 x^{m-1}(a^2-x^2)^{n-1/2}} + \frac{m+2n-2}{(m-1)a^2} \int \frac{dx}{x^{m-2}(a^2-x^2)^{n+1/2}}.$$

3. $$= \frac{1}{(2n-1)x^{m+1}(a^2-x^2)^{n-1/2}} + \frac{m+1}{2n-1} \int \frac{dx}{x^{m+2}(a^2-x^2)^{n-1/2}}.$$

4. $$\int \frac{dx}{x^{2m}(a^2-x^2)^{n+1/2}} =$$
$$= \frac{-1}{a^{2n+2m}} \sum_{k=0}^{m+n-1} \frac{1}{2m-2k-1} \binom{m+n-1}{k} \left(\frac{a^2-x^2}{x^2} \right)^{m-k-1/2}.$$

5. $$\int \frac{dx}{x(a^2-x^2)^{n+1/2}} = \sum_{k=0}^{n-1} \frac{1}{(2n-2k-1)a^{2k+2}(a^2-x^2)^{n-k-1/2}} -$$
$$- \frac{1}{a^{2k+1}} \ln \left| \frac{a+(a^2-x^2)^{1/2}}{x} \right|.$$

6. $$\int \frac{dx}{x(a^2-x^2)^{1/2}} = -\frac{1}{a} \ln \left| \frac{a+(a^2-x^2)^{1/2}}{x} \right|.$$

7. $\int \frac{dx}{x^2 (a^2 - x^2)^{1/2}} = -\frac{1}{a^2 x} (a^2 - x^2)^{1/2}.$

8. $\int \frac{dx}{x^3 (a^2 - x^2)^{1/2}} = -\frac{1}{2a^2 x^2} (a^2 - x^2)^{1/2} - \frac{1}{2a^3} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$

9. $\int \frac{dx}{x (a^2 - x^2)^{3/2}} = \frac{1}{a^2 (a^2 - x^2)^{1/2}} - \frac{1}{a^3} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$

10. $\int \frac{dx}{x^2 (a^2 - x^2)^{3/2}} = \frac{2x^2 - a^2}{a^4 x (a^2 - x^2)^{1/2}}.$

11. $\int \frac{dx}{x^3 (a^2 - x^2)^{3/2}} = -\frac{1}{2a^2 x^2 (a^2 - x^2)^{1/2}} +$

$+ \frac{3}{2a^4 (a^2 - x^2)^{1/2}} - \frac{3}{2a^5} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$

1.2.50. Интегралы вида $\int \frac{dx}{(x+b)^n \sqrt{a^2 - x^2}}$, $\int \frac{dx}{(x^2 \pm b^2)(a^2 - x^2)^{n+1/2}}$.

1. $\int \frac{dx}{(x+b)^n \sqrt{a^2 - x^2}} = \frac{1}{(n-1)(b^2 - a^2)} \times$

$\times \left[\frac{\sqrt{a^2 - x^2}}{(x+b)^{n-1}} + (2n-3) b \int \frac{dx}{(x+b)^{n-1} \sqrt{a^2 - x^2}} - (n-2) \int \frac{dx}{(x+b)^{n-2} \sqrt{a^2 - x^2}} \right]$ $[b \neq -a].$

2. $= \frac{1}{(2n-1) a} \left[\frac{\sqrt{a^2 - x^2}}{(x-a)^n} - (n-1) \int \frac{dx}{(x-a)^{n-1} \sqrt{a^2 - x^2}} \right]$ $[b = -a].$

3. $\int \frac{dx}{(x+b) \sqrt{a^2 - x^2}} = \frac{-1}{\sqrt{a^2 - b^2}} \ln \left| \frac{bx + a^2 + \sqrt{(a^2 - b^2)(a^2 - x^2)}}{x+b} \right|$ $[a^2 > b^2].$

4. $= \frac{1}{\sqrt{b^2 - a^2}} \operatorname{arctg} \frac{\sqrt{(b^2 - a^2)(a^2 - x^2)}}{a^2 + bx}$ $[a^2 < b^2].$

5. $= \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{a^2 + bx}{a \,|\, x + b \,|}$ $[a^2 < b^2].$

6. $\int \frac{dx}{(x+a) \sqrt{a^2 - x^2}} = -\frac{1}{a} \sqrt{\frac{a-x}{a+x}}$ $[a > 0].$

7. $\int \frac{dx}{(x-a) \sqrt{a^2 - x^2}} = -\frac{1}{a} \sqrt{\frac{a+x}{a-x}}$ $[a > 0].$

8. $\int \frac{dx}{(b^2 - x^2)^m (a^2 - x^2)^{n+1/2}} = \frac{x}{2(m-1)(a^2 - b^2) b^2 (b^2 - x^2)^{m-1} (a^2 - x^2)^{n-1/2}} +$

$+ \frac{(2m-3) a^2 - 2(m+2n-3) b^2}{2(m-1)(a^2 - b^2) b^2} \int \frac{dx}{(b^2 - x^2)^{m-1} (a^2 - x^2)^{n-1/2}} +$

$+ \frac{m+n-2}{(m-1)(a^2 - b^2) b^2} \int \frac{dx}{(b^2 - x^2)^{m-2} (a^2 - x^2)^{n+1/2}}.$

9. $$\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{n+1/2}} = \frac{-x}{(2n-1)(a^2-b^2)a^2(a^2-x^2)^{n-1/2}} + \frac{(4n-3)a^2-2(n-1)b^2}{(2n-1)(a^2-b^2)a^2}\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{n-1/2}} - \frac{2(n-1)}{(2n-1)(a^2-b^2)a^2}\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{n-3/2}}.$$

10. $$\int \frac{dx}{(b^2-x^2)\sqrt{a^2-x^2}} = \frac{1}{b\sqrt{a^2-b^2}} \ln \frac{\left| x\sqrt{a^2-b^2}+b\sqrt{a^2-x^2}\right|}{\sqrt{x^2-b^2}} \qquad [a^2>b^2].$$

11. $$= \frac{1}{b\sqrt{b^2-a^2}} \operatorname{arctg} \frac{x\sqrt{b^2-a^2}}{b\sqrt{a^2-x^2}} \qquad [a^2<b^2].$$

12. $$= \frac{1}{b\sqrt{b^2-a^2}} \arcsin \frac{x\sqrt{b^2-a^2}}{a\sqrt{b^2-x^2}} \qquad [a^2<b^2].$$

13. $$\int \frac{dx}{(b^2+x^2)\sqrt{a^2-x^2}} = \frac{1}{b\sqrt{a^2+b^2}} \operatorname{arctg} \frac{x\sqrt{a^2+b^2}}{b\sqrt{a^2-x^2}}.$$

1.2.51. Интегралы вида $\int x^{\pm m}(ax^2+bx+c)^{n+1/2}\,dx$.

Обозначение: $X = ax^2+bx+c$.

1. $$\int x^m X^{n+1/2}\,dx = \frac{x^{m-1}X^{n+3/2}}{(m+2n+2)a} - \frac{(2m+2n+1)b}{2(m+2n+2)a}\int x^{m-1}X^{n+1/2}\,dx - \frac{(m-1)c}{(m+2n+2)a}\int x^{m-2}X^{n+1/2}\,dx.$$

2. $$\int xX^{n+1/2}\,dx = \frac{X^{n+3/2}}{(2n+3)a} - \frac{b}{2a}\int X^{n+1/2}\,dx.$$

3. $$\int X^{n+1/2}\,dx = \frac{2ax+b}{4(n+1)a}X^{n+1/2} + \frac{(2n+1)(4ac-b^2)}{8(n+1)a}\int X^{n-1/2}\,dx.$$

4. $$= \frac{(2ax+b)X^{1/2}}{4(n+1)a} \times \left[X^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{8^{k+1}n(n-1)\ldots(n-k)}\left(\frac{4ac-b^2}{a}\right)^{k+1} X^{n-k-1}\right] + \frac{(2n+1)!!}{8^{n+1}(n+1)!}\left(\frac{4ac-b^2}{a}\right)^{n+1}\int \frac{dx}{X^{1/2}}.$$

5. $$\int x^m X^{1/2}\,dx = \frac{x^{m-1}X^{3/2}}{(m+2)a} - \frac{(2m+1)b}{2(m+2)a}\int x^{m-1}X^{1/2}\,dx - \frac{(m-1)c}{(m+2)a}\int x^{m-2}X^{1/2}\,dx.$$

6. $$\int X^{1/2}\,dx = \frac{2ax+b}{4a}X^{1/2} + \frac{4ac-b^2}{8a^{3/2}} \ln \left| \frac{2ax+b}{2\sqrt{a}} + X^{1/2} \right| \qquad [a>0].$$

7. $$= \frac{2ax+b}{4a}X^{1/2} + \frac{b^2-4ac}{8a\sqrt{|a|}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} \qquad [a<0;\ b^2>4ac].$$

8. $\int xX^{1/2}\,dx = \frac{1}{3a}X^{3/2} - \frac{b}{2a}\int X^{1/2}\,dx.$

9. $\int x^2X^{1/2}\,dx = \frac{6ax-5b}{24a^2}X^{3/2} + \frac{5b^2-4ac}{16a^2}\int X^{1/2}\,dx.$

10. $\int x^3X^{1/2}\,dx = \frac{48a^2x^2-42abx+35b^2-32ac}{240a^3}X^{3/2} - \frac{7b^3-12abc}{32a^3}\int X^{1/2}\,dx.$

11. $\int \frac{X^{n+1/2}}{x^m}\,dx = -\frac{1}{(m-1)cx^{m-1}}X^{n+3/2} +$

$$+\frac{(2n-2m+5)b}{2(m-1)c}\int \frac{X^{n+1/2}}{x^{m-1}}\,dx + \frac{(2n-m+4)a}{(m-1)c}\int \frac{X^{n+1/2}}{x^{m-2}}\,dx.$$

12. $\int \frac{X^{n+1/2}}{x}\,dx = \frac{X^{n+1/2}}{2n+1} + \frac{b}{2}\int X^{n-1/2}\,dx + c\int \frac{X^{n-1/2}}{x}\,dx.$

13. $\int \frac{(ax^2+bx)^{n+1/2}}{x^m}\,dx = \frac{2}{(2n-2m+3)bx^m}(ax^2+bx)^{n+3/2} +$

$$+\frac{2(m-2n-3)a}{(2n-2m+3)b}\int \frac{(ax^2+bx)^{n+1/2}}{x^{m-1}}\,dx.$$

14. $\int \frac{X^{1/2}}{x}\,dx = X^{1/2} + \frac{b}{2}\int \frac{dx}{X^{1/2}} + c\int \frac{dx}{xX^{1/2}}.$

15. $\int \frac{X^{1/2}}{x^2}\,dx = -\frac{X^{1/2}}{x} + a\int \frac{dx}{X^{1/2}} + \frac{b}{2}\int \frac{dx}{xX^{1/2}}.$

16. $\int \frac{(ax^2+bx)^{1/2}}{x^2}\,dx = -\frac{2(ax^2+bx)^{1/2}}{x} + a\int \frac{dx}{(ax^2+bx)^{1/2}}.$

17. $\int \frac{X^{1/2}\,dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4cx}\right)X^{1/2} + \left(\frac{a}{2} - \frac{b^2}{8c}\right)\int \frac{dx}{xX^{1/2}}.$

18. $\int \frac{(ax^2+bx)^{1/2}}{x^3}\,dx = -\frac{2}{3bx^3}(ax^2+bx)^{3/2}.$

19. $\int \frac{X^{3/2}}{x}\,dx = \frac{X^{3/2}}{3} + \frac{2abx+b^2+8ac}{8a}X^{1/2} +$

$$+\frac{(12ac-b^2)b}{16a}\int \frac{dx}{X^{1/2}} + c^2\int \frac{dx}{xX^{1/2}}.$$

20. $\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{5/2}}{cx} + \frac{ax+b}{c}X^{3/2} + \frac{3(2ax+3b)}{4}X^{1/2} +$

$$+\frac{3(4ac+b^2)}{8}\int \frac{dx}{X^{1/2}} + \frac{3bc}{2}\int \frac{dx}{xX^{1/2}}.$$

21. $\int \frac{(ax^2+bx)^{3/2}}{x^2}\,dx = \frac{(ax^2+bx)^{3/2}}{2x} + \frac{3b}{4}(ax^2+bx)^{1/2} + \frac{3b^2}{8}\int \frac{dx}{(ax^2+bx)^{1/2}}.$

22. $$\int \frac{X^{3/2}}{x^3}\,dx = -\left(\frac{1}{2cx^2}+\frac{b}{4c^2x}\right)X^{5/2} + \frac{abx+2ac+b^2}{4c^2}X^{3/2} + \frac{3(abx+2ac+b^2)}{4c}X^{1/2} + \frac{3ab}{2}\int\frac{dx}{X^{1/2}} + \frac{3(4ac+b^2)}{8}\int\frac{dx}{xX^{1/2}}.$$

23. $$\int \frac{(ax^2+bx)^{3/2}}{x^3}\,dx = \left(a-\frac{2b}{x}\right)(ax^2+bx)^{1/2} + \frac{3ab}{2}\int\frac{dx}{(ax^2+bx)^{1/2}}.$$

1.2.52. Интегралы вида $\int \frac{x^{\pm m}\,dx}{(ax^2+bx+c)^{n+1/2}}$.

Обозначение $X = ax^2+bx+c$.

1. $$\int\frac{x^m\,dx}{X^{n+1/2}} = \frac{x^{m-1}}{(m-2n)aX^{n-1/2}} - \frac{(2m-2n-1)b}{2(m-2n)a}\int\frac{x^{m-1}\,dx}{X^{n+1/2}} - \frac{(m-1)c}{(m-2n)a}\int\frac{x^{m-2}\,dx}{X^{n+1/2}}.$$

2. $$= -\frac{2x^m}{(2n-1)bX^{n-1/2}} + \frac{2(2n-m-1)}{(2n-1)b}\int\frac{x^{m-1}\,dx}{X^{n-1/2}} - \frac{2c}{b}\int\frac{x^{m-1}\,dx}{X^{n+1/2}}.$$

3. $$= \frac{1}{a}\int\frac{x^{m-2}\,dx}{X^{n-1/2}} - \frac{c}{a}\int\frac{x^{m-2}\,dx}{X^{n+1/2}} - \frac{b}{a}\int\frac{x^{m-1}\,dx}{X^{n+1/2}}.$$

4. $$\int\frac{x^{2n}\,dx}{X^{n+1/2}} = -\frac{x^{2n-1}}{(2n-1)aX^{n-1/2}} - \frac{b}{2a}\int\frac{x^{2n-1}\,dx}{X^{n+1/2}} + \frac{1}{a}\int\frac{x^{2n-2}\,dx}{X^{n-1/2}}.$$

5. $$\int\frac{x\,dx}{X^{n+1/2}} = -\frac{1}{(2n-1)aX^{n-1/2}} - \frac{b}{2a}\int\frac{dx}{X^{n+1/2}}.$$

6. $$\int\frac{dx}{X^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} + \frac{8(n-1)a}{(2n-1)(4ac-b^2)}\int\frac{dx}{X^{n-1/2}}.$$

7. $$= \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} \times \left[1+\sum_{k=1}^{n-1}\frac{8^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\cdot\frac{a^kX^k}{(4ac-b^2)^k}\right].$$

8. $$\int\frac{dx}{X^{1/2}} = \frac{1}{\sqrt{a}}\ln\left|\frac{2ax+b}{2\sqrt{a}}+\sqrt{X}\right| \qquad [a>0].$$

9. $$= \frac{1}{\sqrt{a}}\operatorname{Arsh}\frac{2ax+b}{\sqrt{4ac-b^2}} \qquad [a>0,\ b^2<4ac].$$

10. $$= \frac{-1}{\sqrt{-a}}\arcsin\frac{2ax+b}{\sqrt{b^2-4ac}} \qquad [a<0,\ b^2>4ac].$$

11. $$= \frac{1}{\sqrt{a}}\ln(2ax+b) \qquad [a>0,\ b^2=4ac].$$

12. $\int \frac{x\,dx}{X^{1/2}} = \frac{X^{1/2}}{a} - \frac{b}{2a}\int \frac{dx}{X^{1/2}}.$

13. $\int \frac{x^2\,dx}{X^{1/2}} = \frac{2ax-3b}{4a^2}X^{1/2} + \frac{3b^2-4ac}{8a^2}\int \frac{dx}{X^{1/2}}.$

14. $\int \frac{x^3\,dx}{X^{1/2}} = \frac{8a^2x^2-10abx+15b^2-16ac}{24a^3}X^{1/2} - \frac{5b^3-12abc}{16a^3}\int \frac{dx}{X^{1/2}}.$

15. $\int \frac{dx}{X^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)X^{1/2}}.$

16. $\int \frac{x\,dx}{X^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)X^{1/2}}.$

17. $\int \frac{x^2\,dx}{X^{3/2}} = \frac{(4ac-2b^2)x-2bc}{a(b^2-4ac)X^{1/2}} + \frac{1}{a}\int \frac{dx}{X^{1/2}}.$

18. $\int \frac{x^3\,dx}{X^{3/2}} = \frac{a(4ac-b^2)x^2+b(10ac-3b^2)x+c(8ac-3b^2)}{a^2(4ac-b^2)X^{1/2}} - \frac{3b}{2a^2}\int \frac{dx}{X^{1/2}}.$

19. $\int \frac{dx}{x^m X^{n+1/2}} = -\frac{1}{(m-1)cx^{m-1}X^{n-1/2}} -$

$$- \frac{(2m+2n-3)b}{2(m-1)c}\int \frac{dx}{x^{m-1}X^{n+1/2}} - \frac{(2n+m-2)a}{(m-1)c}\int \frac{dx}{x^{m-2}X^{n+1/2}}.$$

20. $\int \frac{dx}{xX^{n+1/2}} = \frac{1}{(2n-1)cX^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{X^{n+1/2}} + \frac{1}{c}\int \frac{dx}{xX^{n-1/2}}.$

21. $\int \frac{dx}{x^m(ax^2+bx)^{n+1/2}} = -\frac{2}{(2m+2n-1)bx^m(ax^2+bx)^{n-1/2}} -$

$$- \frac{2(m+2n-1)a}{(2m+2n-1)b}\int \frac{dx}{x^{m-1}(ax^2+bx)^{n+1/2}}.$$

22. $\int \frac{dx}{xX^{1/2}} = -\frac{1}{\sqrt{c}}\ln\left|\frac{2c+bx+2\sqrt{c}\,X^{1/2}}{x}\right|$ $[c>0]$.

23. $= -\frac{1}{\sqrt{c}}\operatorname{Arth}\frac{2c+bx}{2\sqrt{c}\,X^{1/2}}$ $[c>0]$.

24. $= -\frac{1}{\sqrt{c}}\operatorname{Arsh}\frac{2c+bx}{x\sqrt{4ac-b^2}}$ $[c>0,\ b^2<4ac]$.

25. $= \frac{1}{\sqrt{c}}\ln\left|\frac{x}{2c+bx}\right|$ $[c>0,\ b^2=4ac]$

26. $= \frac{1}{\sqrt{-c}}\operatorname{arctg}\frac{2c+bx}{2\sqrt{-c}\,X^{1/2}}$ $[c<0]$.

27. $= \frac{1}{\sqrt{-c}}\arcsin\frac{2c+bx}{x\sqrt{b^2-4ac}}$ $[c<0,\ b^2>4ac]$

28. $= -\frac{2\sqrt{bx+ax^2}}{bx}$ $[c=0,\ b\neq 0]$.

29. $$\int \frac{dx}{x^2X^{1/2}} = -\frac{X^{1/2}}{cx} - \frac{b}{2c}\int \frac{dx}{xX^{1/2}}.$$

30. $$\int \frac{dx}{x^2(ax^2+bx)^{1/2}} = \frac{2}{3}\left(-\frac{1}{bx^2}+\frac{2a}{b^2x}\right)(ax^2+bx)^{1/2}.$$

31. $$\int \frac{dx}{x^3X^{1/2}} = \left(-\frac{1}{2cx^2}+\frac{3b}{4c^2x}\right)X^{1/2}+\left(\frac{3b^2}{8c^2}-\frac{a}{2c}\right)\int \frac{dx}{xX^{1/2}}.$$

32. $$\int \frac{dx}{x^3(ax^2+bx)^{1/2}} = \frac{2}{5}\left(-\frac{1}{bx^3}+\frac{4a}{3b^2x^2}-\frac{8a^2}{3b^3x}\right)(ax^2+bx)^{1/2}.$$

33. $$\int \frac{dx}{xX^{3/2}} = \frac{2(abx-2ac+b^2)}{c(b^2-4ac)X^{1/2}}+\frac{1}{c}\int \frac{dx}{xX^{1/2}}.$$

34. $$\int \frac{dx}{x(ax^2+bx)^{3/2}} = \frac{2}{3}\left(-\frac{1}{bx}+\frac{4a}{b^2}-\frac{8a^2x}{b^3}\right)\frac{1}{(ax^2+bx)^{1/2}}.$$

35. $$\int \frac{dx}{x^2X^{3/2}} = \left[-\frac{1}{cx}+\frac{2ab}{c(4ac-b^2)}+\frac{a(3b^2-3ac)x}{c^2(4ac-b^2)}\right]\frac{1}{X^{1/2}}-\frac{3b}{2c^2}\int \frac{dx}{xX^{1/2}}.$$

36. $$\int \frac{dx}{x^2(ax^2+bx)^{3/2}} = \frac{2}{5}\left(-\frac{1}{bx^2}+\frac{2a}{b^2x}-\frac{8a^2}{b^3}-\frac{16a^3x}{b^4}\right)\frac{1}{(ax^2+bx)^{1/2}}.$$

37. $$\int \frac{dx}{x^3X^{3/2}} = \left[-\frac{1}{cx^2}+\frac{5b}{2c^2x}+\frac{15b^4-62acb^2+24a^2c^2}{2c^3(b^2-4ac)}+\right.$$
$$\left.+\frac{ab(15b^2-52ac)x}{2c^3(b^2-4ac)}\right]\frac{1}{2X^{1/2}}+\frac{15b^2-12ac}{8c^3}\int \frac{dx}{xX^{1/2}}.$$

38. $$\int \frac{dx}{x^3(ax^2+bx)^{3/2}} = \frac{2}{7}\left(-\frac{1}{bx^3}+\frac{8a}{5b^2x^2}-\frac{16a^2}{5b^3x}+\frac{64a^3}{5b^4}+\right.$$
$$\left.+\frac{128a^4x}{5b^5}\right)\frac{1}{(ax^2+bx)^{1/2}}.$$

1.2.53. Интегралы вида $\int R(x+p,\ ax^2+bx+c)\,dx$.

Обозначение $X = ax^2+bx+c$.

1. $$\int \frac{dx}{(x+p)^m X^{n+1/2}} = \frac{-1}{(m-1)(ap^2-bp+c)(x+p)^{m-1}X^{n-1/2}} -$$
$$-\frac{(2m+2n-3)(b-2ap)}{2(m-1)(ap^2-bp+c)}\int \frac{dx}{(x+p)^{m-1}X^{n+1/2}} -$$
$$-\frac{(m+2n-2)a}{(m-1)(ap^2-bp+c)}\int \frac{dx}{(x+p)^{m-2}X^{n+1/2}}.$$

2. $$= \frac{-2}{(2m+2n-1)(b-2ap)(x+p)^m X^{n-1/2}} -$$
$$-\frac{2(m+2n-1)a}{(2m+2n-1)(b-2ap)}\int \frac{dx}{(x+p)^{m-1}X^{n+1/2}}$$

$[ap^2-bp+c=0;\ b-2ap\neq 0]$.

3. $$\int \frac{dx}{(x+p)X^{n+1/2}} = \frac{1}{(2n-1)(ap^2-bp+c)X^{n-1/2}} + \frac{1}{ap^2-bp+c}\int \frac{dx}{(x+p)X^{n-1/2}} + \frac{2ap-b}{2(ap^2-bp+c)}\int \frac{dx}{X^{n+1/2}}.$$

4. $$= \frac{-1}{(2n+1)X^{n+1/2}} + \frac{b-2ap}{2}\int \frac{dx}{X^{n+3/2}} \qquad [ap^2-bp+c=0].$$

5. $$\int \frac{X^{n+1/2}}{(x+p)^m}dx = \frac{-X^{n+3/2}}{(m-1)(ap^2-bp+c)(x+p)^{m-1}} + \frac{(2n-2m+5)(b-2ap)}{2(m-1)(ap^2-bp+c)}\int \frac{X^{n+1/2}}{(x+p)^{m-1}}dx + \frac{(2n-m+4)a}{(m-1)(ap^2-bp+c)}\int \frac{X^{n+1/2}}{(x+p)^{m-2}}.$$

6. $$= \frac{2X^{n+3/2}}{(2n-2m+3)(b-2ap)(x+p)^m} - \frac{2(2n-m+3)a}{(2n-2m+3)(b-2ap)}\int \frac{X^{n+1/2}}{(x+p)^{m-1}}dx$$
$$[ap^2-bp+c=0,\ b-2ap\neq 0].$$

7. $$\int \frac{X^{n+1/2}}{x+p}dx = \frac{X^{n+1/2}}{2n+1} + \frac{b-2ap}{2}\int X^{n-1/2}dx + (ap^2-bp+c)\int \frac{X^{n-1/2}}{x+p}dx.$$

8. $$\int \frac{X^{1/2}dx}{(x+p)^n} = -\frac{X^{1/2}}{(n-1)(x+p)^{n-1}} + \frac{1}{2(n-1)}\int \frac{(b+2ax)dx}{(x+p)^{n-1}X^{1/2}}.$$

9. $$\int \frac{X^{1/2}dx}{x+p} = \int \frac{ax+b-ap}{X^{1/2}}dx + (ap^2-bp+c)\int \frac{dx}{(x+p)X^{1/2}}.$$

10. $$\int \frac{dx}{(x+p)^n X^{1/2}} = \frac{-X^{1/2}}{(n-1)(ap^2-bp+c)(x+p)^{n-1}} - \frac{(2n-3)(b-2ap)}{2(n-1)(ap^2-bp+c)}\int \frac{dx}{(x+p)^{n-1}X^{1/2}} - \frac{(n-2)a}{(n-1)(ap^2-bp+c)}\int \frac{dx}{(x+p)^{n-2}X^{1/2}}.$$

11. $$= \frac{2X^{1/2}}{(2n-1)(2ap-b)(x+p)^n} + \frac{2(n-1)a}{(2n-1)(2ap-b)}\int \frac{dx}{(x+p)^{n-1}X^{1/2}}$$
$$[ap^2-bp+c=0,\ 2ap-b\neq 0].$$

12. $$\int \frac{dx}{(x+p)^n X^{1/2}} = -\int \frac{t^{n-1}dt}{[a+(b-2ap)t+(ap^2-bp+c)t^2]^{1/2}} \qquad \left[t=\frac{1}{x+p}\right].$$

13. $$\int \frac{dx}{(x+p)X^{1/2}} = -\frac{1}{(ap^2-bp+c)^{1/2}}\ln\left|\frac{(ap^2-bp+c)^{1/2}+X^{1/2}}{x+p} + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}}\right| \qquad [ap^2-bp+c>0].$$

14. $$= \frac{1}{2(ap^2-bp+c)^{1/2}}\ln\left|\frac{(ap^2-bp+c)^{1/2}-X^{1/2}}{x+p} + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}}\right| \qquad [ap^2-bp+c>0].$$

15. $$= \frac{1}{(bp-ap^2-c)^{1/2}}\arcsin\frac{(b-2ap)x-bp+2c}{(x+p)(b^2-4ac)^{1/2}} \qquad [ap^2-bp+c<0;\ b^2>4ac].$$

16. $$= -\frac{2X^{1/2}}{(b-2ap)(x+p)} \qquad [ap^2 - bp + c = 0].$$

17. $$\int \frac{dx}{(x^2+px+q)X^{1/2}} = \frac{1}{(p^2-4q)^{1/2}}\left[\int \frac{dx}{\left(x+\frac{p-\sqrt{p^2-4q}}{2}\right)X^{1/2}} - \int \frac{dx}{\left(x+\frac{p+\sqrt{p^2-4q}}{2}\right)X^{1/2}}\right] \qquad [p^2>4q].$$

18. $$\int \frac{dx}{(x^2+p^2)X^{1/2}} = \int \frac{(\alpha-\beta)(t+1)\,dt}{[(\alpha^2+p^2)t^2+(\beta^2+p^2)][(a\alpha^2+b\alpha+c)t^2+a\beta^2+b\beta+c]^{1/2}}$$

$\left[t=\frac{x-\beta}{\alpha-x};\right.$ α и β определяются из системы уравнений $\left. b(\alpha+\beta)+2c-2ap^2=0,\ \alpha\beta+p^2=0\right]$.

19. $$\int \frac{(\alpha x+\beta)\,dx}{(p+X)^n X^{1/2}} = \frac{\alpha}{a}\int \frac{du}{(p+u^2)^n} + \frac{2\beta a-\alpha b}{2a}\int \frac{(1-av^2)^{n-1}\,dv}{(p+c-b^2/4a-apv^2)^n}$$

$$\left[u=X^{1/2};\ v=\frac{b+2ax}{2aX^{1/2}}\right].$$

1.2.54. Интегралы вида $\int f(x, \sqrt{x^2-x+1})\,dx$.

Условие: $x>1$.

Обозначение: $\varphi = \arcsin \frac{1}{\sqrt{x^2-x+1}}$.

1. $$\int_x^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)}} = F\left(\varphi, \frac{\sqrt{3}}{2}\right) \qquad [x\geqslant 1].$$

2. $$\int_x^\infty \frac{dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = \frac{2(2x-1)}{\sqrt{x(x-1)(x^2-x+1)}} - 4E\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

3. $$\int_x^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3}\left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right)\right] \qquad [x\geqslant 1].$$

4. $$\int_x^\infty \frac{dx}{(2x-1)^2\sqrt{x(x-1)(x^2-x+1)}} = \frac{4}{3}E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{2}{2x-1}\sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

5. $$\int_x^\infty \frac{dx}{(2x-1)^4\sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27}E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{27}F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{8(5x^2-5x+2)}{9(2x-1)^3}\sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

6. $\int\limits_x^\infty \frac{(2x-1)^2\,dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}}=4\left[F\left(\varphi,\frac{\sqrt{3}}{2}\right)-E\left(\varphi,\frac{\sqrt{3}}{2}\right)+\right.$
$\left.+\frac{2x-1}{2\sqrt{x(x-1)(x^2-x+1)}}\right].$

7. $\int\limits_x^\infty \frac{(2x-1)^2\,dx}{\sqrt{x(x-1)(x^2-x+1)^3}}=4E\left(\varphi,\frac{\sqrt{3}}{2}\right).$

8. $\int\limits_x^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}}\,dx=\frac{4}{3}E\left(\varphi,\frac{\sqrt{3}}{2}\right)-\frac{1}{3}F\left(\varphi,\frac{\sqrt{3}}{2}\right).$

9. $\int \frac{1}{(2x-1)^2}\sqrt{\frac{x(x-1)}{x^2-x+1}}\,dx=$
$=\frac{1}{3}\left[F\left(\varphi,\frac{\sqrt{3}}{2}\right)-E\left(\varphi,\frac{\sqrt{3}}{2}\right)\right]+\frac{1}{2(2x-1)}\sqrt{\frac{x(x-1)}{x^2-x+1}}.$

10. $\int\limits_x^\infty \frac{1}{(2x-1)^2}\sqrt{\frac{x^2-x+1}{x(x-1)}}\,dx=E\left(\varphi,\frac{\sqrt{3}}{2}\right)-\frac{3}{2(2x-1)}\sqrt{\frac{x(x-1)}{x^2-x+1}}.$

11. $\int\limits_0^x \frac{1}{[1+(\sqrt{3}+1)x]^2}\sqrt{\frac{x^2-x+1}{x(x+1)}}\,dx=$
$=\frac{1}{\sqrt[4]{3}}E\left(\arccos\frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x};\ \frac{\sqrt{2+\sqrt{3}}}{2}\right)$ $[x>0]$.

12. $\int\limits_0^x \sqrt{\frac{x(x+1)}{(x^2-x+1)^3}}\,dx=\frac{4}{\sqrt[4]{27}}E\left(\arccos\frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x},\ \frac{\sqrt{2+\sqrt{3}}}{2}\right)-$
$-\frac{2-\sqrt{3}}{\sqrt[4]{27}}F\left(\arccos\frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x},\ \frac{\sqrt{2+\sqrt{3}}}{2}\right)-$
$-\frac{2(2+\sqrt{3})}{\sqrt{3}}\,\frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}\sqrt{\frac{x(1+x)}{x^2-x+1}}$ $[x>0]$.

1.2.55. Интегралы вида $\int\limits_0^x R\left(x,\sqrt{x^2+a^2},\sqrt{x^2+b^2}\right)dx$.

- Условия: $x>0$, $a>b>0$.

Обозначения: $\varphi=\operatorname{arctg}\frac{x}{b}$, $k=\sqrt{\frac{a^2-b^2}{a}}$.

1. $\int\limits_0^x \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}=\frac{1}{a}F(\varphi,k).$

2. $\int\limits_0^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}=x\sqrt{\frac{x^2+a^2}{x^2+b^2}}-aE(\varphi,k).$

3. $$\int_0^x \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} =$$
$$= \frac{1}{a(p+b^2)}\left[\frac{b^2}{p}\Pi\left(\varphi,\ \frac{p+b^2}{p},\ k\right)+F(\varphi,\ k)\right] \qquad [p \neq 0].$$

4. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)^3}} =$$
$$= \frac{1}{ab^2(a^2-b^2)^2}\left[(a^2+b^2)E(\varphi,\ k)-2b^2F(\varphi,\ k)\right]-\frac{x}{a^2(a^2-b^2)\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

5. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} =$$
$$= \frac{1}{a(a^2-b^2)}\left[F(\varphi,\ k)-E(\varphi,\ k)\right]+\frac{x}{a^2\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

6. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\left[a^2E(\varphi,\ k)-b^2F(\varphi,\ k)\right].$$

7. $$\int_0^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\left[a^2E(\varphi,\ k)-b^2F(\varphi,\ k)\right]-$$
$$-\frac{x}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

8. $$\int_0^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{a}{a^2-b^2}\left[F(\varphi,\ k)-E(\varphi,\ k)\right].$$

9. $$\int_0^x \sqrt{\frac{x^2+a^2}{x^2+b^2}}\,dx = a\left[F(\varphi,\ k)-E(\varphi,\ k)\right]+x\sqrt{\frac{x^2+a^2}{x^2+b^2}}.$$

10. $$\int_0^x \sqrt{\frac{x^2+b^2}{x^2+a^2}}\,dx = \frac{b^2}{a}F(\varphi,\ k)-aE(\varphi,\ k)+x\sqrt{\frac{x^2+a^2}{x^2+b^2}}.$$

11. $$\int_0^x \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\,dx = \frac{a}{b^2}E(\varphi,\ k).$$

12. $$\int_0^x \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\,dx = \frac{1}{a}E(\varphi,\ k)-\frac{a^2-b^2}{a^2}\,\frac{x}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

13. $$\int_0^x \sqrt{(x^2+a^2)(x^2+b^2)}\,dx = \frac{a}{3}\left[2b^2F(\varphi,\ k)-(a^2+b^2)E(\varphi,\ k)\right]+$$
$$+\frac{x}{3}(x^2+a^2+2b^2)\sqrt{\frac{x^2+a^2}{x^2+b^2}}.$$

14. $$\int \frac{dx}{\sqrt{x(x^2+1)(x^2+a^2)}} =$$
$$= \frac{1}{2\sqrt{a(a+1)}} \left\{ F\left(\arccos \frac{(x+\sqrt{a})^2-(a+1)x}{(x+\sqrt{a})^2+(a+1)x},\ \frac{(\sqrt{a}+1)^2}{2(a+1)}\right) \pm \right.$$
$$\left. \pm F\left(\arccos \frac{(x-\sqrt{a})^2-(a+1)x}{(x-\sqrt{a})^2+(a+1)x},\ \frac{(\sqrt{a}-1)^2}{2(a+1)}\right)\right\} \qquad \left[x,\ a>0;\ \begin{Bmatrix} 0\leqslant x\leqslant\sqrt{a} \\ \sqrt{a}\leqslant x<\infty \end{Bmatrix}\right].$$

15. $$\int \frac{\sqrt{x}\,dx}{\sqrt{(x^2+1)(x^2+a^2)}} =$$
$$= \frac{-1}{2\sqrt{a+1}} \left\{ F\left(\arccos \frac{(x+\sqrt{a})^2-(a+1)x}{(x+\sqrt{a})^2+(a+1)x},\ \frac{(\sqrt{a}+1)^2}{2(a+1)}\right) \mp \right.$$
$$\left. \mp F\left(\arccos \frac{(x-\sqrt{a})^2-(a+1)x}{(x-\sqrt{a})^2+(a+1)x},\ \frac{(\sqrt{a}-1)^2}{2(a+1)}\right)\right\} \qquad \left[x,\ a>0;\ \begin{Bmatrix} 0\leqslant x\leqslant\sqrt{a} \\ \sqrt{a}\leqslant x<\infty \end{Bmatrix}\right].$$

16. $$\int_0^x \frac{dx}{\sqrt{x[(x+a)^2+b^2]}} = \frac{1}{\sqrt{c}} F\left(2\,\mathrm{arctg}\sqrt{\frac{x}{c}},\ \sqrt{\frac{c-a}{2c}}\right)$$
$$[x>0;\ c=\sqrt{a^2+b^2}].$$

17. $$\int_0^x \frac{dx}{\sqrt{x(x+a)[(x+b)^2+c^2]}} =$$
$$= \frac{1}{\sqrt{pq}} F\left(2\,\mathrm{arctg}\sqrt{\frac{qx}{p(x+a)}},\ \frac{1}{2}\sqrt{\frac{(p+q)^2+a^2}{pq}}\right)$$
$$[x>0,\ p^2=b^2+c^2;\ q^2=(b+a)^2+c^2].$$

18. $$\int_0^x \frac{dx}{\sqrt{x(a-x)[(x+b)^2+c^2]}} =$$
$$= \frac{1}{\sqrt{pq}} F\left(2\,\mathrm{arcctg}\sqrt{\frac{q(a-x)}{px}},\ \frac{1}{2}\sqrt{\frac{-(p-q)^2+a^2}{pq}}\right)$$
$$[a>x>0,\ p^2=(b-a)^2+c^2;\ q^2=b^2+c^2].$$

1.2.56. Интегралы вида $\int_x^\infty R\left(x,\ \sqrt{x^2+a^2},\ \sqrt{x^2+b^2}\right)dx$.

Условия $x\geqslant 0,\ a>b>0$.

Обозначения: $\varphi=\mathrm{arcctg}\,\frac{x}{a},\ k=\sqrt{\frac{a^2-b^2}{a}}$.

1. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a} F(\varphi,\ k).$$

2. $$\int_x^\infty \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{b^2x}\sqrt{\frac{x^2+b^2}{x^2+a^2}} - \frac{1}{ab^2} E(\varphi,\ k) \qquad [x>0].$$

3. $$\int_x^\infty \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)}\left[\Pi\left(\varphi, \frac{a^2+p}{a^2}, k\right) - F(\varphi, k)\right].$$

4. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}[F(\varphi, k) - E(\varphi, k)].$$

5. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} =$$
$$= \frac{1}{ab^2(a^2-b^2)}[a^2E(\varphi, k) - b^2F(\varphi, k)] - \frac{x}{b^2\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

6. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)^3}} =$$
$$= \frac{1}{ab^2(a^2-b^2)^2}[(a^2+b^2)E(\varphi, k) - 2b^2F(\varphi, k)] - \frac{x}{b^2(a^2-b^2)\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

7. $$\int_x^\infty \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}[a^2E(\varphi, k) - b^2F(\varphi, k)].$$

8. $$\int_x^\infty \frac{x^2\,dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{a}{a^2-b^2}[F(\varphi, k) - E(\varphi, k)] + \frac{x}{\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

9. $$\int_x^\infty \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\,dx = \frac{a}{b^2}E(\varphi, k) - \frac{(a^2-b^2)x}{b^2\sqrt{(x^2+a^2)(x^2+b^2)}}.$$

10. $$\int_x^\infty \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\,dx = \frac{1}{a}E(\varphi, k).$$

11. $$\int_x^\infty \sqrt{\frac{x^2+a^2}{x^2+b^2}}\,\frac{dx}{x^2} = \frac{1}{a}F(\varphi, k) - \frac{a}{b^2}E(\varphi, k) + \frac{a^2}{b^2x}\sqrt{\frac{x^2+b^2}{x^2+a^2}} \quad [x>0].$$

12. $$\int_x^\infty \sqrt{\frac{x^2+b^2}{x^2+a^2}}\,\frac{dx}{x^2} = \frac{1}{a}[F(\varphi, k) - E(\varphi, k)] + \frac{1}{x}\sqrt{\frac{x^2+b^2}{x^2+a^2}} \quad [x>0].$$

2.257. Интегралы вида $\int_b^x R\left(x, \sqrt{x^2+a^2}, \sqrt{x^2-b^2}\right)dx$.

Условие $x > b > 0$.

Обозначения $\varphi = \arccos\frac{b}{x}$, $k = \frac{a}{\sqrt{a^2+b^2}}$.

1. $$\int_b^x \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\varphi, k).$$

2. $$\int_b^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{b^2}{\sqrt{a^2+b^2}}F(\varphi, k) - \sqrt{a^2+b^2}\,E(\varphi, k) + \frac{1}{x}\sqrt{(x^2+a^2)(x^2-b^2)}.$$

3. $$\int_b^x \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left[(a^2+b^2)\,E(\varphi, k) - b^2F(\varphi, k)\right].$$

4. $$\int_b^x \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{p(p-b^2)\sqrt{a^2+b^2}}\left[b^2\Pi\left(\varphi, \frac{p}{p-b^2}, k\right) + (p-b^2)F(\varphi, k)\right] \qquad [p \neq b^2]$$

5. $$\int_b^x \frac{dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\left[F(\varphi, k) - E(\varphi, k)\right] + \frac{1}{(a^2+b^2)\,x}\sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

6. $$\int_b^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}E(\varphi, k) - \frac{a^2}{x(a^2+b^2)}\sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

7. $$\int_b^x \sqrt{\frac{x^2+a^2}{x^2-b^2}}\,dx = \sqrt{a^2+b^2}\left[F(\varphi, k) - E(\varphi, k)\right] + \frac{1}{x}\sqrt{(x^2+a^2)(x^2-b^2)}.$$

8. $$\int_b^x \frac{1}{x^2}\sqrt{\frac{x^2+a^2}{x^2-b^2}}\,dx = \frac{\sqrt{a^2+b^2}}{b^2}E(\varphi, k).$$

9. $$\int_b^x \sqrt{\frac{x^2-b^2}{x^2+a^2}}\,dx = \frac{1}{x}\sqrt{(x^2+a^2)(x^2-b^2)} - \sqrt{a^2+b^2}\,E(\varphi, k).$$

10. $$\int_b^x \frac{1}{x^2}\sqrt{\frac{x^2-b^2}{x^2+a^2}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\left[F(\varphi, k) - E(\varphi, k)\right].$$

11. $$\int_b^x \sqrt{\frac{x^2-b^2}{(x^2+a^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\varphi, k) - \frac{b^2}{a^2\sqrt{a^2+b^2}}F(\varphi, k) - \frac{1}{x}\sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

12. $$\int_b^x \sqrt{(x^2+a^2)(x^2-b^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\,[(b^2-a^2)E(\varphi, k) - b^2F(\varphi, k)] + \frac{x^2+a^2-b^2}{3x}\sqrt{(x^2+a^2)(x^2-b^2)}.$$

1.2.58. Интегралы вида $\int_x^\infty R\left(x, \sqrt{x^2+a^2}, \sqrt{x^2-b^2}\right)dx$.

Условие. $x > b > 0$.

Обозначения. $\varphi = \arcsin\sqrt{\frac{a^2+b^2}{x^2+a^2}}$, $k = \frac{a}{\sqrt{a^2+b^2}}$.

1. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\varphi, k).$$

2. $$\int_x^\infty \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}[(a^2+b^2)E(\varphi, k) - b^2F(\varphi, k)] - \frac{1}{b^2x}\sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

3. $$\int_x^\infty \frac{dx}{(x^2-p)\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{(a^2+p)\sqrt{a^2+b^2}}\left[\Pi\left(\varphi, \frac{a^2+p}{a^2+b^2}, k\right) - F(\varphi, k)\right].$$

4. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}[F(\varphi, k) - E(\varphi, k)].$$

5. $$\int_x^\infty \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}E(\varphi, k).$$

6. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)^3}} = \frac{x}{b^2\sqrt{(x^2+a^2)(x^2-b^2)}} - \frac{1}{b^2\sqrt{a^2+b^2}}E(\varphi, k).$$

7. $$\int_x^\infty \frac{x^2\,dx}{\sqrt{(x^2+a^2)(x^2-b^2)^3}} = \frac{1}{\sqrt{a^2+b^2}}[F(\varphi, k) - E(\varphi, k)] + \frac{x}{\sqrt{(x^2+a^2)(x^2-b^2)}} \quad [x > b].$$

8. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)^3}} = \frac{b^2-a^2}{a^2b^2\sqrt{(a^2+b^2)^3}}E(\varphi, k) - \frac{1}{a^2\sqrt{(a^2+b^2)^3}}F(\varphi, k) + \frac{x}{b^2(a^2+b^2)\sqrt{(x^2+a^2)(x^2-b^2)}} \quad [x > b].$$

9. $$\int_x^\infty \sqrt{\frac{x^2+a^2}{(x^2-b^2)^3}}\,dx=\frac{1}{\sqrt{a^2+b^2}}F(\varphi,\ k)-\frac{\sqrt{a^2+b^2}}{b^2}E(\varphi,\ k)+\frac{(a^2+b^2)x}{b^2\sqrt{(x^2+a^2)(x^2-b^2)}}\qquad [x>b].$$

10. $$\int_x^\infty \frac{1}{x^2}\sqrt{\frac{x^2+a^2}{x^2-b^2}}\,dx=\frac{\sqrt{a^2+b^2}}{b^2}E(\varphi,\ k)-\frac{a^2}{b^2x}\sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

11. $$\int_x^\infty \frac{1}{x^2}\sqrt{\frac{x^2-b^2}{x^2+a^2}}\,dx=\frac{\sqrt{a^2+b^2}}{a^2}[F(\varphi,\ k)-E(\varphi,\ k)]+\frac{1}{x}\sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

12. $$\int_x^\infty \sqrt{\frac{x^2-b^2}{(x^2+a^2)^3}}\,dx=\frac{\sqrt{a^2+b^2}}{a^2}E(\varphi,\ k)-\frac{b^2}{a^2\sqrt{a^2+b^2}}F(\varphi,\ k).$$

1.2.5⁰. Интегралы вида $\int_0^x R\left(x,\ \sqrt{x^2+a^2},\ \sqrt{b^2-x^2}\right)dx$.

Условие: $b \geqslant x > 0$.

Обозначения: $\varphi=\arcsin\frac{x}{b}\sqrt{\frac{a^2+b^2}{x^2+a^2}}$, $k=\frac{b}{\sqrt{a^2+b^2}}$.

1. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}}=\frac{1}{\sqrt{a^2+b^2}}F(\varphi,\ k).$$

2. $$\int_0^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)(b^2-x^2)}}=\sqrt{a^2+b^2}\,E(\varphi,\ k)-\frac{a^2}{\sqrt{a^2+b^2}}F(\varphi,\ k)-x\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

3. $$\int_0^x \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(b^2-x^2)}}=\frac{1}{p(p+a^2)\sqrt{a^2+b^2}}\left\{a^2\Pi\left[\varphi,\ \frac{b^2(p+a^2)}{p(a^2+b^2)},\ k\right]+pF(\varphi,\ k)\right\}\qquad [p\neq 0].$$

4. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(b^2-x^2)}}=\frac{1}{a^2\sqrt{a^2+b^2}}E(\varphi,\ k).$$

5. $$\int_0^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3(b^2-x^2)}}=\frac{1}{\sqrt{a^2+b^2}}[F(\varphi,\ k)-E(\varphi,\ k)].$$

6. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)^3}} = \frac{1}{b^2\sqrt{a^2+b^2}}[F(\varphi, k) - E(\varphi, k)] + \frac{x}{b^2\sqrt{(x^2+a^2)(b^2-x^2)}} \quad [x<b].$$

7. $$\int_0^x \frac{x^2\,dx}{\sqrt{(x^2+a^2)(b^2-x^2)^3}} = \frac{x}{\sqrt{(x^2+a^2)(b^2-x^2)}} - \frac{1}{\sqrt{a^2+b^2}}E(\varphi, k) \quad [x<b].$$

8. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(b^2-x^2)^3}} = \frac{1}{a^2b^2\sqrt{(a^2+b^2)^3}}[a^2F(\varphi, k) - (a^2-b^2)E(\varphi, k)] + \frac{x}{b^2(a^2+b^2)\sqrt{(x^2+a^2)(b^2-x^2)}} \quad [x<b].$$

9. $$\int_0^x \sqrt{\frac{x^2+a^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\varphi, k) - x\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

10. $$\int_0^x \sqrt{\frac{x^2+a^2}{(b^2-x^2)^3}}\,dx = \frac{a^2}{b^2\sqrt{a^2+b^2}}F(\varphi, k) - \frac{\sqrt{a^2+b^2}}{b^2}E(\varphi, k) + \frac{(a^2+b^2)x}{b^2\sqrt{(x^2+a^2)(b^2-x^2)}} \quad [x<b].$$

11. $$\int_0^x \sqrt{\frac{b^2-x^2}{x^2+a^2}}\,dx = \sqrt{a^2+b^2}[F(\varphi, k) - E(\varphi, k)] + x\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

12. $$\int_0^x \sqrt{\frac{b^2-x^2}{(x^2+a^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\varphi, k) - \frac{1}{\sqrt{a^2+b^2}}F(\varphi, k).$$

13. $$\int_0^x \sqrt{(x^2+a^2)(b^2-x^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}[a^2F(\varphi, k) - (a^2-b^2)E(\varphi, k)] + \frac{x}{3}(x^2+2a^2-b^2)\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

1.2.60. Интегралы вида $\int_x^b R\left(x, \sqrt{x^2+a^2}, \sqrt{b^2-x^2}\right)dx$.

Условие: $b > x \geq 0$.

Обозначения: $\varphi = \arccos\frac{x}{b}$, $k = \frac{b}{\sqrt{a^2+b^2}}$.

1. $$\int_x^b \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\varphi, k).$$

2. $$\int_x^b \frac{x^2\,dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \sqrt{a^2+b^2}\,E(\varphi, k) - \frac{a^2}{\sqrt{a^2+b^2}}\,F(\varphi, k).$$

3. $$\int_x^b \frac{dx}{x^2\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left[a^2F(\varphi, k) - (a^2+b^2)\,E(\varphi, k)\right] + \frac{1}{a^2b^2x}\sqrt{(x^2+a^2)(b^2-x^2)} \qquad [x>0].$$

4. $$\int_x^b \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{(p-b^2)\sqrt{a^2+b^2}}\,\Pi\left(\varphi, \frac{b^2}{b^2-p}, k\right) \qquad [p \neq b^2].$$

5. $$\int_x^b \frac{dx}{\sqrt{(x^2+a^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\,E(\varphi, k) - \frac{x}{a^2(a^2+b^2)}\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

6. $$\int_x^b \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\left[F(\varphi, k) - E(\varphi, k)\right] + \frac{x}{a^2+b^2}\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

7. $$\int_x^b \sqrt{\frac{x^2+a^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\varphi, k).$$

8. $$\int_x^b \frac{1}{x^2}\sqrt{\frac{x^2+a^2}{b^2-x^2}}\,dx = \frac{\sqrt{a^2+b^2}}{b^2}\left[F(\varphi, k) - E(\varphi, k)\right] + \frac{\sqrt{(b^2-x^2)(x^2+a^2)}}{b^2x} \qquad [x>0].$$

9. $$\int_x^b \sqrt{\frac{b^2-x^2}{x^2+a^2}}\,dx = \sqrt{a^2+b^2}\left[F(\varphi, k) - E(\varphi, k)\right].$$

10. $$\int_x^b \frac{1}{x^2}\sqrt{\frac{b^2-x^2}{x^2+a^2}}\,dx = \frac{\sqrt{(b^2-x^2)(x^2+a^2)}}{a^2x} - \frac{\sqrt{a^2+b^2}}{a^2}\,E(\varphi, k) \qquad [x>0].$$

11. $$\int_x^b \sqrt{\frac{b^2-x^2}{(x^2+a^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\varphi, k) - \frac{1}{\sqrt{a^2+b^2}}\,F(\varphi, k) - \frac{x}{a^2}\sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

12. $$\int_x^b \sqrt{(x^2+a^2)(b^2-x^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left[a^2F(\varphi, k) + 2(b^2-a^2)\,E(\varphi, k)\right] + \frac{x}{3}\sqrt{(x^2+a^2)(b^2-x^2)}.$$

1.2.61. Интегралы вида $\int\limits_a^x R\left(x, \sqrt{x^2-a^2}, \sqrt{x^2-b^2}\right) dx.$

Условие: $x > a > b > 0$.

Обозначение: $\varphi = \arcsin \sqrt{\dfrac{x^2-a^2}{x^2-b^2}}$.

1. $$\int\limits_a^x \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

2. $$\int\limits_a^x \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = a\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right] + x\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

3. $$\int\limits_a^x \frac{dx}{x^2\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right] + \frac{1}{a^2x}\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

4. $$\int\limits_a^x \frac{dx}{(p-x^2)\sqrt{(x^2-a^2)(x^2-b^2)}} =$$
$$= \frac{1}{a(p-a^2)(p-b^2)}\left[(a^2-b^2)\Pi\left(\varphi, \frac{p-b^2}{p-a^2}, \frac{b}{a}\right) + (p-a^2) F\left(\varphi, \frac{b}{a}\right)\right]$$
$[p \neq a^2, b^2]$.

5. $$\int\limits_a^x \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{b^2(a^2-b^2)} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{ab^2} F\left(\varphi, \frac{b}{a}\right).$$

6. $$\int\limits_a^x \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{a^2-b^2} E\left(\varphi, \frac{b}{a}\right).$$

7. $$\int\limits_a^x \sqrt{\frac{x^2-a^2}{x^2-b^2}}\,dx = x\sqrt{\frac{x^2-a^2}{x^2-b^2}} - aE\left(\varphi, \frac{b}{a}\right).$$

8. $$\int\limits_a^x \frac{1}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}}\,dx = \frac{a}{b^2} E\left(\varphi, \frac{b}{a}\right) - \frac{a^2-b^2}{ab^2} F\left(\varphi, \frac{b}{a}\right) - \frac{1}{x}\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

9. $$\int\limits_a^x \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}}\,dx = \frac{a}{b^2}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right].$$

10. $$\int\limits_a^x \sqrt{\frac{x^2-b^2}{x^2-a^2}}\,dx = \frac{a^2-b^2}{a} F\left(\varphi, \frac{b}{a}\right) - aE\left(\varphi, \frac{b}{a}\right) + x\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

11. $$\int\limits_a^x \frac{1}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}}\,dx = \frac{1}{a} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{x}\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

12. $$\int_a^x \sqrt{(x^2-a^2)(x^2-b^2)}\,dx = \frac{a}{3}\left[(a^2+b^2)E\left(\varphi, \frac{b}{a}\right) - (a^2-b^2)F\left(\varphi, \frac{b}{a}\right)\right] + \frac{x}{3}(x^2-a^2-2b^2)\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

13. $$\int \frac{dx}{\sqrt{x(x^2-1)(x^2-a^2)}} = -\frac{1}{\sqrt{2a(a+1)}}\left[F\left(\arcsin\frac{\sqrt{(x-1)(x-a)}}{x+\sqrt{a}}, \frac{|\sqrt{a}-1|}{\sqrt{2(a+1)}}\right) \pm \pm F\left(\arcsin\frac{\sqrt{(x-1)(x-a)}}{|x-\sqrt{a}|}, \frac{\sqrt{a}+1}{\sqrt{2(a+1)}}\right)\right]$$
$\left[a>0;\ \begin{Bmatrix} 0 \leqslant x \leqslant \min(1, a) \\ \max(1, a) \leqslant x < \infty \end{Bmatrix}\right].$

14. $$\int \frac{\sqrt{x}\,dx}{\sqrt{(x^2-1)(x^2-a^2)}} = \frac{1}{\sqrt{2(a+1)}}\left[F\left(\arcsin\frac{\sqrt{(x-1)(x-a)}}{x+\sqrt{a}}, \frac{|\sqrt{a}-1|}{\sqrt{2(a+1)}}\right) \mp \mp F\left(\arcsin\frac{\sqrt{(x-1)(x-a)}}{|x-\sqrt{a}|}, \frac{\sqrt{a}+1}{\sqrt{2(a+1)}}\right)\right]$$
$\left[a>0;\ \begin{Bmatrix} 0 \leqslant x \leqslant \min(1, a) \\ \max(1, a) \leqslant x \leqslant \infty \end{Bmatrix}\right].$

1.2.62. Интегралы вида $\int_x^\infty R(x, \sqrt{x^2-a^2}, \sqrt{x^2-b^2})\,dx$.

Условие: $x \geqslant a > b > 0$.

Обозначение: $\varphi = \arcsin\frac{a}{x}$.

1. $$\int_x^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}F\left(\varphi, \frac{b}{a}\right).$$

2. $$\int_x^\infty \frac{dx}{x^2\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right].$$

3. $$\int_x^\infty \frac{dx}{(x^2-p)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ap}\left[\Pi\left(\varphi, \frac{p}{a^2}, \frac{b}{a}\right) - F\left(\varphi, \frac{b}{a}\right)\right]$$
$[p \neq 0].$

4. $$\int_x^\infty \frac{dx}{\sqrt{(x^2-a^2)^3(x^2-b^2)}} = \frac{1}{a(b^2-a^2)}\left[E\left(\varphi, \frac{b}{a}\right) - \frac{a}{x}\sqrt{\frac{x^2-b^2}{x^2-a^2}}\right]$$
$[x > a].$

5. $$\int_x^\infty \frac{x^2\,dx}{\sqrt{(x^2-a^2)^3(x^2-b^2)}} = \frac{a}{a^2-b^2}\left[\frac{a}{x}\sqrt{\frac{x^2-b^2}{x^2-a^2}} - E\left(\varphi, \frac{b}{a}\right)\right] + \frac{1}{a}F\left(\varphi, \frac{b}{a}\right)$$
$[x > a].$

6. $$\int_x^{\infty} \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{1}{b^2(a^2-b^2)}\left[aE\left(\varphi, \frac{b}{a}\right) - \frac{b^2}{x}\sqrt{\frac{x^2-a^2}{x^2-b^2}}\right] - \frac{1}{ab^2}F\left(\varphi, \frac{b}{a}\right).$$

7. $$\int_x^{\infty} \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{1}{a^2-b^2}\left[aE\left(\varphi, \frac{b}{a}\right) - \frac{b^2}{x}\sqrt{\frac{x^2-a^2}{x^2-b^2}}\right].$$

8. $$\int_x^{\infty} \frac{dx}{\sqrt{(x^2-a^2)^3(x^2-b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}F\left(\varphi, \frac{b}{a}\right) - \frac{a^2+b^2}{ab^2(a^2-b^2)^2}E\left(\varphi, \frac{b}{a}\right) + \frac{1}{x(a^2-b^2)\sqrt{(x^2-a^2)(x^2-b^2)}} \quad [x>a].$$

9. $$\int_x^{\infty} \frac{1}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}}\,dx = \frac{a}{b^2}E\left(\varphi, \frac{b}{a}\right) - \frac{a^2-b^2}{ab^2}F\left(\varphi, \frac{b}{a}\right).$$

10. $$\int_x^{\infty} \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}}\,dx = \frac{a}{b^2}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right] + \frac{1}{x}\sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

11. $$\int_x^{\infty} \frac{1}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}}\,dx = \frac{1}{a}E\left(\varphi, \frac{b}{a}\right).$$

12. $$\int_x^{\infty} \sqrt{\frac{x^2-b^2}{(x^2-a^2)^3}}\,dx = \frac{1}{a}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right] + \frac{1}{x}\sqrt{\frac{x^2-b^2}{x^2-a^2}} \quad [x>a].$$

1.2.63. Интегралы вида $\int_x^a R\left(x, \sqrt{a^2-x^2}, \sqrt{x^2-b^2}\right)dx$.

Условие: $a > x \geqslant b > 0$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{a^2-x^2}{a^2-b^2}}$, $k = \sqrt{\frac{a^2-b^2}{a^2}}$.

1. $$\int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a}F(\varphi, k).$$

2. $$\int_x^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = aE(\varphi, k).$$

3. $$\int_x^a \frac{dx}{x^2\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ab^2}E(\varphi, k) - \frac{1}{a^2b^2x}\sqrt{(a^2-x^2)(x^2-b^2)}.$$

4. $$\int_x^a \frac{dx}{(x^2-p)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)}\Pi\left(\varphi, \frac{a^2-b^2}{a^2-p}, k\right) \qquad [p \neq a^2].$$

5. $$\int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\left[b^2F(\varphi, k) - a^2E(\varphi, k) + ax\sqrt{\frac{a^2-x^2}{x^2-b^2}}\right] \qquad [x>b].$$

6. $$\int_x^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} = \frac{1}{a^2-b^2}\left[aF(\varphi, k) - aE(\varphi, k) + x\sqrt{\frac{a^2-x^2}{x^2-b^2}}\right] \qquad [x>b].$$

7. $$\int_x^a \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\,[F(\varphi, k) - E(\varphi, k)].$$

8. $$\int_x^a \frac{1}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = \frac{a}{b^2}E(\varphi, k) - \frac{1}{a}F(\varphi, k) - \frac{\sqrt{(a^2-x^2)(x^2-b^2)}}{b^2x}.$$

9. $$\int_x^a \sqrt{\frac{a^2-x^2}{(x^2-b^2)^3}}\,dx = \frac{x}{b^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} - \frac{a}{b^2}E(\varphi, k) \qquad [x>b].$$

10. $$\int_x^a \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = aE(\varphi, k) - \frac{b^2}{a}F(\varphi, k).$$

11. $$\int_x^a \frac{1}{x^2}\sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = \frac{1}{a}[F(\varphi, k) - E(\varphi, k)] + \frac{\sqrt{(a^2-x^2)(x^2-b^2)}}{a^2x}.$$

12. $$\int_x^a \sqrt{(a^2-x^2)(x^2-b^2)}\,dx = \frac{a}{3}\left[(a^2+b^2)E(\varphi, k) - 2b^2F(\varphi, k)\right] - \frac{x}{3}\sqrt{(a^2-x^2)(x^2-b^2)}.$$

1.2.64. Интегралы вида $\int_b^x R\left(x, \sqrt{a^2-x^2}, \sqrt{x^2-b^2}\right)dx$.

Условие $a \geq x > b > 0$.

Обозначения $\varphi = \arcsin \frac{a}{x}\sqrt{\frac{x^2-b^2}{a^2-b^2}}$, $k = \frac{\sqrt{a^2-b^2}}{a}$.

1. $$\int_b^x \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a}F(\varphi, k).$$

2. $$\int\limits_b^x \frac{x^2\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = aE(\varphi, k) - \frac{1}{x}\sqrt{(a^2-x^2)(x^2-b^2)}.$$

3. $$\int\limits_b^x \frac{dx}{x^2\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ab^2}E(\varphi, k).$$

4. $$\int\limits_b^x \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} =$$
$$= \frac{1}{ap(p-b^2)}\left\{b^2\Pi\left[\varphi, \frac{p(a^2-b^2)}{a^2(p-b^2)}, k\right] + (p-b^2)F(\varphi, k)\right\} \qquad [p \neq b^2].$$

5. $$\int\limits_b^x \frac{dx}{\sqrt{(a^2-x^2)^3(x^2-b^2)}} = \frac{1}{a(a^2-b^2)}\left[F(\varphi, k) - E(\varphi, k) + \frac{a}{x}\sqrt{\frac{x^2-b^2}{a^2-x^2}}\right]$$
$$[x < a].$$

6. $$\int\limits_b^x \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3(x^2-b^2)}} =$$
$$= \frac{1}{a(a^2-b^2)}\left[b^2F(\varphi, k) - a^2E(\varphi, k) + \frac{a^3}{x}\sqrt{\frac{x^2-b^2}{a^2-x^2}}\right] \qquad [x < a].$$

7. $$\int\limits_b^x \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a[F(\varphi, k) - E(\varphi, k)] + \frac{1}{x}\sqrt{(a^2-x^2)(x^2-b^2)}.$$

8. $$\int\limits_b^x \frac{1}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = \frac{a}{b^2}E(\varphi, k) - \frac{1}{a}F(\varphi, k).$$

9. $$\int\limits_b^x \frac{1}{x^2}\sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = \frac{1}{a}[F(\varphi, k) - E(\varphi, k)].$$

10. $$\int\limits_b^x \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = aE(\varphi, k) - \frac{b^2}{a}F(\varphi, k) - \frac{1}{x}\sqrt{(a^2-x^2)(x^2-b^2)}.$$

11. $$\int\limits_b^x \sqrt{\frac{x^2-b^2}{(a^2-x^2)^3}}\,dx = \frac{1}{x}\sqrt{\frac{x^2-b^2}{a^2-x^2}} - \frac{1}{a}E(\varphi, k) \qquad [x < a].$$

12. $$\int\limits_b^x \sqrt{(a^2-x^2)(x^2-b^2)}\,dx = \frac{a}{3}[(a^2+b^2)E(\varphi, k) - 2b^2F(\varphi, k)] +$$
$$+ \frac{x^2-a^2-b^2}{3x}\sqrt{(a^2-x^2)(x^2-b^2)}.$$

1.2.65. Интегралы вида $\int\limits_0^x R\left(x, \sqrt{a^2-x^2}, \sqrt{b^2-x^2}\right)dx$.

Условие: $a>b>x>0$.

Обозначение: $\varphi=\arcsin\frac{x}{b}$.

1. $\int\limits_0^x \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}=\frac{1}{a}F\left(\varphi, \frac{b}{a}\right)$.

2. $\int\limits_0^x \frac{x^2\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}=a\left[F\left(\varphi, \frac{b}{a}\right)-E\left(\varphi, \frac{b}{a}\right)\right]$.

3. $\int\limits_0^x \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}}=\frac{1}{ap}\Pi\left(\varphi, \frac{b^2}{p}, \frac{b}{a}\right)$ $[p\neq b]$.

4. $\int\limits_0^x \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}}=\frac{1}{a^2(a^2-b^2)}\left[aE\left(\varphi, \frac{b}{a}\right)-x\sqrt{\frac{b^2-x^2}{a^2-x^2}}\right]$.

5. $\int\limits_0^x \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}}=\frac{1}{a^2-b^2}\left[aE\left(\varphi, \frac{b}{a}\right)-x\sqrt{\frac{b^2-x^2}{a^2-x^2}}\right]-\frac{1}{a}F\left(\varphi, \frac{b}{a}\right)$.

6. $\int\limits_0^x \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)^3}}=\frac{1}{ab^2}F\left(\varphi, \frac{b}{a}\right)-$

$$-\frac{1}{b^2(a^2-b^2)}\left[aE\left(\varphi, \frac{b}{a}\right)-x\sqrt{\frac{a^2-x^2}{b^2-x^2}}\right] \quad [x<b].$$

7. $\int\limits_0^x \frac{x^2\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)^3}}=\frac{1}{a^2-b^2}\left[x\sqrt{\frac{a^2-x^2}{b^2-x^2}}-aE\left(\varphi, \frac{b}{a}\right)\right]$ $[x<b]$.

8. $\int\limits_0^x \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)^3}}=\frac{1}{ab^2(a^2-b^2)}F\left(\varphi, \frac{b}{a}\right)-$

$$-\frac{a^2+b^2}{ab^2(a^2-b^2)^2}E\left(\varphi, \frac{b}{a}\right)+\frac{[a^4+b^4-(a^2+b^2)x^2]\,x}{a^2b^2(a^2-b^2)^2\sqrt{(a^2-x^2)(b^2-x^2)}} \quad [x<b].$$

9. $\int\limits_0^x \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx=aE\left(\varphi, \frac{b}{a}\right)$.

10. $\int\limits_0^x \sqrt{\frac{a^2-x^2}{(b^2-x^2)^3}}\,dx=\frac{a}{b^2}\left[F\left(\varphi, \frac{b}{a}\right)-E\left(\varphi, \frac{b}{a}\right)\right]+\frac{x}{b^2}\sqrt{\frac{a^2-x^2}{b^2-x^2}}$ $[x<b]$.

11. $\int\limits_0^x \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = aE\left(\varphi, \frac{b}{a}\right) - \frac{a^2-b^2}{a}F\left(\varphi, \frac{b}{a}\right).$

12. $\int\limits_0^x \sqrt{\frac{b^2-x^2}{(a^2-x^2)^3}}\,dx = \frac{1}{a}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) + \frac{x}{a}\sqrt{\frac{b^2-x^2}{a^2-x^2}}\right].$

13. $\int\limits_0^x \sqrt{(a^2-x^2)(b^2-x^2)}\,dx = \frac{a}{3}\left[(a^2+b^2)E\left(\varphi, \frac{b}{a}\right) - \right.$

$\left. -(a^2-b^2)F\left(\varphi, \frac{b}{a}\right)\right] + \frac{x}{3}\sqrt{(a^2-x^2)(b^2-x^2)}.$

1.2.66. Интегралы вида $\int\limits_x^b R\left(x, \sqrt{a^2-x^2}, \sqrt{b^2-x^2}\right)dx$.

Условие $a > b > x \geqslant 0$.

Обозначение. $\varphi = \arcsin \frac{a}{b}\sqrt{\frac{b^2-x^2}{a^2-x^2}}$.

1. $\int\limits_x^b \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}F\left(\varphi, \frac{b}{a}\right).$

2. $\int\limits_x^b \frac{x^2\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = a\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right] + x\sqrt{\frac{b^2-x^2}{a^2-x^2}}.$

3. $\int\limits_x^b \frac{dx}{x^2\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ab^2}\left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right)\right] + \frac{1}{b^2x}\sqrt{\frac{b^2-x^2}{a^2-x^2}}$

$[x > 0]$.

4. $\int\limits_x^b \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} =$

$= \frac{1}{a(p-a^2)(p-b^2)}\left\{(b^2-a^2)\Pi\left[\varphi, \frac{b^2(p-a^2)}{a^2(p-b^2)}, \frac{b}{a}\right] + \right.$

$\left. + (p-b^2)F\left(\varphi, \frac{b}{a}\right)\right\}$ $[p \neq b^2]$.

5. $\int\limits_x^b \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{1}{a(a^2-b^2)}E\left(\varphi, \frac{b}{a}\right).$

6. $\int\limits_x^b \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{a}{a^2-b^2}E\left(\varphi, \frac{b}{a}\right) - \frac{1}{a}F\left(\varphi, \frac{b}{a}\right).$

7. $\int\limits_x^b \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = a\left[E\left(\varphi, \frac{b}{a}\right) - \frac{x}{a}\sqrt{\frac{b^2-x^2}{a^2-x^2}}\right].$

8. $$\int_x^b \frac{1}{x^2}\sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx=\frac{a^2-b^2}{ab^2}F\left(\varphi, \frac{b}{a}\right)-\frac{a}{b^2}E\left(\varphi, \frac{b}{a}\right)+\frac{a^2}{b^2x}\sqrt{\frac{b^2-x^2}{a^2-x^2}}$$ $[x>0]$

9. $$\int_x^b \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx=aE\left(\varphi, \frac{b}{a}\right)-\frac{a^2-b^2}{a}F\left(\varphi, \frac{b}{a}\right)-x\sqrt{\frac{b^2-x^2}{a^2-x^2}}.$$

10. $$\int_x^b \sqrt{\frac{b^2-x^2}{(a^2-x^2)^3}}\,dx=\frac{1}{a}\left[F\left(\varphi, \frac{b}{a}\right)-E\left(\varphi, \frac{b}{a}\right)\right].$$

11. $$\int_x^b \frac{1}{x^2}\sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx=\frac{1}{x}\sqrt{\frac{b^2-x^2}{a^2-x^2}}-\frac{1}{a}E\left(\varphi, \frac{b}{a}\right)$$ $[x>0]$.

12. $$\int_x^b \sqrt{(a^2-x^2)(b^2-x^2)}\,dx=\frac{a}{3}\left[(a^2+b^2)E\left(\varphi, \frac{b}{a}\right)-\right.$$
$$\left.-(a^2-b^2)F\left(\varphi, \frac{b}{a}\right)\right]+\frac{x}{3}(x^2-2a^2-b^2)\sqrt{\frac{b^2-x^2}{a^2-x^2}}.$$

1.2.67. Интегралы вида $\int R\left(x, \sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}\right)dx$.

Обозначения: $\varphi=\arccos\frac{x^2-\rho\bar\rho}{x^2+\rho\bar\rho}$, $k=\frac{1}{2}\sqrt{-\frac{(\rho-\bar\rho)^2}{\rho\bar\rho}}$, ρ и $\bar\rho$ — комплексно сопряженные числа.

1. $$\int_x^\infty \frac{dx}{\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}=\frac{1}{\sqrt{\rho\bar\rho}}F(\varphi, k).$$

2. $$\int_x^\infty \frac{x^2\,dx}{(x^2-\rho\bar\rho)^2\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}=$$
$$=\frac{2x\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}{(\rho+\bar\rho)^2(x^4-\rho^2\bar\rho^2)}-\frac{1}{(\rho+\bar\rho)^2\sqrt{\rho\bar\rho}}E(\varphi, k).$$

3. $$\int_x^\infty \frac{x^2\,dx}{(x^2+\rho\bar\rho)^2\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}=-\frac{1}{(\rho-\bar\rho)^2\sqrt{\rho\bar\rho}}[F(\varphi, k)-E(\varphi, k)].$$

4. $$\int_x^\infty \frac{x^2\,dx}{\sqrt{(x^2+\rho^2)^3(x^2+\bar\rho^2)^3}}=-\frac{4\sqrt{\rho\bar\rho}}{(\rho^2-\bar\rho^2)^2}E(\varphi, k)+\frac{1}{(\rho-\bar\rho)^2\sqrt{\rho\bar\rho}}F(\varphi, k)-$$
$$-\frac{2x(x^2-\rho\bar\rho)}{(\rho+\bar\rho)^2(x^2+\rho\bar\rho)\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}.$$

5. $$\int_x^{\infty} \frac{(x^2-\rho\bar\rho)\,dx}{\sqrt{(x^2+\rho^2)^3(x^2+\bar\rho^2)^3}} = -\frac{4\sqrt{\rho\bar\rho}}{(\rho-\bar\rho)^2}[F(\varphi, k)-E(\varphi, k)] + \frac{2x(x^2-\rho\bar\rho)}{(x^2+\rho\bar\rho)\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}.$$

6. $$\int_x^{\infty} \frac{(x^2-\rho\bar\rho)^2\,dx}{(x^2+\rho\bar\rho)^2\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}} = -\frac{4\sqrt{\rho\bar\rho}}{(\rho-\bar\rho)^2}E(\varphi, k) + \frac{(\rho+\bar\rho)^2}{(\rho-\bar\rho)^2\sqrt{\rho\bar\rho}}F(\varphi, k).$$

7. $$\int_x^{\infty} \frac{(x^2+\rho\bar\rho)^2\,dx}{[(x^2+\rho\bar\rho)^2-4p^2\rho\bar\rho x^2]\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}} = \frac{1}{\sqrt{\rho\bar\rho}}\Pi(\varphi, p^2, k).$$

8. $$\int_x^{\infty} \frac{\sqrt{(x^2+\rho^2)(x^2+\bar\rho^2)}}{(x^2+\rho\bar\rho)^2}\,dx = \frac{1}{\sqrt{\rho\bar\rho}}F(\varphi, k).$$

1.2.68. Интегралы вида $\int R\left(x, \sqrt{x^3+1}\right)dx$ *).

Условие: $x \geqslant -1$.

Обозначения: $\varphi = \frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}, \quad k = \sin\frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

1. $$\int \frac{x^n}{\sqrt{x^3+1}}\,dx = \frac{2}{2n-1}x^{n-2}\sqrt{x^3+1} - \frac{2n-4}{2n-1}\int\frac{x^{n-3}}{\sqrt{x^3+1}}\,dx.$$

2. $$\int \frac{dx}{\sqrt{x^3+1}} = -\frac{1}{\sqrt[4]{3}}F(\varphi, k).$$

3. $$\int \frac{x}{\sqrt{x^3+1}}\,dx = \frac{2\sqrt{x^3+1}}{x+1+\sqrt{3}} - \frac{\sqrt{3}-1}{\sqrt[4]{3}}F(\varphi, k) + 2\sqrt[4]{3}\,E(\varphi, k).$$

4. $$\int \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{2}{3}\sqrt{x^3+1}.$$

5. $$\int \frac{dx}{(x-\rho)\sqrt{x^3+1}} = \frac{1}{\sqrt[4]{3}(\rho+1+\sqrt{3})}F(\varphi, k) - \frac{1}{2\sqrt[4]{3}(\rho+1)}\left[\frac{\rho+1-\sqrt{3}}{\rho+1+\sqrt{3}}\Pi(\varphi, -\rho_1, k) + \int\frac{\cos x\,dx}{(1+\rho_1\sin^2 x)\sqrt{1-k^2\sin^2 x}}\right]$$

$$\left[\rho_1 = \frac{-(\rho+1+\sqrt{3})^2}{4\sqrt{3}(\rho+1)};\ \rho \neq -1,\ -1-\sqrt{3},\ \text{см. 1.5.33. 12–14}\right].$$

6. $$\int \frac{dx}{(x+1+\sqrt{3})\sqrt{x^3+1}} = \frac{-\sqrt[4]{3}}{6}F(\varphi, k) + \frac{\sqrt[4]{3}}{6k}\arcsin(k\sin\varphi).$$

7. $$\int \frac{dx}{(x+1)\sqrt{x^3+1}} = \frac{-2\sqrt{x^3+1}}{\sqrt{3}(x+1)(x+1+\sqrt{3})} - \frac{\sqrt[4]{3}}{3}[F(\varphi, k)-2E(\varphi, k)].$$

*) См. также 1.2.73.

8. $$\int \frac{dx}{x\sqrt{x^3+1}} =$$
$$= \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\varphi, k) + \frac{2-\sqrt{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right) - \frac{1}{4} \ln \frac{x^2+2+2\sqrt{x^3+1}}{x^2+2-2\sqrt{x^3+1}}.$$

1.2.69. Интегралы вида $\int R(x, \sqrt{x^3-1})\,dx$ *).

Условие: $x > 1$.

Обозначения. $\varphi = \arccos \frac{\sqrt{3}+1-x}{\sqrt{3}-1+x}$, $\psi = \arccos \frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}$,

$$k = \sin \frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

1. $$\int_1^x \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k).$$

2. $$\int_x^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\psi, k).$$

3. $$\int_1^x \frac{x\,dx}{\sqrt{x^3-1}} = \left(\sqrt[4]{3} + \frac{1}{\sqrt[4]{3}}\right) F(\varphi, k) - 2\sqrt[4]{3}\, E(\varphi, k) + \frac{2\sqrt{x^3-1}}{\sqrt{3}-1+x}.$$

4. $$\int_x^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}} [F(\psi, k) - 2E(\psi, k)] +$$
$$+ \frac{2}{\sqrt{3}} \frac{\sqrt{x^2+x+1}}{(x-1+\sqrt{3})\sqrt{x-1}} \quad [x > 1].$$

5. $$\int_1^x \frac{(x-1)\,dx}{(1+\sqrt{3}-x)^2\sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{x^3-1}}{x^2-2x-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\varphi, k).$$

6. $$\int_x^\infty \frac{(x-1)\,dx}{(1+\sqrt{3}-x)^2\sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{x^3-1}}{x^2-2x-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\psi, k).$$

7. $$\int_1^x \frac{(x-1)\,dx}{(x-1+\sqrt{3})^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\varphi, k) - E(\varphi, k)].$$

8. $$\int_x^\infty \frac{(x-1)\,dx}{(x-1+\sqrt{3})^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\psi, k) - E(\psi, k)].$$

9. $$\int_1^x \frac{(x^2+x+1)\,dx}{(x-1+\sqrt{3})^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\varphi, k).$$

10. $$\int_x^\infty \frac{(x^2+x+1)\,dx}{(x-1+\sqrt{3})^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\psi, k).$$

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*) См. также 1.2.73.

11. $$\int_1^x \frac{(x-1)\,dx}{(x^2+x+1)\sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}} E(\varphi, k) - \frac{2+\sqrt{3}}{\sqrt[4]{27}} F(\varphi, k) - \frac{2-\sqrt{3}}{\sqrt{3}} \frac{2(x-1)(\sqrt{3}+1-x)}{(\sqrt{3}-1+x)\sqrt{x^3-1}}.$$

12. $$\int_1^x \frac{(x-1+\sqrt{3})^2\,dx}{[(x-1+\sqrt{3})^2-4\sqrt{3}p^2(x-1)]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\varphi, p^2, k).$$

13. $$\int_x^\infty \frac{(x-1+\sqrt{3})^2\,dx}{[(x-1+\sqrt{3})^2-4\sqrt{3}p^2(x-1)]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\psi, p^2, k).$$

1.2.70. Интегралы вида $\int R(x, \sqrt{1-x^3})\,dx$ *).
Условие: $x \leqslant 1$.

Обозначения: $\varphi = \arccos\frac{x-1+\sqrt{3}}{x-1-\sqrt{3}}$, $\psi = \arccos\frac{\sqrt{3}-1+x}{\sqrt{3}+1-x}$,

$$k = \sin\frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

1. $$\int \frac{x^m\,dx}{\sqrt{1-x^3}} = \frac{-2x^{m-2}\sqrt{1-x^3}}{2m-1} + \frac{2(m-2)}{2m-1}\int \frac{x^{m-3}\,dx}{\sqrt{1-x^3}}.$$

2. $$\int_{-\infty}^x \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k).$$

3. $$\int_x^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\psi, k).$$

4. $$\int_x^1 \frac{x\,dx}{\sqrt{1-x^3}} = \left(\frac{1}{\sqrt[4]{3}} - \sqrt[4]{3}\right) F(\psi, k) + 2\sqrt[4]{3}\,E(\psi, k) - \frac{2\sqrt{1-x^3}}{\sqrt{3}+1-x}.$$

5. $$\int \frac{x^2\,dx}{\sqrt{1-x^3}} = -\frac{2}{3}\sqrt{1-x^3}.$$

6. $$\int \frac{dx}{(x-\rho)\sqrt{1-x^3}} = \frac{-1}{\sqrt[4]{3}(\rho-1-\sqrt{3})} F(\varphi, k) + \frac{1}{2\sqrt[4]{3}(\rho-1)}\left[\frac{\rho-1+\sqrt{3}}{\rho-1-\sqrt{3}} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x\,dx}{(1+\rho_1 \sin^2 x)\sqrt{1-k^2\sin^2 x}}\right]$$

$$\left[\rho_1 = \frac{(\rho-1-\sqrt{3})^2}{4\sqrt{3}(\rho-1)},\ \rho \neq 1,\ 1+\sqrt{3};\ \text{см. } 1.5.33.\ 12\text{—}14\right].$$

7. $$\int \frac{dx}{(x-1-\sqrt{3})\sqrt{1-x^3}} = \frac{-\sqrt[4]{3}}{6} F(\varphi, k) + \frac{\sqrt[4]{3}}{6k} \arcsin(k \sin\varphi).$$

*) См. также 1.2.74.

8. $$\int \frac{dx}{(x-1)\sqrt{1-x^3}} = \frac{2\sqrt{x^2+x+1}}{\sqrt{3}\,(x-1-\sqrt{3})\sqrt{1-x}} - \frac{1}{\sqrt[4]{27}}\left[F(\varphi, k) - 2E(\varphi, k)\right] \qquad [x < 1].$$

9. $$\int \frac{dx}{x\sqrt{1-x^3}} = \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\varphi, k) + \frac{2-\sqrt{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right) + \frac{1}{4}\ln\frac{x^2+2-2\sqrt{1-x^3}}{x^2+2+2\sqrt{1-x^3}}.$$

10. $$\int_{-\infty}^{x} \frac{(x-1)\,dx}{(x-1+\sqrt{3})^2\sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[E(\varphi, k) - \frac{2\sqrt[4]{3}\sqrt{1-x^3}}{x^2-2x-2}\right].$$

11. $$\int_{-\infty}^{x} \frac{(x-1)\,dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{\sqrt{3}-2}{\sqrt[4]{27}}\left[F(\varphi, k) - E(\varphi, k)\right].$$

12. $$\int_{x}^{1} \frac{(x-1)\,dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{\sqrt{3}-2}{\sqrt[4]{27}}\left[F(\psi, k) - E(\psi, k)\right].$$

13. $$\int_{-\infty}^{x} \frac{(x^2+x+1)\,dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\varphi, k).$$

14. $$\int_{x}^{1} \frac{(x^2+x+1)\,dx}{(x-1+\sqrt{3})^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\psi, k).$$

15. $$\int_{x}^{1} \frac{(x-1-\sqrt{3})^2\,dx}{\left[(x-1-\sqrt{3})^2+4\sqrt{3}\,p^2(x-1)\right]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\psi, p^2, k).$$

16. $$\int_{-\infty}^{x} \frac{(x-1-\sqrt{3})^2\,dx}{\left[(x-1-\sqrt{3})^2+4\sqrt{3}\,p^2(x-1)\right]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\varphi, p^2, k).$$

17. $$\int_{x}^{1} \sqrt{1-x^3}\,dx = \frac{1}{5}\left[\sqrt[4]{27}\,F(\psi, k) - 2x\sqrt{1-x^3}\right].$$

18. $$\int_{0}^{x} \sqrt{\frac{x}{1-x^3}}\frac{dx}{1-x} = \frac{1}{\sqrt[4]{27}}\left[F\left(\arccos\frac{1-(1+\sqrt{3})x}{1+(\sqrt{3}-1)x}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) - 2E\left(\arccos\frac{1-(1+\sqrt{3})x}{1+(\sqrt{3}-1)x}, \frac{\sqrt{2-\sqrt{3}}}{2}\right)\right] + \frac{2}{\sqrt{3}}\frac{\sqrt{x(1+x+x^2)}}{\sqrt{1-x}\,(1+\sqrt{3}\,x-x)} \qquad [x < 1].$$

1.2.71. Интегралы вида $\int R\left(x, \sqrt{x^4+1}\right) dx$.

Обозначения: $\varphi=\arccos\frac{x^2-1}{x^2+1}$, $\psi=\operatorname{arctg}\left[(1+\sqrt{2})\frac{1-x}{1+x}\right]$,

$k=\sqrt[4]{32}\,(\sqrt{2}-1)$.

1. $$\int \frac{x^n\,dx}{\sqrt{x^4+1}}=\frac{x^{n-3}}{n-1}\sqrt{x^4+1}-\frac{n-3}{n-1}\int\frac{x^{n-4}}{\sqrt{x^4+1}}\,dx.$$

2. $$\int\limits_0^x \frac{dx}{\sqrt{x^4+1}}=\frac{1}{2}F\left(\arccos\frac{1-x^2}{1+x^2},\ \frac{\sqrt{2}}{2}\right) \qquad [x\leqslant 1].$$

3. $$\int\limits_x^\infty \frac{dx}{\sqrt{x^4+1}}=\frac{1}{2}F\left(\varphi,\ \frac{\sqrt{2}}{2}\right) \qquad [x\geqslant 1].$$

4. $$\int\limits_x^1 \frac{dx}{\sqrt{x^4+1}}=(2-\sqrt{2})\,F(\psi,\ k) \qquad [0\leqslant x<1].$$

5. $$\int \frac{x\,dx}{\sqrt{x^4+1}}=\frac{1}{2}\ln\left(x^2+\sqrt{x^4+1}\right).$$

6. $$\int\limits_0^x \frac{x^2\,dx}{\sqrt{x^4+1}}=\frac{x\sqrt{x^4+1}}{x^2+1}+\frac{1}{2}F\left(\arccos\frac{1-x^2}{1+x^2},\ \frac{\sqrt{2}}{2}\right)-E\left(\arccos\frac{1-x^2}{1+x^2},\ \frac{\sqrt{2}}{2}\right) \qquad [0\leqslant x\leqslant 1].$$

7. $$\int\limits_1^x \frac{x^2\,dx}{\sqrt{x^4+1}}=\frac{x\sqrt{x^4+1}}{x^2+1}-\frac{1}{2}F\left(\arccos\frac{x^2-1}{x^2+1},\ \frac{\sqrt{2}}{2}\right)+E\left(\arccos\frac{x^2-1}{x^2+1},\ \frac{\sqrt{2}}{2}\right) \qquad [x\geqslant 1].$$

8. $$\int \frac{x^3\,dx}{\sqrt{x^4+1}}=\frac{1}{2}\sqrt{x^4+1}.$$

9. $$\int \frac{dx}{x\sqrt{x^4+1}}=\frac{1}{2}\ln\frac{x^2}{\sqrt{x^4+1}+1}.$$

10. $$\int\limits_x^\infty \frac{dx}{x^2\sqrt{x^4+1}}=\frac{1}{2}\left[F\left(\varphi,\ \frac{\sqrt{2}}{2}\right)-2E\left(\varphi,\ \frac{\sqrt{2}}{2}\right)\right]+\frac{\sqrt{x^4+1}}{x(x^2+1)} \qquad [x\geqslant 1].$$

11. $$\int\limits_x^1 \frac{(1-x)^2\,dx}{(x^2-x\sqrt{2}+1)\sqrt{x^4+1}}=\frac{1}{\sqrt{2}}\left[F(\psi,\ k)-E(\psi,\ k)\right] \qquad [0\leqslant x<1].$$

12. $$\int\limits_x^1 \frac{(1+x)^2\,dx}{(x^2-x\sqrt{2}+1)\sqrt{x^4+1}}=\frac{3\sqrt{2}+4}{2}E(\psi,\ k)-\frac{3\sqrt{2}-4}{2}F(\psi,\ k) \qquad [0\leqslant x<1].$$

13. $\int\limits_x^1 \frac{(x^2+x\sqrt{2}+1)\,dx}{(x^2-x\sqrt{2}+1)\sqrt{x^4+1}} = (2+\sqrt{2})\,E(\psi,\ k)$ $[0 \leqslant x < 1]$.

14. $\int\limits_x^\infty \frac{x^2\,dx}{(x^4+1)\sqrt{x^4+1}} = \frac{1}{2}E\left(\varphi,\ \frac{\sqrt{2}}{2}\right) - \frac{1}{4}F\left(\varphi,\ \frac{\sqrt{2}}{2}\right) - \frac{x(x^2-1)}{2(x^2+1)\sqrt{x^4+1}}$ $[x \geqslant 1]$.

15. $\int\limits_x^\infty \frac{x^2\,dx}{(x^2+1)^2\sqrt{x^4+1}} = \frac{1}{4}\left[F\left(\varphi,\ \frac{\sqrt{2}}{2}\right) - E\left(\varphi,\ \frac{\sqrt{2}}{2}\right)\right]$ $x \geqslant 1]$.

16. $\int\limits_x^\infty \frac{x^2\,dx}{(x^2-1)^2\sqrt{x^4+1}} = \frac{x\sqrt{x^4+1}}{2(x^4-1)} - \frac{1}{4}E\left(\varphi,\ \frac{\sqrt{2}}{2}\right)$ $[x > 1]$.

17. $\int\limits_x^\infty \frac{(x^2+1)^2\,dx}{[(x^2+1)^2-4p^2x^2]\sqrt{x^4+1}} = \frac{1}{2}\Pi\left(\varphi,\ p^2,\ \frac{\sqrt{2}}{2}\right)$ $[x \geqslant 1]$.

18. $\int\limits_x^\infty \frac{(x^2-1)^2\,dx}{(x^2+1)^2\sqrt{x^4+1}} = E\left(\varphi,\ \frac{\sqrt{2}}{2}\right) - \frac{1}{2}F\left(\varphi,\ \frac{\sqrt{2}}{2}\right)$ $[x \geqslant 1]$.

19. $\int\limits_x^\infty \frac{\sqrt{x^4+1}\,dx}{(x^2+1)^2} = \frac{1}{2}E\left(\varphi,\ \frac{\sqrt{2}}{2}\right)$ $[x \geqslant 1]$.

20. $\int\limits_x^\infty \frac{\sqrt{x^4+1}\,dx}{(x^2-1)^2} = \frac{1}{2}\left[F\left(\varphi,\ \frac{\sqrt{2}}{2}\right) - E\left(\varphi,\ \frac{\sqrt{2}}{2}\right)\right] + \frac{x\sqrt{x^4+1}}{x^4-1}$ $[x > 1]$.

1.2.72. Интегралы вида $\int R(x,\ \sqrt{\pm x^4 \mp 1})\,dx$.

Обозначения: $\varphi = \arccos\frac{1}{x}$, $\psi = \arccos x$.

1. $\int\limits_1^x \frac{dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}}F\left(\varphi,\ \frac{\sqrt{2}}{2}\right)$ $[x > 1]$.

2. $\int\limits_1^x \frac{x^2\,dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}}F\left(\varphi,\ \frac{\sqrt{2}}{2}\right) - \sqrt{2}\,E\left(\varphi,\ \frac{\sqrt{2}}{2}\right) + \frac{1}{x}\sqrt{x^4-1}$ $[x > 1]$.

3. $\int\limits_1^x \frac{x^4\,dx}{\sqrt{x^4-1}} = \frac{1}{3}F\left(\varphi,\ \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{3}x\sqrt{x^4-1}$ $[x > 1]$.

4. $\int\limits_x^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}}F\left(\psi,\ \frac{\sqrt{2}}{2}\right)$ $[x < 1]$.

5. $\int_x^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \sqrt{2}\,E\left(\psi, \frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right)$ $[x<1]$.

6. $\int_x^1 \frac{x^4\,dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right) + \frac{x}{3}\sqrt{1-x^4}$ $[x<1]$.

1.2.73. Интегралы вида $\int f(x, \sqrt{x+x^4})\,dx$.
Условие: $x \geqslant 0$ или $x \leqslant -1$.

Обозначения: $\varphi = \arccos \frac{(1-\sqrt{3})\,x+1}{(1+\sqrt{3})\,x+1}$, $k = \sin\frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

1. $\int \frac{dx}{\sqrt{x^4+x}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k)$.

2. $\int \frac{x\,dx}{\sqrt{x^4+x}} = \frac{1}{2}\ln\frac{2x^2+1+2\sqrt{x^4+x}}{2x-1} + \frac{1-\sqrt{3}}{2\sqrt[4]{3}} F(\varphi, k) - \frac{2-\sqrt{3}}{2\sqrt[4]{3}}\,\Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right)$.

3. $\int \frac{x^2\,dx}{\sqrt{x^4+x}} = \frac{(1+\sqrt{3})\sqrt{x^4+x}}{x+1+\sqrt{3}\,x} + \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\varphi, k) - \sqrt[4]{3}\,E(\varphi, k)$.

4. $\int \frac{dx}{(x-\rho)\sqrt{x^4+x}} = \frac{-1-\sqrt{3}}{\sqrt[4]{3}\,(1+\rho+\sqrt{3}\,\rho)} F(\varphi, k) - \frac{1}{2\sqrt[4]{3}\,(1+\rho)\,\rho}\left[\frac{1+\rho-\sqrt{3}\,\rho}{1+\rho+\sqrt{3}\,\rho}\,\Pi(\varphi, -\rho_1, k) + \int \frac{\cos x\,dx}{(1+\rho_1\sin^2 x)\sqrt{1-k^2\sin^2 x}}\right]$

$\left[\rho_1 = \frac{(1+\rho+\sqrt{3}\,\rho)^2}{-4\sqrt{3}\,\rho\,(1+\rho)};\ \rho \neq 0,\ -1,\ \frac{1-\sqrt{3}}{2};\ \text{см. 1.5.33, 12—14}\right]$.

5. $\int \frac{dx}{(x+1)\sqrt{x^4+x}} = \frac{-2\sqrt{x^4+x}}{\sqrt{3}\,(x+1+\sqrt{3}\,x)(x+1)} + \frac{2\sqrt[4]{3}}{3}\left[\frac{\sqrt{3}-1}{2} F(\varphi, k) + E(\varphi, k)\right]$.

6. $\int \frac{dx}{(2x+\sqrt{3}-1)\sqrt{x^4+x}} = \frac{\sqrt[4]{3}}{6}\left[F(\varphi, k) + \frac{2+\sqrt{3}}{k}\arcsin(k\sin\varphi)\right]$.

7. $\int \frac{dx}{x\sqrt[4]{x^4+x}} = -\frac{2\sqrt{x^4+x}}{(x+1+\sqrt{3}\,x)\,x} + \frac{\sqrt{3}-1}{\sqrt[4]{3}} F(\varphi, k) - 2\sqrt[4]{3}\,E(\varphi, k)$.

1.2.74. Интегралы вида $\int R(x, \sqrt{x-x^4})\,dx$.
Условие: $0 \leqslant x \leqslant 1$.

Обозначения: $\varphi = \arccos\frac{1-(\sqrt{3}+1)\,x}{1+(\sqrt{3}-1)\,x}$, $k = \frac{\sqrt{2-\sqrt{3}}}{2}$.

1. $\int \frac{dx}{\sqrt{x-x^4}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k)$.

2. $$\int \frac{x\,dx}{\sqrt{x-x^4}} = -\operatorname{arctg}\frac{x-x^2}{\sqrt{x-x^4}} - \frac{1+\sqrt{3}}{2\sqrt[4]{3}}F(\varphi, k) + \frac{2+\sqrt{3}}{2\sqrt[4]{3}}\Pi\left(\varphi, \frac{3-2\sqrt{3}}{6}, k\right).$$

3. $$\int \frac{x^2\,dx}{\sqrt{x-x^4}} = \frac{(1-\sqrt{3})\sqrt{x-x^4}}{\sqrt{3}x-x+1} - \frac{1+\sqrt{3}}{2\sqrt[4]{3}}F(\varphi, k) + \sqrt[4]{3}E(\varphi, k).$$

4. $$\int \frac{dx}{(x-\rho)\sqrt{x-x^4}} = \frac{-\sqrt{3}+1}{\sqrt[4]{3}(1-\rho+\sqrt{3}\rho)}F(\varphi, k) -$$
$$-\frac{1}{2\sqrt[4]{3}\rho(1-\rho)}\left[\frac{1-\rho-\sqrt{3}\rho}{1-\rho+\sqrt{3}\rho}\Pi(\varphi, -\rho_1, k) + \int\frac{\cos x\,dx}{(1+\rho_1\sin^2 x)\sqrt{1-k^2\sin^2 x}}\right]$$
$$\left[\rho_1 = \frac{(1-\rho+\sqrt{3}\rho)^2}{-4\sqrt{3}\rho(1-\rho)};\ \rho \neq 0,\ 1,\ -\frac{1+\sqrt{3}}{2};\ \text{см. } 1.5.33.\ 12\text{—}14\right].$$

5. $$\int \frac{dx}{(x-1)\sqrt{x-x^4}} = \frac{2\sqrt{x-x^4}}{\sqrt{3}(x-1)(\sqrt{3}x-x+1)} -$$
$$-\frac{2\sqrt[4]{3}}{3}\left[\frac{\sqrt{3}+1}{2}F(\varphi, k) - E(\varphi, k)\right].$$

6. $$\int \frac{dx}{(2x+1+\sqrt{3})\sqrt{x-x^4}} = \frac{\sqrt[4]{3}}{6}\left[F(\varphi, k) + \frac{2-\sqrt{3}}{k}\arcsin(k\sin\varphi)\right].$$

7. $$\int \frac{dx}{x\sqrt{x-x^4}} = \frac{-2\sqrt{x-x^4}}{x(\sqrt{3}x-x+1)} + \frac{1+\sqrt{3}}{\sqrt[4]{3}}F(\varphi, k) - 2\sqrt[4]{3}E(\varphi, k).$$

1.2.75. Интегралы вида $\int R(x, \sqrt{x^4+2b^2x^2+a^4})\,dx$.
Условия: $a > b > 0,\ x \geqslant 0$.

Обозначения: $\varphi = \arccos\frac{x^2-a^2}{x^2+a^2},\ k = \frac{\sqrt{a^2-b^2}}{a\sqrt{2}}$.

1. $$\int_x^\infty \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a}F(\varphi, k).$$

2. $$\int_x^\infty \frac{dx}{x^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a^3}[F(\varphi, k) - 2E(\varphi, k)] + \frac{\sqrt{x^4+2b^2x^2+a^4}}{a^2x(x^2+a^2)}$$
$[x > 0]$.

3. $$\int_x^\infty \frac{x^2\,dx}{(x^2+a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{4a(a^2-b^2)}[F(\varphi, k) - E(\varphi, k)].$$

4. $$\int_x^\infty \frac{x^2\,dx}{(x^2-a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{x\sqrt{x^4+2b^2x^2+a^4}}{2(a^2+b^2)(x^4-a^4)} - \frac{1}{4a(a^2+b^2)}E(\varphi, k).$$
$[x > a]$.

5. $$\int_x^\infty \frac{(x^2-a^2)^2\,dx}{(x^2+a^2)^2\sqrt{x^4+2b^2x^2+a^4}} = \frac{a}{a^2-b^2}E(\varphi, k) - \frac{a^2+b^2}{2a(a^2-b^2)}F(\varphi, k).$$

6. $\int\limits_x^\infty \frac{(x^2+a^2)^2\,dx}{[(x^2+a^2)^2-4a^2p^2x^2]\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a}\Pi(\varphi,\ p^2,\ k).$

7. $\int\limits_x^\infty \frac{x^2\,dx}{\sqrt{(x^4+2b^2x^2+a^4)^3}} = \frac{a}{2(a^4-b^4)}E(\varphi,\ k) - \frac{1}{4a(a^2-b^2)}F(\varphi,\ k) -$

$$- \frac{x(x^2-a^2)}{2(a^2+b^2)(x^2+a^2)\sqrt{x^4+2b^2x^2+a^4}}.$$

8. $\int\limits_x^\infty \frac{(x^2+a^2)^2\,dx}{\sqrt{(x^4+2b^2x^2+a^4)^3}} = \frac{a}{a^2+b^2}E(\varphi,\ k) - \frac{(a^2-b^2)\,x\,(x^2-a^2)}{(a^2+b^2)(x^2+a^2)\sqrt{x^4+2b^2x^2+a^4}}.$

9. $\int\limits_x^\infty \frac{(x^2-a^2)^2\,dx}{\sqrt{(x^4+2b^2x^2+a^4)^3}} =$

$$= \frac{a}{a^2-b^2}[F(\varphi,\ k) - E(\varphi,\ k)] + \frac{x(x^2-a^2)}{(x^2+a^2)\sqrt{x^4+2b^2x^2+a^4}}.$$

10. $\int\limits_x^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{(x^2+a^2)^2}\,dx = \frac{1}{2a}E(\varphi,\ k).$

11. $\int\limits_x^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{(x^2-a^2)^2}\,dx = \frac{1}{2a}[F(\varphi,\ k) - E(\varphi,\ k)] + \frac{x}{x^4-a^4}\sqrt{x^4+2b^2x^2+a^4}.$

12. $\int\limits_x^a \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} =$

$$= \frac{\sqrt{2}}{a\sqrt{2}+\sqrt{a^2+b^2}}F\left[\operatorname{arctg}\left(\frac{a\sqrt{2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}}\,\frac{a-x}{a+x}\right),\ \frac{2\sqrt{a\sqrt{2(a^2-b^2)}}}{a\sqrt{2}+\sqrt{a^2-b^2}}\right]$$

$[0 \leqslant x < a]$.

1.2.76. Интегралы вида $\int R\left(x,\ \sqrt{x^6+1}\right)dx$.

Обозначения: $\varphi = \arccos\frac{(1-\sqrt{3})\,x^2+1}{(1+\sqrt{3})\,x^2+1}$, $\psi = \arccos\frac{x^2+1-\sqrt{3}}{x^2+1+\sqrt{3}}$,

$$k = \sin\frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

1. $\int \frac{dx}{\sqrt{x^6+1}} = \frac{1}{2\sqrt[4]{3}}F(\varphi,\ k).$

2. $\int \frac{x}{\sqrt{x^6+1}}\,dx = \frac{-1}{2\sqrt[4]{3}}F(\psi,\ k).$

3. $\int \frac{x^2}{\sqrt{x^6+1}}\,dx = \frac{1}{3}\ln\left(x^3+\sqrt{x^6+1}\right).$

4. $\int \frac{x^3}{\sqrt{x^6+1}} dx = \frac{\sqrt{x^6+1}}{x^2+1+\sqrt{3}} - \sqrt[4]{3}\left[\frac{3-\sqrt{3}}{6} F(\psi, k) - E(\psi, k)\right].$

5. $\int \frac{x^4}{\sqrt{x^6+1}} dx = \frac{(1+\sqrt{3})\, x\sqrt{x^6+1}}{2(1+\sqrt{3})\, x^2+2} + \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\varphi, k) - \frac{\sqrt[4]{3}}{2} E(\varphi, k).$

6. $\int \frac{x^5}{\sqrt{x^6+1}} dx = \frac{1}{3}\sqrt{x^6+1}.$

7. $\int \frac{dx}{(x-\rho)\sqrt{x^6+1}} = \frac{1}{2\sqrt[4]{3}}\left\{\frac{1}{\rho^2+1+\sqrt{3}} F(\psi, k) - \right.$

$$-\frac{\rho^2+1-\sqrt{3}}{2(\rho^2+1)(\rho^2+1+\sqrt{3})}\Pi(\psi, -\rho_1, k) - \left.\frac{1}{2(\rho^2+1)} I(\psi, \rho_1, k)\right\} +$$

$$+\frac{\rho}{2\sqrt[4]{3}}\left\{\frac{-1-\sqrt{3}}{\rho^2+1+\sqrt{3}\rho^2} F(\varphi, k) - \frac{\rho^2+1-\sqrt{3}\rho^2}{2\rho^2(\rho^2+1)(\rho^2+1+\sqrt{3}\rho^2)}\Pi(\varphi, -\rho_1, k) - \right.$$

$$\left. - \frac{1}{2\rho^2(\rho^2+1)} I(\varphi, \rho_2, k)\right\}$$

$$\left[I(\varphi, \rho, k) = \int_0^\varphi \frac{\cos\varphi\, d\varphi}{(1+\rho\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}};\ x>0,\ \rho\neq 0;\right.$$

$$\left.\rho_1 = \frac{-(\rho^2+1+\sqrt{3})^2}{4\sqrt{3}(\rho^2+1)};\ \rho_2 = \frac{-(\rho^2+1+\sqrt{3}\rho^2)^2}{4\sqrt{3}\rho^2(\rho^2+1)},\ \text{см. } 1.2\,50\right].$$

8. $\int \frac{dx}{x\sqrt{x^6+1}} = \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\psi, k) + \frac{2-\sqrt{3}}{4\sqrt[4]{3}}\Pi\left(\psi, -\frac{3+2\sqrt{3}}{6}, k\right) -$

$$-\frac{1}{8}\ln\frac{x^4+2+2\sqrt{x^6+1}}{x^4+2-2\sqrt{x^6+1}}.$$

1.2.77. Интегралы вида $\int R(x, \sqrt{1-x^6})\, dx$.

Условие: $0 \leqslant x \leqslant 1$.

Обозначения: $\varphi = \arccos\frac{(1+\sqrt{3})x^2-1}{(-1+\sqrt{3})x^2+1}$, $k_1 = \sin\frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$;

$$\psi = \arccos\frac{x^2-1+\sqrt{3}}{x^2-1-\sqrt{3}},\quad k_2 = \sin\frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

1. $\int \frac{dx}{\sqrt{1-x^6}} = \frac{-1}{2\sqrt[4]{3}} F(\varphi, k_1).$ 2. $\int \frac{x}{\sqrt{1-x^6}} dx = \frac{1}{2\sqrt[4]{3}} F(\psi, k_2).$

3. $\int \frac{x^2}{\sqrt{1-x^6}} dx = \frac{1}{3}\arcsin(x^3).$

4. $\int \frac{x^3}{\sqrt{1-x^6}} dx = \frac{-\sqrt{1-x^6}}{x^2-1-\sqrt{3}} - \frac{\sqrt{3}-1}{2\sqrt[4]{3}} F(\psi, k_2) + \sqrt[4]{3}\, E(\psi, k_2).$

5. $\int \frac{x^4}{\sqrt{1-x^6}} dx = \frac{(1-\sqrt{3})\, x\sqrt{1-x^6}}{2(\sqrt{3}x^2-x^2+1)} + \frac{1+\sqrt{3}}{4\sqrt[4]{3}} F(\varphi, k_1) - \frac{\sqrt[4]{3}}{2} E(\varphi, k_1).$

6. $\int \frac{x^5}{\sqrt{1-x^6}} dx = \frac{-1}{3}\sqrt{1-x^6}.$

7. $\int \frac{dx}{(x-\rho)\sqrt{1-x^6}} = \frac{1}{2}\int \frac{dt}{(t-\rho^2)\sqrt{1-t^3}} + \frac{\rho}{2}\int \frac{dt}{(t-\rho^2)\sqrt{t-t^4}}$

$[t=x^2$; см. 1.2.70 и 1.2.74].

8. $\int \frac{dx}{x\sqrt{1-x^6}} = \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\psi, k_2) + \frac{2-\sqrt{3}}{4\sqrt[4]{3}} \Pi\left(\psi, \frac{3+2\sqrt{3}}{6}, k_2\right) + \frac{1}{8}\ln\frac{x^4+2-2\sqrt{1-x^6}}{x^4+2+2\sqrt{1-x^6}}.$

1.2.78. Интегралы вида $\int R(x, \sqrt[3]{x^2 \pm 1})\,dx$.

1. $\int R(x, \sqrt[3]{x^2+1})\,dx =$

$$= -2\sqrt[4]{27}\int R\left(2\sqrt[4]{27}\frac{\sin\varphi\sqrt{1-k^2\sin^2\varphi}}{(1-\cos\varphi)^2}, \frac{(\sqrt{3}-1)\cos\varphi+\sqrt{3}+1}{1-\cos\varphi}\right)\times \frac{2+\cos\varphi-2k^2\sin^2\varphi}{(1-\cos\varphi)^2\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

$$\left[\sin\varphi = \frac{2\sqrt[4]{3}\left(\sqrt[3]{x^2+1}-1\right)^{1/2}}{\sqrt[3]{x^2+1}+\sqrt{3}-1};\quad \cos\varphi = \frac{\sqrt[3]{x^2+1}-\sqrt{3}-1}{\sqrt[3]{x^2+1}+\sqrt{3}-1};\quad k=\frac{\sqrt{2-\sqrt{3}}}{2}\right].$$

2. $\int R(x, \sqrt[3]{x^2-1})\,dx =$

$$= -2\sqrt[4]{27}\int R\left(2\sqrt[4]{27}\frac{\sin\varphi\sqrt{1-k^2\sin^2\varphi}}{(1-\cos\varphi)^2}, \frac{(1+\sqrt{3})\cos\varphi+\sqrt{3}-1}{1-\cos\varphi}\right)\times \frac{2+\cos\varphi-2k^2\sin^2\varphi}{(1-\cos\varphi)^2\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

$$\left[\sin\varphi = \frac{2\sqrt[4]{3}\left(\sqrt[3]{x^2-1}+1\right)^{1/2}}{\sqrt[3]{x^2-1}+1+\sqrt{3}};\quad \cos\varphi = \frac{\sqrt[3]{x^2-1}+1-\sqrt{3}}{\sqrt[3]{x^2-1}+1+\sqrt{3}};\quad k=\frac{\sqrt{2+\sqrt{3}}}{2}\right].$$

1.2.79. Интегралы вида $\int R(x, \sqrt[4]{x^2 \pm 1})\,dx$, $\int R(x, \sqrt[4]{1-x^2})\,dx$.

Обозначения: $\varphi = \arccos\frac{1}{\sqrt[4]{x^2+1}}$, $\psi = \arccos\frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}}$,

$\lambda = \arccos\sqrt[4]{1-x^2}$.

1. $\int_0^x \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{2x}{\sqrt[4]{x^2+1}}$ $[x>0]$.

2. $\int_0^x \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2}F\left(\varphi, \frac{1}{\sqrt{2}}\right)$ $[x>0]$.

3. $\int_0^x \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2}E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \sqrt{2}F\left(\varphi, \frac{1}{\sqrt{2}}\right)$ $[x>0]$.

4. $\int_0^x \frac{x^2\,dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{2x}{\sqrt[4]{x^2+1}}$ $[x>0]$.

5. $\int\limits_1^x \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\psi, \frac{1}{\sqrt{2}}\right) - 2E\left(\psi, \frac{1}{\sqrt{2}}\right) + \frac{2x\sqrt[4]{x^2-1}}{1+\sqrt{x^2-1}}$ $[x>1]$.

6. $\int\limits_1^x \frac{dx}{\sqrt[4]{(x^2-1)^3}} = F\left(\psi, \frac{1}{\sqrt{2}}\right)$ $[x>1]$.

7. $\int\limits_1^x \frac{dx}{x^2\sqrt[4]{x^2-1}} = E\left(\psi, \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\psi, \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}} \cdot \frac{\sqrt[4]{x^2-1}}{x}$ $[x>1]$.

8. $\int\limits_1^x \frac{dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{x^2-1}} = \frac{1}{2}\left[F\left(\psi, \frac{1}{\sqrt{2}}\right) - E\left(\psi, \frac{1}{\sqrt{2}}\right)\right]$ $[x>1]$.

9. $\int\limits_1^x \frac{x^2\,dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{(x^2-1)^3}} = E\left(\psi, \frac{1}{\sqrt{2}}\right)$ $[x>1]$.

10. $\int\limits_0^x \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2}\left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right)\right]$ $[0<x\leqslant 1]$.

11. $\int\limits_0^x \frac{x^2\,dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5}\left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right)\right] - \frac{2x}{5}\sqrt[4]{(1-x^2)^3}$ $[0<x\leqslant 1]$.

12. $\int\limits_0^x \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2}F\left(\lambda, \frac{1}{\sqrt{2}}\right)$ $[0<x\leqslant 1]$.

13. $\int\limits_0^x \frac{x^2\,dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3}F\left(\lambda, \frac{1}{\sqrt{2}}\right) - \frac{2}{3}x\sqrt[4]{1-x^2}$ $[0<x\leqslant 1]$.

14. $\int\limits_0^x \frac{dx}{(1+\sqrt{1-x^2})\sqrt[4]{1-x^2}} = \sqrt{2}\left[F\left(\lambda, \frac{1}{\sqrt{2}}\right) - E\left(\lambda, \frac{1}{\sqrt{2}}\right)\right] + \frac{x\sqrt[4]{1-x^2}}{1+\sqrt{1-x^2}}$ $[0<x\leqslant 1]$.

15. $\int\limits_0^x \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2}\left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right)\right] - \frac{2x\sqrt[4]{1-x^2}}{1+\sqrt{1-x^2}}$ $[0<x\leqslant 1]$.

1.2.80. Интегралы вида $\int R\left(\sqrt[4]{\pm(x-a)},\ \sqrt[4]{(x-b)}\right) dx$.

Обозначения. $\varphi = \arccos \dfrac{a-b-2\sqrt{(x-a)(x-b)}}{a-b+2\sqrt{(x-a)(x-b)}}$,

$$\psi = \arccos \sqrt[4]{\frac{4(a-x)(x-b)}{(a-b)^2}}.$$

1. $$\int_a^x \frac{dx}{\sqrt[4]{(x-a)(x-b)}} = \sqrt{\frac{a-b}{2}} F\left[\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{2(2x-a-b)\sqrt[4]{(x-a)(x-b)}}{a-b+2\sqrt{(x-a)(x-b)}} \qquad [x>a>b].$$

2. $$\int_a^x \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) \qquad [x>a>b].$$

3. $$\int_b^x \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b}\left\{2\left[\mathbf{E}\left(\frac{1}{\sqrt{2}}\right) + E\left(\psi, \frac{1}{\sqrt{2}}\right)\right] - \mathbf{K}\left(\frac{1}{\sqrt{2}}\right) - F\left(\psi, \frac{1}{\sqrt{2}}\right)\right\} \qquad [a \geqslant x > b].$$

4. $$\int_b^x \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}}\left[\mathbf{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\psi, \frac{1}{\sqrt{2}}\right)\right] \qquad [a \geqslant x > b].$$

1.2.81. Интегралы вида $\int R\left(\sqrt[4]{x^4+1}\right) dx$.

1. $$\int \frac{dx}{\sqrt[4]{x^4+1}} = \frac{1}{4} \ln \frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x} - \frac{1}{2} \operatorname{arctg} \frac{\sqrt[4]{x^4+1}}{x}.$$

2. $$\int x\sqrt[4]{x^4+1}\, dx = \frac{1}{3} x^2 \sqrt[4]{x^4+1} + \frac{1}{3\sqrt{2}} F\left(\arccos \frac{1}{\sqrt[4]{x^4+1}},\ \frac{1}{\sqrt{2}}\right).$$

1.3. ПОКАЗАТЕЛЬНАЯ ФУНКЦИЯ

1.3.1. Интегралы вида $\int f(e^{ax})\, dx$.

1. $$\int f(e^{ax})\, dx = \frac{1}{a} \int \frac{f(t)}{t}\, dt \qquad [t=e^{ax}].$$

2. $$\int f(b^x)\, dx = \int f\left(e^{x \ln b}\right) dx \qquad [b>0;\ b \neq 1].$$

3. $$\int e^{ax}\, dx = \frac{1}{a} e^{ax}.$$

4. $$\int b^x\, dx = \frac{b^x}{\ln b} \qquad [b>0;\ b \neq 1].$$

5. $$\int \frac{dx}{e^{ax}+b} = \frac{1}{ab}\left[ax - \ln(e^{ax}+b)\right].$$

6. $\int \frac{dx}{be^{ax}+ce^{-ax}} = \frac{1}{a\sqrt{bc}} \operatorname{arctg}\left(e^{ax}\sqrt{\frac{b}{c}}\right)$ $[bc > 0]$.

$= \frac{1}{2a\sqrt{-bc}} \ln \frac{c+e^{ax}\sqrt{-bc}}{c-e^{ax}\sqrt{-bc}}$ $[bc < 0]$.

7. $\int \frac{e^{ax}-1}{e^{ax}+1}\,dx = \frac{2}{a}\ln\left(e^{ax/2}+e^{-ax/2}\right)$.

8. $\int \frac{dx}{\sqrt{b+ce^{ax}}} = \frac{1}{a\sqrt{b}} \ln \frac{\sqrt{b+ce^{ax}}-\sqrt{b}}{\sqrt{b+ce^{ax}}+\sqrt{b}}$ $[b > 0]$

$= \frac{2}{a\sqrt{-b}} \operatorname{arctg} \frac{\sqrt{b+ce^{ax}}}{\sqrt{-b}}$ $[b < 0]$.

1.3.2. Интегралы вида $\int f(x, e^{ax})\,dx$.

1. $\int x^{\lambda}e^{ax}\,dx = \frac{1}{a}x^{\lambda}e^{ax} - \frac{\lambda}{a}\int x^{\lambda-1}e^{ax}\,dx$.

2. $\int \frac{1}{x^{\lambda}}e^{ax}\,dx = -\frac{e^{ax}}{(\lambda-1)x^{\lambda-1}} + \frac{a}{\lambda-1}\int \frac{e^{ax}}{x^{\lambda-1}}\,dx$.

3. $\int_0^x x^{\lambda-1}e^{-ax}\,dx = \frac{1}{a^{\lambda}}\gamma(\lambda, ax)$ $[\operatorname{Re}\lambda > 0]$.

4. $\int_x^{\infty} x^{\lambda-1}e^{-ax}\,dx = \frac{1}{a^{\lambda}}\Gamma(\lambda, ax)$ $[\operatorname{Re} a > 0]$.

5. $\int_x^{\infty} x^{\lambda-n-1}e^{-ax}\,dx = \frac{(-1)^n a^{n}}{(n-\lambda)(n-\lambda-1)\dots(1-\lambda)}\int_x^{\infty} x^{\lambda-1}e^{-ax}\,dx +$

$+ \frac{e^{-ax}}{x^{n-\lambda}}\sum_{k=0}^{n-1}\frac{(-1)^k(ax)^k}{(n-\lambda)(n-\lambda-1)\dots(n-\lambda-k)}$ $[\operatorname{Re} a > 0]$

6. $\int x^n e^{ax}\,dx = e^{ax}\left[\frac{x^n}{a} + \sum_{k=1}^{n}(-1)^k\frac{n(n-1)\dots(n-k+1)}{a^{k+1}}x^{n-k}\right]$.

7. $\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right)$.

8. $\int x^2e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)$.

9. $\int P(x)e^{ax}\,dx = \frac{e^{ax}}{a}\sum_{k=0}^{m}\frac{(-1)^k}{a^k}\frac{d^k}{dx^k}P(x)$,

где $P(x)$ — многочлен степени m.

10. $$\int x^{n-1/2}e^{-ax}\,dx=(2n-1)!!\,\sqrt{\pi}\,2^{-n}a^{-n-1/2}\,\operatorname{erf}(\sqrt{ax})-\frac{1}{a}x^{n-1/2}e^{-ax}\sum_{k=0}^{n-1}\frac{(2n-1)!!}{(2n-2k-1)!!}(2ax)^{-k}.$$

11. $$\int\frac{e^{ax}}{x^n}\,dx=-e^{ax}\sum_{k=1}^{n-1}\frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}}+\frac{a^{n-1}}{(n-1)!}\operatorname{Ei}(ax).$$

12. $$\int\frac{e^{ax}}{x}\,dx=\operatorname{Ei}(ax)\qquad [a\neq 0].$$

13. $$=\ln|x|+\sum_{k=1}^{\infty}\frac{(ax)^k}{k!k}.$$

14. $$\int_x^{\infty}\frac{e^{-ax}}{x}\,dx=-\operatorname{Ei}(-ax)\qquad [a>0].$$

15. $$\int\frac{e^{ax}}{\sqrt{x}}\,dx=\sqrt{\frac{\pi}{a}}\operatorname{erfi}(\sqrt{ax})\qquad [a>0].$$

16. $$\int\frac{e^{-ax}}{\sqrt{x}}\,dx=\sqrt{\frac{\pi}{a}}\operatorname{erf}(\sqrt{ax})\qquad [a>0].$$

17. $$\int\frac{e^{-ax}}{x^{3/2}}\,dx=-\frac{2}{\sqrt{x}}e^{-ax}-2\sqrt{\pi a}\operatorname{erf}(\sqrt{ax})\qquad [a>0].$$

18. $$\int_x^{\infty}x^{-n-1/2}e^{-ax}\,dx=\frac{(-2)^n a^{n-1/2}\sqrt{\pi}}{(2n-1)!!}\operatorname{erfc}(\sqrt{ax})+2\frac{e^{-ax}}{x^{n-1/2}}\sum_{k=0}^{n-1}\frac{(2n-2k-3)!!\,(-2ax)^k}{(2n-1)!!}\qquad [a>0].$$

19. $$\int_x^{\infty}\frac{e^{-ax}}{\sqrt{x}}\,dx=\sqrt{\frac{\pi}{a}}\operatorname{erfc}(\sqrt{ax})\qquad [a>0].$$

20. $$\int\frac{e^{ax}}{x^2}\,dx=-\frac{e^{ax}}{x}+a\operatorname{Ei}(ax).$$

21. $$\int\frac{e^{ax}}{x^3}\,dx=-\frac{e^{ax}}{2x^2}-\frac{ae^{ax}}{2x}+\frac{a^2}{2}\operatorname{Ei}(ax).$$

22. $$\int\frac{e^{ax}\,dx}{x+b}=e^{-ab}\operatorname{Ei}(ab+ax).$$

23. $$\int\frac{x^m e^{ax}\,dx}{(x+b)^n}=(-1)^m b^m e^{-ab}\sum_{k=0}^{m}\frac{(-1)^k}{b^k}\binom{m}{k}\int t^{k-n}e^{at}\,dt\qquad [t=x+b].$$

24. $\int \frac{xe^{ax}\,dx}{x+b} = \frac{e^{ax}}{a} - be^{-ab}\operatorname{Ei}(ax+ab).$

25. $\int \frac{x^2e^{ax}\,dx}{x+b} = \frac{ax-ab-1}{a^2}e^{ax} + b^2e^{-ab}\operatorname{Ei}(ax+ab).$

26. $\int \frac{e^{ax}\,dx}{(x+b)^2} = -\frac{e^{ax}}{x+b} + ae^{-ab}\operatorname{Ei}(ax+ab).$

27. $\int \frac{xe^{ax}\,dx}{(x+b)^2} = \frac{be^{ax}}{x+b} + (1-ab)e^{-ab}\operatorname{Ei}(ax+ab).$

28. $\int \frac{x^2e^{ax}\,dx}{(x+b)^2} = \left(\frac{1}{a} - \frac{b^2}{x+b}\right)e^{ax} + (ab^2-2b)e^{-ab}\operatorname{Ei}(ax+ab).$

29. $\int \frac{e^{ix}\,dx}{x^2+a^2} = \frac{i}{2a}[e^{a}\operatorname{Ei}(-a+ix) - e^{-a}\operatorname{Ei}(a+ix)].$

30. $\int \frac{xe^{ix}\,dx}{x^2+a^2} = \frac{1}{2}[e^{-a}\operatorname{Ei}(a+ix) + e^{a}\operatorname{Ei}(-a+ix)].$

31. $\int \frac{e^{x}\,dx}{x^2+a^2} = \frac{1}{a}\operatorname{Im}[e^{ia}\operatorname{Ei}(x-ia)].$

32. $\int \frac{xe^{x}\,dx}{x^2+a^2} = \operatorname{Re}[e^{ia}\operatorname{Ei}(x-ia)].$

33. $\int\limits_0^x \frac{e^x-1}{x}\,dx = \operatorname{Ei}(x) - C - \ln x$ $[x>0].$

34. $\int\limits_0^x \frac{(1-e^{-ix})\,dx}{x} = C + \ln x - \operatorname{ci}(x) + i\operatorname{Si}(x).$

1.3.3. Интегралы вида $\int f(x,\ e^{-a^2x^2})\,dx$.

1. $\int\limits_0^x e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}\operatorname{erf}(ax).$

2. $\int\limits_0^x e^{a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}\operatorname{erfi}(ax).$

3. $\int\limits_x^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}\operatorname{erfc}(ax).$

4. $\int x^pe^{-ax^2}\,dx = -\frac{x^{p-1}}{2a}e^{-ax^2} + \frac{p-1}{2a}\int x^{p-2}e^{-ax^2}\,dx.$

5. $\int x^{2n}e^{-a^2x^2}\,dx =$

$$= -\frac{x^{2n-1}e^{-a^2x^2}(2n-1)!!}{2a^2}\sum_{k=0}^{n-1}\frac{1}{2^k(2n-2k-1)!!\,(ax)^{2k}} + \frac{\sqrt{\pi}\,(2n-1)!!}{2^{n+1}a^{2n+1}}\operatorname{erf}(ax).$$

6. $\int x^{2n+1}e^{-ax^2}\,dx = -\frac{x^{2n}e^{-ax^2}}{2a}\sum_{k=0}^{n}\frac{n!}{(n-k)!a^k x^{2k}}.$

7. $\int xe^{-ax^2}\,dx = -\frac{1}{2a}e^{-ax^2}.$

8. $\int x^2e^{-a^2x^2}\,dx = -\frac{1}{2a^2}xe^{-a^2x^2} + \frac{\sqrt{\pi}}{4a^3}\operatorname{erf}(ax).$

9. $\int x^3e^{-ax^2}\,dx = -\frac{ax^2+1}{2a^2}e^{-ax^2}.$

10. $\int \frac{e^{-ax^2}}{x^p}\,dx = -\frac{e^{-ax^2}}{(p-1)x^{p-1}} - \frac{2a}{p-1}\int \frac{e^{-ax^2}}{x^{p-2}}\,dx.$

11. $\int \frac{e^{-a^2x^2}}{x^{2n}}\,dx = \frac{1}{2a^2\Gamma(n+1/2)}e^{-a^2x^2}\sum_{k=1}^{n}(-1)^k\Gamma\left(n-k+\frac{1}{2}\right)a^{2k}x^{2k-2n-1} + \frac{(-1)^n\pi a^{2n-1}}{2\Gamma(n+1/2)}\operatorname{erf}(ax).$

12. $\int \frac{e^{-a^2x^2}}{x^{2n+1}}dx = \frac{1}{2a^2n!}e^{-a^2x^2}\sum_{k=1}^{n}(-1)^k(n-k)!a^{2k}x^{2k-2n-2} + \frac{(-a^2)^n}{2(n!)}\operatorname{Ei}(-a^2x^2).$

13. $\int \frac{e^{-a^2x^2}}{x}\,dx = \frac{1}{2}\operatorname{Ei}(-a^2x^2).$

14. $\int \frac{e^{-a^2x^2}}{x^2}dx = -\frac{1}{x}e^{-a^2x^2} - \sqrt{\pi}\,a\operatorname{erf}(ax).$

15. $\int \frac{e^{-a^2x^2}}{x^3}dx = -\frac{1}{2x^2}e^{-a^2x^2} - \frac{a^2}{2}\operatorname{Ei}(-a^2x^2).$

16. $\int f(x)\,e^{-(ax^2+bx+c)}\,dx =$

$$= \frac{1}{\sqrt{a}}e^{(b^2-4ac)/(4a)}\int f\left(\frac{2\sqrt{a}t-b}{2a}\right)e^{-t^2}\,dt \qquad \left[t=\sqrt{a}\left(x+\frac{b}{2a}\right),\ a>0\right].$$

17. $\int e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/(4a)}\operatorname{erf}\left(x\sqrt{a}+\frac{b}{2\sqrt{a}}\right).$

18. $\int (x-a)^n e^{-c^2(x-b)^2}\,dx =$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{(b-a)^{n-k}}{c^{k+1}}\int t^k e^{-t^2}\,dt \qquad [t=c(x-b),\ c\neq 0].$$

19. $\int x^n e^{-a^2x^2+bx}\,dx =$

$$= \frac{1}{a^{n+1}}e^{b^2/(4a^2)}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b}{2a}\right)^{n-k}\int t^k e^{-t^2}\,dt \qquad \left[t=ax-\frac{b}{2a}\right].$$

20. $$\int_0^x e^{-a^2x^2-b^2/x^2}\,dx=\frac{\sqrt{\pi}}{4a}\left[e^{2ab}\operatorname{erf}\left(ax+\frac{b}{x}\right)+e^{-2ab}\operatorname{erf}\left(ax-\frac{b}{x}\right)-\right.$$
$$\left.-e^{2ab}+e^{-2ab}\right]\qquad [\operatorname{Re} b^2>0].$$

1.4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1.4.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\operatorname{sh}x,\ \operatorname{ch}x,\ \operatorname{th}x,\ \operatorname{cth}x)\,dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

1. $$\int R(\operatorname{sh}x, \operatorname{ch}x, \operatorname{th}x, \operatorname{cth}x)\,dx=\int R\left(\frac{t^2-1}{2t}, \frac{t^2+1}{2t}, \frac{t^2-1}{t^2+1}, \frac{t^2+1}{t^2-1}\right)\frac{dt}{t}\qquad [t=e^x].$$

2. $$\int R(\operatorname{sh}x,\ \operatorname{ch}^2x)\operatorname{ch}x\,dx=\int R(t,\ 1+t^2)\,dt\qquad [t=\operatorname{sh}x].$$

3. $$\int R(\operatorname{sh}^2x,\ \operatorname{ch}x)\operatorname{sh}x\,dx=\int R(t^2-1,\ t)\,dt\qquad [t=\operatorname{ch}x].$$

4. $$\int R(\operatorname{th}x,\ \operatorname{cth}x)\,dx=\int R\left(t, \frac{1}{t}\right)\frac{dt}{1-t^2}\qquad [t=\operatorname{th}x].$$

1.4.2. Интегралы вида $\int \operatorname{sh}^p x\,dx$, $\int \operatorname{ch}^p x\,dx$.

1. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^p dx=\frac{1}{p}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^{p-1}\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}\mp\frac{p-1}{p}\int\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^{p-2}dx.$$

2. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^{2m} dx=(\mp 1)^m\binom{2m}{m}\frac{x}{2^{2m}}+$$
$$+\frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\frac{(\mp 1)^k}{2m-2k}\binom{2m}{k}\begin{Bmatrix}\operatorname{sh}(2m-2k)x\\ \operatorname{ch}(2m-2k)x\end{Bmatrix}.$$

3. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^{2m+1} dx=\sum_{k=0}^{m}\frac{(\mp 1)^{m+k}}{2k+1}\binom{m}{k}\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}^{2k+1}.$$

4. $$=\frac{1}{2^{2m}}\sum_{k=0}^{m}\frac{(\mp 1)^k}{2m-2k+1}\binom{2m+1}{k}\begin{Bmatrix}\operatorname{ch}(2m-2k+1)x\\ \operatorname{sh}(2m-2k+1)x\end{Bmatrix}.$$

5. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix} dx=\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}.$$

6. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^2 dx=\frac{1}{4}\operatorname{sh}2x\mp\frac{x}{2}.$$

7. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^3 dx=\mp\frac{3}{4}\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}+\frac{1}{12}\begin{Bmatrix}\operatorname{ch}3x\\ \operatorname{sh}3x\end{Bmatrix}=\mp\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}+\frac{1}{3}\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}^3.$$

8. $$\int \begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^4 dx=\frac{3}{8}x\mp\frac{1}{4}\operatorname{sh}2x+\frac{1}{32}\operatorname{sh}4x=$$
$$=\frac{3}{8}x\mp\frac{3}{8}\operatorname{sh}x\operatorname{ch}x+\frac{1}{4}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^3\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}.$$

9. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-p} dx = \mp \frac{1}{p-1}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-p+1}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} \mp \frac{p-2}{p-1}\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-p+2} dx.$

10. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2m} dx = \frac{1}{2m-1}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}\left[\mp \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2m+1} + \sum_{k=1}^{m-1} (\mp 1)^{k-1} \frac{2^k (m-1)(m-2)\ldots(m-k)}{(2m-3)(2m-5)\ldots(2m-2k-1)}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2m+2k+1}\right].$

11. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2m-1} dx = \frac{1}{2m}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}\left[\mp \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2m} + \sum_{k=1}^{m-1} (\mp 1)^{k-1} \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k (m-1)(m-2)\ldots(m-k)}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2m+2k}\right] + (\mp 1)^m \frac{(2m-1)!!}{(2m)!!}\begin{Bmatrix}\ln\left|\operatorname{th}\frac{x}{2}\right|\\ \operatorname{arctg} \operatorname{sh} x\end{Bmatrix}.$

12. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-1} dx = \begin{Bmatrix}\ln\left|\operatorname{th}\frac{x}{2}\right|\\ \operatorname{arctg} \operatorname{sh} x\end{Bmatrix} = \begin{Bmatrix}\frac{1}{2}\ln\frac{\operatorname{ch} x-1}{\operatorname{ch} x+1}\\ 2\operatorname{arctg} e^x\end{Bmatrix}.$

13. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2} dx = \mp \begin{Bmatrix}\operatorname{cth} x\\ \operatorname{th} x\end{Bmatrix}.$

14. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-3} dx = \mp \frac{1}{2}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2} \mp \frac{1}{2}\begin{Bmatrix}\ln\left|\operatorname{th}\frac{x}{2}\right|\\ \operatorname{arctg} \operatorname{sh} x\end{Bmatrix}.$

15. $\displaystyle\int \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-4} dx = -\frac{1}{3}\begin{Bmatrix}\operatorname{cth} x\\ \operatorname{th} x\end{Bmatrix}^{3} + \begin{Bmatrix}\operatorname{cth} x\\ \operatorname{th} x\end{Bmatrix}.$

1.4.3. Интегралы вида $\int \operatorname{sh}^p x \operatorname{ch}^q x\, dx$.

1. $\displaystyle\int \operatorname{sh}^p \operatorname{ch}^q x\, dx = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \operatorname{sh}^p x \operatorname{ch}^{q-2} x\, dx.$

2. $\displaystyle = \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{p+q} - \frac{p-1}{p+q}\int \operatorname{sh}^{p-2} x \operatorname{ch}^q x\, dx.$

3. $\displaystyle = \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{q+1} - \frac{p-1}{q+1}\int \operatorname{sh}^{p-2} x \operatorname{ch}^{q+2} x\, dx.$

4. $\displaystyle = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+1} - \frac{q-1}{p+1}\int \operatorname{sh}^{p+2} x \operatorname{ch}^{q-2} x\, dx.$

5. $\displaystyle = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{p+1} - \frac{p+q+2}{p+1}\int \operatorname{sh}^{p+2} x \operatorname{ch}^q x\, dx.$

6. $\displaystyle = -\frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \operatorname{sh}^p x \operatorname{ch}^{q+2} x\, dx$

7. $\displaystyle\int \begin{Bmatrix}\operatorname{sh}^n x \operatorname{ch} x\\ \operatorname{ch}^n x \operatorname{sh} x\end{Bmatrix} dx = \frac{1}{n+1}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{n+1}.$

8. $$\int \begin{Bmatrix} \operatorname{sh}^p x \operatorname{ch}^{2n} x \\ \operatorname{ch}^p x \operatorname{sh}^{2n} x \end{Bmatrix} dx = \frac{1}{2n+p} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{p+1} \left[\begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2n-1} + \right.$$
$$+ \left. \sum_{k=1}^{n-1} \frac{(\pm 1)^k (2n-1)(2n-3)\ldots(2n-2k+1)}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2n-2k-1} \right] +$$
$$+ \frac{(\pm 1)^n (2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)} \int \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^p dx \qquad [p \neq -2, -4, \ldots, -2n].$$

9. $$\int \begin{Bmatrix} \operatorname{sh}^p x \operatorname{ch}^{2n+1} x \\ \operatorname{ch}^p x \operatorname{sh}^{2n+1} x \end{Bmatrix} dx = \frac{1}{2n+p+1} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{p+1} \left[\begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2n} + \right.$$
$$+ \left. \sum_{k=1}^{n} \frac{(\pm 1)^k 2^k n (n-1)\ldots(n-k+1)}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2n-2k} \right]$$
$$[p \neq -1, -3, \ldots, -(2n+1)].$$

10. $$\int \operatorname{sh} ax \operatorname{ch} bx \, dx = \frac{\operatorname{ch}(a+b)x}{2(a+b)} + \frac{\operatorname{ch}(a-b)x}{2(a-b)}.$$

11. $$\int \operatorname{sh} ax \operatorname{ch} ax \, dx = \frac{1}{4a} \operatorname{ch} 2ax.$$

12. $$\int \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^p x \\ \operatorname{ch} x \operatorname{sh}^p x \end{Bmatrix} dx = \frac{1}{p+1} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{p+1}.$$

13. $$\int \operatorname{sh}^2 x \operatorname{ch}^2 x \, dx = -\frac{x}{8} + \frac{1}{32} \operatorname{sh} 4x.$$

14. $$\int \begin{Bmatrix} \operatorname{sh}^2 x \operatorname{ch}^3 x \\ \operatorname{ch}^2 x \operatorname{sh}^3 x \end{Bmatrix} dx = \frac{1}{5} \begin{Bmatrix} \operatorname{ch}^2 x \operatorname{sh}^3 x \\ \operatorname{sh}^2 x \operatorname{ch}^3 x \end{Bmatrix} \pm \frac{2}{15} \begin{Bmatrix} \operatorname{sh}^3 x \\ \operatorname{ch}^3 x \end{Bmatrix}.$$

15. $$\int \begin{Bmatrix} \operatorname{sh}^2 x \operatorname{ch}^4 x \\ \operatorname{ch}^2 x \operatorname{sh}^4 x \end{Bmatrix} dx = \mp \frac{x}{16} - \frac{1}{64} \operatorname{sh} 2x \pm \frac{1}{64} \operatorname{sh} 4x + \frac{1}{192} \operatorname{sh} 6x.$$

16. $$\int \operatorname{sh}^3 x \operatorname{ch}^3 x \, dx = \frac{1}{6} \operatorname{sh}^6 x + \frac{1}{4} \operatorname{sh}^4 x = \frac{1}{6} \operatorname{ch}^6 x - \frac{1}{4} \operatorname{ch}^4 x.$$

17. $$\int \begin{Bmatrix} \operatorname{sh}^3 x \operatorname{ch}^4 x \\ \operatorname{ch}^3 x \operatorname{sh}^4 x \end{Bmatrix} dx = \frac{1}{7} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^2 \mp \frac{2}{35} \begin{Bmatrix} \operatorname{sh}^2 x \operatorname{ch}^5 x \\ \operatorname{ch}^2 x \operatorname{sh}^5 x \end{Bmatrix}.$$

18. $$\int \operatorname{sh}^4 x \operatorname{ch}^4 x \, dx = \frac{3x}{128} - \frac{1}{128} \operatorname{sh} 4x + \frac{1}{1024} \operatorname{sh} 8x.$$

19. $$\int \operatorname{sh} ax \operatorname{sh} bx \operatorname{sh} cx \, dx = \frac{\operatorname{ch}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{ch}(-a+b+c)x}{4(-a+b+c)} -$$
$$- \frac{\operatorname{ch}(a-b+c)x}{4(a-b+c)} - \frac{\operatorname{ch}(a+b-c)x}{4(a+b-c)}.$$

20. $$\int \operatorname{sh} ax \operatorname{sh} bx \operatorname{ch} cx \, dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} -$$
$$- \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

21. $$\int \operatorname{sh} ax \operatorname{ch} bx \operatorname{ch} cx \, dx = \frac{\operatorname{ch}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{ch}(-a+b+c)x}{4(-a+b+c)} +$$
$$+ \frac{\operatorname{ch}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{ch}(a+b-c)x}{4(a+b-c)}.$$

22. $$\int \operatorname{ch} ax \operatorname{ch} bx \operatorname{ch} cx \, dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} + \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} + \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

1.4.4. Интегралы вида $\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^q x}\,dx$, $\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x}\,dx$.

1. $$\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^q x}\,dx = \frac{1}{p-q}\,\frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} - \frac{p-1}{p-q}\int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^q x}\,dx.$$

2. $$= \frac{1}{q-1}\,\frac{\operatorname{sh}^{p+1} x}{\operatorname{ch}^{q-1} x} - \frac{p-q+2}{q-1}\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{q-2} x}\,dx.$$

3. $$= -\frac{1}{q-1}\,\frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} + \frac{p-1}{q-1}\int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^{q-2} x}\,dx.$$

4. $$\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x}\,dx = \frac{1}{q-p}\,\frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{q-p}\int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^p x}\,dx.$$

5. $$= -\frac{1}{p-1}\,\frac{\operatorname{ch}^{q+1} x}{\operatorname{sh}^{p-1} x} + \frac{q-p+2}{p-1}\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^{p-2} x}\,dx.$$

6. $$= -\frac{1}{p-1}\,\frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{p-1}\int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^{p-2} x}\,dx.$$

7. $$\int \left\{\begin{matrix}\operatorname{sh}^p x \operatorname{ch}^{-2n} x\\ \operatorname{ch}^p x \operatorname{sh}^{-2n} x\end{matrix}\right\} dx = \pm\frac{1}{2n-1}\left\{\begin{matrix}\operatorname{sh} x\\ \operatorname{ch} x\end{matrix}\right\}^{p+1}\left[\left\{\begin{matrix}\operatorname{ch} x\\ \operatorname{sh} x\end{matrix}\right\}^{-2n+1} + \sum_{k=1}^{n-1}\frac{(\pm 1)^k (2n-p-2)(2n-p-4)\ldots(2n-p-2k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\left\{\begin{matrix}\operatorname{ch} x\\ \operatorname{sh} x\end{matrix}\right\}^{-2n+2k+1}\right] + (\pm 1)^n\frac{(2n-p-2)(2n-p-4)\ldots(-p+2)(-p)}{(2n-1)!!}\int \left\{\begin{matrix}\operatorname{sh} x\\ \operatorname{ch} x\end{matrix}\right\}^p dx.$$

8. $$\int \left\{\begin{matrix}\operatorname{sh}^p x \operatorname{ch}^{-2n-1} x\\ \operatorname{ch}^p x \operatorname{sh}^{-2n-1} x\end{matrix}\right\} dx = \pm\frac{1}{2n}\left\{\begin{matrix}\operatorname{sh} x\\ \operatorname{ch} x\end{matrix}\right\}^{p+1}\left[\left\{\begin{matrix}\operatorname{ch} x\\ \operatorname{sh} x\end{matrix}\right\}^{-2n} + \sum_{k=1}^{n-1}\frac{(\pm 1)^k (2n-p-1)(2n-p-3)\ldots(2n-p-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)}\left\{\begin{matrix}\operatorname{ch} x\\ \operatorname{sh} x\end{matrix}\right\}^{-2n+2k}\right] + (\pm 1)^n\frac{(2n-p-1)(2n-p-3)\ldots(3-p)(1-p)}{2^n n!}\int \left\{\begin{matrix}\operatorname{sh}^p x \operatorname{ch}^{-1} x\\ \operatorname{ch}^p x \operatorname{sh}^{-1} x\end{matrix}\right\} dx.$$

9. $$\int \left\{\begin{matrix}\operatorname{sh}^{2m} x \operatorname{ch}^{-1} x\\ \operatorname{ch}^{2m} x \operatorname{sh}^{-1} x\end{matrix}\right\} dx = \sum_{k=1}^{m}\frac{(\mp 1)^{m+k}}{2k-1}\left\{\begin{matrix}\operatorname{sh} x\\ \operatorname{ch} x\end{matrix}\right\}^{2k-1} + (\mp 1)^m\left\{\begin{matrix}\operatorname{arctg} \operatorname{sh} x\\ \ln|\operatorname{th}(x/2)|\end{matrix}\right\}.$$

10. $$\int \left\{\begin{matrix}\operatorname{sh}^{2m+1} x \operatorname{ch}^{-1} x\\ \operatorname{ch}^{2m+1} x \operatorname{sh}^{-1} x\end{matrix}\right\} dx = \sum_{k=1}^{m}\frac{(\mp 1)^{m+k}}{2k}\left\{\begin{matrix}\operatorname{sh} x\\ \operatorname{ch} x\end{matrix}\right\}^{2k} + (\mp 1)^m \ln\left\{\begin{matrix}\operatorname{ch} x\\ |\operatorname{sh} x|\end{matrix}\right\} = \sum_{k=1}^{m}\frac{(\mp 1)^{m+k}}{2k}\binom{m}{k}\left\{\begin{matrix}\operatorname{ch} x\\ \operatorname{sh} x\end{matrix}\right\}^{2k} + (\mp 1)^m \ln\left\{\begin{matrix}\operatorname{ch} x\\ |\operatorname{sh} x|\end{matrix}\right\}.$$

11. $$\int \begin{Bmatrix} \operatorname{sh}^{2m+1} x \operatorname{ch}^{-2n} x \\ \operatorname{ch}^{2m+1} x \operatorname{sh}^{-2n} x \end{Bmatrix} dx = \sum_{k=0}^{m} \frac{(\mp 1)^{m+k}}{2k-2n+1} \binom{m}{k} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2k-2n+1}.$$

12. $$\int \begin{Bmatrix} \operatorname{sh}^{2m+1} x \operatorname{ch}^{-2n-1} x \\ \operatorname{ch}^{2m+1} x \operatorname{sh}^{-2n-1} x \end{Bmatrix} dx = \sum_{\substack{k=0 \\ k\neq n}}^{m} \frac{(\mp 1)^{m+k}}{2k-2n} \binom{m}{k} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2k-2n} +$$
$$+ (\mp 1)^{m+n} \binom{m}{n} \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix} \qquad [n \leqslant m].$$

13. $$= \sum_{k=0}^{m} \frac{(\mp 1)^{m+k}}{2k-2n} \binom{m}{k} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{2k-2n} \qquad [n > m].$$

14. $$\int \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-p} x \\ \operatorname{ch} x \operatorname{sh}^{-p} x \end{Bmatrix} dx = -\frac{1}{p-1} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-p+1}.$$

15. $$\int \begin{Bmatrix} \operatorname{sh}^{p} x \operatorname{ch}^{-p-2} x \\ \operatorname{ch}^{p} x \operatorname{sh}^{-p-2} x \end{Bmatrix} dx = \pm \frac{1}{p+1} \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{p+1}.$$

16. $$\int \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-1} x \\ \operatorname{ch} x \operatorname{sh}^{-1} x \end{Bmatrix} dx = \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

17. $$\int \begin{Bmatrix} \operatorname{sh}^2 x \operatorname{ch}^{-1} x \\ \operatorname{ch}^2 x \operatorname{sh}^{-1} x \end{Bmatrix} dx = \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} \mp \begin{Bmatrix} \operatorname{arctg}(\operatorname{sh} x) \\ \ln|\operatorname{th}(x/2)| \end{Bmatrix}.$$

18. $$\int \begin{Bmatrix} \operatorname{sh}^3 x \operatorname{ch}^{-1} x \\ \operatorname{ch}^3 x \operatorname{sh}^{-1} x \end{Bmatrix} dx = \frac{1}{2} \operatorname{ch}^2 x \mp \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

19. $$\int \begin{Bmatrix} \operatorname{sh}^4 x \operatorname{ch}^{-1} x \\ \operatorname{ch}^4 x \operatorname{sh}^{-1} x \end{Bmatrix} dx = \frac{1}{3} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^3 \mp \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} + \begin{Bmatrix} \operatorname{arctg}(\operatorname{sh} x) \\ \ln|\operatorname{th}(x/2)| \end{Bmatrix}.$$

20. $$\int \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-2} x \\ \operatorname{ch} x \operatorname{sh}^{-2} x \end{Bmatrix} dx = -\begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-1}.$$

21. $$\int \begin{Bmatrix} \operatorname{sh}^2 x \operatorname{ch}^{-2} x \\ \operatorname{ch}^2 x \operatorname{sh}^{-2} x \end{Bmatrix} dx = x - \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}.$$

22. $$\int \begin{Bmatrix} \operatorname{sh}^3 x \operatorname{ch}^{-2} x \\ \operatorname{ch}^3 x \operatorname{sh}^{-2} x \end{Bmatrix} dx = \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} \pm \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-1}.$$

23. $$\int \begin{Bmatrix} \operatorname{sh}^4 x \operatorname{ch}^{-2} x \\ \operatorname{ch}^4 x \operatorname{sh}^{-2} x \end{Bmatrix} dx = \mp \frac{3}{2} x + \frac{1}{4} \operatorname{sh} 2x \pm \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}.$$

24. $$\int \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-3} x \\ \operatorname{ch} x \operatorname{sh}^{-3} x \end{Bmatrix} dx = \pm \frac{1}{2} \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^2.$$

25. $$\int \begin{Bmatrix} \operatorname{sh}^2 x \operatorname{ch}^{-3} x \\ \operatorname{ch}^2 x \operatorname{sh}^{-3} x \end{Bmatrix} dx = -\frac{1}{2} \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-2} x \\ \operatorname{ch} x \operatorname{sh}^{-2} x \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} \operatorname{arctg}(\operatorname{sh} x) \\ \ln|\operatorname{th}(x/2)| \end{Bmatrix}.$$

26. $$\int \begin{Bmatrix} \operatorname{sh}^3 x \operatorname{ch}^{-3} x \\ \operatorname{ch}^3 x \operatorname{sh}^{-3} x \end{Bmatrix} dx = -\frac{1}{2} \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^2 + \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

27. $$\int \begin{Bmatrix} \operatorname{sh}^4 x \operatorname{ch}^{-3} x \\ \operatorname{ch}^4 x \operatorname{sh}^{-3} x \end{Bmatrix} dx = \pm \frac{1}{2} \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-2} x \\ \operatorname{ch} x \operatorname{sh}^{-2} x \end{Bmatrix} + \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} \mp \frac{3}{2} \begin{Bmatrix} \operatorname{arctg}(\operatorname{sh} x) \\ \ln|\operatorname{th}(x/2)| \end{Bmatrix}.$$

28. $$\int \begin{Bmatrix} \operatorname{sh} x \operatorname{ch}^{-4} x \\ \operatorname{ch} x \operatorname{sh}^{-4} x \end{Bmatrix} dx = -\frac{1}{3} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-3}.$$

29. $$\int \left\{\begin{matrix} \operatorname{sh}^2 x \operatorname{ch}^{-4} x \\ \operatorname{ch}^2 x \operatorname{sh}^{-4} x \end{matrix}\right\} dx = \pm \frac{1}{3} \left\{\begin{matrix} \operatorname{th} x \\ \operatorname{cth} x \end{matrix}\right\}^3.$$

30. $$\int \left\{\begin{matrix} \operatorname{sh}^3 x \operatorname{ch}^{-4} x \\ \operatorname{ch}^3 x \operatorname{sh}^{-4} x \end{matrix}\right\} dx = -\left\{\begin{matrix} \operatorname{ch} x \\ \operatorname{sh} x \end{matrix}\right\}^{-1} \pm \frac{1}{3} \left\{\begin{matrix} \operatorname{ch} x \\ \operatorname{sh} x \end{matrix}\right\}^{-3}.$$

31. $$\int \left\{\begin{matrix} \operatorname{sh}^4 x \operatorname{ch}^{-4} x \\ \operatorname{ch}^4 x \operatorname{sh}^{-4} x \end{matrix}\right\} dx = -\frac{1}{3} \left\{\begin{matrix} \operatorname{th} x \\ \operatorname{cth} x \end{matrix}\right\}^3 - \left\{\begin{matrix} \operatorname{th} x \\ \operatorname{cth} x \end{matrix}\right\} + x.$$

1.45. Интегралы вида $\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x}$.

1. $$\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x} = -\frac{1}{(p-1) \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} - \frac{p+q-2}{p-1} \int \frac{dx}{\operatorname{sh}^{p-2} x \operatorname{ch}^q x}.$$

2. $$= \frac{1}{(q-1) \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} + \frac{p+q-2}{q-1} \int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^{q-2} x}.$$

3. $$\int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch}^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2m-2k-1} \binom{m+n-1}{k} \operatorname{th}^{2k-2m+1} x.$$

4. $$\int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch}^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \operatorname{th}^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln|\operatorname{th} x|.$$

5. $$\int \left\{\begin{matrix} \operatorname{sh}^{-2m} x \operatorname{ch}^{-1} x \\ \operatorname{ch}^{-2m} x \operatorname{sh}^{-1} x \end{matrix}\right\} dx = \sum_{k=1}^{m} \frac{(\mp 1)^k}{2m-2k+1} \left\{\begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix}\right\}^{-2m+2k-1} + (\mp 1)^m \left\{\begin{matrix} \operatorname{arctg} (\operatorname{sh} x) \\ \ln|\operatorname{th} (x/2)| \end{matrix}\right\}.$$

6. $$\int \left\{\begin{matrix} \operatorname{sh}^{-2m-1} x \operatorname{ch}^{-1} x \\ \operatorname{ch}^{-2m-1} x \operatorname{sh}^{-1} x \end{matrix}\right\} dx = \sum_{k=1}^{m} \frac{(\mp 1)^k}{2m-2k+2} \left\{\begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix}\right\}^{-2m+2k-2} + (\mp 1)^m \ln|\operatorname{th} x|.$$

7. $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch} x} = \ln|\operatorname{th} x|.$$

8. $$\int \left\{\begin{matrix} \operatorname{sh}^{-1} x \operatorname{ch}^{-2} x \\ \operatorname{ch}^{-1} x \operatorname{sh}^{-2} x \end{matrix}\right\} dx = \pm \left\{\begin{matrix} \operatorname{ch} x \\ \operatorname{sh} x \end{matrix}\right\}^{-1} \pm \left\{\begin{matrix} \ln|\operatorname{th} (x/2)| \\ \operatorname{arctg} (\operatorname{sh} x) \end{matrix}\right\}.$$

9. $$\int \left\{\begin{matrix} \operatorname{sh}^{-1} x \operatorname{ch}^{-3} x \\ \operatorname{ch}^{-1} x \operatorname{sh}^{-3} x \end{matrix}\right\} dx = -\frac{1}{2} \left\{\begin{matrix} \operatorname{th} x \\ \operatorname{cth} x \end{matrix}\right\}^2 + \ln\left|\left\{\begin{matrix} \operatorname{th} x \\ \operatorname{cth} x \end{matrix}\right\}\right|.$$

10. $$\int \left\{\begin{matrix} \operatorname{sh}^{-1} x \operatorname{ch}^{-4} x \\ \operatorname{ch}^{-1} x \operatorname{sh}^{-4} x \end{matrix}\right\} dx = \left\{\begin{matrix} \operatorname{ch} x \\ \operatorname{sh} x \end{matrix}\right\}^{-1} \pm \frac{1}{3} \left\{\begin{matrix} \operatorname{ch} x \\ \operatorname{sh} x \end{matrix}\right\}^{-3} + \left\{\begin{matrix} \ln|\operatorname{th} (x/2)| \\ \operatorname{arctg} (\operatorname{sh} x) \end{matrix}\right\}.$$

11. $$\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^2 x} = -2 \operatorname{cth} 2x.$$

12. $$\int \left\{\begin{matrix} \operatorname{sh}^{-2} x \operatorname{ch}^{-3} x \\ \operatorname{ch}^{-2} x \operatorname{sh}^{-3} x \end{matrix}\right\} dx = -\left\{\begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix}\right\}^{-1} - \frac{1}{2} \left\{\begin{matrix} \operatorname{sh} x \operatorname{ch}^{-2} x \\ \operatorname{ch} x \operatorname{sh}^{-2} x \end{matrix}\right\} - \frac{3}{2} \left\{\begin{matrix} \operatorname{arctg} (\operatorname{sh} x) \\ \ln|\operatorname{th} (x/2)| \end{matrix}\right\}.$$

13. $$\int \left\{\begin{matrix} \operatorname{sh}^{-2} x \operatorname{ch}^{-4} x \\ \operatorname{ch}^{-2} x \operatorname{sh}^{-4} x \end{matrix}\right\} dx = \pm \frac{1}{3} \left\{\begin{matrix} \operatorname{sh}^{-1} x \operatorname{ch}^{-3} x \\ \operatorname{ch}^{-1} x \operatorname{sh}^{-3} x \end{matrix}\right\} \mp \frac{8}{3} \operatorname{cth} 2x.$$

14. $$\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^3 x} = \frac{1}{2} \operatorname{th}^2 x - \frac{1}{2} \operatorname{cth}^2 x - 2 \ln|\operatorname{th} x|.$$

15. $$\int \begin{Bmatrix} \operatorname{sh}^{-3} x \operatorname{ch}^{-4} x \\ \operatorname{ch}^{-3} x \operatorname{sh}^{-4} x \end{Bmatrix} dx =$$
$$= \mp 2 \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-1} - \frac{1}{3} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-3} \mp \frac{1}{2} \begin{Bmatrix} \operatorname{ch} x \operatorname{sh}^{-2} x \\ \operatorname{sh} x \operatorname{ch}^{-2} x \end{Bmatrix} \mp \frac{5}{2} \begin{Bmatrix} \ln |\operatorname{th}(x/2)| \\ \operatorname{arctg}(\operatorname{sh} x) \end{Bmatrix}.$$

16. $$\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^4 x} = 8 \operatorname{cth} 2x - \frac{8}{3} \operatorname{cth}^3 2x.$$

1.4.6. Интегралы вида $\int \operatorname{th}^p x\, dx$, $\int \operatorname{cth}^p x\, dx$.

1. $$\int \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^p dx = -\frac{1}{p-1} \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{p-1} + \int \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{p-2} dx.$$

2. $$\int \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{2n} dx = -\sum_{k=1}^{n} \frac{1}{2n-2k+1} \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{2n-2k+1} + x.$$

3. $$\int \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{2n+1} dx = -\sum_{k=1}^{n} \frac{1}{2n-2k+2} \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^{2n-2k+2} + \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

4. $$= -\frac{1}{2} \sum_{k=1}^{n} \frac{(\mp 1)^k}{k} \binom{n}{k} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix}^{-2k} + \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

5. $$\int \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix} dx = \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

6. $$\int \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}^2 dx = x - \begin{Bmatrix} \operatorname{th} x \\ \operatorname{cth} x \end{Bmatrix}.$$

1.4.7. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x)\, dx$.

1. $$\int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} dx =$$
$$= \frac{Bc - Cb + (Ac - Ca) \operatorname{ch} x + (Ab - Ba) \operatorname{sh} x}{(1-n)(a^2 - b^2 + c^2)(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)} \times$$
$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba) \operatorname{ch} x - (n-2)(Ac - Ca) \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} dx$$
$[a^2 + c^2 \neq b^2]$.

2. $$= \frac{Bc - Cb - Ca \operatorname{ch} x - Ba \operatorname{sh} x}{(n-1)\, a\, (a + b \operatorname{ch} x + c \operatorname{sh} x)^n} +$$
$$+ \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)\, a^2}\right] (c \operatorname{ch} x + b \operatorname{sh} x) \frac{(n-1)!}{(2n-1)!!} \times$$
$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!\, a^k} \cdot \frac{1}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-k}}$$
$[a^2 + c^2 = b^2]$.

3. $$\int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x + c \operatorname{sh} x} dx = \frac{Cb - Bc}{b^2 - c^2} \ln (a + b \operatorname{ch} x + c \operatorname{sh} x) +$$
$$+ \frac{Bb - Cc}{b^2 - c^2} x + \left(A - a \frac{Bb - Cc}{b^2 - c^2}\right) \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x}$$
$[b^2 \neq c^2]$.

4. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{a+b\operatorname{ch}x\pm b\operatorname{sh}x}dx=\frac{C\mp B}{2a}(\operatorname{ch}x\mp\operatorname{sh}x)+$$
$$+\left[\frac{A}{a}-\frac{(B\mp C)b}{2a^2}\right]x+\left[\frac{C\pm B}{2b}\pm\frac{A}{a}-\frac{(C\mp B)b}{2a^2}\right]\ln(a+b\operatorname{ch}x\pm b\operatorname{sh}x)$$ $[ab\neq 0]$.

5. $$\int \frac{dx}{a+b\operatorname{ch}x+c\operatorname{sh}x}=\frac{2}{\sqrt{b^2-a^2-c^2}}\operatorname{arctg}\frac{(b-a)\operatorname{th}\frac{x}{2}+c}{\sqrt{b^2-a^2-c^2}}$$ $[b^2>a^2+c^2;\ a\neq b]$.

6. $$=\frac{1}{\sqrt{a^2-b^2+c^2}}\ln\frac{(a-b)\operatorname{th}\frac{x}{2}-c+\sqrt{a^2-b^2+c^2}}{(a-b)\operatorname{th}\frac{x}{2}-c-\sqrt{a^2-b^2+c^2}}$$ $[b^2<a^2+c^2;\ a\neq b]$.

7. $$=\frac{1}{c}\ln\left(a+c\operatorname{th}\frac{x}{2}\right)$$ $[a=b,\ c\neq 0]$.

8. $$=\frac{2}{(a-b)\operatorname{th}\frac{x}{2}+c}$$ $[b^2=a^2+c^2]$.

9. $$\int \frac{dx}{(a\operatorname{ch}x+b\operatorname{sh}x)^n}=\frac{1}{(a^2-b^2)^{n/2}}\int \frac{dx}{\operatorname{ch}^n\left(x+\operatorname{Arth}\frac{b}{a}\right)}$$ $[a>|b|]$.

10. $$=\frac{1}{(b^2-a^2)^{n/2}}\int \frac{dx}{\operatorname{sh}^n\left(x+\operatorname{Arth}\frac{a}{b}\right)}$$ $[b>|a|]$.

11. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{a\operatorname{ch}x+b\operatorname{sh}x}dx=\frac{A}{\sqrt{a^2-b^2}}\operatorname{arctg}\left|\operatorname{sh}\left(x+\operatorname{Arth}\frac{b}{a}\right)\right|+$$
$$+\frac{1}{a^2-b^2}\left[(Ca-Bb)\ln\operatorname{ch}\left(x+\operatorname{Arth}\frac{b}{a}\right)+(Ba-Cb)x\right]$$ $[a>|b|]$.

12. $$=\frac{A}{\sqrt{b^2-a^2}}\ln\left|\operatorname{th}\frac{x+\operatorname{Arth}\frac{a}{b}}{2}\right|+$$
$$+\frac{1}{b^2-a^2}\left[(Cb-Ba)x+(Bb-Ca)\ln\operatorname{sh}\left(x+\operatorname{Arth}\frac{a}{b}\right)\right]$$ $[b>|a|]$.

13. $$=\frac{1}{a}\left[\frac{B+C}{2}x-\frac{B-C}{4}e^{-2x}-Ae^{-x}\right]$$ $[a=b]$.

14. $$=\frac{1}{a}\left[\frac{B-C}{2}x+\frac{B+C}{4}e^{2x}+Ae^{x}\right]$$ $[a=-b]$.

15. $$\int \frac{dx}{a+b\operatorname{sh}x}=\frac{1}{\sqrt{a^2+b^2}}\ln\frac{a\operatorname{th}\frac{x}{2}-b+\sqrt{a^2+b^2}}{a\operatorname{th}\frac{x}{2}-b-\sqrt{a^2+b^2}}.$$

16. $$= \frac{2}{\sqrt{a^2+b^2}} \operatorname{Arth} \frac{a \operatorname{th} \frac{x}{2} - b}{\sqrt{a^2+b^2}}.$$

17. $$\int \frac{dx}{a+b \operatorname{ch} x} = \frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{b+a \operatorname{ch} x}{a+b \operatorname{ch} x} \qquad [b^2>a^2;\ x<0].$$

18. $$= -\frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{b+a \operatorname{ch} x}{a+b \operatorname{ch} x} \qquad [b^2>a^2;\ x>0].$$

19. $$= \frac{1}{\sqrt{a^2-b^2}} \ln \frac{a+b+\sqrt{a^2-b^2} \operatorname{th} \frac{x}{2}}{a+b-\sqrt{a^2-b^2} \operatorname{th} \frac{x}{2}} \qquad [a^2>b^2].$$

20. $$\int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{(1 \pm \operatorname{ch} x)^n} dx = \frac{B \operatorname{sh} x}{(1-n)(1 \pm \operatorname{ch} x)^n} \pm \frac{C}{1-n} \frac{1}{(1 \pm \operatorname{ch} x)^{n-1}} + \left(\frac{n}{n-1} B \pm A\right) \frac{(n-1)!}{(2n-1)!!} \operatorname{sh} x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \frac{1}{(1 \pm \operatorname{ch} x)^{n-k}}.$$

21. $$\int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{1 \pm \operatorname{ch} x} dx = \pm Bx \pm C \ln|1 \pm \operatorname{ch} x| + (A \mp B) \frac{\operatorname{ch} x \mp 1}{\operatorname{sh} x}.$$

22. $$\int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{(a_1+b_1 \operatorname{ch} x+c_1 \operatorname{sh} x)(a_2+b_2 \operatorname{ch} x+c_2 \operatorname{sh} x)} dx = A_0 \ln \frac{a_1+b_1 \operatorname{ch} x+c_1 \operatorname{sh} x}{a_2+b_2 \operatorname{ch} x+c_2 \operatorname{sh} x} + A_1 \int \frac{dx}{a_1+b_1 \operatorname{ch} x+c_1 \operatorname{sh} x} + A_2 \int \frac{dx}{a_2+b_2 \operatorname{ch} x+c_2 \operatorname{sh} x},$$

где

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} b_1 & c_1 \\ B & C \end{vmatrix} & \begin{vmatrix} c_1 & a_1 \\ C & A \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ A & B \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} B & A \\ b_2 & a_2 \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}$$

$$\left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2\right].$$

23. $$\int \frac{A\operatorname{ch}^2 x+2B\operatorname{sh}x\operatorname{ch}x+C\operatorname{sh}^2 x}{a\operatorname{ch}^2 x+2b\operatorname{sh}x\operatorname{ch}x+c\operatorname{sh}^2 x}dx=$$
$$=\frac{1}{4b^2-(a+c)^2}\{[4Bb-(A+C)(a+c)]x+$$
$$+[(A+C)b-B(a+c)]\ln(a\operatorname{ch}^2 x+2b\operatorname{sh}x\operatorname{ch}x+c\operatorname{sh}^2 x)+$$
$$+[2(A-C)b^2+2Bb(a-c)+(Ca-Ac)(a+c)]f(x)\},$$

где

$$f(x)=\frac{1}{2\sqrt{b^2-ac}}\ln\frac{c\operatorname{th}x+b-\sqrt{b^2-ac}}{c\operatorname{th}x+b+\sqrt{b^2-ac}} \qquad [b^2>ac],$$
$$=\frac{1}{\sqrt{ac-b^2}}\operatorname{arctg}\frac{c\operatorname{th}x+b}{\sqrt{ac-b^2}} \qquad [b^2<ac],$$
$$=-\frac{1}{c\operatorname{th}x+b} \qquad [b^2=ac].$$

24. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{\operatorname{sh}x\,(a+b\operatorname{sh}x)}dx=$$
$$=\frac{1}{a}\left[A\ln\left|\operatorname{th}\frac{x}{2}\right|+B\ln\left|\frac{\operatorname{sh}x}{a+b\operatorname{sh}x}\right|+(Ca-Ab)\int\frac{dx}{a+b\operatorname{sh}x}\right].$$

25. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{\operatorname{sh}x\,(a+b\operatorname{ch}x)}dx=$$
$$=\frac{1}{a^2-b^2}\left[(Aa+Bb)\ln\left|\operatorname{th}\frac{x}{2}\right|+(Ab-Ba)\ln\left|\frac{a+b\operatorname{ch}x}{\operatorname{sh}x}\right|\right]+C\int\frac{dx}{a+b\operatorname{ch}x}.$$

26. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{\operatorname{sh}x\,(1\pm\operatorname{ch}x)}dx=$$
$$=\frac{B\pm A}{2}\ln\left|\begin{Bmatrix}\operatorname{th}(x/2)\\ \operatorname{cth}(x/2)\end{Bmatrix}\right|\mp\frac{A\mp B}{4}\begin{Bmatrix}\operatorname{th}(x/2)\\ \operatorname{cth}(x/2)\end{Bmatrix}^2+C\frac{\operatorname{ch}x\mp1}{\operatorname{sh}x}.$$

27. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{\operatorname{ch}x\,(a+b\operatorname{sh}x)}dx=$$
$$=\frac{1}{a^2+b^2}\left[(Aa+Cb)\operatorname{arctg}(\operatorname{sh}x)+(Ab-Ca)\ln\left|\frac{a+b\operatorname{sh}x}{\operatorname{ch}x}\right|\right]+B\int\frac{dx}{a+b\operatorname{sh}x}.$$

28. $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{\operatorname{ch}x\,(a+b\operatorname{ch}x)}dx=$$
$$=\frac{1}{a}\left[A\operatorname{arctg}(\operatorname{sh}x)-C\ln\left|\frac{a+b\operatorname{ch}x}{\operatorname{ch}x}\right|-(Ab-Ba)\int\frac{dx}{a+b\operatorname{ch}x}\right].$$

29. $$\int\frac{dx}{a+b\operatorname{sh}^2 x}=\frac{1}{\sqrt{a(b-a)}}\operatorname{arctg}\left(\sqrt{\frac{b}{a}-1}\operatorname{th}x\right) \qquad [b/a>1].$$

30. $$=\frac{1}{\sqrt{a(a-b)}}\operatorname{Arth}\left(\sqrt{1-\frac{b}{a}}\operatorname{th}x\right)$$
$[0<b/a<1]$ или $[b/a<0,\ \operatorname{sh}^2 x<-a/b]$.

31. $$=\frac{1}{\sqrt{a(a-b)}}\operatorname{Arcth}\left(\sqrt{1-\frac{b}{a}}\operatorname{th}x\right) \qquad [b/a<0,\ \operatorname{sh}^2 x>-a/b].$$

32. $$=\frac{1}{a}\operatorname{th}x \qquad [a=b].$$

33. $= \frac{1}{a\sqrt{2}} \begin{Bmatrix} \operatorname{Arth}(\sqrt{2}\,\operatorname{th} x) \\ \operatorname{Arcth}(\sqrt{2}\,\operatorname{th} x) \end{Bmatrix}$ $[a=-b,\ \operatorname{sh}^2 x \lessgtr 1]$.

34. $\int \frac{dx}{a+b\operatorname{ch}^2 x} = \frac{1}{\sqrt{-a(a+b)}} \operatorname{arctg}\left(\sqrt{-\left(1+\frac{b}{a}\right)}\operatorname{cth} x\right)$ $[b/a<-1]$.

35. $= \frac{1}{\sqrt{a(a+b)}} \operatorname{Arth}\left(\sqrt{1+\frac{b}{a}}\operatorname{cth} x\right)$ $[-1<b/a<0,\ \operatorname{ch}^2 x>-a/b]$.

36. $= \frac{1}{\sqrt{a(a+b)}} \operatorname{Arcth}\left(\sqrt{1+\frac{b}{a}}\operatorname{cth} x\right)$

$[b/a>0]$ или $[-1<b/a<0,\ \operatorname{ch}^2 x < -a/b]$.

37. $= \frac{1}{a\sqrt{2}} \operatorname{Arcth}(\sqrt{2}\operatorname{cth} x)$ $[a=b]$.

38. $= \frac{1}{a}\operatorname{cth} x$ $[a=-b]$.

39. $\int \left(a+b\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}\right)^{-2} dx =$

$= \frac{1}{2a(b \mp a)}\left[\pm b\operatorname{sh} x\operatorname{ch} x\left(a+b\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^2\right)^{-1} + (b \mp 2a)\int\left(a+b\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^2\right)^{-1} dx\right]$.

40. $\int \frac{dx}{a+b\operatorname{th}^2 x} = \frac{1}{a+b}\left[x+\sqrt{\frac{b}{a}}\operatorname{arctg}\left(\sqrt{\frac{b}{a}}\operatorname{th} x\right)\right]$ $[ab>0,\ a+b\neq 0]$.

41. $= \frac{1}{a+b}\left[x+\frac{b}{2\sqrt{-ab}}\ln\frac{\sqrt{-ab}-b\operatorname{th} x}{\sqrt{-ab}+b\operatorname{th} x}\right]$ $[ab<0,\ a+b\neq 0]$.

42. $= \frac{1}{2a}x + \frac{1}{4a}\operatorname{sh} 2x$ $[a+b=0]$.

1.4.8. Интегралы вида $\int R(\operatorname{sh}(ax+b),\ \operatorname{ch}(cx+d))\,dx$.

1. $\int \begin{Bmatrix}\operatorname{sh}(ax+b)\ \operatorname{sh}(cx+d)\\ \operatorname{ch}(ax+b)\ \operatorname{ch}(cx+d)\end{Bmatrix} dx =$

$= \frac{1}{2(a+c)}\operatorname{sh}[(a+c)x+b+d] \mp \frac{1}{2(a-c)}\operatorname{sh}[(a-c)x+b-d]$.

2. $\int \operatorname{sh}(ax+b)\operatorname{ch}(cx+d)\,dx =$

$= \frac{1}{2(a+c)}\operatorname{ch}[(a+c)x+b+d] + \frac{1}{2(a-c)}\operatorname{ch}[(a-c)x+b-d]$.

3. $\int \begin{Bmatrix}\operatorname{sh}(ax+b)\operatorname{sh}(ax+d)\\ \operatorname{ch}(ax+b)\operatorname{ch}(ax+d)\end{Bmatrix} dx = \mp\frac{x}{2}\operatorname{ch}(b-d) + \frac{1}{4a}\operatorname{sh}(2ax+b+d)$.

4. $\int \operatorname{sh}(ax+b)\operatorname{ch}(ax+d)\,dx = \frac{x}{2}\operatorname{sh}(b-d) + \frac{1}{4a}\operatorname{ch}(2ax+b+d)$.

1.4.9. Интегралы вида $\int \operatorname{sh}^p x\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix}dx$, $\int \operatorname{ch}^p x\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix}dx$.

1. $\int \operatorname{sh}^p x\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix}dx = \frac{1}{p+a}\left[\operatorname{sh}^p x\begin{Bmatrix}\operatorname{ch} ax\\ \operatorname{sh} ax\end{Bmatrix} - p\int \operatorname{sh}^{p-1} x\begin{Bmatrix}\operatorname{ch}(a-1)x\\ \operatorname{sh}(a-1)x\end{Bmatrix}dx\right]$.

2. $$\int \operatorname{ch}^p x \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = \frac{1}{p+a}\left[\operatorname{ch}^p x \begin{Bmatrix} \operatorname{ch} ax \\ \operatorname{sh} ax \end{Bmatrix} + p \int \operatorname{ch}^{p-1} x \begin{Bmatrix} \operatorname{sh}(a-1)x \\ \operatorname{ch}(a-1)x \end{Bmatrix} dx\right].$$

3. $$\int \operatorname{sh} 2nx \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^p dx = \begin{Bmatrix} 2n \\ (-1)^{n+1} \end{Bmatrix}\left[\frac{1}{p+2}\begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{p+2} + \right.$$
$$\left. + \sum_{k=1}^{n-1} (\pm 1)^k \frac{(4n^2-2^2)(4n^2-4^2)\dots(4n^2-4k^2)}{(2k+1)!\,(2k+p+2)} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2k+p+2}\right].$$ *)

$[p \neq -2, -4, \dots, -2n]$.

4. $$\int \operatorname{ch} 2nx \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^p dx = (\pm 1)^n \int \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^p dx +$$
$$+ \sum_{k=1}^{n} (\pm 1)^k \frac{4n^2(4n^2-2^2)\dots[4n^2-(2k-2)^2]}{(2k)!} \int \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2k+p} dx.$$

5. $$\int \operatorname{sh}^p x \begin{Bmatrix} \operatorname{sh}(p+2)x \\ \operatorname{ch}(p+2)x \end{Bmatrix} dx = \frac{1}{p+1} \operatorname{sh}^{p+1} x \begin{Bmatrix} \operatorname{sh}(p+1)x \\ \operatorname{ch}(p+1)x \end{Bmatrix}.$$

6. $$\int \operatorname{ch}^p x \begin{Bmatrix} \operatorname{sh}(p+2)x \\ \operatorname{ch}(p+2)x \end{Bmatrix} dx = \frac{1}{p+1} \operatorname{ch}^{p+1} x \begin{Bmatrix} \operatorname{ch}(p+1)x \\ \operatorname{sh}(p+1)x \end{Bmatrix}.$$

7. $$\int \begin{Bmatrix} \operatorname{sh}^p x \operatorname{sh}(2n+1)x \\ \operatorname{ch}^p x \operatorname{ch}(2n+1)x \end{Bmatrix} dx = (\pm 1)^n (2n+1)\left[\int \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{p+1} dx + \right.$$
$$+ \sum_{k=1}^{n} (\pm 1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times$$
$$\left. \times \int \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2k+p+1} dx\right].$$

8. $$\int \begin{Bmatrix} \operatorname{sh}^p x \operatorname{ch}(2n+1)x \\ \operatorname{ch}^p x \operatorname{sh}(2n+1)x \end{Bmatrix} dx = \frac{(\pm 1)^n}{p+1} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{p+1} +$$
$$+ \sum_{k=1}^{n} (\pm 1)^{n+k} \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!\,(2k+p+1)} \times$$
$$\times \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2k+p+1} \qquad [p \neq -3, -5, \dots, -(2n+1)].$$

9. $$\int \begin{Bmatrix} \operatorname{sh}(2n+1)x \operatorname{sh}^{-1} x \\ \operatorname{ch}(2n+1)x \operatorname{ch}^{-1} x \end{Bmatrix} dx = 2 \sum_{k=0}^{n-1} (\pm 1)^k \frac{\operatorname{sh}(2n-2k)x}{2n-2k} + (\pm 1)^n x.$$

10. $$\int \begin{Bmatrix} \operatorname{sh}(2n+1)x \operatorname{ch}^{-1} x \\ \operatorname{ch}(2n+1)x \operatorname{sh}^{-1} x \end{Bmatrix} dx = 2 \sum_{k=0}^{n-1} (\mp 1)^k \frac{\operatorname{ch}(2n-2k)x}{2n-2k} + (\mp 1)^n \ln \begin{Bmatrix} \operatorname{ch} x \\ |\operatorname{sh} x| \end{Bmatrix}.$$

11. $$\int \operatorname{sh} 2nx \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-1} dx = 2 \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{2n-2k-1} \begin{Bmatrix} \operatorname{sh}(2n-2k-1)x \\ \operatorname{ch}(2n-2k-1)x \end{Bmatrix}.$$

12. $\int \operatorname{ch} 2nx \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-1} dx =$

$$= 2\sum_{k=0}^{n-1} \frac{(\pm 1)^k}{2n-2k-1} \begin{Bmatrix}\operatorname{ch}(2n-2k-1)x\\ \operatorname{sh}(2n-2k-1)x\end{Bmatrix} + (\pm 1)^n \begin{Bmatrix}\ln|\operatorname{th}(x/2)|\\ \arcsin \operatorname{th} x\end{Bmatrix}.$$

13. $\int \operatorname{sh} 2x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-n} dx = \frac{2}{2-n} \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2-n}.$

14. $\int \operatorname{sh} 2x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2} dx = 2\ln \begin{Bmatrix}|\operatorname{sh} x|\\ \operatorname{ch} x\end{Bmatrix}.$

15. $\int \operatorname{ch} 2x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-1} dx = 2\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} \pm \begin{Bmatrix}\ln|\operatorname{th}(x/2)|\\ \arcsin \operatorname{th} x\end{Bmatrix}.$

16. $\int \operatorname{ch} 2x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2} dx = -\begin{Bmatrix}\operatorname{cth} x\\ \operatorname{th} x\end{Bmatrix} + 2x.$

17. $\int \operatorname{ch} 2x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-3} dx = -\frac{1}{2} \begin{Bmatrix}\operatorname{ch} x \operatorname{sh}^{-2} x\\ \operatorname{sh} x \operatorname{ch}^{-2} x\end{Bmatrix} + \frac{3}{2} \begin{Bmatrix}\ln|\operatorname{th}(x/2)|\\ \arcsin \operatorname{th} x\end{Bmatrix}.$

18. $\int \begin{Bmatrix}\operatorname{sh} 3x \operatorname{sh}^{-1} x\\ \operatorname{ch} 3x \operatorname{ch}^{-1} x\end{Bmatrix} dx = \operatorname{sh} 2x \pm x.$

19. $\int \begin{Bmatrix}\operatorname{sh} 3x \operatorname{sh}^{-2} x\\ \operatorname{ch} 3x \operatorname{ch}^{-2} x\end{Bmatrix} dx = 4\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} \pm 3\begin{Bmatrix}\ln|\operatorname{th}(x/2)|\\ \arcsin \operatorname{th} x\end{Bmatrix}.$

20. $\int \begin{Bmatrix}\operatorname{sh} 3x \operatorname{sh}^{-3} x\\ \operatorname{ch} 3x \operatorname{ch}^{-3} x\end{Bmatrix} dx = 4x - 3\begin{Bmatrix}\operatorname{cth} x\\ \operatorname{th} x\end{Bmatrix}.$

21. $\int \begin{Bmatrix}\operatorname{sh} 3x \operatorname{ch}^{-n} x\\ \operatorname{ch} 3x \operatorname{sh}^{-n} x\end{Bmatrix} dx = \frac{4}{3-n} \begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{3-n} \mp \frac{1}{1-n} \begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{1-n}.$

22. $\int \begin{Bmatrix}\operatorname{sh} 3x \operatorname{ch}^{-1} x\\ \operatorname{ch} 3x \operatorname{sh}^{-1} x\end{Bmatrix} dx = 2\operatorname{sh}^2 x \mp \ln \begin{Bmatrix}\operatorname{ch} x\\ |\operatorname{sh} x|\end{Bmatrix}.$

23. $\int \begin{Bmatrix}\operatorname{sh} 3x \operatorname{ch}^{-3} x\\ \operatorname{ch} 3x \operatorname{sh}^{-3} x\end{Bmatrix} dx = -\frac{1}{2} \begin{Bmatrix}\operatorname{th} x\\ \operatorname{cth} x\end{Bmatrix}^2 + 4\ln \begin{Bmatrix}\operatorname{ch} x\\ |\operatorname{sh} x|\end{Bmatrix}.$

1.4.10. Интегралы вида $\int R(\operatorname{sh} 2ax, \operatorname{ch} 2ax, \sqrt{\operatorname{sh} 2ax})\,dx$.

Условие: $ax > 0$.

Обозначение: $\varphi = \arccos \frac{1-\operatorname{sh} 2ax}{1+\operatorname{sh} 2ax}$.

1. $\int_0^x \sqrt{\operatorname{sh} 2ax}\,dx = \frac{1}{2a}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{a} \frac{\sqrt{\operatorname{sh} 2ax\,(1+\operatorname{sh}^2 2ax)}}{1+\operatorname{sh} 2ax}.$

2. $\int_0^x \frac{\sqrt{\operatorname{sh} 2ax}\,dx}{(1+\operatorname{sh} 2ax)^2} = \frac{1}{4a}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right].$

3. $\int\limits_0^x \frac{dx}{\sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$

4. $\int\limits_0^x \frac{\operatorname{ch}^2 2ax\, dx}{(1+\operatorname{sh} 2ax)^2 \sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$

5. $\int\limits_0^x \frac{(1-\operatorname{sh} 2ax)^2\, dx}{(1+\operatorname{sh} 2ax)^2 \sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a}\left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right].$

6. $\int\limits_0^x \frac{(1+\operatorname{sh} 2ax)^2\, dx}{[(1+\operatorname{sh} 2ax)^2 - 4p^2 \operatorname{sh} 2ax]\sqrt{\operatorname{sh} 2ax}} = \frac{1}{2a}\Pi\left(\varphi, p^2, \frac{1}{\sqrt{2}}\right).$

1.4.11. Интегралы вида $\int R(\operatorname{sh} 2ax, \operatorname{ch} 2ax, \sqrt{\operatorname{ch} 2ax})\, dx$.

Условие: $ax > 0$.

Обозначение: $\varphi = \arcsin \sqrt{\frac{\operatorname{ch} 2ax - 1}{\operatorname{ch} 2ax}}$.

1. $\int\limits_0^x \sqrt{\operatorname{ch} 2ax}\, dx = \frac{1}{a\sqrt{2}}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{\operatorname{sh} 2ax}{a\sqrt{\operatorname{ch} 2ax}}.$

2. $\int\limits_0^x \frac{\sqrt{\operatorname{ch} 2ax}\, dx}{p^2 + (1-p^2)\operatorname{ch} 2ax} = \frac{1}{a\sqrt{2}}\Pi\left(\varphi, p^2, \frac{1}{\sqrt{2}}\right).$

3. $\int\limits_0^x \frac{dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$

4. $\int\limits_0^x \frac{\operatorname{sh} 2ax\, dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{1}{a}[\sqrt{\operatorname{ch} 2ax} - 1].$

5. $\int\limits_0^x \frac{\operatorname{sh}^2 2ax\, dx}{\sqrt{\operatorname{ch} 2ax}} = -\frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) + \frac{1}{3a}\operatorname{sh} 2ax \sqrt{\operatorname{ch} 2ax}.$

6. $\int\limits_0^x \frac{\operatorname{th}^2 2ax\, dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{th} 2ax}{3a\sqrt{\operatorname{ch} 2ax}}.$

7. $\int\limits_0^x \frac{dx}{\sqrt{\operatorname{ch}^3 2ax}} = \frac{1}{a\sqrt{2}}\left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right].$

1.4.12. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a+b\operatorname{sh} x})\,dx$.

Условия: $a, b > 0$, $x > x_1$.

Обозначения: $\varphi = \arccos\dfrac{\sqrt{a^2+b^2}-a-b\operatorname{sh} x}{\sqrt{a^2+b^2}+a+b\operatorname{sh} x}$,

$$k = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}},\quad x_1 = -\operatorname{Arsh}\frac{a}{b}.$$

1. $$\int_{x_1}^{x}\sqrt{a+b\operatorname{sh} x}\,dx = \sqrt[4]{a^2+b^2}\,[F(\varphi, k)-2E(\varphi, k)] + \frac{2b\operatorname{ch} x\sqrt{a+b\operatorname{sh} x}}{\sqrt{a^2+b^2}+a+b\operatorname{sh} x}.$$

2. $$\int_{x_1}^{x}\frac{\sqrt{a+b\operatorname{sh} x}}{\operatorname{ch}^2 x}dx = \sqrt[4]{a^2+b^2}\,E(\varphi, k) - \frac{\sqrt{a^2+b^2}-a}{2\sqrt[4]{a^2+b^2}}F(\varphi, k) -$$
$$-\frac{a+\sqrt{a^2+b^2}}{b}\,\frac{\sqrt{a^2+b^2}-a-b\operatorname{sh} x}{\sqrt{a^2+b^2}+a+b\operatorname{sh} x}\,\frac{\sqrt{a+b\operatorname{sh} x}}{\operatorname{ch} x}.$$

3. $$\int\frac{\operatorname{ch} x\sqrt{a+b\operatorname{sh} x}}{p+q\operatorname{sh} x}dx = 2\sqrt{\frac{aq-bp}{q}}\operatorname{Arcth}\sqrt{\frac{q(a+b\operatorname{sh} x)}{aq-bp}}$$
$[b\operatorname{sh} x > 0;\ (aq-bp)/q > 0]$.

4. $$= 2\sqrt{\frac{aq-bp}{q}}\operatorname{Arth}\sqrt{\frac{q(a+b\operatorname{sh} x)}{aq-bp}}$$
$[b\operatorname{sh} x < 0;\ (aq-bp)/q > 0]$.

5. $$= 2\sqrt{\frac{bp-aq}{q}}\operatorname{Arth}\sqrt{\frac{q(a+b\operatorname{sh} x)}{bp-aq}}$$ $[(aq-bp)/q < 0]$.

6. $$\int_{x_1}^{x}\frac{\sqrt{a+b\operatorname{sh} x}\,dx}{[\sqrt{a^2+b^2}-a-b\operatorname{sh} x]^2} =$$
$$= -\frac{1}{\sqrt[4]{a^2+b^2}\,(\sqrt{a^2+b^2}-a)}E(\varphi, k) + \frac{b}{\sqrt{a^2+b^2}-a}\,\frac{\operatorname{ch} x\sqrt{a+b\operatorname{sh} x}}{a^2+b^2-(a+b\operatorname{sh} x)^2}.$$

7. $$\int_{x_1}^{x}\frac{dx}{\sqrt{a+b\operatorname{sh} x}} = \frac{1}{\sqrt[4]{a^2+b^2}}F(\varphi, k).$$

8. $$\int_{x_1}^{x}\frac{\operatorname{ch} x\,dx}{\sqrt{a+b\operatorname{sh} x}} = \frac{2}{b}\sqrt{a+b\operatorname{sh} x}.$$

9. $$\int_{x_1}^{x}\frac{\operatorname{ch}^2 x\,dx}{(\sqrt{a^2+b^2}+a+b\operatorname{sh} x)^2\sqrt{a+b\operatorname{sh} x}} = \frac{1}{b^2\sqrt[4]{a^2+b^2}}E(\varphi, k).$$

10. $$\int_{x_1}^{x}\frac{\operatorname{cth} x\,dx}{\sqrt{a+b\operatorname{sh} x}} = \frac{2}{\sqrt{a}}\operatorname{Arcth}\sqrt{1+\frac{b}{a}\operatorname{sh} x}$$
$[b\operatorname{sh} x > 0;\ a > 0]$.

11. $$=\frac{2}{\sqrt{a}}\operatorname{Arth}\sqrt{1+\frac{b}{a}\operatorname{sh}x} \qquad [b\operatorname{sh}x<0;\ a>0].$$

12. $$=\frac{2}{\sqrt{-a}}\operatorname{Arth}\sqrt{-\left(1+\frac{b}{a}\operatorname{sh}x\right)} \qquad [a<0].$$

1.4.13. Интегралы вида $\int R\left(\operatorname{sh}x,\ \operatorname{ch}x,\ \sqrt{b\operatorname{ch}x-a}\right)dx$.

Условия: $b>a>0,\ x>0$.

Обозначения: $\varphi=\arcsin\sqrt{\frac{b(\operatorname{ch}x-1)}{b\operatorname{ch}x-a}},\quad k=\sqrt{\frac{a+b}{2b}}$.

1. $$\int_0^x \sqrt{b\operatorname{ch}x-a}\,dx=(b-a)\sqrt{\frac{2}{b}}F(\varphi,k)-2\sqrt{2b}\,E(\varphi,k)+\frac{2b\operatorname{sh}x}{\sqrt{b\operatorname{ch}x-a}}.$$

2. $$\int_0^x \frac{\sqrt{b\operatorname{ch}x-a}}{\operatorname{ch}x-1}dx=\sqrt{a+b}\,[F(\varphi,k)-E(\varphi,k)].$$

3. $$\int_0^x \frac{\sqrt{b\operatorname{ch}x-a}\,dx}{p^2b-a+b(1-p^2)\operatorname{ch}x}=\sqrt{\frac{2}{b}}\,\Pi(\varphi,p^2,k).$$

4. $$\int_0^x \frac{dx}{\sqrt{b\operatorname{ch}x-a}}=\sqrt{\frac{2}{b}}F(\varphi,k).$$

5. $$\int_0^x \frac{\operatorname{ch}x\,dx}{\sqrt{b\operatorname{ch}x-a}}=\sqrt{\frac{2}{b}}\,[F(\varphi,k)-2E(\varphi,k)]+\frac{2\operatorname{sh}x}{\sqrt{b\operatorname{ch}x-a}}.$$

6. $$\int_0^x \frac{\operatorname{sh}x\,dx}{\sqrt{b\operatorname{ch}x-a}}=\frac{2}{b}\left[\sqrt{b\operatorname{ch}x-a}-\sqrt{b-a}\right].$$

7. $$\int_0^x \frac{dx}{\sqrt{(b\operatorname{ch}x-a)^3}}=\frac{1}{b^2-a^2}\sqrt{\frac{2}{b}}\,[2bE(\varphi,k)-(b-a)F(\varphi,k)].$$

8. $$\int_0^x \frac{(\operatorname{ch}x+1)\,dx}{\sqrt{(b\operatorname{ch}x-a)^3}}=\frac{2}{b-a}\sqrt{\frac{2}{b}}\,E(\varphi,k).$$

1.4.14. Интегралы вида $\int R\left(\operatorname{sh}x,\ \operatorname{ch}x,\ \sqrt{b\operatorname{ch}x-a}\right)dx$.

Условия: $a>b>0,\ x>x_1$.

Обозначения: $\varphi=\arcsin\sqrt{\frac{b\operatorname{ch}x-a}{b(\operatorname{ch}x-1)}},\quad k=\sqrt{\frac{2b}{a+b}},\quad x_1=\operatorname{Arch}\frac{a}{b}$.

1. $$\int_{x_1}^x \sqrt{b\operatorname{ch}x-a}\,dx=-2\sqrt{a+b}\,E(\varphi,k)+2\operatorname{cth}\frac{x}{2}\sqrt{b\operatorname{ch}x-a}.$$

2. $$\int_{x_1}^{x} \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

3. $$\int_{x_1}^{x} \frac{\operatorname{cth}^2 \frac{x}{2}\, dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2\sqrt{a+b}}{a-b} E(\varphi, k).$$

4. $$\int_{x_1}^{x} \frac{dx}{(\operatorname{ch} x + 1)\sqrt{b \operatorname{ch} x - a}} = \frac{1}{\sqrt{a+b}} [F(\varphi, k) - E(\varphi, k)] + \frac{2\sqrt{b \operatorname{ch} x - a}}{(a+b)\operatorname{sh} x}.$$

5. $$\int_{x_1}^{x} \frac{dx}{(\operatorname{ch} x - 1)\sqrt{b \operatorname{ch} x - a}} = \frac{\sqrt{a+b}}{a-b} E(\varphi, k) - \frac{1}{\sqrt{a+b}} F(\varphi, k).$$

6. $$\int_{x_1}^{x} \frac{dx}{(\operatorname{ch} x + 1)^2\sqrt{b \operatorname{ch} x - a}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+2b) F(\varphi, k) - (a+3b) E(\varphi, k)] +$$
$$+ \frac{\sqrt{b \operatorname{ch} x - a}}{3(a+b)\operatorname{sh} x}\left(2\frac{a+3b}{a+b} - \operatorname{th}^2 \frac{x}{2}\right).$$

7. $$\int_{x_1}^{x} \frac{dx}{(\operatorname{ch} x - 1)^2\sqrt{b \operatorname{ch} x - a}} =$$
$$= \frac{1}{3(a-b)^2\sqrt{a+b}} [(a-2b)(a-b) F(\varphi, k) + (3a-b)(a+b) E(\varphi, k)] +$$
$$+ \frac{a+b}{6b(a-b)} \frac{\operatorname{ch}(x/2)}{\operatorname{sh}^3(x/2)} \sqrt{b \operatorname{ch} x - a}.$$

1.4.15. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a - b \operatorname{ch} x})\, dx$.

Условия: $a > b > 0$, $0 < x < x_1$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a - b \operatorname{ch} x}{a-b}}$, $k = \sqrt{\frac{a-b}{a+b}}$, $x_1 = \operatorname{Arch} \frac{a}{b}$.

1. $$\int_{x}^{x_1} \sqrt{a - b \operatorname{ch} x}\, dx = 2\sqrt{a+b}\, [F(\varphi, k) - E(\varphi, k)].$$

2. $$\int_{x}^{x_1} \frac{dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

3. $$\int_{x}^{x_1} \frac{\operatorname{sh} x\, dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2}{b}\sqrt{a - b \operatorname{ch} x}.$$

4. $$\int_{x}^{x_1} \frac{\operatorname{ch} x\, dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\varphi, k) - \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

5. $$\int_x^{x_1} \frac{\operatorname{ch}^2 x\,dx}{\sqrt{a-b\operatorname{ch}x}} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\varphi, k) + \frac{4a\sqrt{a+b}}{3b^2} E(\varphi, k) + \frac{2}{3b}\operatorname{sh}x\sqrt{a-b\operatorname{ch}x}.$$

6. $$\int_x^{x_1} \frac{dx}{\operatorname{ch}x\sqrt{a-b\operatorname{ch}x}} = \frac{2b}{a\sqrt{a+b}} \Pi\left(\varphi, \frac{a-b}{a}, k\right).$$

7. $$\int_x^{x_1} \frac{dx}{(1+\operatorname{ch}x)\sqrt{a-b\operatorname{ch}x}} = \frac{1}{\sqrt{a+b}} E(\varphi, k) - \frac{1}{a+b}\operatorname{th}\frac{x}{2}\sqrt{a-b\operatorname{ch}x}.$$

8. $$\int_x^{x_1} \frac{dx}{(1+\operatorname{ch}x)^2\sqrt{a-b\operatorname{ch}x}} = \frac{1}{3\sqrt{(a+b)^3}}[(a+3b)E(\varphi, k) - bF(\varphi, k)] - \frac{1}{3(a+b)^2}\,\frac{\operatorname{th}\frac{x}{2}\sqrt{a-b\operatorname{ch}x}}{\operatorname{ch}x+1}[2a+4b+(a+3b)\operatorname{ch}x].$$

9. $$\int_x^{x_1} \frac{dx}{(a-b-ap^2+bp^2\operatorname{ch}x)\sqrt{a-b\operatorname{ch}x}} = \frac{2}{(a-b)\sqrt{a+b}} \Pi(\varphi, p^2, k).$$

1.4.16. Интегралы вида $\int R(\operatorname{sh}x, \operatorname{ch}x, \sqrt{a+b\operatorname{ch}x})\,dx$.

Условия: $a > b > 0$, $x > 0$.

Обозначения: $\varphi = \arcsin\left(\operatorname{th}\frac{x}{2}\right)$, $k = \sqrt{\frac{a-b}{a+b}}$.

1. $$\int_0^x \sqrt{a+b\operatorname{ch}x}\,dx = 2\sqrt{a+b}\,[F(\varphi, k) - E(\varphi, k)] + 2\operatorname{th}\frac{x}{2}\sqrt{a+b\operatorname{ch}x}.$$

2. $$\int_0^x \frac{\sqrt{a+b\operatorname{ch}x}}{\operatorname{ch}x+1}dx = \sqrt{a+b}\,E(\varphi, k).$$

3. $$\int \frac{\operatorname{sh}x\sqrt{a+b\operatorname{ch}x}}{p+q\operatorname{ch}x}dx = 2\sqrt{\frac{aq-bp}{q}}\operatorname{Arcth}\sqrt{\frac{q(a+b\operatorname{ch}x)}{aq-bp}} \quad [(aq-bp)/q > 0].$$

4. $$= 2\sqrt{\frac{bp-aq}{q}}\operatorname{Arth}\sqrt{\frac{q(a+b\operatorname{ch}x)}{bp-aq}} \quad [(aq-bp)/q < 0].$$

5. $$\int_0^x \frac{dx}{\sqrt{a+b\operatorname{ch}x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

6. $$\int_0^x \frac{\operatorname{sh}x\,dx}{\sqrt{a+b\operatorname{ch}x}} = \frac{2}{b}\left(\sqrt{a+b\operatorname{ch}x} - \sqrt{a+b}\right).$$

7. $$\int_0^x \frac{\operatorname{ch} x\,dx}{\sqrt{a+b\operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k) - \frac{2\sqrt{a+b}}{b} E(\varphi, k) + \frac{2}{b} \operatorname{th}\frac{x}{2}\sqrt{a+b\operatorname{ch} x}.$$

8. $$\int_0^x \frac{\operatorname{ch} x-1}{\sqrt{a+b\operatorname{ch} x}}\,dx = \frac{2}{b}\left[\operatorname{th}\frac{x}{2}\sqrt{a+b\operatorname{ch} x} - \sqrt{a+b}\,E(\varphi, k)\right].$$

9. $$\int_0^x \frac{(\operatorname{ch} x-1)^2}{\sqrt{a+b\operatorname{ch} x}}\,dx = \frac{4\sqrt{a+b}}{3b^2}\left[(a+3b)E(\varphi, k) - bF(\varphi, k)\right] + \frac{4}{3b^2}\left[b\operatorname{ch}^2\frac{x}{2} - (a+3b)\right]\operatorname{th}\frac{x}{2}\sqrt{a+b\operatorname{ch} x}.$$

10. $$\int_0^x \frac{\operatorname{th} x\,dx}{\sqrt{a+b\operatorname{ch} x}} = 2\sqrt{a}\operatorname{Arcth}\sqrt{1+\frac{b}{a}\operatorname{ch} x}.$$

11. $$\int_0^x \frac{\operatorname{th}^2(x/2)}{\sqrt{a+b\operatorname{ch} x}}\,dx = \frac{2\sqrt{a+b}}{a-b}\left[F(\varphi, k) - E(\varphi, k)\right].$$

12. $$\int_0^x \frac{dx}{(\operatorname{ch} x+1)\sqrt{a+b\operatorname{ch} x}} = \frac{\sqrt{a+b}}{a-b} E(\varphi, k) - \frac{2b}{(a-b)\sqrt{a+b}} F(\varphi, k).$$

13. $$\int_0^x \frac{dx}{(\operatorname{ch} x+1)^2\sqrt{a+b\operatorname{ch} x}} = \frac{1}{3(a-b)^2\sqrt{a+b}}\left[b(5b-a)F(\varphi, k) + (a-3b)(a+b)E(\varphi, k)\right] + \frac{1}{6(a-b)}\frac{\operatorname{sh}(x/2)}{\operatorname{ch}^3(x/2)}\sqrt{a+b\operatorname{ch} x}.$$

14. $$\int_0^x \frac{(1+\operatorname{ch} x)\,dx}{\left[1+p^2+(1-p^2)\operatorname{ch} x\right]\sqrt{a+b\operatorname{ch} x}} = \frac{2}{\sqrt{a+b}}\Pi(\varphi, p^2, k).$$

1.4.17. Интегралы вида

$$\int R\left(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a^2\operatorname{sh}^2 x \pm b^2}, \sqrt{b^2-a^2\operatorname{sh}^2 x}, \sqrt{a^2\operatorname{ch}^2 x \pm b^2}, \sqrt{b^2-a^2\operatorname{ch}^2 x}\right) dx.$$

1. $$\int \frac{1}{\sqrt{\operatorname{sh}^2 x+a^2}}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = \ln\left(\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} + \sqrt{\operatorname{sh}^2 x+a^2}\right).$$

2. $$\int \frac{1}{\sqrt{\operatorname{sh}^2 x-a^2}}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = \ln\left(\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} + \sqrt{\operatorname{sh}^2 x-a^2}\right) \qquad [\operatorname{sh}^2 x > a^2].$$

3. $$\int \frac{1}{\sqrt{a^2-\operatorname{sh}^2 x}}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = \begin{Bmatrix}\arcsin\left[(\operatorname{ch} x)/\sqrt{a^2+1}\right]\\ \arcsin\left[(\operatorname{sh} x)/a\right]\end{Bmatrix} \qquad [\operatorname{sh}^2 x < a^2].$$

4. $$\int \frac{1}{\sqrt{\operatorname{ch}^2 x+a^2}}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = \ln\left(\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} + \sqrt{\operatorname{ch}^2 x+a^2}\right).$$

5. $$\int \frac{1}{\sqrt{\operatorname{ch}^2 x-a^2}}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = \ln\left(\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix} + \sqrt{\operatorname{ch}^2 x-a^2}\right) \qquad [\operatorname{ch}^2 x > a^2].$$

6. $\int \frac{1}{\sqrt{a^2-\operatorname{ch}^2 x}} \left\{ \begin{matrix} \operatorname{sh} x \\ \operatorname{ch} x \end{matrix} \right\} dx = \left\{ \begin{matrix} \arcsin[(\operatorname{ch} x)/a] \\ \arcsin\left[(\operatorname{sh} x)/\sqrt{a^2-1}\right] \end{matrix} \right\}$ $[\operatorname{ch}^2 x < a^2]$.

7. $\int\limits_0^x \frac{dx}{\sqrt{1+k^2 \operatorname{sh}^2 x}} = F\left(\arcsin \operatorname{th} x, \sqrt{1-k^2}\right)$ $[x>0;\ k^2<1]$.

8. $\int\limits_0^x \frac{dx}{\sqrt{k^2+(1-k^2)\operatorname{ch}^2 x}} = F(\arcsin \operatorname{th} x, k)$ $[x>0;\ k^2<1]$.

9. $\int\limits_0^x \frac{dx}{\sqrt{1-k^2 \operatorname{ch}^2 x}} = F\left(\arcsin \frac{\operatorname{th} x}{\sqrt{1-k^2}}, \sqrt{1-k^2}\right)$

$[0<x<\operatorname{ch}(1/\sqrt{1-k^2});\ k^2<1]$.

10. $\int\limits_x^\infty \frac{dx}{\sqrt{\operatorname{ch}^2 x-k^2}} = F(\arcsin \operatorname{sech} x, k)$ $[x>0;\ k^2<1]$.

11. $\int\limits_x^\infty \frac{dx}{\sqrt{\operatorname{sh}^2 x+k^2}} = F\left(\arcsin \operatorname{sech} x, \sqrt{1-k^2}\right)$ $[x>0;\ k^2<1]$.

1.4.18. Интегралы вида $\int R\left(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a \operatorname{sh} x + b \operatorname{ch} x}\right) dx$.

Условия: $b>a>0$, $x>x_1$.

Обозначения: $\varphi = \arccos \frac{\sqrt[4]{b^2-a^2}}{\sqrt{a \operatorname{sh} x+b \operatorname{ch} x}}$, $x_1 = -\operatorname{Arsh} \frac{a}{\sqrt{b^2-a^2}}$.

1. $\int\limits_{x_1}^x \sqrt{a \operatorname{sh} x+b \operatorname{ch} x}\, dx = \sqrt[4]{4(b^2-a^2)} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{2(a \operatorname{ch} x+b \operatorname{sh} x)}{\sqrt{a \operatorname{sh} x+b \operatorname{ch} x}}$.

2. $\int\limits_{x_1}^x \frac{dx}{\sqrt{a \operatorname{sh} x+b \operatorname{ch} x}} = \sqrt[4]{\frac{4}{b^2-a^2}}\, F\left(\varphi, \frac{1}{\sqrt{2}}\right)$.

3. $\int\limits_{x_1}^x \frac{dx}{\sqrt{(a \operatorname{sh} x+b \operatorname{ch} x)^3}} = \sqrt[4]{\frac{4}{(b^2-a^2)^3}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right]$.

4. $\int\limits_{x_1}^x \frac{\left(\sqrt{b^2-a^2}+a \operatorname{sh} x+b \operatorname{ch} x\right) dx}{\sqrt{(a \operatorname{sh} x+b \operatorname{ch} x)^3}} = 2 \sqrt[4]{\frac{4}{b^2-a^2}}\, E\left(\varphi, \frac{1}{\sqrt{2}}\right)$.

1.4.19. Интегралы вида $\int \sqrt{\operatorname{th} x}\, dx$, $\int \sqrt{\operatorname{cth} x}\, dx$.

1. $\int \sqrt{\operatorname{th} x}\, dx = \operatorname{Arth} \sqrt{\operatorname{th} x} - \operatorname{arctg} \sqrt{\operatorname{th} x}$.

2. $\int \sqrt{\operatorname{cth} x}\, dx = \operatorname{Arcth} \sqrt{\operatorname{cth} x} - \operatorname{arctg} \sqrt{\operatorname{cth} x}$.

1.4.20. Интегралы вида $\int x^p \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^q dx$.

1. $$\int x^p \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^q dx = \frac{1}{q} x^p \begin{Bmatrix} \operatorname{sh}^{q-1} x \operatorname{ch} x \\ \operatorname{ch}^{q-1} x \operatorname{sh} x \end{Bmatrix} - \frac{p}{q^2} x^{p-1} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^q + \frac{p(p-1)}{q^2} \int x^{p-2} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^q dx \mp \frac{q-1}{q} \int x^p \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{q-2} dx.$$

2. $$\int x^m \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2n} dx = (\mp 1)^n \binom{2n}{n} \frac{x^{m+1}}{2^{2n}(m+1)} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (\mp 1)^k \binom{2n}{k} \int x^m \operatorname{ch}(2n-2k) x \, dx.$$

3. $$\int x^m \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (\mp 1)^k \binom{2n+1}{k} \int x^m \begin{Bmatrix} \operatorname{sh}(2n-2k+1) x \\ \operatorname{ch}(2n-2k+1) x \end{Bmatrix} dx.$$

4. $$\int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2n} dx = \frac{1}{2n+1} \sum_{k=0}^{n-1} (\mp 1)^k \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1} n(n-1)\ldots(n-k)} \times \left[\frac{-1}{2n-2k} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2n-2k} + x \begin{Bmatrix} \operatorname{sh}^{2n-2k-1} x \operatorname{ch} x \\ \operatorname{ch}^{2n-2k-1} x \operatorname{sh} x \end{Bmatrix} \right] + (\mp 1)^n \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2}.$$

5. $$\int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2n+1} dx = \frac{1}{n+1} \sum_{k=0}^{n} (\mp 1)^k \frac{2^k (n+1) n \ldots (n-k+1)}{(2n+1)(2n-1)\ldots(2n-2k+1)} \left[\frac{-1}{2n-2k+1} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{2n-2k+1} + x \begin{Bmatrix} \operatorname{sh}^{2n-2k} x \operatorname{ch} x \\ \operatorname{ch}^{2n-2k} x \operatorname{sh} x \end{Bmatrix} \right].$$

6. $$\int x^p \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} dx = x^p \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} - p \int x^{p-1} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} dx.$$

7. $$\int x^{2n} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} dx = (2n)! \left[\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} - \sum_{k=0}^{n-1} \frac{x^{2k+1}}{(2k+1)!} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} \right].$$

8. $$\int x^{2n+1} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} dx = (2n+1)! \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} - \frac{x^{2k}}{(2k)!} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} \right].$$

9. $$\int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} dx = x \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} - \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}.$$

10. $$\int x^2 \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix} dx = (x^2+2) \begin{Bmatrix} \operatorname{ch} x \\ \operatorname{sh} x \end{Bmatrix} - 2x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}.$$

11. $$\int x^n \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^2 dx = \mp \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \left[\sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \sum_{k=0}^{[(n-1)/2]} \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x \right].$$

12. $$\int x\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^2 dx=\frac{x}{4}\operatorname{sh}2x-\frac{1}{8}\operatorname{ch}2x\mp\frac{x^2}{4}.$$

13. $$\int_0^x x^{\alpha-1}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}dx=\frac{x^{\alpha+\delta}}{\alpha+\delta}\,{}_1F_2\left(\frac{\alpha+\delta}{2};\ \frac{\alpha+\delta}{2}+1,\ \frac{1}{2}+\delta;\ \frac{x^2}{4}\right)$$
$$\left[\operatorname{Re}\alpha>-\delta;\ \delta=\begin{Bmatrix}1\\0\end{Bmatrix}\right].$$

14. $$\int x^2\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^2 dx=\frac{1}{4}\left(x^2+\frac{1}{2}\right)\operatorname{sh}2x-\frac{x}{4}\operatorname{ch}2x\mp\frac{x^3}{6}.$$

15. $$\int x^n\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^3 dx=\frac{n!}{4}\sum_{k=0}^{[n/2]}\frac{x^{n-2k}}{(n-2k)!}\left[\frac{1}{3^{2k+1}}\begin{Bmatrix}\operatorname{ch}3x\\ \operatorname{sh}3x\end{Bmatrix}\mp\right.$$
$$\left.\mp 3\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}\right]-\frac{n!}{4}\sum_{k=0}^{[(n-1)/2]}\frac{x^{n-2k-1}}{(n-2k-1)!}\left[\frac{1}{3^{2k+2}}\begin{Bmatrix}\operatorname{sh}3x\\ \operatorname{ch}3x\end{Bmatrix}\mp 3\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}\right].$$

16. $$\int x\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^3 dx=\pm\frac{3}{4}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}-\frac{1}{36}\begin{Bmatrix}\operatorname{sh}3x\\ \operatorname{ch}3x\end{Bmatrix}\mp\frac{3}{4}x\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}+\frac{x}{12}\begin{Bmatrix}\operatorname{ch}3x\\ \operatorname{sh}3x\end{Bmatrix}.$$

17. $$\int x^2\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^3 dx=\mp\left(\frac{3}{4}x^2+\frac{3}{2}\right)\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}+\left(\frac{x^2}{12}+\frac{1}{54}\right)\begin{Bmatrix}\operatorname{ch}3x\\ \operatorname{sh}3x\end{Bmatrix}\pm$$
$$\pm\frac{3}{2}x\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}-\frac{x}{18}\begin{Bmatrix}\operatorname{sh}3x\\ \operatorname{ch}3x\end{Bmatrix}.$$

1.4.21. Интегралы вида $\int\frac{1}{x^p}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^q dx$.

1. $$\int\frac{1}{x^p}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^q dx=-\frac{1}{(p-1)x^{p-1}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^q-\frac{q}{(p-1)(p-2)x^{p-2}}\begin{Bmatrix}\operatorname{sh}^{q-1}x\operatorname{ch}x\\ \operatorname{ch}^{q-1}x\operatorname{sh}x\end{Bmatrix}\pm$$
$$\pm\frac{q(q-1)}{(p-1)(p-2)}\int\frac{1}{x^{p-2}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^{q-2}dx+\frac{q^2}{(p-1)(p-2)}\int\frac{1}{x^{p-2}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}^q dx.$$

2. $$\int\frac{1}{x^{2n}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}dx=-\frac{1}{x(2n-1)!}\left[\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}+\right.$$
$$\left.+\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}\right]+\frac{1}{(2n-1)!}\begin{Bmatrix}\operatorname{chi}x\\ \operatorname{shi}x\end{Bmatrix}.$$

3. $$\int\frac{1}{x^{2n+1}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}dx=-\frac{1}{x(2n)!}\left[\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\begin{Bmatrix}\operatorname{ch}x\\ \operatorname{sh}x\end{Bmatrix}+\right.$$
$$\left.+\sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\begin{Bmatrix}\operatorname{sh}x\\ \operatorname{ch}x\end{Bmatrix}\right]+\frac{1}{(2n)!}\begin{Bmatrix}\operatorname{shi}x\\ \operatorname{chi}x\end{Bmatrix}.$$

4. $$\int \frac{1}{x}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2n} dx = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}(\mp 1)^k \binom{2n}{k}\operatorname{chi}(2n-2k)x + \frac{(\mp 1)^n}{2^{2n}}\binom{2n}{n}\ln|x|.$$

5. $$\int \frac{1}{x}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2n+1} dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}(\mp 1)^k \binom{2n+1}{k}\begin{Bmatrix}\operatorname{shi}(2n-2k+1)x\\ \operatorname{chi}(2n-2k+1)x\end{Bmatrix}.$$

6. $$\int \frac{1}{x^2}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2n} dx = -\frac{(\mp 1)^n}{2^{2n}x}\binom{2n}{n} - \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}(\mp 1)^k \binom{2n}{k}\left[\frac{\operatorname{ch}(2n-2k)x}{x} - (2n-2k)\operatorname{shi}(2n-2k)x\right].$$

7. $$\int \frac{1}{x^2}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2n+1} dx = -\frac{1}{2^{2n}}\sum_{k=0}^{n}(\mp 1)^k \binom{2n+1}{k}\times \left[\frac{1}{x}\begin{Bmatrix}\operatorname{sh}(2n-2k+1)x\\ \operatorname{ch}(2n-2k+1)x\end{Bmatrix} - (2n-2k+1)\begin{Bmatrix}\operatorname{chi}(2n-2k+1)x\\ \operatorname{shi}(2n-2k+1)x\end{Bmatrix}\right].$$

8. $$\int \frac{1}{x}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = \begin{Bmatrix}\operatorname{shi} x\\ \operatorname{chi} x\end{Bmatrix}.$$

9. $$\int_0^x \frac{\operatorname{sh} x}{x}\,dx = \operatorname{shi} x.$$

10. $$\int_0^x \frac{\operatorname{ch} x - 1}{x}\,dx = \operatorname{chi} x - \ln x - C.$$

11. $$\int \frac{1}{x^2}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} dx = -\frac{1}{x}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix} + \begin{Bmatrix}\operatorname{chi} x\\ \operatorname{shi} x\end{Bmatrix}.$$

1.4.22. Интегралы вида $\int x^p \begin{Bmatrix}\operatorname{th} x\\ \operatorname{cth} x\end{Bmatrix}^m dx$.

1. $$\int x^p \operatorname{th} x\,dx = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k+p)(2k)!}x^{2k+p} \qquad [|x|<\pi/2;\ p\geqslant -1].$$

2. $$\int x^p \operatorname{cth} x\,dx = \sum_{k=0}^{\infty}\frac{2^{2k}B_{2k}}{(2k+p)(2k)!}x^{2k+p} \qquad [|x|<\pi;\ p\geqslant 1].$$

3. $$\int x^p \begin{Bmatrix}\operatorname{th} x\\ \operatorname{cth} x\end{Bmatrix}^{2n} dx = \frac{1}{p+1}x^{p+1} + \sum_{k=0}^{n-1}(\mp 1)^{n+k}\binom{n}{k}\int x^p \begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{2k-2n} dx.$$

4. $$\int x^p \begin{Bmatrix}\operatorname{th} x\\ \operatorname{cth} x\end{Bmatrix}^{2n+1} dx = \int x^p \begin{Bmatrix}\operatorname{th} x\\ \operatorname{cth} x\end{Bmatrix} dx + \sum_{k=0}^{n-1}\frac{(\mp 1)^{n+k}}{2n-2k}\binom{n}{k}\left[-x^p\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{2k-2n} + p\int x^{p-1}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{2k-2n} dx\right].$$

1.4.23. Интегралы вида $\int x^r \operatorname{sh}^p x \operatorname{ch}^q x\,dx$.

1. $$\int x^r \operatorname{sh}^p x \operatorname{ch}^q x\,dx = \frac{1}{(p+q)^2}\Big[(p+q)\,x^r \operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r+1)\int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x\,dx + rp\int x^{r-1}\operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x\,dx + (q-1)(p+q)\int x^r \operatorname{sh}^p x \operatorname{ch}^{q-2} x\,dx\Big].$$

2. $$= \frac{1}{(p+q)^2}\Big[(p+q)\,x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r-1)\int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x\,dx - rq\int x^{r-1}\operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x\,dx - (p-1)(p+q)\int x^r \operatorname{sh}^{p-2} x \operatorname{ch}^q x\,dx\Big].$$

3. $$\int x^p \begin{Bmatrix}\operatorname{sh}^{2m} x \operatorname{ch}^{-n} x\\ \operatorname{ch}^{2m} x \operatorname{sh}^{-n} x\end{Bmatrix} dx = \sum_{k=0}^{m} (\mp 1)^{m+k}\binom{m}{k}\int x^p \begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{2k-n} dx.$$

4. $$\int x^p \begin{Bmatrix}\operatorname{sh}^{2m+1} x \operatorname{ch}^{-n} x\\ \operatorname{ch}^{2m+1} x \operatorname{sh}^{-n} x\end{Bmatrix} dx = \sum_{k=0}^{m} (\mp 1)^{m+k}\binom{m}{k}\int x^p \begin{Bmatrix}\operatorname{sh} x \operatorname{ch}^{2k-n} x\\ \operatorname{ch} x \operatorname{sh}^{2k-n} x\end{Bmatrix} dx.$$

5. $$\int x^p \begin{Bmatrix}\operatorname{sh} x \operatorname{ch}^{-n} x\\ \operatorname{ch} x \operatorname{sh}^{-n} x\end{Bmatrix} dx = -\frac{x^p}{n-1}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{-n+1} + \frac{p}{n-1}\int x^{p-1}\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{-n+1} dx.$$

6. $$\int x \begin{Bmatrix}\operatorname{sh} x \operatorname{ch}^{-2} x\\ \operatorname{ch} x \operatorname{sh}^{-2} x\end{Bmatrix} dx = -x\begin{Bmatrix}\operatorname{ch} x\\ \operatorname{sh} x\end{Bmatrix}^{-1} + \begin{Bmatrix}\operatorname{arctg} \operatorname{sh} x\\ \ln \operatorname{th}(x/2)\end{Bmatrix}.$$

1.4.24. Интегралы вида $\int \frac{x^p}{\operatorname{sh}^q x}\,dx$, $\int \frac{x^p}{\operatorname{ch}^q x}\,dx$.

1. $$\int x^p \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q} dx = \mp\frac{p x^{p-1}}{(q-1)(q-2)}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q+2} \mp \frac{x^p}{q-1}\begin{Bmatrix}\operatorname{ch} x \operatorname{sh}^{-q+1} x\\ \operatorname{sh} x \operatorname{ch}^{-q+1} x\end{Bmatrix} \pm \frac{p(p-1)}{(q-1)(q-2)}\int x^{p-2}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q+2} dx \mp \frac{q-2}{q-1}\int x^p \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q+2} dx.$$

2. $$\int \frac{x^p}{\operatorname{sh} x}\,dx = \sum_{k=0}^{\infty}\frac{(2-2^{2k})B_{2k}}{(2k)!\,(p+2k)}x^{p+2k} \qquad [|x|<\pi;\ p>0].$$

3. $$\int \frac{x^p}{\operatorname{ch} x}\,dx = \sum_{k=0}^{\infty}\frac{E_{2k}}{(2k)!\,(p+2k+1)}x^{p+2k+1} \qquad [|x|<\pi/2;\ p\geqslant 0].$$

4. $$\int \frac{x^p}{\operatorname{sh}^2 x}\,dx = -x^p \operatorname{cth} x + p\sum_{k=0}^{\infty}\frac{2^{2k}B_{2k}}{(p+2k-1)\,(2k)!}x^{p+2k-1} \qquad [|x|<\pi;\ p>1].$$

5. $$\int \frac{x^p}{\operatorname{ch}^2 x}\,dx = x^p \operatorname{th} x - p\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(p+2k-1)\,(2k)!}x^{p+2k-1} \qquad [|x|<\pi/2;\ p>1].$$

6. $$\int x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2n} dx = \frac{1}{2n}\sum_{k=1}^{n-1} (\mp 1)^k \frac{2n(2n-2)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times$$
$$\times \left[x \begin{Bmatrix}\operatorname{ch} x \operatorname{sh}^{2k-2n-1} x\\ \operatorname{sh} x \operatorname{ch}^{2k-2n-1} x\end{Bmatrix} + \frac{1}{2n-2k}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2k-2n}\right] + (\mp 1)^{n-1}\frac{(2n-2)!!}{(2n-1)!!}\int x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2} dx.$$

7. $$\int x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-2n-1} dx = \frac{1}{2n+1}\sum_{k=1}^{n} (\mp 1)^k \frac{(2n+1)(2n-1)\dots(2n-2k+3)}{2n(2n-2)\dots(2n-2k+2)} \times$$
$$\times \left[x \begin{Bmatrix}\operatorname{ch} x \operatorname{sh}^{2k-2n-2} x\\ \operatorname{sh} x \operatorname{ch}^{2k-2n-2} x\end{Bmatrix} + \frac{1}{2n-2k+1}\begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{2k-2n-1}\right] +$$
$$+ (\mp 1)^{n}\frac{(2n-1)!!}{(2n)!!}\int x \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-1} dx.$$

8. $$\int \frac{1}{x^p} \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q} dx = \pm \frac{p}{(q-1)(q-2)x^{p+1}} \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q+2} \mp$$
$$\mp \frac{1}{(q-1)x^p} \begin{Bmatrix}\operatorname{ch} x \operatorname{sh}^{-q+1} x\\ \operatorname{sh} x \operatorname{ch}^{-q+1} x\end{Bmatrix} - \frac{q-2}{q-1}\int \frac{1}{x^p} \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q+2} dx \pm$$
$$\pm \frac{p(p+1)}{(q-1)(q-2)}\int \frac{1}{x^{p+2}} \begin{Bmatrix}\operatorname{sh} x\\ \operatorname{ch} x\end{Bmatrix}^{-q+2} dx.$$

9. $$\int \frac{dx}{x^n \operatorname{sh} x} = -[1+(-1)^n]\frac{2^{n-1}-1}{n!} B_n \ln|x| + \sum_{\substack{k=0\\ k\neq n/2}}^{\infty} \frac{2-2^{2k}}{(2k)!(2k-n)} B_{2k} x^{2k-n}$$
$$[|x|<\pi;\ n\geqslant 1].$$

10. $$\int \frac{dx}{x^n \operatorname{ch} x} = \sum_{\substack{k=0\\ k\neq (n-1)/2}}^{\infty} \frac{E_{2k}}{(2k)!(2k-n+1)} x^{2k-n+1} +$$
$$+\frac{1}{2}[1-(-1)^n]\frac{E_{n-1}}{(n-1)!}\ln|x| \qquad [|x|<\pi/2].$$

11. $$\int \frac{dx}{x^n \operatorname{sh}^2 x} = -\frac{\operatorname{cth} x}{x^n} - [1-(-1)^n]\frac{2^n n}{(n+1)!} B_{n+1}\ln|x| -$$
$$-\frac{n}{x^{n+1}} \sum_{\substack{k=0\\ k\neq (n+1)/2}}^{\infty} \frac{B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \qquad [|x|<\pi].$$

12. $$\int \frac{dx}{x^n \operatorname{ch}^2 x} = \frac{\operatorname{th} x}{x^n} + [1-(-1)^n]\frac{2^n(2^{n+1}-1)n}{(n+1)!} B_{n+1}\ln|x| +$$
$$+\frac{n}{x^{n+1}} \sum_{\substack{k=1\\ k\neq (n+1)/2}}^{\infty} \frac{(2^{2k}-1)B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \qquad [|x|<\pi/2].$$

13. $$\int \frac{x\,dx}{\operatorname{sh} x} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k)!(2k+1)} B_{2k} x^{2k+1} \qquad [|x|<\pi].$$

14. $\int \frac{x\,dx}{\operatorname{ch} x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k)!\,(2k+2)}$ $[|x| < \pi/2]$.

15. $\int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-2} dx = \mp x \begin{Bmatrix} \operatorname{cth} x \\ \operatorname{th} x \end{Bmatrix} \pm \ln \begin{Bmatrix} |\operatorname{sh} x| \\ \operatorname{ch} x \end{Bmatrix}.$

16. $\int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-3} dx = \mp \frac{x}{2} \begin{Bmatrix} \operatorname{ch} x \operatorname{sh}^{-2} x \\ \operatorname{sh} x \operatorname{ch}^{-2} x \end{Bmatrix} \mp \frac{1}{2} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-1} \mp \frac{1}{2} \int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-1} dx.$

17. $\int x \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-4} dx =$

$$= \mp \frac{x}{3} \begin{Bmatrix} \operatorname{ch} x \operatorname{sh}^{-3} x \\ \operatorname{sh} x \operatorname{ch}^{-3} x \end{Bmatrix} \mp \frac{1}{6} \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}^{-2} + \frac{2}{3} x \begin{Bmatrix} \operatorname{cth} x \\ \operatorname{th} x \end{Bmatrix} - \frac{2}{3} \ln \begin{Bmatrix} |\operatorname{sh} x| \\ \operatorname{ch} x \end{Bmatrix}.$$

1.4.25. Интегралы вида $\int R\left(x^p, \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}, a + b \begin{Bmatrix} \operatorname{sh} x \\ \operatorname{ch} x \end{Bmatrix}\right) dx$.

1. $\int \frac{x^p \operatorname{ch} x\,dx}{(a + b \operatorname{sh} x)^q} = -\frac{x^p}{(q-1)\,b\,(a + b \operatorname{sh} x)^{q-1}} + \frac{p}{(q-1)\,b} \int \frac{x^{p-1}\,dx}{(a + b \operatorname{sh} x)^{q-1}}.$

2. $\int \frac{x^p \operatorname{sh} x\,dx}{(a + b \operatorname{ch} x)^q} = -\frac{x^p}{(q-1)\,b\,(a + b \operatorname{ch} x)^{q-1}} + \frac{p}{(q-1)\,b} \int \frac{x^{p-1}\,dx}{(a + b \operatorname{ch} x)^{q-1}}.$

3. $\int \frac{x\,dx}{1 \pm \operatorname{ch} x} = x \begin{Bmatrix} \operatorname{th}(x/2) \\ \operatorname{cth}(x/2) \end{Bmatrix} - 2 \ln \begin{Bmatrix} \operatorname{ch}(x/2) \\ |\operatorname{sh}(x/2)| \end{Bmatrix}.$

4. $\int \frac{x \operatorname{sh} x\,dx}{(1 \pm \operatorname{ch} x)^2} = -\frac{x}{\operatorname{ch} x \pm 1} \pm \begin{Bmatrix} \operatorname{th}(x/2) \\ \operatorname{cth}(x/2) \end{Bmatrix}.$

1.4.26. Интегралы вида $\int (bx+c)^{\pm n} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx$.

1. $\int (bx+c)^n \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx =$

$$= \frac{b^n}{a^{n+1}} \left(\operatorname{ch} \frac{ac}{b} \int t^n \begin{Bmatrix} \operatorname{sh} t \\ \operatorname{ch} t \end{Bmatrix} dt - \operatorname{sh} \frac{ac}{b} \int t^n \begin{Bmatrix} \operatorname{ch} t \\ \operatorname{sh} t \end{Bmatrix} dt \right) \quad \left[\frac{b}{a} t = bx + c \right].$$

2. $\int (bx+c) \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = \frac{1}{a} (bx+c) \begin{Bmatrix} \operatorname{ch} ax \\ \operatorname{sh} ax \end{Bmatrix} - \frac{b}{a^2} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix}.$

3. $\int (bx+c)^2 \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = \frac{1}{a} \left[(bx+c)^2 + \frac{2b^2}{a^2} \right] \begin{Bmatrix} \operatorname{ch} ax \\ \operatorname{sh} ax \end{Bmatrix} - \frac{2b\,(bx+c)}{a^2} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix}.$

4. $\int (bx+c)^3 \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx =$

$$= \frac{bx+c}{a} \left[(bx+c)^2 + \frac{6b^2}{a^2} \right] \begin{Bmatrix} \operatorname{ch} ax \\ \operatorname{sh} ax \end{Bmatrix} - \frac{3b}{a^2} \left[(bx+c)^2 + \frac{2b^2}{a^2} \right] \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix}.$$

5. $\int \frac{1}{(bx+c)^n} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx =$

$$= \frac{a^{n-1}}{b^n} \left(\operatorname{ch} \frac{ac}{b} \int \frac{1}{t^n} \begin{Bmatrix} \operatorname{sh} t \\ \operatorname{ch} t \end{Bmatrix} dt - \operatorname{sh} \frac{ac}{b} \int \frac{1}{t^n} \begin{Bmatrix} \operatorname{ch} t \\ \operatorname{sh} t \end{Bmatrix} dt \right) \quad \left[\frac{b}{a} t = bx + c \right].$$

6. $\int \frac{1}{bx+c} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = \frac{1}{b} \operatorname{ch} \frac{ac}{b} \begin{Bmatrix} \operatorname{shi}(ax + ac/b) \\ \operatorname{chi}(ax + ac/b) \end{Bmatrix} - \frac{1}{b} \operatorname{sh} \frac{ac}{b} \begin{Bmatrix} \operatorname{chi}(ax + ac/b) \\ \operatorname{shi}(ax + ac/b) \end{Bmatrix} =$

$$= \frac{1}{2b} e^{-ac/b} \operatorname{Ei}(ax + ac/b) \mp \frac{1}{2b} e^{ac/b} \operatorname{Ei}\left(-ax - \frac{ac}{b}\right).$$

7. $$\int \frac{1}{(bx+c)^2}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} dx = -\frac{1}{b(bx+c)}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} + \frac{a}{b}\int \frac{1}{bx+c}\begin{Bmatrix}\operatorname{ch} ax\\ \operatorname{sh} ax\end{Bmatrix} dx.$$

8. $$\int \frac{1}{(bx+c)^3}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} dx = -\frac{1}{2b(bx+c)^2}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} - \frac{a}{2b^2(bx+c)}\begin{Bmatrix}\operatorname{ch} ax\\ \operatorname{sh} ax\end{Bmatrix} + \frac{a^2}{2b^2}\int \frac{1}{bx+c}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} dx.$$

1.4.27. Интегралы вида $\int R(x,\ e^{ax},\ \operatorname{sh} bx,\ \operatorname{ch} bx)\,dx$.

1. $$\int e^{ax}\begin{Bmatrix}\operatorname{sh}(bx+c)\\ \operatorname{ch}(bx+c)\end{Bmatrix} dx = \frac{e^{ax}}{a^2-b^2}\left[a\begin{Bmatrix}\operatorname{sh}(bx+c)\\ \operatorname{ch}(bx+c)\end{Bmatrix} - b\begin{Bmatrix}\operatorname{ch}(bx+c)\\ \operatorname{sh}(bx+c)\end{Bmatrix}\right].$$

2. $$\int e^{ax}\begin{Bmatrix}\operatorname{sh}(ax+c)\\ \operatorname{ch}(ax+c)\end{Bmatrix} dx = \mp\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}.$$

3. $$\int e^{-ax}\begin{Bmatrix}\operatorname{sh}(ax+c)\\ \operatorname{ch}(ax+c)\end{Bmatrix} dx = \frac{1}{2}xe^{c} \pm \frac{1}{4a}e^{-(2ax+c)}.$$

4. $$\int x^p e^{ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = \frac{1}{2}\int x^p e^{(a+b)x}\,dx \mp \frac{1}{2}\int x^p e^{(a-b)x}\,dx$$ [см. 1.3.2].

5. $$\int\limits_0^x x^{\alpha-1}e^{-ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = \frac{1}{2}\left[(a-b)^{-\alpha}\gamma(\alpha,\ ax-bx) \mp (a+b)^{-\alpha}\gamma(\alpha,\ ax+bx)\right]$$ $\left[\operatorname{Re}\alpha > -\delta,\ \delta = \begin{Bmatrix}1\\ 0\end{Bmatrix}\right]$.

6. $$\int\limits_x^\infty x^{\alpha-1}e^{-ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = \frac{1}{2}\left[(a-b)^{-\alpha}\Gamma(\alpha,\ ax-bx) \mp (a+b)^{-\alpha}\Gamma(\alpha,\ ax+bx)\right]$$ $[a > b > 0]$.

7. $$\int xe^{ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} - \left(bx - \frac{2ab}{a^2-b^2}\right)\begin{Bmatrix}\operatorname{ch} bx\\ \operatorname{sh} bx\end{Bmatrix}\right].$$

8. $$\int x^2e^{ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = \frac{e^{ax}}{a^2-b^2}\left[ax^2 - \frac{2(a^2+b^2)}{a^2-b^2}x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2}\right]\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} - \frac{e^{ax}}{a^2-b^2}\left[bx^2 - \frac{4ab}{a^2-b^2}x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2}\right]\begin{Bmatrix}\operatorname{ch} bx\\ \operatorname{sh} bx\end{Bmatrix}.$$

9. $$\int xe^{ax}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} dx = \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right) \mp \frac{x^2}{4}.$$

10. $$\int x^2e^{ax}\begin{Bmatrix}\operatorname{sh} ax\\ \operatorname{ch} ax\end{Bmatrix} dx = \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right) \mp \frac{x^3}{6}.$$

11. $$\int \frac{1}{x}e^{ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = \frac{1}{2}\left[\operatorname{Ei}(ax+bx) \mp \operatorname{Ei}(ax-bx)\right]$$ $[a^2 \neq b^2]$.

12. $$\int\limits_x^\infty \frac{1}{x}e^{-ax}\begin{Bmatrix}\operatorname{sh} bx\\ \operatorname{ch} bx\end{Bmatrix} dx = -\frac{1}{2}\left[\operatorname{Ei}(bx-ax) \mp \operatorname{Ei}(-ax-bx)\right]$$ $[a > |b|]$.

13. $\int\limits_0^x \frac{1}{x} e^{-ax} \operatorname{sh} bx\, dx = \frac{1}{2} \ln \frac{a+b}{a-b} + \frac{1}{2} [\operatorname{Ei}(bx-ax) - \operatorname{Ei}(-ax-bx)]$ $[a > |b|]$.

14. $\int \frac{1}{x} e^{ax} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = \frac{1}{2} [\operatorname{Ei}(2ax) \mp \ln x]$.

15. $\int\limits_0^x \frac{1}{x} e^{\pm x} \operatorname{sh} x\, dx = \pm \frac{1}{2} [\operatorname{Ei}(\pm 2x) - (C + \ln 2x)]$.

16. $\int \frac{1}{x} e^{-ax} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = \frac{1}{2} [\ln x \mp \operatorname{Ei}(-2ax)]$.

17. $\int \frac{1}{x^2} e^{ax} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = a \operatorname{Ei}(2ax) - \frac{1}{2x} (e^{2ax} \mp 1)$.

18. $\int \frac{1}{x^2} e^{-ax} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix} dx = -\frac{1}{2x} \pm \left[\frac{1}{2x} e^{-2ax} + a \operatorname{Ei}(-2ax)\right]$.

19. $\int \frac{x\, dx}{\operatorname{ch}^2 ax} = \frac{x}{a} \operatorname{th} ax - \frac{1}{a^2} \ln |\operatorname{ch} ax|$.

20. $\int\limits_x^\infty \frac{1}{x} \begin{Bmatrix} \operatorname{sh} ax \\ \operatorname{ch} ax \end{Bmatrix}^{-1} dx = -2 \sum_{k=0}^{\infty} (\pm 1)^k \operatorname{Ei}[-(2k+1)ax]$.

21. $\int\limits_0^x \frac{x\, dx}{\cos t - \operatorname{ch} ax} = \frac{2}{a^2 \sin t} \left[L\left(\theta - \frac{t}{2}\right) - L\left(\theta + \frac{t}{2}\right) + 2L\left(\frac{t}{2}\right)\right]$

$\left[\theta = \operatorname{arctg}\left(\operatorname{th} \frac{ax}{2} \operatorname{ctg} \frac{t}{2}\right);\quad t \neq 2\pi k;\quad k = 0, \pm 1, \pm 2, \ldots\right]$.

22. $\int\limits_0^x \frac{x \operatorname{ch}(ax/2)}{\cos t - \operatorname{ch} ax} dx = \frac{2}{a^2 \sin(t/2)} \left[L\left(\frac{\theta - t}{4}\right) - L\left(\frac{\theta + t}{4}\right) - \right.$

$\left. - L\left(\pi - \frac{\psi + t}{4}\right) - L\left(\frac{\psi - t}{4}\right) + 2L\left(\frac{t}{4}\right) + 2L\left(\frac{2\pi - t}{4}\right)\right]$

$\left[\theta = 4 \operatorname{arctg}\left(\operatorname{th} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right),\quad \psi = 4 \operatorname{arctg}\left(\operatorname{cth} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right);\quad t \neq 2\pi k;\quad k = 0, \pm 1, \pm 2, \ldots\right]$.

1.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1.5.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x)\, dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

1. $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x)\, dx =$

$$= \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}, \frac{1-t^2}{2t}\right) \frac{2\, dt}{1+t^2} \qquad \left[t = \operatorname{tg} \frac{x}{2}\right].$$

Если

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

то

2. $\int R(\sin x, \cos x)\,dx = -\int R\left(\sqrt{1-t^2}, t\right)\frac{dt}{\sqrt{1-t^2}}$ $[t=\cos x]$.

Если

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

то

3. $\int R(\sin x, \cos x)\,dx = \int R\left(t, \sqrt{1-t^2}\right)\frac{dt}{\sqrt{1-t^2}}$ $[t=\sin x]$.

Если

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

то

4. $\int R(\sin x, \cos x) = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right)\frac{dt}{1+t^2}$ $[t=\operatorname{tg} x]$.

1.5.2. Интегралы вида $\int \sin^p x\,dx$.

1. $\int \sin^p x\,dx = -\frac{1}{p}\sin^{p-1}x\cos x + \frac{p-1}{p}\int \sin^{p-2}x\,dx.$

2. $\int \sin^{2n} x\,dx =$

$$= -\frac{\cos x}{2n}\left[\sin^{2n-1}x + \sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)}\sin^{2n-2k-1}x\right] + \frac{(2n-1)!!}{2^n n!}x.$$

3. $$= \frac{1}{2^{2n}}\binom{2n}{n}x + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{\sin(2n-2k)x}{2n-2k}.$$

4. $\int \sin^{2n+1} x\,dx =$

$$= -\frac{\cos x}{2n+1}\left[\sin^{2n}x + \sum_{k=0}^{n-1}\frac{2^{k+1}n(n-1)\ldots(n-k)}{(2n-1)(2n-3)\ldots(2n-2k-1)}\sin^{2n-2k-2}x\right].$$

5. $$= (-1)^{n+1}\frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\frac{\cos(2n-2k+1)x}{2n-2k+1}.$$

6. $$= \sum_{k=0}^{n}\frac{(-1)^{k+1}}{2k+1}\binom{n}{k}\cos^{2k+1}x.$$

7. $\int \sin x\,dx = -\cos x.$

8. $\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{1}{2}x = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x.$

9. $\int \sin^3 x\,dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x = \frac{1}{3}\cos^3 x - \cos x.$

10. $\int \sin^4 x\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x =$

$$= -\frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x.$$

11. $\int \sin^5 x\,dx = -\frac{5}{8}\cos x + \frac{5}{48}\cos 3x - \frac{1}{80}\cos 5x =$

$$= -\frac{1}{5}\sin^4 x\cos x + \frac{4}{15}\cos^3 x - \frac{4}{5}\cos x.$$

12. $\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x =$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x.$$

13. $\int \frac{dx}{\sin^p x} = -\frac{\cos x}{(p-1)\sin^{p-1} x} + \frac{p-2}{p-1}\int \frac{dx}{\sin^{p-2} x}.$

14. $\int \frac{dx}{\sin^{2n} x} =$

$$= -\frac{\cos x}{2n-1}\left[\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\operatorname{cosec}^{2n-2k-1} x\right].$$

15. $\int \frac{dx}{\sin^{2n+1} x} =$

$$= -\frac{\cos x}{2n}\left[\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k (n-1)(n-2)\dots(n-k)}\operatorname{cosec}^{2n-2k} x\right] +$$

$$+ \frac{(2n-1)!!}{2^n n!}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

16. $\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right| = \ln\left|\frac{1-\cos x}{\sin x}\right| = \ln\left|\frac{\sin x}{1+\cos x}\right| = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|.$

17. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$

18. $\int \frac{dx}{\sin^3 x} = -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

19. $\int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3\sin^3 x} - \frac{2}{3}\operatorname{ctg} x = -\frac{1}{3}\operatorname{ctg}^3 x - \operatorname{ctg} x.$

20. $\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{8}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

21. $\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5\sin^5 x} - \frac{4}{15}\operatorname{ctg}^3 x - \frac{4}{5}\operatorname{ctg} x = -\frac{1}{5}\operatorname{ctg}^5 x - \frac{2}{3}\operatorname{ctg}^3 x - \operatorname{ctg} x.$

1.5.3. Интегралы вида $\int \cos^p x\,dx$.

1. $\int \cos^p x\,dx = \frac{1}{p}\sin x\cos^{p-1} x + \frac{p-1}{p}\int \cos^{p-2} x\,dx.$

2. $\int \cos^{2n} x\,dx =$

$$= \frac{\sin x}{2n}\left[\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k (n-1)(n-2)\dots(n-k)} \cos^{2n-2k-1} x\right] + \frac{(2n-1)!!}{2^n n!}\,x;$$

3. $$= \frac{1}{2^{2n}}\binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}.$$

4. $\int \cos^{2n+1} x\,dx =$

$$= \frac{\sin x}{2n+1}\left[\cos^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1)\dots(n-k)}{(2n-1)(2n-3)\dots(2n-2k-1)} \cos^{2n-2k-2} x\right].$$

5. $$= \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}.$$

6. $$= \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \sin^{2k+1} x.$$

7. $\int \cos x\,dx = \sin x.$

8. $\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\sin x \cos x + \frac{1}{2}x.$

9. $\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x = \sin x - \frac{1}{3}\sin^3 x.$

10. $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x =$

$$= \frac{3}{8}x + \frac{3}{8}\sin x \cos x + \frac{1}{4}\sin x \cos^3 x.$$

11. $\int \cos^5 x\,dx = \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x =$

$$= \frac{4}{5}\sin x - \frac{4}{15}\sin^3 x + \frac{1}{5}\cos^4 x \sin x.$$

12. $\int \cos^6 x\,dx = \frac{5}{16}x + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x =$

$$= \frac{5}{16}x + \frac{5}{16}\sin x \cos x + \frac{5}{24}\sin x \cos^3 x + \frac{1}{6}\sin x \cos^5 x.$$

13. $$\int \frac{dx}{\cos^p x} = \frac{\sin x}{(p-1)\cos^{p-1} x} + \frac{p-2}{p-1}\int \frac{dx}{\cos^{p-2} x}.$$

14. $$\int \frac{dx}{\cos^{2n} x} = \frac{\sin x}{2n-1}\left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \sec^{2n-2k-1} x\right].$$

15. $$\int \frac{dx}{\cos^{2n+1} x} = \frac{\sin x}{2n}\left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)} \sec^{2n-2k} x\right] + \frac{(2n-1)!!}{2^n n!} \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$$

16. $$\int \frac{dx}{\cos x} = \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| = \ln\left|\operatorname{ctg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|.$$

17. $$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

18. $$\int \frac{dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$$

19. $$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3\cos^3 x} + \frac{2}{3}\operatorname{tg} x = \frac{1}{3}\operatorname{tg}^3 x + \operatorname{tg} x.$$

20. $$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3}{8}\frac{\sin x}{\cos^2 x} + \frac{3}{8}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|.$$

21. $$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5\cos^5 x} + \frac{4}{15}\operatorname{tg}^3 x + \frac{4}{5}\operatorname{tg} x = \frac{1}{5}\operatorname{tg}^5 x + \frac{2}{3}\cdot\operatorname{tg}^3 x + \operatorname{tg} x.$$

1.5.4. Интегралы вида $\int \sin^p x \cos^q x\, dx$.

1. $$\int \sin^p x \cos^q x\, dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x \cos^{q+2} x\, dx.$$

2. $$= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx.$$

3. $$= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1}\int \sin^{p+2} x \cos^q x\, dx.$$

4. $$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1}\int \sin^{p+2} x \cos^{q-2} x\, dx.$$

5. $$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sin^p x \cos^{q-2} x\, dx.$$

6. $$= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \sin^p x \cos^{q+2} x\, dx.$$

7. $$= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q}\left(\sin^2 x - \frac{q-1}{p+q-2}\right) + \frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int \sin^{p-2} x \cos^{q-2} x\, dx.$$

8. $\int \sin^p x \cos^{2n} dx =$

$$= \frac{\sin^{p+1} x}{2n+p}\left[\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)\cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\right] +$$

$$+ \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)} \int \sin^p x\, dx \qquad [p \neq -2, -4, \ldots, -2n].$$

9. $\int \sin^p x \cos^{2n+1} x\, dx =$

$$= \frac{\sin^{p+1} x}{2n+p+1}\left[\cos^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1)\cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)}\right]$$

$$[p \neq -1, -3, \ldots, -(2n+1)].$$

10. $\int \cos^p x \sin^{2n} x\, dx =$

$$= -\frac{\cos^{p+1} x}{2n+p}\left[\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)\sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\right] +$$

$$+ \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)} \int \cos^p x\, dx \qquad [p \neq -2, -4, \ldots, -2n].$$

11. $\int \cos^p x \sin^{2n+1} x\, dx =$

$$= -\frac{\cos^{p+1} x}{2n+p+1}\left[\sin^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1)\sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)}\right]$$

$$[p \neq -1, -3, \ldots, -(2n+1)]$$

12. $\int \sin x \cos^p x\, dx = -\frac{1}{p+1}\cos^{p+1} x \qquad [p \neq -1].$

13. $\int \sin^p x \cos x\, dx = \frac{1}{p+1}\sin^{p+1} x \qquad [p \neq -1].$

14. $\int \sin^2 x \cos^2 x\, dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right).$

15. $\int \sin^2 x \cos^3 x\, dx = -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right) =$

$$= \frac{\sin^3 x}{5}\left(\cos^2 x + \frac{2}{3}\right) = \frac{\sin^3 x}{5}\left(\frac{5}{3} - \sin^2 x\right).$$

16. $\int \sin^2 x \cos^4 x\, dx = \frac{x}{16} + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x.$

17. $\int \sin^3 x \cos^2 x\, dx = \frac{1}{16}\left(\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x - 2\cos x\right) = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x.$

18. $\int \sin^3 x \cos^3 x\, dx = \frac{1}{32}\left(\frac{1}{6}\cos 6x - \frac{3}{2}\cos 2x\right).$

19. $\int \sin^3 x \cos^4 x\, dx = \frac{1}{7}\cos^3 x\left(-\frac{2}{5} - \frac{3}{5}\sin^2 x + \sin^4 x\right).$

20. $\int \sin^4 x \cos^2 x \, dx = \frac{1}{16} x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x.$

21. $\int \sin^4 x \cos^3 x \, dx = \frac{1}{7} \sin^3 x \left(\frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right).$

22. $\int \sin^4 x \cos^4 x \, dx = \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x.$

23. $\int \sin ax \sin bx \sin cx \, dx =$

$$= -\frac{1}{4} \left[\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right].$$

24. $\int \sin ax \cos bx \cos cx \, dx =$

$$= -\frac{1}{4} \left[\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right].$$

25. $\int \cos ax \sin bx \sin cx \, dx =$

$$= \frac{1}{4} \left[\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right].$$

26. $\int \cos ax \cos bx \cos cx \, dx =$

$$= \frac{1}{4} \left[\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right].$$

1.55. Интегралы вида $\int \frac{\sin^p x}{\cos^q x} dx$

1. $\int \frac{\sin^p x}{\cos^q x} dx = -\frac{\sin^{p-1} x}{(p-q)\cos^{q-1} x} + \frac{p-1}{p-q} \int \frac{\sin^{p-2} x}{\cos^q x} dx.$

2. $\quad = \frac{\sin^{p+1} x}{(q-1)\cos^{q-1} x} - \frac{p-q+2}{q-1} \int \frac{\sin^p x}{\cos^{q-2} x} dx.$

3. $\quad = \frac{\sin^{p-1} x}{(q-1)\cos^{q-1} x} - \frac{p-1}{q-1} \int \frac{\sin^{p-2} x}{\cos^{q-2} x} dx.$

4. $\int \frac{\sin^p x}{\cos^{2n+1} x} dx = \frac{\sin^{p+1} x}{2n} \times$

$$\times \left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\ldots(2n-p-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)} \sec^{2n-2k} x \right] +$$

$$+ \frac{(2n-p-1)(2n-p-3)\ldots(3-p)(1-p)}{2^n n!} \int \frac{\sin^p x}{\cos x} dx.$$

5. $\int \frac{\sin^p x}{\cos^{2n} x} dx = \frac{\sin^{p+1} x}{2n-1} \times$

$$\times \left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\ldots(2n-p-2k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \sec^{2n-2k-1} x \right] +$$

$$+ \frac{(2n-p-2)(2n-p-4)\ldots(-p+2)(-p)}{(2n-1)!!} \int \sin^p x \, dx.$$

6. $\displaystyle\int \frac{\sin^{2m+1} x}{\cos^{2n+1} x}\,dx = \frac{1}{2}\sum_{\substack{k=0\\k\neq n}}^{m} (-1)^k \binom{m}{k} \frac{\cos^{2k-2n} x}{n-k} + (-1)^{n+1}\binom{m}{n} \ln|\cos x|$ $[n \leqslant m]$.

7. $\displaystyle = \frac{1}{2}\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\cos^{2k-2n} x}{n-k}$ $[n > m]$.

8. $\displaystyle\int \frac{\sin^{2m+1} x}{\cos^{2n} x}\,dx = \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \frac{\cos^{2k-2n+1} x}{2k-2n+1}.$

9. $\displaystyle\int \frac{\sin x}{\cos^q x}\,dx = \frac{1}{(q-1)\cos^{q-1} x}.$

10. $\displaystyle\int \frac{\sin^p x}{\cos^{p+2} x}\,dx = \frac{1}{p+1}\operatorname{tg}^{p+1} x.$

11. $\displaystyle\int \frac{\sin^{2n+1} x}{\cos x}\,dx = -\sum_{k=1}^{n} \frac{\sin^{2k} x}{2k} - \ln|\cos x|.$

12. $\displaystyle = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k}\binom{n}{k}\cos^{2k} x - \ln|\cos x|.$

13. $\displaystyle\int \frac{\sin^{2n} x}{\cos x}\,dx = -\sum_{k=1}^{n} \frac{\sin^{2k-1} x}{2k-1} + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

14. $\displaystyle\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x|.$

15. $\displaystyle\int \frac{\sin^2 x}{\cos x}\,dx = -\sin x + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

16. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx = -\frac{\sin^2 x}{2} - \ln|\cos x| = \frac{1}{2}\cos^2 x - \ln|\cos x|.$

17. $\displaystyle\int \frac{\sin^4 x}{\cos x}\,dx = -\frac{1}{3}\sin^3 x - \sin x + \ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|.$

18. $\displaystyle\int \frac{\sin^p x}{\cos^2 x}\,dx = \frac{\sin^{p-1} x}{\cos x} - (p-1)\int \sin^{p-2} x\,dx.$

19. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \frac{1}{\cos x}.$

20. $\displaystyle\int \frac{\sin^2 x}{\cos^2 x}\,dx = \operatorname{tg} x - x.$

21. $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx = \cos x + \frac{1}{\cos x}.$

22. $\displaystyle\int \frac{\sin^4 x}{\cos^2 x}\,dx = \operatorname{tg} x + \frac{1}{2}\sin x\cos x - \frac{3}{2}x.$

23. $\int \frac{\sin x}{\cos^3 x} dx = \frac{1}{2\cos^2 x} = \frac{1}{2}\operatorname{tg}^2 x.$

24. $\int \frac{\sin^2 x}{\cos^3 x} dx = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

25. $\int \frac{\sin^3 x}{\cos^3 x} dx = \frac{1}{2\cos^2 x} + \ln|\cos x|.$

26. $\int \frac{\sin^4 x}{\cos^3 x} dx = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|.$

27. $\int \frac{\sin x}{\cos^4 x} dx = \frac{1}{3\cos^3 x}.$ 28. $\int \frac{\sin^2 x}{\cos^4 x} dx = \frac{1}{3}\operatorname{tg}^3 x.$

29. $\int \frac{\sin^3 x}{\cos^4 x} dx = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x}.$

30. $\int \frac{\sin^4 x}{\cos^4 x} dx = \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x.$

1.5.6. Интегралы вида $\int \frac{\cos^q x}{\sin^p x} dx.$

1. $\int \frac{\cos^q x}{\sin^p x} dx = \frac{\cos^{q-1} x}{(q-p)\sin^{p-1} x} + \frac{q-1}{q-p}\int \frac{\cos^{q-2} x}{\sin^p x} dx.$

2. $= -\frac{\cos^{q+1} x}{(p-1)\sin^{p-1} x} - \frac{q-p+2}{p-1}\int \frac{\cos^q x}{\sin^{p-2} x} dx.$

3. $= -\frac{\cos^{q-1} x}{(p-1)\sin^{p-1} x} - \frac{q-1}{p-1}\int \frac{\cos^{q-2} x}{\sin^{p-2} x}.$

4. $$\int \frac{\cos^p x}{\sin^{2n+1} x} dx = -\frac{\cos^{p+1} x}{2n}\Bigg[\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\ldots(2n-p-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)}\operatorname{cosec}^{2n-2k} x\Bigg] + \frac{(2n-p-1)(2n-p-3)\ldots(3-p)(1-p)}{2^n n!}\int \frac{\cos^p x}{\sin x} dx.$$

5. $$\int \frac{\cos^p x}{\sin^{2n} x} dx = -\frac{\cos^{p+1} x}{2n-1}\Bigg[\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\ldots(2n-p-2k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\operatorname{cosec}^{2n-2k-1} x\Bigg] + \frac{(2n-p-2)(2n-p-4)\ldots(2-p)(-p)}{(2n-1)!!}\int \cos^p x\, dx.$$

6. $$\int \frac{\cos^{2n+1} x}{\sin^{2m+1} x} dx = \frac{1}{2}\sum_{\substack{k=0\\ k\neq m}}^{n} (-1)^{k+1}\binom{n}{k}\frac{\sin^{2k-2m}}{m-k} + (-1)^m\binom{n}{m}\ln|\sin x|$$

$[m \leqslant n].$

7. $$= \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{\sin^{2k-2m} x}{m-k} \qquad [m > n].$$

8. $$\int \frac{\cos^{2n+1} x}{\sin^{2m} x} dx = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\sin^{2k-2m+1} x}{2k-2m+1}.$$

9. $$\int \frac{\cos x}{\sin^p x} dx = -\frac{1}{(p-1)\sin^{p-1} x}.$$

10. $$\int \frac{\cos^q x}{\sin^{q+2} x} dx = -\frac{1}{q+1} \operatorname{ctg}^{q+1} x.$$

11. $$\int \frac{\cos^{2n+1} x}{\sin x} dx = \sum_{k=1}^{n} \frac{\cos^{2k} x}{2k} + \ln|\sin x|.$$

12. $$= \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} \sin^{2k} x + \ln|\sin x|.$$

13. $$\int \frac{\cos^{2n} x}{\sin x} dx = \sum_{k=1}^{n} \frac{\cos^{2k-1} x}{2k-1} + \ln\left|\operatorname{tg} \frac{x}{2}\right|.$$

14. $$\int \frac{\cos x}{\sin x} dx = \ln|\sin x|.$$

15. $$\int \frac{\cos^2 x}{\sin x} dx = \cos x + \ln\left|\operatorname{tg} \frac{x}{2}\right|.$$

16. $$\int \frac{\cos^3 x}{\sin x} dx = \frac{\cos^2 x}{2} + \ln|\sin x|.$$

17. $$\int \frac{\cos^4 x}{\sin x} dx = \frac{1}{3} \cos^3 x + \cos x + \ln\left|\operatorname{tg} \frac{x}{2}\right|.$$

18. $$\int \frac{\cos^p x}{\sin^2 x} dx = -\frac{\cos^{p-1} x}{\sin x} - (p-1) \int \cos^{p-2} x \, dx.$$

19. $$\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}.$$

20. $$\int \frac{\cos^2 x}{\sin^2 x} dx = -\operatorname{ctg} x - x.$$

21. $$\int \frac{\cos^3 x}{\sin^2 x} dx = -\sin x - \frac{1}{\sin x}.$$

22. $$\int \frac{\cos^4 x}{\sin^2 x} dx = -\operatorname{ctg} x - \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

23. $$\int \frac{\cos x}{\sin^3 x} dx = -\frac{1}{2\sin^2 x}.$$

24. $$\int \frac{\cos^2 x}{\sin^3 x} dx = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2} \ln\left|\operatorname{tg} \frac{x}{2}\right|.$$

25. $$\int \frac{\cos^3 x}{\sin^3 x} dx = -\frac{1}{2\sin^2 x} - \ln|\sin x|.$$

26. $\int \frac{\cos^4 x}{\sin^3 x}\,dx = -\frac{1}{2}\frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

27. $\int \frac{\cos x}{\sin^4 x}\,dx = -\frac{1}{3\sin^3 x}.$ 28. $\int \frac{\cos^2 x}{\sin^4 x}\,dx = -\frac{1}{3}\operatorname{ctg}^3 x.$

29. $\int \frac{\cos^3 x}{\sin^4 x}\,dx = \frac{1}{\sin x} - \frac{1}{3\sin^3 x}.$

30. $\int \frac{\cos^4 x}{\sin^4 x}\,dx = -\frac{1}{3}\operatorname{ctg}^3 x + \operatorname{ctg} x + x.$

1.5.7. Интегралы вида $\int \frac{dx}{\sin^p x \cos^q x}$.

1. $\int \frac{dx}{\sin^p x\cos^q x} = \frac{-1}{(p-1)\sin^{p-1}x\cos^{q-1}x} + \frac{p+q-2}{p-1}\int\frac{dx}{\sin^{p-2}x\cos^q x}$

2. $= \frac{1}{(q-1)\sin^{p-1}x\cos^{q-1}x} + \frac{p+q-2}{q-1}\int\frac{dx}{\sin^p x\cos^{q-2}x}$

[см. также 1.5.4].

3. $\int \frac{dx}{\sin^{2m}x\cos^{2n}x} = \sum_{k=0}^{m+n-1}\binom{m+n-1}{k}\frac{\operatorname{tg}^{2k-2m+1}x}{2k-2m+1}.$

4. $\int \frac{dx}{\sin^{2m+1}x\cos^{2n+1}x} = \sum_{\substack{k=0\\k\neq m}}^{m+n}\binom{m+n}{k}\frac{\operatorname{tg}^{2k-2m}x}{2k-2m} + \binom{m+n}{m}\ln|\operatorname{tg}x|.$

5. $\int \frac{dx}{\sin^{2m+1}x\cos x} = -\sum_{k=1}^{m}\frac{1}{(2m-2k+2)\sin^{2m-2k+2}x} + \ln|\operatorname{tg}x|.$

6. $\int \frac{dx}{\sin^{2m}x\cos x} = -\sum_{k=1}^{m}\frac{1}{(2m-2k+1)\sin^{2m-2k+1}x} + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|.$

7. $\int \frac{dx}{\sin x\cos^{2m+1}x} = \sum_{k=1}^{m}\frac{1}{(2m-2k+2)\cos^{2m-2k+2}x} + \ln|\operatorname{tg}x|.$

8. $\int \frac{dx}{\sin x\cos^{2m}x} = \sum_{k=1}^{m}\frac{1}{(2m-2k+1)\cos^{2m-2k+1}x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

9. $\int \frac{dx}{\sin x\cos x} = \ln|\operatorname{tg}x|.$ 10. $\int \frac{dx}{\sin x\cos^2 x} = \frac{1}{\cos x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

11. $\int \frac{dx}{\sin x\cos^3 x} = \frac{1}{2\cos^2 x} + \ln|\operatorname{tg}x|.$

12. $\int \frac{dx}{\sin x\cos^4 x} = \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

13. $\int \frac{dx}{\sin^2 x\cos x} = \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| - \operatorname{cosec}x.$

14. $\int \frac{dx}{\sin^2 x \cos^2 x} = -2 \operatorname{ctg} 2x.$

15. $\int \frac{dx}{\sin^2 x \cos^3 x} = \left(\frac{1}{2\cos^2 x} - \frac{3}{2}\right)\frac{1}{\sin x} + \frac{3}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$

16. $\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3\sin x \cos^3 x} - \frac{8}{3}\operatorname{ctg} 2x.$

17. $\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2\sin^2 x} + \ln|\operatorname{tg} x|.$

18. $\int \frac{dx}{\sin^3 x \cos^2 x} = -\frac{1}{\cos x}\left(\frac{1}{2\sin^2 x} - \frac{3}{2}\right) + \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

19. $\int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2\cos 2x}{\sin^2 2x} + 2\ln|\operatorname{tg} x|.$

20. $\int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3\cos^3 x} - \frac{\cos x}{2\sin^2 x} + \frac{5}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

21. $\int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$

22. $\int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3\cos x \sin^3 x} - \frac{8}{3}\operatorname{ctg} 2x.$

23. $\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3\sin^3 x} + \frac{\sin x}{2\cos^2 x} + \frac{5}{2}\ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$

24. $\int \frac{dx}{\sin^4 x \cos^4 x} = -8\operatorname{ctg} 2x - \frac{8}{3}\operatorname{ctg}^3 2x.$

1.5.8. Интегралы вида $\int \operatorname{tg}^p x\, dx$, $\int \operatorname{ctg}^p x\, dx$.

1. $\int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^p dx = \pm\frac{1}{p-1}\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{p-1} - \int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{p-2} dx.$

2. $\int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{2n} dx = \mp \sum_{k=1}^{n} (-1)^k \frac{1}{2n-2k+1}\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{2n-2k+1} + (-1)^n x.$

3. $\int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{2n+1} dx = \mp \sum_{k=1}^{n} (-1)^k \frac{1}{2n-2k+2}\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{2n-2k+2} \mp$

$\mp (-1)^n \ln\begin{Bmatrix}|\cos x|\\ |\sin x|\end{Bmatrix}.$

4. $= \pm \sum_{k=1}^{n} \frac{(-1)^{n+k}}{2k}\binom{n}{k}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}^{-2k} \mp (-1)^n \ln\begin{Bmatrix}|\cos x|\\ |\sin x|\end{Bmatrix}.$

5. $\int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix} dx = \mp \ln\begin{Bmatrix}|\cos x|\\ |\sin x|\end{Bmatrix}.$ 6. $\int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^2 dx = \pm\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix} - x.$

7. $\int \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^3 dx = \pm\frac{1}{2}\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^2 \pm \ln\begin{Bmatrix}|\cos x|\\ |\sin x|\end{Bmatrix}.$

8. $\int \sqrt{\operatorname{tg} x}\, dx = \frac{1}{\sqrt{2}} [\ln (\sin x + \cos x - \sqrt{\sin 2x}) + \arcsin (\sin x - \cos x)]$.

9. $= \frac{1}{\sqrt{2}} \left[\frac{1}{2} \ln \frac{1 - \sqrt{2 \operatorname{tg} x} + \operatorname{tg} x}{1 + \sqrt{2 \operatorname{tg} x} + \operatorname{tg} x} + \operatorname{arctg} \frac{\sqrt{2 \operatorname{tg} x}}{1 - \operatorname{tg} x} \right]$.

1.5.9. Интегралы вида $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x)\, dx$.

1. $\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n}\, dx =$

$$= \frac{(Bc - Cb) + (Ac - Ca)\cos x - (Ab - Ba)\sin x}{(n-1)(a^2 - b^2 - c^2)(a + b\cos x + c\sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \times$$

$$\times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba)\cos x - (Ac - Ca)\sin x]}{(a + b\cos x + c\sin x)^{n-1}}\, dx$$

$[n \geqslant 2,\ a^2 \neq b^2 + c^2]$.

2. $= \frac{Cb - Bc + Ca\cos x - Ba\sin x}{(n-1)a(a + b\cos x + c\sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right) \times$

$$\times (-c\cos x + b\sin x) \frac{(n-1)!}{(2n-1)!!} \sum_{k=0}^{n-1} \frac{(2n - 2k - 3)!!}{(n - k - 1)!\, a^k} \frac{1}{(a + b\cos x + c\sin x)^{n-k}}$$

$[n \geqslant 2,\ a^2 = b^2 + c^2]$.

3. $\int \frac{A + B\cos x + C\sin x}{a + b\cos x + c\sin x}\, dx = \frac{Bc - Cb}{b^2 + c^2} \ln (a + b\cos x + c\sin x) +$

$$+ \frac{Bb + Cc}{b^2 + c^2} x + \left(A - \frac{Bb + Cc}{b^2 + c^2} a\right) \int \frac{dx}{a + b\cos x + c\sin x}.$$

4. $\int \frac{dx}{(a + b\cos x + c\sin x)^n} = \int \frac{d(x - \alpha)}{[a + r\cos (x - \alpha)]^n}$ $[b = r\cos\alpha,\ c = r\sin\alpha]$.

5. $\int \frac{dx}{a + b\cos x + c\sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \operatorname{arctg} \frac{(a - b)\operatorname{tg} \frac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}}$ $[a^2 > b^2 + c^2]$.

6. $= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln \frac{(a - b)\operatorname{tg} \frac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a - b)\operatorname{tg} \frac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}}$ $[a^2 < b^2 + c^2]$.

7. $= \frac{1}{c} \ln \left(a + c\operatorname{tg} \frac{x}{2}\right)$ $[a = b]$.

8. $= \frac{-2}{c + (a - b)\operatorname{tg} \frac{x}{2}}$ $[a^2 = b^2 + c^2]$.

9. $\int \frac{dx}{(a\cos x + b\sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}} \int \frac{dx}{\sin^n \left(x + \operatorname{arctg} \frac{a}{b}\right)}$ [см. 1.5.2.].

10. $\int \frac{A + B\cos x + C\sin x}{a\cos x + b\sin x}\, dx = \frac{Ba + Cb}{a^2 + b^2} x +$

$$+ \frac{Bb - Ca}{a^2 + b^2} \ln \sin \left(x + \operatorname{arctg} \frac{a}{b}\right) + \frac{A}{\sqrt{a^2 + b^2}} \ln \operatorname{tg} \left[\frac{1}{2}\left(x + \operatorname{arctg} \frac{a}{b}\right)\right].$$

11. $\int \frac{1}{a\cos x+b\sin x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}dx=\frac{1}{a^2+b^2}\left[\begin{Bmatrix}b\\ a\end{Bmatrix}x\mp\begin{Bmatrix}a\\ b\end{Bmatrix}\ln(a\cos x+b\sin x)\right].$

12. $\int \frac{dx}{(a\cos x+b\sin x)^2}=-\frac{1}{a^2+b^2}\cdot\frac{a\sin x-b\cos x}{a\cos x+b\sin x}.$

13. $\int \frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\operatorname{arctg}\frac{a\operatorname{tg}\frac{x}{2}+b}{\sqrt{a^2-b^2}}$ $[a^2>b^2]$.

14. $=\frac{1}{\sqrt{b^2-a^2}}\ln\frac{a\operatorname{tg}\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\operatorname{tg}\frac{x}{2}+b+\sqrt{b^2-a^2}}$ $[a^2<b^2]$.

15. $\int \frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\operatorname{arctg}\frac{\sqrt{a^2-b^2}\operatorname{tg}\frac{x}{2}}{a+b}$ $[a^2>b^2]$.

16. $=\frac{1}{\sqrt{b^2-a^2}}\ln\frac{\sqrt{b^2-a^2}\operatorname{tg}\frac{x}{2}+a+b}{\sqrt{b^2-a^2}\operatorname{tg}\frac{x}{2}-a-b}$ $[a^2<b^2]$.

17. $\int \frac{A+B\cos x+C\sin x}{(1\pm\sin x)^n}dx=-\frac{1}{2^{n-1}}\left[2C\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\operatorname{tg}^{2k+1}\left(\frac{\pi}{4}\mp\frac{x}{2}\right)}{2k+1}\pm\right.$

$\left.\pm(A\mp C)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\operatorname{tg}^{2k+1}\left(\frac{\pi}{4}\mp\frac{x}{2}\right)}{2k+1}\right]\mp\frac{B}{n-1}\frac{1}{(1\pm\sin x)^{n-1}}$ $[n\geq 2]$

18. $=\pm Cx+(A\mp C)\operatorname{tg}\left(\frac{\pi}{4}\mp\frac{x}{2}\right)\pm B\ln(1\pm\sin x)$ $[n=1]$.

19. $\int \frac{A+B\cos x+C\sin x}{(1\pm\cos x)^n}dx=\frac{1}{2^{n-1}}\left[2B\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\operatorname{tg}^{\pm(2k+1)}\frac{x}{2}}{2k+1}\pm\right.$

$\left.\pm(A\mp B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\operatorname{tg}^{\pm(2k+1)}\frac{x}{2}}{2k+1}\right]\pm\frac{C}{n-1}\frac{1}{(1\pm\cos x)^{n-1}}$ $[n\geq 2]$.

20. $=\pm Bx\pm(A\mp B)\operatorname{tg}^{\pm 1}\frac{x}{2}\mp C\ln(1\pm\cos x)$ $[n=1]$.

21. $\int \frac{A+B\cos x+C\sin x}{\sin x(a+b\sin x)}dx=$

$=\frac{A}{a}\ln\operatorname{tg}\frac{x}{2}-\frac{B}{a}\ln\frac{a+b\sin x}{\sin x}+\frac{Ca-Ab}{a}\int\frac{dx}{a+b\sin x}.$

22. $\int \frac{A+B\cos x+C\sin x}{\sin x(a+b\cos x)}dx=$

$=\frac{1}{a^2-b^2}\left[(Aa-Bb)\ln\operatorname{tg}\frac{x}{2}+(Ab-Ba)\ln\frac{a+b\cos x}{\sin x}\right]+C\int\frac{dx}{a+b\cos x}.$

23. $$\int \frac{A+B\cos x+C\sin x}{\sin x\,(1\pm\cos x)}\,dx=\frac{A\pm B}{2}\ln\operatorname{tg}\frac{x}{2}-\frac{B\mp A}{2}\,\frac{1}{1\pm\cos x}\pm C\left(\operatorname{tg}\frac{x}{2}\right)^{\pm 1}.$$

24. $$\int \frac{A+B\cos x+C\sin x}{\cos x\,(a+b\sin x)}\,dx=$$
$$=\frac{1}{a^2-b^2}\left[(Aa-Cb)\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)-(Ab-Ca)\ln\frac{a+b\sin x}{\cos x}\right]+B\int\frac{dx}{a+b\sin x}.$$

25. $$\int \frac{A+B\cos x+C\sin x}{\cos x\,(1\pm\sin x)}\,dx=$$
$$=\frac{A\pm C}{2}\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\mp\frac{A\mp C}{2}\cdot\frac{1}{(1\pm\sin x)}+B\operatorname{tg}\left(\frac{\pi}{4}\pm\frac{x}{2}\right).$$

26. $$\int \frac{A+B\cos x+C\sin x}{\cos x\,(a+b\cos x)}\,dx=$$
$$=\frac{A}{a}\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)+\frac{C}{a}\ln\frac{a+b\cos x}{\cos x}+\left(B-\frac{Ab}{a}\right)\int\frac{dx}{a+b\cos x}.$$

27. $$\int\frac{dx}{a+b\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2}=\frac{\pm\operatorname{sign}a}{\sqrt{a(a+b)}}\operatorname{arctg}\left(\sqrt{\frac{a+b}{a}}\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}\right)\qquad\left[\frac{b}{a}>-1\right];$$

28. $$=\pm\frac{\operatorname{sign}a}{\sqrt{-a(a+b)}}\operatorname{Arth}\left(\sqrt{-\frac{a+b}{a}}\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}\right)$$
$$\left[\frac{b}{a}<-1;\ \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2<-\frac{a}{b}\right].$$

29. $$=\pm\frac{\operatorname{sign}a}{\sqrt{-a(a+b)}}\operatorname{Arcth}\left(\sqrt{-\frac{a+b}{a}}\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}\right)$$
$$\left[\frac{b}{a}<-1;\ \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2>-\frac{a}{b}\right].$$

30. $$\int\frac{dx}{1+\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2}=\pm\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\sqrt{2}\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}\right).$$

31. $$\int\frac{dx}{1-\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2}=\pm\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}.$$

32. $$\int\frac{dx}{\left(a+b\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2\right)^2}=\frac{1}{2a(a+b)}\left[(2a+b)\int\frac{dx}{a+b\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2}\pm\frac{b\sin x\cos x}{a+b\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2}\right].$$

33. $$\int\frac{dx}{\left(1+\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2\right)^2}=\frac{1}{4}\left[\frac{3}{\sqrt{2}}\operatorname{arctg}\left(\sqrt{2}\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}\right)\pm\frac{\sin x\cos x}{1+\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2}\right].$$

34. $$\int\frac{dx}{(a+b\sin^2 x)^3}=\frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p\operatorname{tg}x)+\right.$$
$$\left.+\left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\operatorname{tg}x}{1+p^2\operatorname{tg}^2x}+\left(1-\frac{2}{p^2}-\frac{1}{p^2}\operatorname{tg}^2x\right)\frac{2p\operatorname{tg}x}{(1+p^2\operatorname{tg}^2x)^2}\right]$$
$$\left[p^2=1+\frac{b}{a}>0\right].$$

35. $$= \frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{Arth}(q \operatorname{tg} x)+\left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q \operatorname{tg} x}{1-q^2 \operatorname{tg}^2 x}+\left(1+\frac{2}{q^2}+\frac{1}{q^2}\operatorname{tg}^2 x\right)\frac{2q \operatorname{tg} x}{(1-q^2 \operatorname{tg}^2 x)^2}\right]$$

$\left[q^2=-1-\frac{b}{a}>0,\ \sin^2 x<-\frac{a}{b};\right.$ при $\sin^2 x>-\frac{a}{b}$ следует $\operatorname{Arth}(q \operatorname{tg} x)$ заменить на $\left.\operatorname{Arcth}(q \operatorname{tg} x)\right]$.

36. $$\int \frac{dx}{(a+b\cos^2 x)^3} = -\frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p \operatorname{ctg} x)+\left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p \operatorname{ctg} x}{1+p^2 \operatorname{ctg}^2 x}+\left(1-\frac{2}{p^2}-\frac{1}{p^2}\operatorname{ctg}^2 x\right)\frac{2p \operatorname{ctg} x}{(1+p^2 \operatorname{ctg}^2 x)^2}\right]$$

$\left[p^2=1+\frac{b}{a}>0\right]$.

37. $$= -\frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{Arth}(q \operatorname{ctg} x)+\left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q \operatorname{ctg} x}{1-q^2 \operatorname{ctg}^2 x}+\left(1+\frac{2}{q^2}+\frac{1}{q^2}\operatorname{ctg}^2 x\right)\frac{2q \operatorname{ctg} x}{(1-q^2 \operatorname{ctg}^2 x)^2}\right]$$

$\left[q^2=-1-\frac{b}{a}>0;\ \cos^2 x<-\frac{a}{b};\right.$ при $\cos^2 x>-\frac{a}{b}$ следует $\operatorname{Arth}(q \operatorname{ctg} x)$ заменить на $\left.\operatorname{Arcth}(q \operatorname{ctg} x)\right]$.

38. $$\int \frac{A+B\cos x+C\sin x}{(a_1+b_1\cos x+c_1\sin x)(a_2+b_2\cos x+c_2\sin x)}dx = A_0 \ln\frac{a_1+b_1\cos x+c_1\sin x}{a_2+b_2\cos x+c_2\sin x}+A_1\int\frac{dx}{a_1+b_1\cos x+c_1\sin x}+A_2\int\frac{dx}{a_2+b_2\cos x+c_2\sin x},$$

где

$$A_0=\frac{\begin{vmatrix} A & B & C\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2-\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2+\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2},$$

$$A_1=\frac{\begin{vmatrix} \begin{vmatrix} B & C\\ b_1 & c_1\end{vmatrix} & \begin{vmatrix} A & C\\ a_1 & c_1\end{vmatrix} & \begin{vmatrix} B & A\\ b_1 & a_1\end{vmatrix}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2-\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2+\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2},$$

$$A_2=\frac{\begin{vmatrix} \begin{vmatrix} C & B\\ c_2 & b_2\end{vmatrix} & \begin{vmatrix} C & A\\ c_2 & a_2\end{vmatrix} & \begin{vmatrix} A & B\\ a_2 & b_2\end{vmatrix}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2-\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2+\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2}$$

$\left[\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2+\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2\neq\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2;\right.$ см. 1.5.9.5$\left.\right]$.

39. $$\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x} = \frac{1}{\sqrt{b^2-4ac}} \ln \frac{2c\,\mathrm{tg}\,x + b - \sqrt{b^2-4ac}}{2c\,\mathrm{tg}\,x + b + \sqrt{b^2-4ac}} \quad [b^2 > 4ac].$$

40. $$= \frac{2}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{2c\,\mathrm{tg}\,x + b}{\sqrt{4ac-b^2}} \quad [b^2 < 4ac].$$

41. $$= -\frac{2}{2c\,\mathrm{tg}\,x + b} \quad [b^2 = 4ac].$$

42. $$\int \frac{A\cos^2 x + 2B\sin x\cos x + C\sin^2 x}{a\cos^2 x + 2b\sin x\cos x + c\sin^2 x}\,dx = \frac{1}{4b^2+(a-c)^2}\{[4Bb + (A-C)(a-c)]\,x + [(A-C)\,b - B\,(a-c)] \times$$
$$\times \ln(a\cos^2 x + 2b\sin x\cos x + c\sin^2 x) +$$
$$+ [2\,(A+C)\,b^2 - 2Bb\,(a+c) + (aC - Ac)\,(a-c)]\,f(x)\},$$

где

$$f(x) = \frac{1}{2\sqrt{b^2-ac}} \ln \frac{c\,\mathrm{tg}\,x + b - \sqrt{b^2-ac}}{c\,\mathrm{tg}\,x + b + \sqrt{b^2-ac}} \quad [b^2 > ac].$$

$$= \frac{1}{\sqrt{ac-b^2}} \operatorname{arctg} \frac{c\,\mathrm{tg}\,x + b}{\sqrt{ac-b^2}} \quad [b^2 < ac].$$

$$= -\frac{1}{c\,\mathrm{tg}\,x + b} \quad [b^2 = ac].$$

43. $$\int \frac{dx}{(a\cos^2 x + b\sin x\cos x + c\sin^2 x)^2} = \frac{b\cos 2x + (c-a)\sin 2x}{(4ac-b^2)(a\cos^2 x + b\sin x\cos x + c\sin^2 x)} +$$
$$+ \frac{2\,(a+c)}{4ac-b^2} \int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x} \quad [b^2 \neq 4ac].$$

44. $$= 16a^3 \int \frac{dx}{(2a\cos x + b\sin x)^4} \quad [b^2 = 4ac].$$

45. $$\int \frac{\mathrm{tg}\,x\,dx}{\mathrm{tg}\,x + a} = \frac{1}{a^2+1}\,[x - a\ln(a\cos x + \sin x)].$$

46. $$\int \frac{\mathrm{tg}\,x - a}{\mathrm{tg}\,x + a}\,dx = \frac{1-a^2}{1+a^2}\,x - \frac{2a}{1+a^2} \ln\sin(x + \operatorname{arctg} a).$$

47. $$\int \frac{dx}{a^2 + \mathrm{tg}^2 x} = \frac{x}{a^2-1} - \frac{1}{(a^2-1)\,a} \operatorname{arctg} \frac{\mathrm{tg}\,x}{a}.$$

48. $$\int \frac{dx}{a^2 - \mathrm{tg}^2 x} = \frac{x}{a^2+1} - \frac{1}{2\,(a^2+1)\,a} \ln\left|\frac{\mathrm{tg}\,x - a}{\mathrm{tg}\,x + a}\right|.$$

49. $$\int \frac{dx}{1 + \mathrm{tg}^2 x} = \frac{x}{2} + \frac{1}{4}\sin 2x.$$

50. $$\int \frac{\mathrm{tg}\,x\,dx}{a^2 + \mathrm{tg}^2 x} = \frac{\ln(a^2\cos^2 x + \sin^2 x)}{2\,(1-a^2)}.$$

1.5.10. Интегралы вида $\int \begin{Bmatrix} \sin(ax+b) & \sin(cx+d) \\ \cos(ax+b) & \cos(cx+d) \end{Bmatrix} dx$,

$\int \sin(ax+b)\cos(cx+d)\,dx$.

1. $\int \begin{Bmatrix} \sin(ax+b) \\ \cos(ax+b) \end{Bmatrix} dx = \mp \frac{1}{a} \begin{Bmatrix} \cos(ax+b) \\ \sin(ax+b) \end{Bmatrix}$.

2. $\int \begin{Bmatrix} \sin(ax+b) & \sin(cx+d) \\ \cos(ax+b) & \cos(cx+d) \end{Bmatrix} dx =$
$$= \frac{1}{2(a-c)} \sin[(a-c)x+b-d] \mp \frac{\sin[(a+c)x+b+d]}{2(a+c)}.$$

3. $\int \sin(ax+b)\cos(cx+d)\,dx =$
$$= -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)}.$$

4. $\int \begin{Bmatrix} \sin(ax+b) & \sin(ax+d) \\ \cos(ax+b) & \cos(ax+d) \end{Bmatrix} dx = \frac{x}{2}\cos(b-d) \mp \frac{\sin(2ax+b+d)}{4a}$.

5. $\int \sin(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\sin(b-d) - \frac{\cos(2ax+b+d)}{4a}$.

1.5.11. Интегралы вида $\int \sin^p x \sin ax\,dx$.

1. $\int \sin^p x \sin ax\,dx = \frac{-1+\sin^p x \cos ax}{p+a} + \frac{p}{p+a}\int \sin^{p-1} x \cos(a-1)x\,dx$.

2. $$= -\frac{\sin^p x \cos ax}{p+a} + \frac{p\sin^{p-1} x \sin(a-1)x}{(p+a)(p+a-2)} - \\ - \frac{p(p-1)}{(p+a)(p+a-2)}\int \sin^{p-2} x \sin(a-2)x\,dx.$$

3. $\int \sin^p x \sin 2nx\,dx =$
$$= 2n\left\{\frac{\sin^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2)\ldots[4n^2-(2k)^2]}{(2k+1)!\,(2k+p+2)} \sin^{2k+p+2} x\right\}$$
$[p \neq -2, -4, \ldots, -2n]$

4. $\int \sin^p x \sin(2n+1)x\,dx = (2n+1)\left\{\int \sin^{p+1} x\,dx + \right.$
$$+ \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \\ \left. \times \int \sin^{2k+p+1} x\,dx\right\}.$$

5. $\int \sin^{p-1} x \sin(p+1)x\,dx = \frac{1}{p}\sin^p x \sin px$.

6. $\int \sin^{p-1} x \sin\left[(p+1)\left(\frac{\pi}{2}-x\right)\right] dx = \frac{1}{p}\sin^p x \cos p\left(\frac{\pi}{2}-x\right)$.

7. $\int \frac{\sin ax}{\sin^p x}dx = 2\int \frac{\cos(a-1)x}{\sin^{p-1} x}dx + \int \frac{\sin(a-2)x}{\sin^p x}dx$.

8. $\int \frac{\sin 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\sin (2k-1)x}{2k-1}.$

9. $\int \frac{\sin (2n+1)x}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\sin 2kx}{2k} + x.$

10. $\int \frac{\sin 2x}{\sin^n x}\,dx = -\frac{2}{(n-2)\sin^{n-2} x}.$

11. $\int \frac{\sin 2x}{\sin^2 x}\,dx = 2\ln|\sin x|.$ 12. $\int \frac{\sin 3x}{\sin x}\,dx = x + \sin 2x.$

13. $\int \frac{\sin 3x}{\sin^2 x}\,dx = 3\ln\left|\operatorname{tg}\frac{x}{2}\right| + 4\cos x.$

14. $\int \frac{\sin 3x}{\sin^3 x}\,dx = -3\operatorname{ctg} x - 4x.$

1.5.12. Интегралы вида $\int \sin^p x \cos ax\,dx.$

1. $\int \sin^p x \cos ax\,dx = \frac{\sin^p x \sin ax}{p+a} - \frac{p}{p+a}\int \sin^{p-1} x \sin (a-1)x\,dx.$

2. $$= \frac{\sin^p x \sin ax}{p+a} + \frac{p\sin^{p-1} x \cos (a-1)x}{(p+a)(p+a-2)} - \frac{p(p-1)}{(p+a)(p+a-2)}\int \sin^{p-2} x \cos (a-2)x\,dx.$$

3. $$\int \sin^p x \cos 2nx\,dx = \int \sin^p x\,dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2(4n^2-2^2)\ldots[4n^2-(2k-2)^2]}{(2k)!}\int \sin^{2k+p} x\,dx.$$

4. $$\int \sin^p x \cos (2n+1)x\,dx = \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!\,(2k+p+1)}\sin^{2k+p+1} x$$

$[p \neq -1, -3, -5, \ldots, -(2n+1)].$

5. $\int \sin^{p-1} x \cos (p+1)x\,dx = \frac{1}{p}\sin^p x \cos px.$

6. $\int \sin^{p-1} x \cos\left[(p+1)\left(\frac{\pi}{2}-x\right)\right]dx = -\frac{1}{p}\sin^p x \sin p\left(\frac{\pi}{2}-x\right).$

7. $\int \frac{\cos ax}{\sin^p x}\,dx = -2\int \frac{\sin (a-1)x}{\sin^{p-1} x}\,dx + \int \frac{\cos (a-2)x}{\sin^p x}\,dx.$

8. $\int \frac{\cos 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos (2k-1)x}{2k-1} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

9. $\int \frac{\cos(2n+1)x}{\sin x}\,dx = \sum_{k=1}^{n} \frac{\cos 2kx}{k} + \ln|\sin x|.$

10. $\int \frac{\cos 2x}{\sin x}\,dx = 2\cos x + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$ 11. $\int \frac{\cos 2x}{\sin^2 x}\,dx = -\operatorname{ctg} x - 2x.$

12. $\int \frac{\cos 2x}{\sin^3 x}\,dx = -\frac{\cos x}{2\sin^2 x} - \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

13. $\int \frac{\cos 3x}{\sin^n x}\,dx = \frac{4}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}.$

14. $\int \frac{\cos 3x}{\sin x}\,dx = -2\sin^2 x + \ln|\sin x|.$

15. $\int \frac{\cos 3x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - 4\ln|\sin x|.$

1.5.13. Интегралы вида $\int \cos^p x \sin ax\,dx$.

1. $\int \cos^p x \sin ax\,dx = -\frac{\cos^p x \cos ax}{p+a} + \frac{p}{p+a}\int \cos^{p-1} x \sin(a-1)x\,dx.$

2. $$\int \cos^p x \sin 2nx\,dx = (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2)\ldots[4n^2-(2k)^2]}{(2k+1)!\,(2k+p+2)} \cos^{2k+p+2} x \right\};$$
$[p \neq -2, -4, \ldots, -2n].$

3. $$\int \cos^p x \sin(2n+1)x\,dx = (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!\,(2k+p+1)} \times \cos^{2k+p+1} x \right\}$$
$[p \neq -1, -3, -5, \ldots, -(2n+1)].$

4. $\int \cos^{p-1} x \sin(p+1)x\,dx = -\frac{1}{p}\cos^p x \cos px.$

5. $\int \frac{\sin ax}{\cos^p x}\,dx = 2\int \frac{\sin(a-1)x}{\cos^{p-1} x}\,dx - \int \frac{\sin(a-2)x}{\cos^p x}\,dx.$

6. $\int \frac{\sin(2n+1)x}{\cos x}\,dx = \sum_{k=1}^{n} (-1)^{n-k+1}\frac{\cos 2kx}{k} + (-1)^{n+1}\ln|\cos x|.$

7. $\int \frac{\sin 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k+1}\frac{\cos(2k-1)x}{2k-1}.$

8. $\int \frac{\sin 2x}{\cos^n x}\,dx = \frac{2}{(n-2)\cos^{n-2} x}.$ 9. $\int \frac{\sin 2x}{\cos^2 x}\,dx = -2\ln|\cos x|.$

10. $\int \frac{\sin 3x}{\cos^n x}\,dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x}$.

11. $\int \frac{\sin 3x}{\cos x}\,dx = 2\sin^2 x + \ln|\cos x|$.

12. $\int \frac{\sin 3x}{\cos^3 x}\,dx = -\frac{1}{2\cos^2 x} - 4\ln|\cos x|$.

1.5.14. Интегралы вида $\int \cos^p x \cos ax\,dx$.

1. $\int \cos^p x \cos ax\,dx = \frac{\cos^p x \sin ax}{p+a} + \frac{p}{p+a}\int \cos^{p-1} x \cos(a-1)x\,dx$.

2. $$\int \cos^p x \cos 2nx\,dx = (-1)^n \left\{\int \cos^p x\,dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2(2n^2-2^2)\dots[4n^2-(2k-2)^2]}{(2k)!}\int \cos^{2k+p} x\,dx\right\}.$$

3. $$\int \cos^p x \cos(2n+1)x\,dx = (-1)^n (2n+1)\left\{\int \cos^{p+1} x\,dx + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\times \int \cos^{2k+p+1} x\,dx\right\}.$$

4. $\int \cos^{p-1} x \cos(p+1)x\,dx = \frac{1}{p}\cos^p x \sin px$.

5. $\int \frac{\cos ax}{\cos^p x}\,dx = 2\int \frac{\cos(a-1)x}{\cos^{p-1} x}\,dx - \int \frac{\cos(a-2)x}{\cos^p x}\,dx$.

6. $\int \frac{\cos 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

7. $\int \frac{\cos(2n+1)x}{\cos x}\,dx = \sum_{k=1}^{n} (-1)^{n-k} \frac{\sin 2kx}{k} + (-1)^n x$.

8. $\int \frac{\cos 2x}{\cos x}\,dx = 2\sin x - \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

9. $\int \frac{\cos 2x}{\cos^2 x}\,dx = 2x - \operatorname{tg} x$.

10. $\int \frac{\cos 2x}{\cos^3 x}\,dx = -\frac{\sin x}{2\cos^2 x} + \frac{3}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

11. $\int \frac{\cos 3x}{\cos x}\,dx = \sin 2x - x$.

12. $\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\sin x - 3\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$.

13. $\int \frac{\cos 3x}{\cos^3 x}\,dx = 4x - 3\operatorname{tg} x$.

1.5.15. Интегралы вида $\int \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^m \begin{Bmatrix}\sin nx\\ \cos nx\end{Bmatrix}^{-1} dx$.

1. $$\int \frac{\sin^m x\,dx}{\sin(2n+1)x}=$$
$$=\frac{1}{2n+1}\sum_{k=0}^{2n}(-1)^{n+k}\cos^m\left[\frac{2k+1}{2(2n+1)}\pi\right]\ln\left|\frac{\sin\left[\frac{(k-n)\pi}{2(2n+1)}+\frac{x}{2}\right]}{\sin\left[\frac{k+n+1}{2(2n+1)}\pi-\frac{x}{2}\right]}\right| \quad [m\leqslant 2n].$$

2. $$\int \frac{\sin^{2m}x}{\sin 2nx}dx=\frac{(-1)^n}{2n}\left\{\ln|\cos x|+\sum_{k=1}^{n-1}(-1)^k\cos^{2m}\frac{k\pi}{2n}\ln\left|\cos^2 x-\sin^2\frac{k\pi}{2n}\right|\right\} \quad [m\leqslant n].$$

3. $$\int \frac{\sin^{2m+1}x}{\sin 2nx}dx=\frac{(-1)^n}{2n}\left\{\ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+\right.$$
$$\left.+\sum_{k=1}^{n-1}(-1)^k\cos^{2m+1}\frac{k\pi}{2n}\ln\left|\operatorname{tg}\left(\frac{n+k}{4n}\pi-\frac{x}{2}\right)\operatorname{tg}\left(\frac{n-k}{4n}\pi-\frac{x}{2}\right)\right|\right\} \quad [m<n].$$

4. $$\int \frac{\sin^{2m}x\,dx}{\cos(2n+1)x}=\frac{(-1)^{n+1}}{2n+1}\left\{\ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+\right.$$
$$\left.+\sum_{k=1}^{n}(-1)^k\cos^{2m}\frac{k\pi}{2n+1}\ln\left|\operatorname{tg}\left(\frac{2n+2k+1}{4(2n+1)}\pi-\frac{x}{2}\right)\operatorname{tg}\left(\frac{2n-2k+1}{4(2n+1)}\pi-\frac{x}{2}\right)\right|\right\} \quad [m\leqslant n].$$

5. $$\int \frac{\sin^{2m+1}x\,dx}{\cos(2n+1)x}=$$
$$=\frac{(-1)^{n+1}}{2n+1}\left\{\ln|\cos x|+\sum_{k=1}^{n}(-1)^k\cos^{2m+1}\frac{k\pi}{2n+1}\ln\left|\cos^2 x-\sin^2\frac{k\pi}{2n+1}\right|\right\} \quad [m\leqslant n].$$

6. $$\int \frac{\sin^m x\,dx}{\cos 2nx}=\frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^{n+k}\cos^m\left(\frac{2k+1}{4n}\pi\right)\times$$
$$\times\ln\left|\frac{\sin\left(\frac{2k-2n+1}{8n}\pi+\frac{x}{2}\right)}{\sin\left(\frac{2k+2n+1}{8n}\pi-\frac{x}{2}\right)}\right| \quad [m<2n].$$

7. $$\int \frac{\cos^{2m+1}x\,dx}{\sin(2n+1)x}=$$
$$=\frac{1}{2n+1}\left\{\ln|\sin x|+\sum_{k=1}^{n}(-1)^k\cos^{2m+1}\frac{k\pi}{2n+1}\ln\left|\sin^2 x-\sin^2\frac{k\pi}{2n+1}\right|\right\} \quad [m\leqslant n].$$

8. $$\int \frac{\cos^{2m} x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\left\{\ln\left|\operatorname{tg}\frac{x}{2}\right| + \sum_{k=1}^{n} (-1)^k \cos^{2m}\frac{k\pi}{2n+1} \ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{k\pi}{4n+2}\right)\operatorname{tg}\left(\frac{x}{2}-\frac{k\pi}{4n+2}\right)\right|\right\}$$ $[m \leqslant n]$.

9. $$\int \frac{\cos^{2m+1} x}{\sin 2nx}\,dx = \frac{1}{2n}\left\{\ln\left|\operatorname{tg}\frac{x}{2}\right| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1}\frac{k\pi}{2n} \ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{k\pi}{4n}\right)\operatorname{tg}\left(\frac{x}{2}-\frac{k\pi}{4n}\right)\right|\right\}$$ $[m < n]$.

10. $$\int \frac{\cos^{2m} x}{\sin 2nx}\,dx = \frac{1}{2n}\left\{\ln|\sin x| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m}\frac{k\pi}{2n} \ln\left|\sin^2 x - \sin^2\frac{k\pi}{2n}\right|\right\}$$ $[m \leqslant n]$.

11. $$\int \frac{\cos^m x}{\cos nx}\,dx = \frac{1}{n}\sum_{k=0}^{n-1} (-1)^k \cos^m\frac{2k+1}{2n}\pi \ln\left|\frac{\sin\left(\frac{2k+1}{4n}\pi+\frac{x}{2}\right)}{\sin\left(\frac{2k+1}{4n}\pi-\frac{x}{2}\right)}\right|$$ $[m \leqslant n]$.

1.5.16. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax})\,dx$.

Обозначение: $\varphi = \arcsin\sqrt{\dfrac{2\sin cx}{1+\sin ax+\cos cx}}$.

1. $$\int \sin^{m+1/2} 2x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n+1} dx = \pm 2^{m+3/2}\int \frac{t^{2(2n+m+2)}}{(1+t^4)^{n+m+2}}\,dt$$ $\left[t = \begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{1/2}\right]$

$$= \mp 2^{m+3/2}\int \frac{t^{2(m+1)}\,dt}{(1+t^4)^{n+m+2}}$$ $\left[t = \begin{Bmatrix}\operatorname{ctg} x\\ \operatorname{tg} x\end{Bmatrix}^{1/2}\right]$

2. $$\int \sin^{m+1/2} 2x \begin{Bmatrix}\sec x\\ \operatorname{cosec} x\end{Bmatrix}^{2n+1} dx = \mp 2^{m+3/2}\begin{Bmatrix}\operatorname{ctg} x\\ \operatorname{tg} x\end{Bmatrix}^{1/2}\sum_{k=0}^{n-m-1}\binom{n-m-1}{k}\frac{1}{2m-4n+4k+1}\begin{Bmatrix}\operatorname{ctg} x\\ \operatorname{tg} x\end{Bmatrix}^{m-2n+2k}$$ $[n \geqslant m+1]$.

3. $$\int \sin^{m+1/2} 2x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix} dx = \mp 2^{m+3/2}\int \frac{t^{2(m+2)}\,dt}{(1+t^4)^{n+m+2}}$$ $\left[t = \begin{Bmatrix}\operatorname{ctg} x\\ \operatorname{tg} x\end{Bmatrix}^{1/2}\right]$.

4. $$\int \sqrt{\sin 2x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = \mp\frac{1}{2}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}\sqrt{\sin 2x} + \frac{1}{4}\left[\ln\left(\sin x+\cos x \pm \sqrt{\sin 2x}\right) + \arcsin(\sin x - \cos x)\right].$$

5. $\int\limits_0^x \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$

6. $\int \frac{1}{\sin^{n+1/2} 2x} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2m+1} dx =$

$$= \pm 2^{-n+1/2} \begin{Bmatrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{Bmatrix}^{3/2} \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{1}{4m-2n+4k+3} \begin{Bmatrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{Bmatrix}^{2m-n+2k}$$

$[n \geqslant m+1]$.

7. $\int \frac{1}{\sqrt{\sin 2x}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx =$

$$= \frac{1}{2} \left[\ln\left(\sin x + \cos x \mp \sqrt{\sin 2x}\right) + \arcsin(\sin x - \cos x)\right].$$

8. $\int \frac{1}{\sin^{n+1/2} 2x} \begin{Bmatrix} \sec x \\ \operatorname{cosec} x \end{Bmatrix}^{2m+1} dx =$

$$= \pm 2^{1/2-n} \begin{Bmatrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{Bmatrix}^{1/2} \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{1}{4m+2n-4k+1} \begin{Bmatrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{Bmatrix}^{2m+n-2k}.$$

9. $\int \frac{\sin x}{\sin^{n+1/2} 2x} \cos^{\pm 2m} x\, dx =$

$$= 2^{-n+1/2} \operatorname{tg}^{3/2} x \sum_{k=0}^{n \mp m-1} \binom{n \mp m-1}{k} \frac{\operatorname{tg}^{2k-n} x}{4k-2n+3} \qquad [n \geqslant m+1].$$

10. $\int \frac{\cos x}{\sin^{n+1/2} 2x} \sin^{\pm 2m} x\, dx =$

$$= -2^{-n+1/2} \operatorname{ctg}^{3/2} x \sum_{k=0}^{n \mp m-1} \binom{n \mp m-1}{k} \frac{\operatorname{ctg}^{2k-n} x}{4k-2n+3} \qquad [n \geqslant m+1].$$

11. $\int \frac{\operatorname{tg}^{\pm l} x\, dx}{\sin^{2m+1} x \sin^{n+1/2} 2x} =$

$$= -2^{-n+1/2} \operatorname{ctg}^{1/2} x \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{ctg}^{2m+n \mp l-2k} x}{4m+2n \mp 2l-4k+1}.$$

12. $$\int \frac{\operatorname{tg}^{\pm l} x\, dx}{\cos^{2m+1} x \sin^{n+1/2} 2x} = 2^{-n+1/2} \operatorname{tg}^{1/2} x \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{tg}^{2k \pm l-n} x}{4k \pm 2l-2n+1}.$$

13. $$\int\limits_0^x \frac{\sin ax\, dx}{(1+\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right].$$

14. $$\int_0^x \frac{\sin ax\,dx}{(1-\sin ax+\cos ax)\sqrt{\sin 2ax}}=\frac{\sqrt{2}}{a}\left[\sqrt{\operatorname{tg} ax}-E\left(\varphi,\ \frac{1}{\sqrt{2}}\right)\right]$$

$[ax\neq\pi/2]$.

15. $$\int_0^x \frac{(1+\cos ax)\,dx}{(1+\sin ax+\cos ax)\sqrt{\sin 2ax}}=\frac{\sqrt{2}}{a}E\left(\varphi,\ \frac{1}{\sqrt{2}}\right).$$

16. $$\int_0^x \frac{(1+\cos ax)\,dx}{(1-\sin ax+\cos ax)\sqrt{\sin 2ax}}=\frac{\sqrt{2}}{a}\left[F\left(\varphi,\ \frac{1}{\sqrt{2}}\right)-E\left(\varphi,\ \frac{1}{\sqrt{2}}\right)+\sqrt{\operatorname{tg} ax}\right]$$

$[ax\neq\pi/2]$.

17. $$\int_0^x \frac{(1-\sin ax+\cos ax)\,dx}{(1+\sin ax+\cos ax)\sqrt{\sin 2ax}}=\frac{\sqrt{2}}{a}\left[2E\left(\varphi,\ \frac{1}{\sqrt{2}}\right)-F\left(\varphi,\ \frac{1}{\sqrt{2}}\right)\right].$$

18. $$\int_0^x \frac{(1+\sin ax+\cos ax)\,dx}{[1-\cos ax+(1-2r^2)\sin ax]\sqrt{\sin 2ax}}=\frac{\sqrt{2}}{a}\Pi\left(\varphi,\ r^2,\ \frac{1}{\sqrt{2}}\right).$$

1.5.17. Интегралы вида $\int R(\sin ax,\ \cos ax,\ \sqrt{\cos 2ax})\,dx$.

Условие: $0<ax\leqslant\pi/4$.

Обозначение: $\varphi=\arcsin(\sqrt{2}\sin ax)$.

1. $$\int R(\sin ax,\ \cos ax,\ \sqrt{\cos 2ax})\,dx=\frac{1}{a}\int R(\sin t,\ \cos t,\ \sqrt{1-2\sin^2 t})\,dt$$

$[t=ax$; см. 1.5.36].

2. $$\int_0^x \sqrt{\cos 2ax}\,dx=\frac{\sqrt{2}}{a}E\left(\varphi,\ \frac{1}{\sqrt{2}}\right)-\frac{1}{a\sqrt{2}}F\left(\varphi,\ \frac{1}{\sqrt{2}}\right).$$

3. $$\int \cos^{n+1/2} 2x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx=\mp\frac{\sqrt{\cos 2x}}{2(n+1)}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}\times$$
$$\times\left[\cos^n 2x+\sum_{k=0}^{n-1}(\mp 1)^{k+1}\frac{(2n+1)(2n-1)\dots(2n-2k+1)}{2^{k+1}n(n-1)\dots(n-k)}\cos^{n-k-1}2x\right]+$$
$$+(-1)^n\frac{(2n+1)!!}{(2n+2)!!\sqrt{2}}\begin{Bmatrix}\ln\left(\cos x+\sqrt{\dfrac{\cos 2x}{2}}\right)\\ \arcsin(\sqrt{2}\sin x)\end{Bmatrix}.$$

4. $$\int \sqrt{\cos 2x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx=\mp\frac{1}{2}\sqrt{\cos 2x}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}+\frac{1}{2\sqrt{2}}\begin{Bmatrix}\ln\left(\cos x+\sqrt{\dfrac{\cos 2x}{2}}\right)\\ \arcsin(\sqrt{2}\sin x)\end{Bmatrix}.$$

5. $$\int_0^x \frac{\sqrt{\cos 2ax}}{\cos^2 ax}\,dx=\frac{\sqrt{2}}{a}\left[F\left(\varphi,\ \frac{1}{\sqrt{2}}\right)-E\left(\varphi,\ \frac{1}{\sqrt{2}}\right)\right]+\frac{1}{a}\operatorname{tg} ax\sqrt{\cos 2ax}.$$

6. $$\int_0^x \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

7. $$\int \frac{1}{\cos^{n+1/2} 2x} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \mp \frac{1}{(2n-1)\cos^{n-1/2} 2x} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \times$$
$$\times \left[1 + \sum_{k=0}^{n-2} (\mp 1)^{k+1} \frac{2^{k+1}(n-1)(n-2)\ldots(n-k-1)}{(2n-3)(2n-5)\ldots(2n-2k-3)} \cos^{k+1} 2x\right].$$

8. $$= \frac{1}{\sqrt{\cos 2x}} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \sum_{k=0}^{n-1} \frac{(\mp 1)^{n+k+1} 2^k}{2k+1} \binom{n-1}{k} \frac{1}{\cos^k 2x} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix}^{2k}.$$

9. $$\int \frac{1}{\sqrt{\cos 2x}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \frac{1}{\sqrt{2}} \begin{Bmatrix} \ln\left(\cos x - \sqrt{\dfrac{\cos 2x}{2}}\right) \\ \arcsin(\sqrt{2}\sin x) \end{Bmatrix}.$$

10. $$\int_0^x \frac{\sin^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

11. $$\int_0^x \frac{\cos^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

12. $$\int_0^x \frac{\operatorname{tg}^2 ax}{\sqrt{\cos 2ax}} dx = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \operatorname{tg} ax \sqrt{\cos 2ax}.$$

13. $$\int_0^x \frac{\operatorname{tg}^4 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{3a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a\cos^3 ax} \sqrt{\cos 2ax}.$$

14. $$\int_0^x \frac{dx}{\cos^2 ax \sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{tg} ax}{a} \sqrt{\cos 2ax}.$$

15. $$\int_0^x \frac{dx}{(1-2r^2\sin^2 ax)\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$$

16. $$\int_0^x \frac{dx}{\cos^4 ax \sqrt{\cos 2ax}} =$$
$$= \frac{2\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{(6\cos^2 ax + 1)\sin ax}{3a\cos^3 ax} \sqrt{\cos 2ax}.$$

17. $$\int_0^x \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}}.$$

18. $\int\limits_0^x \frac{\sin^2 ax}{\sqrt{\cos^3 2ax}}\,dx = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$

1.5.18. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax})\,dx$.

Условие: $\pi/4 \leqslant ax < \pi/2$.

Обозначение: $\varphi = \arcsin(\sqrt{2}\cos ax)$.

1. $\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax})\,dx =$

$$= \frac{1}{a}\int R(\sin t, \cos t, \sqrt{2\sin^2 t - 1})\,dt \qquad [t = ax;\ \text{см. } 1.5.38].$$

2. $\int\limits_x^{\pi/(2a)} \sqrt{-\cos 2ax}\,dx = \frac{1}{a\sqrt{2}}\left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right].$

3. $\int\limits_x^{\pi/(2a)} \frac{dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$

4. $\int\limits_x^{\pi/(2a)} \frac{\sin^2 ax\,dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$

5. $\int\limits_x^{\pi/(2a)} \frac{\cos^2 ax\,dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right]$

6. $\int\limits_x^{\pi/(2a)} \frac{\operatorname{ctg}^2 ax\,dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}}\left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] -$

$$- \frac{1}{a}\operatorname{ctg} ax\sqrt{-\cos 2ax}.$$

7. $\int\limits_x^{\pi/(2a)} \frac{\cos^4 ax\,dx}{\sqrt{-\cos 2ax}} =$

$$= \frac{1}{3a\sqrt{2}}\left[\frac{5}{2}E\left(\varphi, \frac{1}{\sqrt{2}}\right) - 3F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{12a}\sin 2ax\sqrt{-\cos 2ax}.$$

8. $\int\limits_x^{\pi/(2a)} \frac{dx}{\sin^2 ax\sqrt{-\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a}\operatorname{ctg} ax\sqrt{-\cos 2ax}.$

9. $\int\limits_x^{\pi/(2a)} \frac{dx}{(1 - 2r^2\cos^2 ax)\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$

10. $\int\limits_x^{\pi/(2a)} \frac{dx}{\sin^4 ax\sqrt{-\cos 2ax}} = \frac{-2}{3a\sqrt{2}}\left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 6E\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] -$

$$- \frac{\cos ax}{3a\sin^3 ax}(6\sin^2 ax + 1)\sqrt{-\cos 2ax}.$$

11. $$\int_x^{\pi/(2a)} \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}}\left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] - \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}.$$

12. $$\int_x^{\pi/(2a)} \frac{\cos^2 ax\, dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}}.$$

1.5.19. Интегралы вида $\int R\left(\sin x, \cos x, \sqrt{a \pm b \sin x}\right) dx$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{1-\sin x}{2}}$,

$$\psi = \arcsin\sqrt{\frac{b(1-\sin x)}{a+b}}, \quad k = \sqrt{\frac{2b}{a+b}}.$$

1. $$\int_x^{\pi/2} \sqrt{a+b\sin x}\, dx = 2\sqrt{a+b}\, E(\varphi, k) \qquad \left[a>b>0;\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}\right].$$

2. $$= (a-b)\sqrt{\frac{2}{b}}\, F\left(\psi, \frac{1}{k}\right) + 2\sqrt{2b}\, E\left(\psi, \frac{1}{k}\right)$$
$$\left[0<|a|<b;\ -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

3. $$\int_x^{\pi/2} \frac{dx}{\sqrt{a+b\sin x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k) \qquad \left[a>b>0;\ -\frac{\pi}{2} \leqslant x < \frac{\pi}{2}\right].$$

4. $$= \sqrt{\frac{2}{b}}\, F\left(\psi, \frac{1}{k}\right) \qquad \left[0<|a|<b;\ -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

5. $$\int_x^{\pi/2} \frac{\sin x\, dx}{\sqrt{a+b\sin x}} = \frac{2\sqrt{a+b}}{b} E(\varphi, k) - \frac{2a}{b\sqrt{a+b}} F(\varphi, k)$$
$$\left[a>b>0;\ -\frac{\pi}{2} \leqslant x < \frac{\pi}{2}\right].$$

6. $$= \sqrt{\frac{2}{b}}\left\{2E\left(\psi, \frac{1}{k}\right) - F\left(\psi, \frac{1}{k}\right)\right\} \qquad \left[0<|a|<b;\ -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

7. $$\int_x^{\pi/2} \frac{\sin^2 x\, dx}{\sqrt{a+b\sin x}} = -\frac{4a\sqrt{a+b}}{3b^2} E(\varphi, k) + \frac{2(2a^2+b^2)}{3b^2\sqrt{a+b}} F(\varphi, k) +$$
$$+ \frac{2}{3b}\cos x\sqrt{a+b\sin x} \qquad \left[a>b>0;\ -\frac{\pi}{2} \leqslant x < \frac{\pi}{2}\right].$$

8. $$= \sqrt{\frac{2}{b}}\left\{\frac{2a+b}{3b} F\left(\psi, \frac{1}{k}\right) - \frac{4a}{3b} E\left(\psi, \frac{1}{k}\right)\right\} + \frac{2}{3b}\cos x\sqrt{a+b\sin x}$$
$$\left[0<|a|<b;\ -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right].$$

9. $$\int\limits_x^{\pi/2} \frac{\operatorname{tg}^2 x\,dx}{\sqrt{a+b\sin x}} = \frac{-1}{\sqrt{a+b}} F(\varphi, k) - \frac{a}{(a-b)\sqrt{a+b}} E(\varphi, k) + \frac{b-a\sin x}{(a^2-b^2)\cos x}\sqrt{a+b\sin x} \qquad \left[0<b<a,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\right].$$

10. $$= -\sqrt{\frac{2}{b}}\left\{\frac{2a+b}{2(a+b)} F\left(\psi, \frac{1}{k}\right) + \frac{ab}{a^2-b^2} E\left(\psi, \frac{1}{k}\right)\right\} + \frac{b-a\sin x}{(a^2-b^2)\cos x}\sqrt{a+b\sin x} \qquad \left[0<|a|<b;\ -\arcsin\frac{a}{b}<x<\frac{\pi}{2}\right].$$

11. $$\int\limits_x^{\pi/2} \frac{dx}{(2-p^2+p^2\sin x)\sqrt{a+b\sin x}} = \frac{1}{\sqrt{a+b}} \Pi(\varphi, p^2, k) \qquad \left[0<b<a;\ -\frac{\pi}{2}\leqslant x<\frac{\pi}{2}\right].$$

12. $$\int\limits_x^{\pi/2} \frac{dx}{(a+b-p^2b+p^2b\sin x)\sqrt{a+b\sin x}} = \frac{1}{a+b}\sqrt{\frac{2}{b}}\,\Pi\left(\psi, p^2, \frac{1}{k}\right) \qquad \left[0<|a|<b;\ -\arcsin\frac{a}{b}<x<\frac{\pi}{2}\right].$$

13. $$\int\limits_x^{\pi/2} \frac{1-\sin x}{1+\sin x}\,\frac{dx}{\sqrt{a+b\sin x}} = \frac{2}{a-b}\left\{-\sqrt{a+b}\,E(\varphi, k) + \operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\sqrt{a+b\sin x}\right\} \qquad \left[0<b<a;\ -\frac{\pi}{2}\leqslant x<\frac{\pi}{2}\right].$$

14. $$\int\limits_x^{\pi/2} \frac{dx}{\sqrt{(a+b\sin x)^3}} = \frac{-2b\cos x}{(a^2-b^2)\sqrt{a+b\sin x}} + \frac{2}{(a-b)\sqrt{a+b}} E'(\varphi, k) \qquad \left[0<b<a;\ -\frac{\pi}{2}\leqslant x<\frac{\pi}{2}\right].$$

15. $$= \sqrt{\frac{2}{b}}\left\{\frac{1}{a+b} F\left(\psi, \frac{1}{k}\right) - \frac{2b}{b^2-a^2} E\left(\psi, \frac{1}{k}\right)\right\} - \frac{2b}{b^2-a^2}\,\frac{\cos x}{\sqrt{a+b\sin x}} \qquad \left[0<|a|<b;\ -\arcsin\frac{a}{b}<x<\frac{\pi}{2}\right].$$

16. $$\int (1\pm\sin ax)^{n+1/2}\,dx = \mp\frac{1}{a}\sqrt{1\mp\sin x}\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k+1}\frac{(1\mp\sin ax)^k}{2k+1}.$$

17. $$\int \frac{dx}{(1\pm\sin ax)^{n+1/2}} = \frac{\cos ax}{na2^n(1\pm\sin ax)^{n+1/2}}\left[2^{n-1} + \sum_{k=0}^{n-2}\frac{(2n-1)(2n-3)\ldots(2n-2k-1)}{(2n-2)(2n-4)\ldots(2n-2k-2)} \times 2^{n-k-2}(1\pm\sin ax)^{k+1}\right] \mp \frac{(2n-1)!!}{(2n)!!}\,\frac{1}{2^n\sqrt{2}\,a}\ln\frac{\sqrt{2}+\sqrt{1\mp\sin ax}}{\sqrt{2}-\sqrt{1\mp\sin ax}}.$$

1.5.20. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a \pm b\cos x})\,dx$.

Обозначения: $\varphi = \arcsin\sqrt{\frac{b(1-\cos x)}{a+b}}$,

$$\psi = \arcsin\sqrt{\frac{(a+b)(1-\cos x)}{2(a-b\cos x)}}, \quad k = \sqrt{\frac{2b}{a+b}}.$$

1. $\int_0^x \sqrt{a+b\cos x}\,dx = 2\sqrt{a+b}\,E\left(\frac{x}{2}, k\right)$ $[a>b>0;\ 0 \leqslant x \leqslant \pi]$.

2. $= \sqrt{\frac{2}{b}}\left\{(a-b)F\left(\varphi, \frac{1}{k}\right) + 2bE\left(\varphi, \frac{1}{k}\right)\right\}$ $\left[b \geqslant |a| > 0,\ 0 \leqslant x < \arccos\left(-\frac{a}{b}\right)\right]$.

3. $\int_0^x \sqrt{a-b\cos x}\,dx = 2\sqrt{a+b}\,E(\psi, k) - \frac{2b\sin x}{\sqrt{a-b\cos x}}$ $[a>b>0;\ 0 \leqslant x \leqslant \pi]$.

4. $\int_0^x \sqrt{\frac{a-b\cos x}{1+p\cos x}}\,dx = \frac{2(a-b)}{(1+p)\sqrt{a+b}}\,\Pi\left(\psi, \frac{2ap}{(a+b)(1+p)}, k\right)$ $[a>b>0,\ 0 \leqslant x \leqslant \pi;\ p \neq -1]$.

5. $\int_0^x \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{\sqrt{a+b}}F\left(\frac{x}{2}, k\right)$ $[a>b>0;\ 0 \leqslant x \leqslant \pi]$.

6. $= \sqrt{\frac{2}{b}}F\left(\varphi, \frac{1}{k}\right)$ $\left[b \geqslant |a| > 0;\ 0 \leqslant x < \arccos\left(-\frac{a}{b}\right)\right]$.

7. $\int_0^x \frac{dx}{\sqrt{a-b\cos x}} = \frac{2}{\sqrt{a+b}}F(\psi, k)$ $[a>b>0;\ 0 \leqslant x \leqslant \pi]$.

8. $\int_0^x \frac{\cos x\,dx}{\sqrt{a+b\cos x}} = \frac{2}{b\sqrt{a+b}}\left\{(a+b)E\left(\frac{x}{2}, k\right) - aF\left(\frac{x}{2}, k\right)\right\}$ $[a>b>0;\ 0 \leqslant x \leqslant \pi]$.

9. $= \sqrt{\frac{2}{b}}\left\{2E\left(\varphi, \frac{1}{k}\right) - F\left(\varphi, \frac{1}{k}\right)\right\}$ $\left[b > |a| > 0;\ 0 \leqslant x < \arccos\left(-\frac{a}{b}\right)\right]$.

10. $\int_0^x \frac{\cos x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{b\sqrt{a+b}}\{(b-a)\Pi(\psi, k^2, k) + aF(\psi, k)\}$ $[a>b>0;\ 0 \leqslant x \leqslant \pi]$.

11. $$\int_0^x \frac{\cos^2 x\,dx}{\sqrt{a+b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\left\{(2a^2+b^2)\,F\left(\frac{x}{2},\ k\right) - 2a(a+b)\,E\left(\frac{x}{2},\ k\right)\right\} + \frac{2}{3b}\sin x\sqrt{a+b\cos x} \qquad [a>b>0;\ 0\leqslant x\leqslant\pi].$$

12. $$= \frac{1}{3b}\sqrt{\frac{2}{b}}\left\{(2a+b)\,F\left(\varphi,\ \frac{1}{k}\right) - 4aE\left(\varphi,\ \frac{1}{k}\right)\right\} + \frac{2}{3b}\sin x\sqrt{a+b\cos x} \qquad \left[b\geqslant|a|>0;\ 0\leqslant x<\arccos\left(-\frac{a}{b}\right)\right].$$

13. $$\int_0^x \frac{\cos^2 x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\{(2a^2+b^2)\,F(\psi,\ k) - 2a(a+b)\,E(\psi,\ k)\} + \frac{2}{3b}\sin x\,\frac{a+b\cos x}{\sqrt{a-b\cos x}} \qquad [a>b>0;\ 0\leqslant x<\pi].$$

14. $$\int (1\pm\cos ax)^{n+1/2}\,dx = \pm\sqrt{1\mp\cos ax}\sum_{k=0}^{n}(-1)^n\binom{n}{k}2^{n-k+1}\frac{(1\mp\cos ax)^k}{2k+1}.$$

15. $$\int_0^x \frac{dx}{(2-p^2+p^2\cos x)\sqrt{a+b\cos x}} = \frac{1}{\sqrt{a+b}}\,\Pi\left(\frac{x}{2},\ p^2,\ k\right) \qquad [a>b>0;\ 0\leqslant x<\pi].$$

16. $$\int_0^x \frac{dx}{(a+b-p^2b+p^2b\cos x)\sqrt{a+b\cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}}\,\Pi\left(\varphi,\ p^2,\ \frac{1}{k}\right) \qquad \left[b\geqslant|a|>0;\ 0\leqslant x<\arccos\left(-\frac{a}{b}\right)\right].$$

17. $$\int_0^x \frac{1-\cos x}{1+\cos x}\,\frac{dx}{\sqrt{a+l\cos x}} = \frac{2}{a-b}\,\mathrm{tg}\,\frac{x}{2}\sqrt{a+b\cos x} - \frac{2\sqrt{a+b}}{a-b}\,E\left(\frac{x}{2},\ k\right) \qquad [a>b>0;\ 0\leqslant x<\pi].$$

18. $$\int_0^x \frac{dx}{\sqrt{(a+b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}\,E\left(\frac{x}{2},\ k\right) - \frac{2b}{a^2-b^2}\,\frac{\sin x}{\sqrt{a+b\cos x}} \qquad [a>b>0,\ 0\leqslant x\leqslant\pi].$$

19. $$= \frac{1}{a^2-b^2}\sqrt{\frac{2}{b}}\left\{(a-b)\,F\left(\varphi,\ \frac{1}{k}\right) + 2bE\left(\varphi,\ \frac{1}{k}\right)\right\} + \frac{2b}{b^2-a^2}\,\frac{\sin x}{\sqrt{a+b\cos x}} \qquad \left[b\geqslant|a|>0,\ 0\leqslant x<\arccos\left(-\frac{a}{b}\right)\right].$$

20. $$\int_0^x \frac{dx}{\sqrt{(a-b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}\,E(\psi,\ k) \qquad [a>b>0;\ 0\leqslant x\leqslant\pi].$$

21. $$\int \frac{dx}{(1 \pm \cos ax)^{n+1/2}} = \pm \frac{\sin ax}{na2^n (1 \pm \cos ax)^{n+1/2}} \times$$
$$\times \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3)\dots(2n-2k-1)}{(2n-2)(2n-4)\dots(2n-2k-2)} 2^{n-k-2} (1 \pm \cos ax)^{k+1} \right] \pm$$
$$\pm \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{a2^n \sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 \mp \cos ax}}{\sqrt{2} - \sqrt{1 \mp \cos ax}}.$$

1.5.21. Интегралы вида $\int R\left(\sin x,\ \cos x,\ \sqrt{a + b\sin x + c\cos x}\right) dx$.

Обозначения: $$\varphi = \arcsin \sqrt{\frac{\sqrt{b^2+c^2} - b\sin x - c\cos x}{2\sqrt{b^2+c^2}}},$$
$$k = \sqrt{\frac{2\sqrt{b^2+c^2}}{a + \sqrt{b^2+c^2}}}, \quad x_1 = \arcsin \frac{b}{\sqrt{b^2+c^2}}.$$

1. $$\int_x^{x_1} \sqrt{a + b\sin x + c\cos x}\, dx = 2\sqrt{a + \sqrt{b^2+c^2}}\, E(\varphi, k)$$
$$\left[0 < \sqrt{b^2+c^2} < a;\ x_1 - \pi \leqslant x < x_1\right].$$

2. $$= 2\sqrt{2}\sqrt[4]{b^2+c^2}\, E(\varphi, k) - \frac{\sqrt{2}\left(\sqrt{b^2+c^2} - a\right)}{\sqrt[4]{b^2+c^2}} F(\varphi, k)$$
$$\left[0 < |a| < \sqrt{b^2+c^2},\ x_1 - \arccos\left(\frac{-a}{\sqrt{b^2+c^2}}\right) \leqslant x < x_1\right].$$

3. $$\int_x^{x_1} \frac{dx}{\sqrt{a + b\sin x + c\cos x}} = \frac{2}{\sqrt{a + \sqrt{b^2+c^2}}} F(\varphi, k)$$
$$\left[0 < \sqrt{b^2+c^2} < a;\ x_1 - \pi \leqslant x < x_1\right].$$

4. $$= \frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}} F(\varphi, k)$$
$$\left[0 < |a| < \sqrt{b^2+c^2};\ x_1 - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \leqslant x < x_1\right].$$

5. $$\int_x^{x_1} \frac{\sin x\, dx}{\sqrt{a + b\sin x + c\cos x}} =$$
$$= \frac{\sqrt{2}\, b}{\sqrt[4]{(b^2+c^2)^3}} \{2E(\varphi, k) - F(\varphi, k)\} + \frac{2c}{b^2+c^2} \sqrt{a + b\sin x + c\cos x}$$
$$\left[0 < |a| < \sqrt{b^2+c^2};\ x_1 - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \leqslant x < x_1\right].$$

6. $$\int \frac{(b\cos x - c\sin x)\, dx}{\sqrt{a + b\sin x + c\cos x}} = 2\sqrt{a + b\sin x + c\cos x}.$$

7. $$\int_x^{x_1} \frac{\sqrt{b^2+c^2} + b\sin x + c\cos x}{\sqrt{a + b\sin x + c\cos x}} dx = 2\sqrt{a + \sqrt{b^2+c^2}}\, E(\varphi, k) -$$
$$- \frac{2\left(a - \sqrt{b^2+c^2}\right)}{\sqrt{a + \sqrt{b^2+c^2}}} F(\varphi, k) \qquad \left[0 < \sqrt{b^2+c^2} < a;\ x_1 - \pi \leqslant x < x_1\right].$$

8. $$\int_x^{x_1} \frac{\sqrt{b^2+c^2}+b\sin x+c\cos x}{\sqrt{a+b\sin x+c\cos x}}\,dx = 2\sqrt{2}\sqrt[4]{b^2+c^2}\,E(\varphi, k)$$

$$\left[0<|a|<\sqrt{b^2+c^2};\ x_1 - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \leqslant x < x_1\right].$$

1.5.22. Интегралы вида $\int R\left(\sqrt{1-k^2\sin^2 x}\right)dx$.

Условие: $0<k^2<1$.

Обозначение: $\Delta=\sqrt{1-k^2\sin^2 x}$.

1. $$\int \Delta^n\,dx = \frac{n-1}{n}(2-k^2)\int \Delta^{n-2}\,dx - \frac{n-2}{n}(1-k^2)\int \Delta^{n-4}\,dx + \frac{k^2}{n}\Delta^{n-2}\sin x\cos x.$$

2. $$\int \frac{dx}{\Delta^{n+1}} = -\frac{k^2\sin x\cos x}{(n-1)(1-k^2)\Delta^{n-1}} + \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)}\int\frac{dx}{\Delta^{n-1}} - \frac{n-3}{(n-1)(1-k^2)}\int\frac{dx}{\Delta^{n-3}}.$$

3. $$\int \Delta\,dx = E(x, k).$$

4. $$\int \Delta^3\,dx = \frac{2}{3}(2-k^2)E(x, k) - \frac{1-k^2}{3}F(x, k) + \frac{k^2}{3}\Delta\sin x\cos x.$$

5. $$\int\frac{dx}{\Delta} = F(x, k).$$ 6. $$\int\frac{dx}{\Delta^3} = \frac{1}{1-k^2}E(x, k) - \frac{k^2}{1-k^2}\,\frac{\sin x\cos x}{\Delta}.$$

1.5.23. Интегралы вида $\int R\left(\sin x, \sqrt{1-k^2\sin^2 x}\right)dx$.

Условие: $0<k^2<1$.

Обозначение: $\Delta=\sqrt{1-k^2\sin^2 x}$.

1. $$\int\frac{\sin x\,dx}{\Delta^p} = \frac{-\cos x}{(p-2)(1-k^2)\Delta^{p-1}} + \frac{p-3}{(p-2)(1-k^2)}\int\frac{\sin x\,dx}{\Delta^{p-1}}.$$

2. $$\int\frac{\sin x\,dx}{\Delta^{2n+1}} = -\cos x\sum_{l=0}^{n-1}\frac{2^l(n-1)(n-2)\ldots(n-l-1)}{(2n-1)(2n-3)\ldots(2n-2l-1)(1-k^2)^{l+1}\Delta^{2n-2l-1}}.$$

3. $$= -\frac{1}{(1-k^2)^n}\sum_{l=0}^{n-1}\binom{n-1}{k}\frac{(-k^2)^l}{2l+1}\,\frac{\cos^{2l+1}x}{\Delta^{2l+1}}.$$

4. $$\int\frac{\sin^p x\,dx}{\Delta^{p+3}} = \frac{-\sin^{p-3}x\cos x}{(p+1)k^2(1-k^2)\Delta^{p+1}} + \frac{2(p-1)k^2-p+2}{(p+1)k^2(1-k^2)}\int\frac{\sin^{p-2}x\,dx}{\Delta^{p+1}} + \frac{p-3}{(p+1)k^2(1-k^2)}\int\frac{\sin^{p-4}x\,dx}{\Delta^{p-1}}.$$

5. $$\int\Delta\sin x\,dx = -\frac{\Delta\cos x}{2} - \frac{1-k^2}{2k}\ln(k\cos x+\Delta).$$

6. $$\int\Delta\sin^2 x\,dx = -\frac{\Delta}{3}\sin x\cos x + \frac{1-k^2}{3k^2}F(x, k) + \frac{2k^2-1}{3k^2}E(x, k).$$

7. $\int \Delta \sin^3 x\,dx = -\frac{2k^2\sin^2 x + 3k^2 - 1}{8k^2}\Delta\cos x + \frac{3k^4 - 2k^2 - 1}{8k^3}\ln(k\cos x + \Delta).$

8. $\int \Delta \sin^4 x\,dx = -\frac{3k^2\sin^2 x + 4k^2 - 1}{15k^2}\Delta\sin x\cos x -$

$$-\frac{2(2k^4 - k^2 - 1)}{15k^4}F(x, k) + \frac{8k^4 - 3k^2 - 2}{15k^4}E(x, k).$$

9. $\int \Delta^3 \sin x\,dx = \frac{2k^2\sin^2 x + 3k^2 - 5}{8}\Delta\cos x - \frac{3(1-k^2)^2}{8k}\ln(k\cos x + \Delta).$

10. $\int \Delta^3 \sin^2 x\,dx = \frac{3k^2\sin^2 x + 4k^2 - 6}{15}\Delta\sin x\cos x +$

$$+\frac{(1-k^2)(3-4k^2)}{15k^2}F(x, k) - \frac{8k^4 - 13k^2 + 3}{15k^2}E(x, k).$$

11. $\int \Delta^3 \sin^3 x\,dx = \frac{8k^4\sin^4 x + 2k^2(5k^2 - 7)\sin^2 x + 15k^4 - 22k^2 + 3}{48k^2}\Delta\cos x -$

$$-\frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3}\ln(k\cos x + \Delta).$$

12. $\int \frac{\sin^n x\,dx}{\Delta} = \frac{\sin^{n-3} x}{(n-1)k^2}\Delta\cos x +$

$$+\frac{n-2}{n-1}\frac{1+k^2}{k^2}\int \frac{\sin^{n-2} x}{\Delta}dx - \frac{n-3}{(n-1)k^2}\int \frac{\sin^{n-4} x}{\Delta}dx.$$

13. $\int \frac{\sin x\,dx}{\Delta} = \frac{1}{2k}\ln\frac{\Delta - k\cos x}{\Delta + k\cos x} = -\frac{1}{k}\ln(k\cos x + \Delta) = \frac{1}{k}\ln(\Delta - k\cos x).$

14. $\int \frac{\sin^2 x\,dx}{\Delta} = \frac{1}{k^2}F(x, k) - \frac{1}{k^2}E(x, k).$

15. $\int \frac{\sin^3 x\,dx}{\Delta} = \frac{\Delta\cos x}{2k^2} - \frac{1+k^2}{2k^3}\ln(k\cos x + \Delta).$

16. $\int \frac{\sin^4 x\,dx}{\Delta} = \frac{\Delta\sin x\cos x}{3k^2} + \frac{2+k^2}{3k^4}F(x, k) - \frac{2(1+k^2)}{3k^4}E(x, k).$

17. $\int \frac{\sin x\,dx}{\Delta^3} = -\frac{\cos x}{(1-k^2)\Delta}.$

18. $\int \frac{\sin^2 x\,dx}{\Delta^3} = \frac{1}{k^2(1-k^2)}E(x, k) - \frac{1}{k^2}F(x, k) - \frac{1}{1-k^2}\frac{\sin x\cos x}{\Delta}.$

19. $\int \frac{\sin^3 x\,dx}{\Delta^3} = -\frac{\cos x}{k^2(1-k^2)\Delta} + \frac{1}{k^3}\ln(k\cos x + \Delta).$

20. $\int \frac{\sin^4 x\,dx}{\Delta^3} = \frac{2-k^2}{k^4(1-k^2)}E(x, k) - \frac{2}{k^4}F(x, k) - \frac{\sin x\cos x}{k^2(1-k^2)\Delta}.$

21. $\int \frac{\Delta\,dx}{\sin x} = -\frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x} + k\ln(k\cos x + \Delta).$

22. $\int \frac{\Delta\,dx}{\sin^2 x} = (1-k^2)F(x, k) - E(x, k) - \Delta\operatorname{ctg} x.$

23. $\int \frac{\Delta\,dx}{\sin^3 x} = -\frac{\Delta\cos x}{2\sin^2 x} + \frac{1-k^2}{4}\ln\frac{\Delta + \cos x}{\Delta - \cos x}.$

24. $\int \frac{\Delta\, dx}{\sin^4 x} =$

$$= \frac{1}{3}\left[-\Delta \operatorname{ctg}^3 x + (k^2-3)\Delta \operatorname{ctg} x + 2(1-k^2) F(x, k) + (k^2-2) E(x, k)\right].$$

25. $\int \frac{dx}{\Delta \sin^n x} =$

$$= -\frac{\Delta \cos x}{(n-1)\sin^{n-1} x} + \frac{(n-2)(1+k^2)}{n-1}\int \frac{dx}{\Delta \sin^{n-2} x} - \frac{(n-3)k^2}{n-1}\int \frac{dx}{\Delta \sin^{n-4} x}.$$

26. $\int \frac{dx}{\Delta \sin x} = -\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x} = \ln\frac{\Delta+\cos x}{\sin x}.$

27. $\int \frac{dx}{\Delta \sin^2 x} = \int \frac{1+\operatorname{ctg}^2 x}{\Delta}\, dx = F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$

28. $\int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2\sin^2 x} - \frac{1+k^2}{4}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$

29. $\int \frac{dx}{\Delta \sin^4 x} =$

$$= \frac{1}{3}\left[-\Delta \operatorname{ctg}^3 x - \Delta(2k^2+3)\operatorname{ctg} x + (k^2+2) F(x, k) - 2(k^2+1) E(x, k)\right].$$

1.5.24. Интегралы вида $\int R\left(\cos x, \sqrt{1-k^2\sin^2 x}\right) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$.

1. $\int \frac{\cos x\, dx}{\Delta^p} = \frac{\sin x}{(p-2)\Delta^{p-2}} + \frac{p-3}{p-2}\int \frac{\cos x\, dx}{\Delta^{p-2}}.$

2. $\int \frac{\cos x\, dx}{\Delta^{2n+1}} = -\sin x \sum_{l=0}^{n-1} \frac{2^l (n-1)(n-2)\dots(n-l+1)}{(2n-1)(2n-3)\dots(2n-2l-1)\,\Delta^{2n-2l-1}}.$

3. $\qquad = \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{k^{2l}}{2l+1}\,\frac{\sin^{2l+1} x}{\Delta^{2l+1}}.$

4. $\int \Delta \cos x\, dx = \frac{\Delta \sin x}{2} + \frac{1}{2k}\arcsin(k\sin x).$

5. $\int \Delta \cos^2 x\, dx = \frac{\Delta}{3}\sin x\cos x - \frac{1-k^2}{3k^2} F(x, k) + \frac{k^2+1}{3k^2} E(x, k).$

6. $\int \Delta \cos^3 x\, dx = \frac{2k^2\cos^2 x + 2k^2 + 1}{8k^2}\,\Delta \sin x + \frac{4k^2-1}{8k^3}\arcsin(k\sin x).$

7. $\int \Delta \cos^4 x\, dx = \frac{3k^2\cos^2 x + 3k^2 + 1}{15k^2}\,\Delta \sin x\cos x +$

$$+ \frac{2(1-k^2)(1-3k^2)}{15k^4} F(x, k) + \frac{3k^4+7k^2-2}{15k^4} E(x, k).$$

8. $\int \Delta^3 \cos x\, dx = \frac{-2k^2\sin^2 x + 5}{8}\,\Delta \sin x + \frac{3}{8k}\arcsin(k\sin x).$

9. $$\int \Delta^3 \cos^2 x\, dx = \frac{-3k^2 \sin^2 x + k^2 + 6}{15} \Delta \sin x \cos x - \frac{(1-k^2)(k^2+3)}{15k^2} F(x, k) - \frac{2k^4 - 7k^2 - 3}{15k^2} E(x, k).$$

10. $$\int \Delta^3 \cos^3 x\, dx = \frac{8k^4 \sin^4 x - 2k^2 (6k^2+7) \sin^2 x + 30k^2 + 3}{48k^3} \Delta \sin x + \frac{6k^2 - 1}{16k^3} \arcsin (k \sin x).$$

11. $$\int \frac{\cos^n x\, dx}{\Delta} = \frac{\cos^{n-3} x}{(n-1) k^2} \Delta \sin x + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} dx + \frac{n-3}{n-1} \frac{1-k^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} dx.$$

12. $$\int \frac{\cos x\, dx}{\Delta} = \frac{1}{k} \arcsin (k \sin x) = \frac{1}{k} \operatorname{arctg} \frac{k \sin x}{\Delta}.$$

13. $$\int \frac{\cos^2 x\, dx}{\Delta} = \frac{1}{k^2} E(x, k) - \frac{1-k^2}{k^2} F(x, k).$$

14. $$\int \frac{\cos^3 x\, dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{2k^2-1}{2k^3} \arcsin (k \sin x).$$

15. $$\int \frac{\cos x\, dx}{\Delta^3} = \frac{\sin x}{\Delta}.$$

16. $$\int \frac{\cos^2 x\, dx}{\Delta^3} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta}.$$

17. $$\int \frac{\cos^3 x\, dx}{\Delta^3} = -\frac{(1-k^2) \sin x}{k^2 \Delta} + \frac{1}{k^3} \arcsin (k \sin x).$$

18. $$\int \frac{\cos^4 x\, dx}{\Delta^3} = \frac{2-k^2}{k^4} E(x, k) - \frac{2(1-k^2)}{k^4} F(x, k) - \frac{(1-k^2) \sin x \cos x}{k^2 \Delta}.$$

19. $$\int \frac{\cos^4 x\, dx}{\Delta} = \frac{\Delta \sin x \cos x}{3k^2} + \frac{4k^2-2}{3k^4} E(x, k) + \frac{3k^4 - 5k^2 + 2}{3k^4} F(x, k).$$

20. $$\int \frac{\Delta\, dx}{\cos x} = \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} + k \arcsin (k \sin x).$$

21. $$\int \frac{\Delta\, dx}{\cos^2 x} = F(x, k) - E(x, k) + \Delta \operatorname{tg} x.$$

22. $$\int \frac{\Delta\, dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$$

23. $$\int \frac{\Delta\, dx}{\cos^4 x} = \frac{1}{3} \left\{ \left[\operatorname{tg}^3 x - \frac{2k^2-3}{1-k^2} \operatorname{tg} x \right] \Delta + 2F(x, k) + \frac{k^2-2}{1-k^2} E(x, k) \right\}.$$

24. $$\int \frac{dx}{\Delta \cos x} = -\frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$$

25. $$\int \frac{dx}{\Delta \cos^2 x} = F(x, k) - \frac{1}{1-k^2} E(x, k) + \frac{1}{1-k^2} \Delta \operatorname{tg} x.$$

26. $$\int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2(1-k^2) \cos^2 x} + \frac{2k^2-1}{4(1-k^2)^{3/2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$$

27. $$\int \frac{dx}{\Delta \cos^4 x} = \frac{1}{3(1-k^2)}\left[\Delta \operatorname{tg}^3 x - \frac{5k^2-3}{1-k^2}\Delta \operatorname{tg} x - (3k^2-2)F(x, k) + \frac{2(2k^2-1)}{1-k^2}E(x, k)\right].$$

28. $$\int \frac{dx}{\Delta \cos^n x} = \frac{\Delta \sin x}{(n-1)(1-k^2)\cos^{n-1} x} - \frac{(n-2)(2k^2-1)}{(n-1)(1-k^2)}\int \frac{dx}{\Delta \cos^{n-2} x} + \frac{(n-3)k^2}{(n-1)(1-k^2)}\int \frac{dx}{\Delta \cos^{n-4} x}.$$

1.5.25. Интегралы вида $\int \sin^m x \cos^n x \sqrt{(1-k^2 \sin^2 x)^p}\, dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $$\int \Delta^p \sin^m x \cos^n x\, dx = \frac{1}{(m+n+p)k^2}\Big\{\Delta^{p+2} \sin^{m-3} x \cos^{n+1} x + [m+n-2+(m+p-1)k^2]\int \Delta^p \sin^{m-2} x \cos^n x\, dx - (m-3)\int \Delta^p \sin^{m-4} x \cos^n x\, dx\Big\}.$$

2. $$= \frac{1}{(m+n+p)k^2}\Big\{\Delta^{p+2} \sin^{m+1} x \cos^{n-3} x + [(n+p-1)k^2 - (m+n-2)(1-k^2)]\int \Delta^p \sin^m x \cos^{n-2} x\, dx + (n-3)(1-k^2)\int \Delta^p \sin^m x \cos^{n-4} x\, dx\Big\} \quad [m+n+p \neq 0].$$

3. $$\int \Delta^p \sin x \cos^n x\, dx = -\frac{\Delta^{p+2} \cos^{n-1} x}{(n+p+1)k^2} - \frac{(n-1)(1-k^2)}{(n+p+1)k^2}\int \Delta^p \cos^{n-2} x \sin x\, dx.$$

4. $$\int \Delta^p \sin^m x \cos x\, dx = -\frac{\Delta^{p+2} \sin^{m-1} x}{(m+p+1)k^2} + \frac{m-1}{(m+p+1)k^2}\int \Delta^p \sin^{m-2} x \cos x\, dx.$$

5. $$\int \Delta^p \sin^3 x \cos^n x\, dx = \frac{(n+p+1)k^2 \cos^2 x + [(p+2)k^2+n+1]}{(n+p+1)(n+p+3)k^4}\Delta^{p+2} \cos^{n-1} x - \frac{[(p+2)k^2+n+1](n-1)(1-k^2)}{(n+p+1)(n+p+3)k^4}\int \Delta^p \cos^{n-2} x \sin x\, dx.$$

6. $$\int \Delta^p \sin^m x \cos^3 x\, dx = \frac{(m+p+1)k^2 \sin^2 x - [(p-m+1)k^2+m+1]}{(m+p+1)(m+p+3)k^4}\Delta^{p+2} \sin^{m-1} x + \frac{[(p-m+1)k^2+m+1](m-1)}{(m+p+1)(m+p+3)k^4}\int \Delta^p \sin^{m-2} x \cos x\, dx.$$

7. $$\int \Delta \sin x \cos x\, dx = -\frac{\Delta^3}{3k^2}.$$

8. $$\int \Delta \sin x \cos^2 x\, dx = -\frac{2k^2 \cos^2 x + 1 - k^2}{8k^2}\Delta \cos x + \frac{(1-k^2)^2}{8k^3}\ln(k \cos x + \Delta).$$

9. $\int \Delta \sin x \cos^3 x\, dx = -\frac{3k^4 \sin^4 x - k^2 (5k^2+1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta.$

10. $\int \Delta \sin x \cos^4 x\, dx =$

$$= \frac{-8k^4 \sin^4 x + 2k^2 (7k^2+1) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x - \frac{(1-k^2)^3}{16k^5} \ln (k \cos x + \Delta).$$

11. $\int \Delta \sin^2 x \cos x\, dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin (k \sin x).$

12. $\int \Delta \sin^2 x \cos^2 x\, dx = -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x -$

$$- \frac{(1-k^2)(2-k^2)}{15k^4} F(x, k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x, k).$$

13. $\int \Delta \sin^2 x \cos^3 x\, dx =$

$$= \frac{-8k^4 \sin^4 x + 2k^2 (6k^2+1) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x + \frac{2k^2 - 1}{16k^5} \arcsin (k \sin x).$$

14. $\int \Delta \sin^3 x \cos x\, dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta.$

15. $\int \Delta \sin^3 x \cos^2 x\, dx =$

$$= \frac{8k^4 \sin^4 x - 2k^2 (k^2+1) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x + \frac{(1-k^2)^2 (k^2+1)}{16k^5} \ln (k \cos x + \Delta).$$

16. $\int \Delta \sin^4 x \cos x\, dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin (k \sin x).$

17. $\int \Delta^3 \sin x \cos x\, dx = -\frac{\Delta^5}{5k^2}.$

18. $\int \Delta^3 \sin x \cos^2 x\, dx =$

$$= \frac{-8k^4 \sin^4 x + 2k^2 (k^2+7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \Delta \cos x + \frac{(1-k^2)^3}{16k^3} \ln (k \cos x + \Delta).$$

19. $\int \Delta^3 \sin^2 x \cos x\, dx =$

$$= \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x + \frac{1}{16k^3} \arcsin (k \sin x).$$

1.5.26. Интегралы вида $\int \frac{\sin^m x}{\cos^n x}\sqrt{1-k^2\sin^2 x}\,dx$.

Условие: $0<k^2<1$.

Обозначение: $\Delta=\sqrt{1-k^2\sin^2 x}$.

1. $\int \frac{\sin x}{\cos x}\Delta\,dx=\int \Delta\,\operatorname{tg} x\,dx=-\Delta+\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}$.

2. $\int \frac{\sin x}{\cos^2 x}\Delta\,dx=\frac{\Delta}{\cos x}-k\ln(k\cos x+\Delta)$.

3. $\int \frac{\sin x}{\cos^3 x}\Delta\,dx=\frac{\Delta}{2\cos^2 x}+\frac{k^2}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}$.

4. $\int \frac{\sin x}{\cos^4 x}\Delta\,dx=\frac{-(2k^2+1)k^2\sin^2 x+3k^4-k^2+1}{3(1-k^2)\cos^3 x}\Delta$.

5. $\int \frac{\sin^2 x}{\cos x}\Delta\,dx=$

$$=-\frac{\Delta\sin x}{2}+\frac{2k^2-1}{2k}\arcsin(k\sin x)+\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}.$$

6. $\int \frac{\sin^2 x}{\cos^2 x}\Delta\,dx=\int \Delta\,\operatorname{tg}^2 x\,dx=\Delta\operatorname{tg} x+F(x,k)-2E(x,k)$.

7. $\int \frac{\sin^2 x}{\cos^3 x}\Delta\,dx=\frac{\sin x}{2\cos^2 x}\Delta+\frac{2k^2-1}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}-k\arcsin(k\sin x)$.

8. $\int \frac{\sin^3 x}{\cos x}\Delta\,dx=-\frac{k^2\sin x+3k^2-1}{3k^2}\Delta+\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}$.

9. $\int \frac{\sin^3 x}{\cos^2 x}\Delta\,dx=-\frac{\sin^2 x-3}{2\cos x}\Delta-\frac{3k^2-1}{2k}\ln(k\cos x+\Delta)$.

10. $\int \frac{\sin^4 x}{\cos x}\Delta\,dx=-\frac{2k^2\sin^2 x+4k^2-1}{8k^2}\Delta\sin x+$

$$+\frac{8k^4-4k^2-1}{8k^3}\arcsin(k\sin x)+\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}.$$

11. $\int \frac{\cos x}{\sin x}\Delta\,dx=\int \Delta\operatorname{ctg} x\,dx=\Delta+\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$.

12. $\int \frac{\cos^2 x}{\sin x}\Delta\,dx=\frac{\Delta\cos x}{2}+\frac{k^2+1}{2k}\ln(k\cos x+\Delta)+\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}$.

13. $\int \frac{\cos^3 x}{\sin x}\Delta\,dx=-\frac{k^2\sin^2 x-3k^2-1}{3k^2}\Delta+\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$.

14. $\int \frac{\cos^4 x}{\sin x}\Delta\,dx=\frac{-2k^2\sin^2 x+5k^2+1}{8k^2}\Delta\cos x+$

$$+\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}+\frac{3k^4+6k^2-1}{8k^3}\ln(k\cos x+\Delta).$$

15. $\int \frac{\cos x}{\sin^2 x}\Delta\,dx=-\frac{\Delta}{\sin x}-k\arcsin(k\sin x)$.

16. $\int \frac{\cos^2 x}{\sin^2 x} \Delta\, dx = \int \operatorname{ctg}^2 x \Delta\, dx = -\Delta \operatorname{ctg} x + (1-k^2) F(x, k) - 2E(x, k).$

17. $\int \frac{\cos^3 x}{\sin^2 x} \Delta\, dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2+1}{2k} \arcsin (k \sin x).$

18. $\int \frac{\cos x}{\sin^3 x} \Delta\, dx = -\frac{\Delta}{2 \sin^2 x} + \frac{k^2}{4} \ln \frac{1+\Delta}{1-\Delta}.$

19. $\int \frac{\cos x}{\sin^4 x} \Delta\, dx = -\frac{\Delta^3}{3 \sin^3 x}.$

20. $\int \frac{\cos^2 x}{\sin^3 x} \Delta\, dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2+1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln (k \cos x + \Delta).$

1.5.27. Интегралы вида $\int \frac{\sin^p x \cos^q x}{\sqrt{(1-k^2 \sin^2 x)^r}}\, dx.$

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $\int \frac{\sin^p x \cos^q x\, dx}{\Delta^r} =$

$$= \frac{-k^2 \sin^{p+1} x \cos^{q+1} x}{(r-2)(1-k^2)\Delta^{r-2}} + \frac{p-q-(2-k^2)(p-r+3)}{(r-2)(1-k^2)} \int \frac{\sin^p x \cos^q x}{\Delta^{r-2}}\, dx + \frac{p+q-r+4}{(r-2)(1-k^2)} \int \frac{\sin^p x \cos^q x}{\Delta^{r-4}}\, dx.$$

2. $$= \frac{1}{k^2} \int \frac{\sin^{p-2} x \cos^q x}{\Delta^r}\, dx - \frac{1}{k^2} \int \frac{\sin^{p-2} x \cos^q x}{\Delta^{r-2}}\, dx.$$

3. $$= \frac{k^2-1}{k^2} \int \frac{\sin^p x \cos^{q-2} x\, dx}{\Delta^r} + \frac{1}{k^2} \int \frac{\sin^p x \cos^{q-2} x}{\Delta^{r-2}}\, dx.$$

4. $\int \frac{\sin x \cos x\, dx}{\Delta^r} = \frac{1}{(r-2) k^2 \Delta^{r-2}}.$

5. $\int \frac{\sin x \cos x\, dx}{\Delta} = -\frac{\Delta}{k^2}.$

6. $\int \frac{\sin x \cos^2 x\, dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{1-k^2}{2k^3} \ln (k \cos x + \Delta).$

7. $\int \frac{\sin x \cos^3 x\, dx}{\Delta} = -\frac{1}{3k^4} (k^2 \cos^2 x - 2 + 2k^2) \Delta.$

8. $\int \frac{\sin x \cos^4 x\, dx}{\Delta} = \frac{3 - 5k^2 + 2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4 - 6k^2 + 3}{8k^5} \ln (k \cos x + \Delta).$

9. $\int \frac{\sin^2 x \cos x\, dx}{\Delta} = -\frac{\Delta \sin x}{2k^2} + \frac{\arcsin (k \sin x)}{2k^3}.$

10. $\int \frac{\sin^2 x \cos^2 x\, dx}{\Delta} = -\frac{\Delta \sin x \cos x}{3k^2} + \frac{2-k^2}{3k^4} E(x, k) + \frac{2k^2-2}{3k^4} F(x, k).$

11. $\int \frac{\sin^2 x \cos^3 x\, dx}{\Delta} = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \Delta \sin x + \frac{4k^2-3}{8k^5} \arcsin (k \sin x).$

12. $\int \frac{\sin^2 x \cos^4 x\,dx}{\Delta} = -\frac{3k^2\cos^2 x + 3k^2 - 4}{15k^4}\Delta \sin x \cos x + \frac{9k^4 - 17k^2 + 8}{15k^6}F(x, k) - \frac{3k^4 - 13k^2 + 8}{15k^6}E(x, k).$

13. $\int \frac{\sin^3 x \cos x\,dx}{\Delta} = -\frac{1}{3k^4}(2 + k^2 \sin^2 x)\Delta.$

14. $\int \frac{\sin^3 x \cos^2 x\,dx}{\Delta} = \frac{2k^2\cos^2 x - k^2 - 3}{8k^4}\Delta \cos x - \frac{k^4 + 2k^2 - 3}{8k^5}\ln(k\cos x + \Delta).$

15. $\int \frac{\sin^3 x \cos^3 x\,dx}{\Delta} = \frac{3k^4 \sin^4 x - (5k^4 - 4k^2)\sin^2 x - 10k^2 + 8}{15k^6}\Delta.$

16. $\int \frac{\sin^3 x \cos^4 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2(7k^2 - 5)\sin^2 x + 3k^4 - 22k^2 + 15}{48k^6}\Delta \cos x - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7}\ln(k\cos x + \Delta).$

17. $\int \frac{\sin^4 x \cos x\,dx}{\Delta} = -\frac{2k^2\sin^2 x + 3}{8k^4}\Delta \sin x + \frac{3}{8k^5}\arcsin(k \sin x).$

18. $\int \frac{\sin^4 x \cos^2 x\,dx}{\Delta} = \frac{3k^2\cos^2 x - 2k^2 - 4}{15k^4}\Delta \sin x \cos x + \frac{k^4 + 7k^2 - 8}{15k^6}F(x, k) - \frac{2k^4 + 3k^2 - 8}{15k^6}E(x, k).$

19. $\int \frac{\sin^4 x \cos^3 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2(6k^2 - 5)\sin^2 x - 18k^2 + 15}{48k^6}\Delta \sin x + \frac{6k^2 - 5}{16k^7}\arcsin(k \sin x).$

20. $\int \frac{\sin^m x \cos^n x\,dx}{\Delta^3} = \frac{\sin^{m-1} x \cos^{n-1} x}{k^2\Delta} - \frac{m-1}{k^2}\int \frac{\sin^{m-2} x \cos^n x\,dx}{\Delta} + \frac{n-1}{k^2}\int \frac{\sin^m x \cos^{n-2} x}{\Delta}dx.$

21. $\int \frac{\sin x \cos x\,dx}{\Delta^3} = \frac{1}{k^2\Delta}.$

22. $\int \frac{\sin x \cos^2 x\,dx}{\Delta^3} = \frac{\cos x}{k^2\Delta} - \frac{1}{k^3}\ln(k \cos x + \Delta).$

23. $\int \frac{\sin x \cos^3 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4\Delta}.$

24. $\int \frac{\sin x \cos^4 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4\Delta}\cos x + \frac{3(1 - k^2)}{2k^5}\ln(k \cos x + \Delta).$

25. $\int \frac{\sin^2 x \cos x\,dx}{\Delta^3} = \frac{\sin x}{k^2\Delta} - \frac{1}{k^3}\arcsin(k \sin x).$

26. $\int \frac{\sin^2 x \cos^2 x\,dx}{\Delta^3} = \frac{2 - k^2}{k^4}F(x, k) - \frac{2}{k^4}E(x, k) + \frac{\sin x \cos x}{k^2\Delta}.$

27. $\int \frac{\sin^2 x \cos^3 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4\Delta}\sin x - \frac{2k^2 - 3}{2k^5}\arcsin(k \sin x).$

28. $$\int \frac{\sin^3 x \cos x \, dx}{\Delta^3} = \frac{2-k^2 \sin^2 x}{k^4 \Delta}.$$

29. $$\int \frac{\sin^3 x \cos^2 x \, dx}{\Delta^3} = \frac{-k^2 \sin^2 x+3}{2k^4 \Delta} \cos x + \frac{k^2-3}{2k^5} \ln (k \cos x + \Delta).$$

30. $$\int \frac{\sin^4 x \cos x \, dx}{\Delta^3} = \frac{-k^2 \sin^2 x+3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin (k \sin x).$$

1.5.28. Интегралы вида $\int \frac{\sin^p x}{\cos^q x} \frac{dx}{\sqrt{(1-k^2 \sin^2 x)^r}}$, $\int \frac{\cos^p x}{\sin^q x} \frac{dx}{\sqrt{(1-k^2 \sin^2 x)^r}}$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $$\int \frac{\sin^p x}{\cos^q x} \frac{dx}{\Delta} = \frac{\sin^{p+1} x}{(q-1)(1-k^2) \cos^{q-1} x \, \Delta^{r-2}} - \frac{p-q+2-(p-2q-r+5) k^2}{(q-1)(1-k^2)} \int \frac{\sin^p x}{\cos^{q-2} x} \frac{dx}{\Delta^r} - \frac{(p-q-r+4) k^2}{(q-1)(1-k^2)} \int \frac{\sin^p x}{\cos^{q-4} x} \frac{dx}{\Delta^r}.$$

2. $$\int \frac{\cos^q x}{\sin^p x} \frac{dx}{\Delta^r} = \frac{-\cos^{q+1} x}{(p-1) \sin^{p-1} x \Delta^{r-2}} + \frac{p-q-2+(p+r-3) k^2}{p-1} \times \int \frac{\cos^q x}{\sin^{p-2} x} \frac{dx}{\Delta^r} + \frac{(p-q+r-4) k^2}{p-1} \int \frac{\cos^q x}{\sin^{p-4} x} \frac{dx}{\Delta^r}.$$

3. $$\int \frac{\sin x}{\cos^{2m} x} \frac{dx}{\Delta^{2n+1}} = (-1)^{m+n} \frac{k^{2(m+n-1)}}{(1-k^2)^{m+n}} \sum_{l=0}^{m+n-1} \binom{m+n-1}{l} \frac{(-1)^l}{(2n-2l-1)^{2l}} \left(\frac{\cos x}{\Delta}\right)^{2n-2l-1}.$$

4. $$\int \frac{\cos x}{\sin^{2m}} \frac{dx}{\Delta^{2n+1}} = \sum_{l=0}^{m+n-1} \binom{m+n-1}{l} \frac{k^{2(m+n-l-1)}}{2n-2l-1} \left(\frac{\sin x}{\Delta}\right)^{2n-2l-1}.$$

5. $$\int \frac{\sin x}{\cos x} \frac{dx}{\Delta} = \int \operatorname{tg} x \frac{dx}{\Delta} = \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

6. $$\int \frac{\sin x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{(1-k^2) \cos x}.$$

7. $$\int \frac{\sin x}{\cos^3 x} \frac{dx}{\Delta} = \int \operatorname{tg} x (1+\operatorname{tg}^2 x) \frac{dx}{\Delta} = \frac{\Delta}{2(1-k^2) \cos^2 x} - \frac{k^2}{4(1-k^2)^{3/2}} \ln \frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

8. $$\int \frac{\sin x}{\cos^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \cos^2 x-1+k^2}{3(1-k^2)^2 \cos^3 x} \Delta.$$

9. $$\int \frac{\sin^2 x}{\cos x} \frac{dx}{\Delta} = \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta+\sqrt{1-k^2} \sin x}{\Delta-\sqrt{1-k^2} \sin x} - \frac{1}{k} \arcsin (k \sin x).$$

10. $$\int \frac{\sin^2 x}{\cos^2 x}\frac{dx}{\Delta} = \int \frac{\operatorname{tg}^2 x}{\Delta}\,dx = \frac{\Delta}{(1-k^2)}\operatorname{tg} x - \frac{1}{(1-k^2)}E(x, k).$$

11. $$\int \frac{\sin^2 x}{\cos^3 x}\frac{dx}{\Delta} = \frac{\Delta \sin x}{2(1-k^2)\cos^2 x} - \frac{1}{4(1-k^2)^{3/2}}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}.$$

12. $$\int \frac{\sin^3 x}{\cos x}\frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

13. $$\int \frac{\sin^3 x}{\cos^2 x}\frac{dx}{\Delta} = \frac{\Delta}{(1-k^2)\cos x} + \frac{1}{k}\ln(k\cos x+\Delta).$$

14. $$\int \frac{\sin^4 x}{\cos x}\frac{dx}{\Delta} =$$
$$= \frac{\Delta \sin x}{2k^2} + \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x} - \frac{2k^2+1}{2k^3}\arcsin(k\sin x).$$

15. $$\int \frac{\cos x}{\sin x}\frac{dx}{\Delta} = \int \operatorname{ctg} x\,\frac{dx}{\Delta} = \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}.$$

16. $$\int \frac{\cos^2 x}{\sin x}\frac{dx}{\Delta} = \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x} + \frac{1}{k}\ln(k\cos x+\Delta).$$

17. $$\int \frac{\cos^3 x}{\sin x}\frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2}\ln\frac{1+\Delta}{1-\Delta}.$$

18. $$\int \frac{\cos^4 x}{\sin x}\frac{dx}{\Delta} = \frac{\Delta\cos x}{2k^2} + \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x} + \frac{3k^2-1}{2k^3}\ln(k\cos x+\Delta).$$

19. $$\int \frac{\cos x}{\sin^2 x}\frac{dx}{\Delta} = -\frac{\Delta}{\sin x}.$$

20. $$\int \frac{\cos^2 x}{\sin^2 x}\frac{dx}{\Delta} = \int \frac{\operatorname{ctg}^2 x}{\Delta}\,dx = -\Delta\operatorname{ctg} x - E(x, k).$$

21. $$\int \frac{\cos^3 x}{\sin^2 x}\frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k}\arcsin(k\sin x).$$

22. $$\int \frac{\cos x}{\sin^3 x}\frac{dx}{\Delta} = -\frac{\Delta}{2\sin^2 x} - \frac{k^2}{4}\ln\frac{1+\Delta}{1-\Delta}.$$

23. $$\int \frac{\cos^2 x}{\sin^3 x}\frac{dx}{\Delta} = -\frac{\Delta\cos x}{2\sin^2 x} + \frac{1-k^2}{4}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

24. $$\int \frac{\cos x}{\sin^4 x}\frac{dx}{\Delta} = -\frac{2k^2\sin^2 x+1}{3\sin^3 x}\Delta.$$

25. $$\int \operatorname{tg}^n x\,\frac{dx}{\Delta} = \frac{\operatorname{tg}^{n-3} x}{(n-1)(1-k^2)}\frac{\Delta}{\cos^2 x} -$$
$$- \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)}\int \frac{\operatorname{tg}^{n-2} x}{\Delta}\,dx - \frac{n-3}{(n-1)(1-k^2)}\int \frac{\operatorname{tg}^{n-4} x}{\Delta}\,dx.$$

26. $$\int \operatorname{ctg}^n x\,\frac{dx}{\Delta} = -\frac{\operatorname{ctg}^{n-1} x}{n-1}\frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1}(2-k^2)\times$$
$$\times \int \frac{\operatorname{ctg}^{n-2} x}{\Delta}\,dx - \frac{n-3}{n-1}(1-k^2)\int \frac{\operatorname{ctg}^{n-4} x}{\Delta}\,dx.$$

1.5.29. Интегралы вида $\int \frac{\sqrt{(1-k^2\sin^2 x)^r}}{\sin^m x\cos^n x}\,dx$.

Условие: $0<k^2<1$.

Обозначение: $\Delta=\sqrt{1-k^2\sin^2 x}$.

1. $$\int \frac{\Delta\,dx}{\sin x\cos x}=\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}+\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

2. $$\int \frac{\Delta\,dx}{\sin x\cos^2 x}=\frac{\Delta}{\cos x}+\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

3. $$\int \frac{\Delta\,dx}{\sin x\cos^3 x}=\frac{\Delta}{2\cos^2 x}-\frac{1}{2}\ln\frac{1+\Delta}{1-\Delta}+\frac{2-k^2}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

4. $$\int \frac{\Delta\,dx}{\sin x\cos^4 x}=\frac{(2k^2-3)\sin^2 x-3k^2+4}{3(1-k^2)\cos^3 x}\Delta+\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

5. $$\int \frac{\Delta\,dx}{\sin^2 x\cos x}=\frac{-\Delta}{\sin x}-\frac{1+k^2}{2\sqrt{1-k^2}}\ln\frac{\Delta-\sqrt{1-k^2}\sin x}{\Delta+\sqrt{1-k^2}\sin x}.$$

6. $$\int \frac{\Delta\,dx}{\sin^2 x\cos^2 x}=\left(\frac{1}{1-k^2}\operatorname{tg} x-\operatorname{ctg} x\right)\Delta+2F(x,\ k)-\frac{2-k^2}{1-k^2}E(x,\ k).$$

7. $$\int \frac{\Delta\,dx}{\sin^2 x\cos^3 x}=\frac{3\sin^2 x-2}{2\sin x\cos^2 x}\Delta-\frac{2k^2-3}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}.$$

8. $$\int \frac{\Delta\,dx}{\sin^3 x\cos x}=-\frac{\Delta}{2\sin^2 x}+\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}+\frac{k^2-2}{4}\ln\frac{1+\Delta}{1-\Delta}.$$

9. $$\int \frac{\Delta\,dx}{\sin^3 x\cos^2 x}=\frac{3\sin^2 x-1}{2\sin^2 x\cos x}\Delta+\frac{k^2-3}{4}\ln\frac{\Delta-\cos x}{\Delta+\cos x}.$$

10. $$\int \frac{\Delta\,dx}{\sin^4 x\cos x}=-\frac{(3-k^2)\sin^2 x+1}{3\sin^3 x}\Delta-\frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta-\sqrt{1-k^2}\sin x}{\Delta+\sqrt{1-k^2}\sin x}.$$

1.5.30. Интегралы вида $\int \frac{dx}{\sin^m x\cos^n x\sqrt{(1-k^2\sin^2 x)^r}}$.

Условие: $0<k^2<1$.

Обозначение: $\Delta=\sqrt{1-k^2\sin^2 x}$.

1. $$\int \frac{dx}{\Delta\sin x\cos x}=\int(\operatorname{tg} x+\operatorname{ctg} x)\frac{dx}{\Delta}=\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}+\frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

2. $$\int \frac{dx}{\Delta\sin x\cos^2 x}=\frac{\Delta}{(1-k^2)\cos x}+\frac{1}{2}\ln\frac{\Delta-\cos x}{\Delta+\cos x}.$$

3. $$\int \frac{dx}{\Delta\sin x\cos^3 x}=\int(\operatorname{ctg} x+2\operatorname{tg} x+\operatorname{tg}^3 x)\frac{dx}{\Delta}=\frac{\Delta}{2(1-k^2)\cos^2 x}-\frac{1}{2}\ln\frac{1+\Delta}{1-\Delta}+\frac{2-3k^2}{4(1-k^2)^{3/2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

4. $$\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{(5k^2-3)\sin^2 x - 6k^2+4}{3(1-k^2)^2 \cos^3 x}\Delta - \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

5. $$\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta-\sqrt{1-k^2}\sin x}{\Delta+\sqrt{1-k^2}\sin x}.$$

6. $$\int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \left(\frac{\operatorname{tg} x}{1-k^2} - \operatorname{ctg} x\right)\Delta + \frac{k^2-2}{1-k^2}E(x,\ k) + 2F(x,\ k).$$

7. $$\int \frac{dx}{\Delta \sin^2 x \cos^3 x} =$$
$$= \frac{(3-2k^2)\sin^2 x - 2(1-k^2)}{2(1-k^2)\sin x \cos^2 x}\Delta - \frac{4k^2-3}{4(1-k^2)^{3/2}}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}.$$

8. $$\int \frac{dx}{\Delta \sin^3 x \cos x} = -\frac{\Delta}{2\sin^2 x} + \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}} - \frac{k^2+2}{4}\ln\frac{1+\Delta}{1-\Delta}.$$

9. $$\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{(3-k^2)\sin^2 x - 1 + k^2}{2(1-k^2)\sin^2 x \cos x}\Delta + \frac{k^2+3}{4}\ln\frac{\Delta-\cos x}{\Delta+\cos x}.$$

10. $$\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3+2k^2)\sin^2 x + 1}{3\sin^3 x}\Delta - \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta-\sqrt{1-k^2}\sin x}{\Delta+\sqrt{1-k^2}\sin x}.$$

1.5.31. Интегралы вида
$\int R\left(\sin x,\ \cos x,\ \sqrt{1-p^2\sin^2 x},\ \sqrt{1-q^2\sin^2 x}\right)dx.$

Условия: $0 < p^2 < q^2 < 1,\ 0 < x \leqslant \pi/2.$

Обозначение: $\varphi = \arcsin\dfrac{\sqrt{1-p^2}\sin x}{\sqrt{1-p^2\sin^2 x}}.$

1. $$\int_0^x \frac{dx}{\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)}} = \frac{1}{\sqrt{1-p^2}}F\left(\varphi,\ \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$$

2. $$\int_0^x \frac{\operatorname{tg}^2 x\,dx}{\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)}} =$$
$$= \frac{\operatorname{tg} x\sqrt{1-q^2\sin^2 x}}{(1-q^2)\sqrt{1-p^2\sin^2 x}} - \frac{}{(1-q^2)\sqrt{1-p^2}}E\left(\varphi,\ \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$$

3. $$\int_0^x \frac{\sin^2 x\,dx}{\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)^3}} = \frac{\sqrt{1-p^2}}{(1-q^2)(q^2-p^2)}E\left(\varphi,\ \sqrt{\frac{q^2-p^2}{1-p^2}}\right) -$$
$$- \frac{1}{(q^2-p^2)\sqrt{1-p^2}}F\left(\varphi,\ \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{\sin x\cos x}{(1-q^2)\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)}}.$$

4. $$\int_0^x \frac{\cos^2 x\,dx}{\sqrt{(1-p^2\sin^2 x)^3(1-q^2\sin^2 x)}} =$$
$$= \frac{\sqrt{1-p^2}}{q^2-p^2}E\left(\varphi,\ \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1-q^2}{(q^2-p^2)\sqrt{1-p^2}}F\left(\varphi,\ \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$$

5. $\int\limits_0^x \frac{1}{1-p^2\sin^2 x}\sqrt{\frac{1-q^2\sin^2 x}{1-p^2\sin^2 x}}\,dx = \frac{1}{\sqrt{1-p^2}}E\left(\varphi, \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$

6. $\int\limits_0^x \sqrt{\frac{1-p^2\sin^2 x}{(1-q^2\sin^2 x)^3}}\,dx =$

$$= \frac{\sqrt{1-p^2}}{1-q^2}E\left(\varphi, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2}\frac{\sin x\cos x}{\sqrt{(1-p^2\sin^2 x)(1-q^2\sin^2 x)}}.$$

7. $\int\limits_0^x \frac{1}{1+(p^2r^2-p^2-r^2)\sin^2 x}\sqrt{\frac{1-p^2\sin^2 x}{1-q^2\sin^2 x}}\,dx =$

$$= \frac{1}{\sqrt{1-p^2}}\Pi\left(\varphi, r^2, \sqrt{\frac{q^2-p^2}{1-p^2}}\right).$$

1.5.32. Интегралы вида $\int \frac{(a+\sin x)^p}{\sqrt{1-k^2\sin^2 x}}\,dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$.

1. $\int \frac{(a+\sin x)^p}{\Delta}\,dx =$

$$= \frac{1}{(p-1)k^2}\left[(a+\sin x)^{p-3}\cos x\Delta + 2(2p-3)ak^2\int\frac{(a+\sin x)^{p-1}}{\Delta}dx + \right.$$
$$+(p-2)(1+k^2-6a^2k^2)\int\frac{(a+\sin x)^{p-2}}{\Delta}dx - a(2p-5)(1+k^2-2a^2k^2)\times$$
$$\left.\times\int\frac{(a+\sin x)^{p-3}}{\Delta}dx - (p-3)(1-a^2)(1-a^2k^2)\int\frac{(a+\sin x)^{p-4}}{\Delta}dx\right]$$
$$\left[a^2\neq 1, \ \frac{1}{k^2} \text{ при } p=-1, -2, \ldots\right].$$

2. $\int \frac{a+\sin x}{\Delta}\,dx = aF(x, k) + \frac{1}{2k}\ln\frac{\Delta-k\cos x}{\Delta+k\cos x}.$

3. $\int \frac{(a+\sin x)^2}{\Delta}\,dx = \frac{1+k^2a^2}{k^2}F(x, k) - \frac{1}{k^2}E(x, k) + \frac{a}{k}\ln\frac{\Delta-k\cos x}{\Delta+k\cos x}.$

4. $\int \frac{dx}{(a+\sin x)^n\Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)}\left[-\frac{\cos x\Delta}{(a+\sin x)^{n-1}} - \right.$
$$-(2n-3)(1+k^2-2a^2k^2)a\int\frac{dx}{(a+\sin x)^{n-1}\Delta} -$$
$$-(n-2)(6a^2k^2-k^2-1)\int\frac{dx}{(a+\sin x)^{n-2}\Delta} -$$
$$\left.-(10-4n)ak^2\int\frac{dx}{(a+\sin x)^{n-3}\Delta} - (n-3)k^2\int\frac{dx}{(a+\sin x)^{n-4}\Delta}\right].$$

5. $\int \frac{dx}{(a+\sin x)\Delta} =$

$$= \frac{1}{a}\Pi\left(x, \frac{1}{a^2}, k\right) + \frac{1}{2\sqrt{(1-a^2)(1-a^2k^2)}}\ln\frac{\sqrt{1-a^2}\,\Delta - \sqrt{1-k^2a^2}\cos x}{\sqrt{1-a^2}\,\Delta + \sqrt{1-k^2a^2}\cos x}.$$

6. $$\int \frac{dx}{(a+\sin x)^2 \Delta} = \frac{1}{(1-a^2)(1-a^2k^2)}\left[-\frac{\Delta \cos x}{a+\sin x} - \right.$$
$$\left. -a(1+k^2-2a^2k^2)\int \frac{dx}{(a+\sin x)\Delta} - 2ak^2 \int \frac{(a+\sin x)\,dx}{\Delta} + k^2 \int \frac{(a+\sin x)^2\,dx}{\Delta}\right].$$

7. $$\int \frac{dx}{(a+\sin x)^3 \Delta} =$$
$$= \frac{1}{2(1-a^2)(1-a^2k^2)}\left[-\frac{\Delta \cos x}{(a+\sin x)^2} - 3a(1+k^2-2a^2k^2)\int \frac{dx}{(a+\sin x)^2 \Delta} - \right.$$
$$\left. -(6a^2k^2-k^2-1)\int \frac{dx}{(a+\sin x)\Delta} + 2ak^2 F(x, k)\right].$$

8. $$\int \frac{dx}{(1\pm\sin x)^n \Delta} = \frac{1}{(2n-1)(1-k^2)}\left[\mp \frac{\Delta \cos x}{(1\pm\sin x)^n} + \right.$$
$$+(n-1)(1-5k^2)\int \frac{dx}{(1\pm\sin x)^{n-1}\Delta} + 2(2n-3)k^2 \int \frac{dx}{(1\pm\sin x)^{n-2}\Delta} -$$
$$\left. -(n-2)k^2 \int \frac{dx}{(1\pm\sin x)^{n-3}\Delta}\right].$$

9. $$\int \frac{dx}{(1\pm\sin x)\Delta} = \frac{\mp \Delta \cos x}{(1-k^2)(1\pm\sin x)} + F(x, k) - \frac{1}{1-k^2} E(x, k).$$

10. $$\int \frac{dx}{(1\pm\sin x)^2 \Delta} = \frac{1}{3(1-k^2)^2}\left[\mp \frac{(1-k^2)\Delta \cos x}{(1\pm\sin x)^2} \mp \frac{(1-5k^2)\Delta \cos x}{1\pm\sin x} + \right.$$
$$\left. +(1-3k^2)(1-k^2)F(x, k) - (1-5k^2)E(x, k)\right].$$

11. $$\int \frac{dx}{(1\pm k\sin x)^n \Delta} \overset{?}{=}$$
$$= \frac{1}{(2n-1)(1-k^2)}\left[\pm \frac{k\Delta \cos x}{(1\pm k\sin x)^n} + (n-1)(5-k^2)\int \frac{dx}{(1\pm k\sin x)^{n-1}\Delta} - \right.$$
$$\left. -2(2n-3)\int \frac{dx}{(1\pm k\sin x)^{n-2}\Delta} + (n-2)\int \frac{dx}{(1\pm k\sin x)^{n-2}\Delta}\right].$$

12. $$\int \frac{dx}{(1\pm k\sin x)\Delta} = \pm \frac{k\Delta \cos x}{(1-k^2)(1\pm k\sin x)} + \frac{1}{1-k^2} E(x, k).$$

13. $$\int \frac{dx}{(1\pm k\sin x)^2 \Delta} = \frac{1}{3(1-k^2)^2}\left[\pm \frac{k(1-k^2)\Delta \cos x}{(1\pm k\sin x)^2} \pm \right.$$
$$\left. \pm \frac{k(5-k^2)\Delta \cos x}{1\pm k\sin x} - 2(1-k^2)F(x, k) + (5-k^2)E(x, k)\right].$$

1.5.33. Интегралы вида $\int \frac{(1+a\sin^2 x)^p}{\sqrt{1-k^2\sin^2 x}}\,dx.$

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$.

1. $$\int \frac{(1+a\sin^2 x)^p}{\Delta}\,dx = \frac{1}{(2p-1)k^2}\left[a^2(1+a\sin^2 x)^{p-2}\Delta \sin x \cos x + \right.$$
$$+(2p-2)(3k^2+k^2a+a)\int \frac{(1+a\sin^2 x)^{p-1}}{\Delta}\,dx -$$
$$-(2p-3)(3k^2+2ak^2+2a+a^2)\int \frac{(1+a\sin^2 x)^{p-2}}{\Delta}\,dx +$$
$$\left. +2(p-2)a(a+1)(k^2+a)\int \frac{(1+a\sin^2 x)^{p-3}}{\Delta}\,dx\right].$$

2. $\int \frac{1+a\sin^2 x}{\Delta}\,dx = \left(1+\frac{a}{k^2}\right)F(x,k) - \frac{a}{k^2}E(x,k).$

3. $\int \frac{(1+a\sin^2 x)^2}{\Delta}\,dx = \frac{a^2}{3k^2}\Delta\sin x\cos x +$
$+\left[1+\frac{2a}{k^2}+\frac{(2+k^2)a^2}{3k^4}\right]F(x,k) - \left[\frac{2a}{k^2}+\frac{2(k^2+1)a^2}{3k^4}\right]E(x,k).$

4. $\int \frac{dx}{(a+\sin^2 x)^n\Delta} = \frac{1}{2(n-1)(1+a)(a+k^2)}\left[\frac{\Delta\sin x\cos x}{(1+a\sin^2 x)^{n-1}} +\right.$
$+(2n-3)(a^2+2a+2ak^2+3k^2)\int\frac{dx}{(1+a\sin^2 x)^{n-1}\Delta} -$
$-(2n-4)(k^2a+a+3k^2)\int\frac{dx}{(1+a\sin^2 x)^{n-2}\Delta} +$
$\left.+(2n-5)k^2\int\frac{dx}{(1+a\sin^2 x)^{n-3}\Delta}\right].$

5. $\int \frac{dx}{(1+a\sin^2 x)\Delta} = \Pi(x,-a,k).$

6. $\int \frac{dx}{(1+a\sin^2 x)^2\Delta} =$
$= \frac{1}{2(1+a)(k^2+a^2)}\left[\frac{a^2\Delta\sin x\cos x}{1+a\sin^2 x} + (a+k^2)F(x,k) + aE(x,k) +\right.$
$\left.+(a^2+2a+2ak^2+3k^2)\Pi(x,-a,k)\right].$

7. $\int_0^x \frac{\sin x\,dx}{(1+a\sin^2 x)\Delta} = \frac{1}{\sqrt{(1+a)(k^2+a)}}\ln\frac{\sqrt{1+a}\,\Delta - \sqrt{k^2+a}\cos x}{(\sqrt{1+a}-\sqrt{k^2+a})\sqrt{1+a\sin^2 x}}$
$[a>-k^2].$

8. $= \frac{1}{2\sqrt{-(1+a)(k^2+a)}}\times$
$\times\left[\arcsin\frac{(1+a)\Delta^2+(k^2+a)\cos^2 x}{(1-k^2)(1+a\sin^2 x)} - \arcsin\frac{1+2a+k^2}{1-k^2}\right]$ $[-1<a<-k^2].$

9. $= -\frac{1}{\sqrt{(1+a)(k^2+a)}}\ln\frac{\sqrt{-(k^2+a)}\cos x - \sqrt{-(1+a)}\,\Delta}{[\sqrt{-(k^2+a)}-\sqrt{-(1+a)}]\sqrt{1+a\sin^2 x}}$
$[a<-1;\ 0\leqslant\sin^2 x<-1/a].$

10. $\int_0^x \frac{\sin x\,dx}{(1-\sin^2 x)\Delta} = \frac{\Delta}{(1-k^2)\cos x} - \frac{1}{1-k^2}.$

11. $\int_0^x \frac{\sin x\,dx}{(1-k^2\sin^2 x)\Delta} = -\frac{\cos x}{(1-k^2)\Delta} + \frac{1}{1-k^2}.$

12. $\int_0^x \frac{\cos x\,dx}{(1+a\sin^2 x)\Delta} = \frac{1}{\sqrt{k^2+a}}\operatorname{arctg}\frac{\sqrt{k^2+a}\sin x}{\Delta}$ $[a>-k^2].$

13. $$= \frac{1}{2\sqrt{-(k^2+a)}} \ln \frac{\Delta + \sqrt{-(k^2+a)} \sin x}{\Delta - \sqrt{-(k^2+a)} \sin x}$$ $[a < -k^2;\ 0 \leqslant \sin^2 x < -1/a]$.

14. $$\int_0^x \frac{\cos x\, dx}{(1-k^2 \sin^2 x)\,\Delta} = \frac{\sin x}{\Delta}.$$

1.5.34. Интегралы вида $\int \frac{(a+\cos x)^p}{\sqrt{1-k^2 \sin^2 x}}\, dx$.

Условие $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $$\int \frac{(a+\cos x)^p}{\Delta}\, dx = \frac{1}{(p-1)k^2} \Big[(a+\cos x)^{p-3} \Delta \sin x + 2(2p-3)ak^2 \int \frac{(a+\cos x)^{p-1}}{\Delta}\, dx - (p-2)(1-2k^2+6a^2k^2) \times \int \frac{(a+\cos x)^{p-2}}{\Delta}\, dx + (2p-5)a(1-2k^2+a^2k^2) \int \frac{(a+\cos x)^{p-3}}{\Delta}\, dx + (p-3)(1-a^2)(1-k^2a^2) \int \frac{(a+\cos x)^{p-4}}{\Delta}\, dx \Big]$$ $[p \neq 1;\ a^2 \neq 1,\ -(1-k^2)/k^2]$.

2. $$\int \frac{a+\cos x}{\Delta}\, dx = aF(x, k) + \frac{1}{k} \arcsin (k \sin x).$$

3. $$\int \frac{(a+\cos x)^2}{\Delta}\, dx = \frac{a^2k^2+k^2-1}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2a}{k} \arcsin (k \sin x).$$

4. $$\int \frac{dx}{(a+\cos x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-k^2+a^2k^2)} \Big[\frac{\Delta \sin x}{(a+\cos x)^{n-1}} - (2n-3)(1-2k^2+2a^2k^2)a \int \frac{dx}{(a+\cos x)^{n-1} \Delta} - (n-2)(2k^2-1-6a^2k^2) \int \frac{dx}{(a+\cos x)^{n-2} \Delta} - (4n-10)ak^2 \int \frac{dx}{(a+\cos x)^{n-3} \Delta} + (n-3)k^2 \int \frac{dx}{(a+\cos x)^{n-4} \Delta} \Big].$$

5. $$= \frac{1}{(2n-1)a(1-2k^2+2a^2k^2)} \Big[\frac{\sin x \Delta}{(a+\cos x)^n} + (n-1)(1-2k^2+6a^2k^2) \int \frac{dx}{(a+\cos x)^{n-1} \Delta} - 2(2n-3)k^2a \int \frac{dx}{(a+\cos x)^{n-2} \Delta} + (n-2)k^2 \int \frac{dx}{(a+\cos x)^{n-3} \Delta} \Big]$$ $[a = \pm 1,\ \pm\sqrt{1-k^2}/(ki)]$.

6. $$\int \frac{dx}{(1 \pm \cos x)\Delta} = \frac{\pm 1 - \cos x}{\sin x} \Delta + F(x, k) - E(x, k).$$

7. $$\int \frac{dx}{(1 \pm \cos x\, ki/\sqrt{1-k^2})\,\Delta} = -\frac{k^2 \sin x \cos x \left(1 \mp \cos ki/x\sqrt{1-k^2}\right)}{\Delta} \mp \frac{ki}{\sqrt{1-k^2}} \Delta \sin x + E(x, k).$$

8. $$\int \frac{dx}{(a+\cos x)\,\Delta} = \frac{a}{a^2-1}\,\Pi\left(x,\ -\frac{1}{a^2-1},\ k\right) + \frac{1}{2\sqrt{(1-a^2)(1+k^2a^2-k^2)}} \ln \frac{\sqrt{1-a^2}\,\Delta + k\sqrt{1+k^2a^2-k^2}\sin x}{\sqrt{1-a^2}\,\Delta - k\sqrt{1+k^2a^2-k^2}\sin x}.$$

9. $$\int \frac{dx}{(a+\cos x)^2\,\Delta} = \frac{1}{(1-a^2)(1-k^2+a^2k^2)}\left[\frac{\sin x\Delta}{a+\cos x} - (1-2k^2+2a^2k^2)\,a \int \frac{dx}{(a+\cos x)\,\Delta} + 2ak^2 \int \frac{a+\cos x}{\Delta}\,dx - k^2 \int \frac{(a+\cos x)^2}{\Delta}\,dx\right].$$

10. $$\int \frac{dx}{(a+\cos x)^3\,\Delta} = \frac{1}{2(1-a^2)(1-k^2+a^2k^2)}\left[\frac{\Delta \sin x}{(a+\cos x)^2} - 3a(1-2k^2+2k^2a^2) \int \frac{dx}{(a+\cos x)^2\,\Delta} - (2k^2-1-6a^2k^2) \int \frac{dx}{(a+\cos x)\,\Delta} - 2ak^2F(x,\ k)\right].$$

1.5.35. Интегралы вида $\int \frac{(a+\operatorname{tg} x)^p}{\sqrt{1-k^2\sin^2 x}}\,dx$.

Условие. $0 < k^2 < 1$.

Обозначение. $\Delta = \sqrt{1-k^2\sin^2 x}$.

1. $$\int \frac{(a+\operatorname{tg} x)^p}{\Delta}\,dx = \frac{1}{(p-1)(1-k^2)}\left[\frac{(a+\operatorname{tg} x)^{p-3}\,\Delta}{\cos^2 x} + 2(2p-3)\,a(1-k^2) \int \frac{(a+\operatorname{tg} x)^{p-1}}{\Delta}\,dx - (p-2)(2+6a^2-k^2-6k^2a^2) \int \frac{(a+\operatorname{tg} x)^{p-2}}{\Delta}\,dx + (2p-5)\,a(2+2a^2-k^2-6k^2a^2) \int \frac{(a+\operatorname{tg} x)^{p-3}}{\Delta} - (p-3)(1+a^2)(1+a^2-k^2a^2) \int \frac{(a+\operatorname{tg} x)^{p-4}}{\Delta}\,dx\right].$$

2. $$\int \frac{a+\operatorname{tg} x}{\Delta}\,dx = aF(x,\ k) + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

3. $$\int \frac{(a+\operatorname{tg} x)^2}{\Delta}\,dx = \frac{1}{1-k^2}\,\Delta \operatorname{tg} x + a^2F(x,\ k) - \frac{1}{1-k^2}\,E(x,\ k) + \frac{a}{\sqrt{1-k^2}} \ln \frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

4. $$\int \frac{dx}{(a+\operatorname{tg} x)^n\,\Delta} = \frac{1}{(n-1)(1+a^2)(1+a^2-k^2a^2)} \times \left[-\frac{\Delta}{(a+\operatorname{tg} x)^{n-1}\cos^2 x} + (2n-3)\,a(2+2a^2-k^2-2k^2a^2) \times \int \frac{dx}{(a+\operatorname{tg} x)^{n-1}\,\Delta} - (n-2)(2+6a^2-k^2-6k^2a^2) \int \frac{dx}{(a+\operatorname{tg} x)^{n-2}\,\Delta} + (4n-10)\,a(1-k^2) \int \frac{dx}{(a+\operatorname{tg} x)^{n-3}\,\Delta} - (n-3)(1-k^2) \int \frac{dx}{(a+\operatorname{tg} x)^{n-4}\,\Delta}\right]$$

$[n \neq 1,\ a^2 \neq -1,\ -1\,(1-k^2)]$.

5. $$\int \frac{dx}{(a+\operatorname{tg} x)\Delta} = \frac{a}{1+a^2} F(x, k) + \frac{1}{a(1+a^2)} \Pi\left(x, \frac{1+a^2}{a^2}, k\right) - \frac{1}{2\sqrt{(1+a^2)(1+a^2-k^2a^2)}} \ln \frac{\sqrt{1+a^2-k^2a^2}+\sqrt{1+a^2}\,\Delta}{\sqrt{1+a^2-k^2a^2}-\sqrt{1+a^2}\,\Delta}.$$

6. $$\int \frac{dx}{(a+\operatorname{ctg} x)\Delta} = \frac{a}{1+a^2} \Pi\left(x, \frac{a^2}{1+a^2}, k\right) - \frac{1}{\sqrt{(1+a^2)(a^2+1-k^2)}} \ln \frac{\sqrt{1+a^2}\,\Delta \cos x + a\sqrt{a^2+1-k^2}}{\left(\sqrt{1+a^2}\,\Delta+\sqrt{a^2+1-k^2}\right)\sqrt{1+a^2-k^2\sin^2 x}}$$ $[a > 1]$.

7. $$\int \frac{dx}{(a+\operatorname{tg} x)^2\Delta} = \frac{1}{(1+a^2)(1+a^2-k^2a^2)} \left[\frac{-\Delta}{(a+\operatorname{tg} x)\cos^2 x} + a(2+2a^2-k^2-2k^2a^2)\int \frac{dx}{(a+\operatorname{tg} x)\Delta} - 2a(1-k^2)\int \frac{a+\operatorname{tg} x}{\Delta}dx + (1-k^2)\int \frac{(a+\operatorname{tg} x)^2}{\Delta}dx\right].$$

8. $$\int \frac{dx}{(a+\operatorname{tg} x)^3\Delta} = \frac{1}{2(1+a^2)(1+a^2-k^2a^2)} \left[\frac{-\Delta}{(a+\operatorname{tg} x)^2\cos^2 x} + 3a(2+2a^2-k^2-2k^2a^2)\int \frac{dx}{(a+\operatorname{tg} x)^2\Delta} - (2+6a^2-k^2-6k^2a^2)\int \frac{dx}{(a+\operatorname{tg} x)\Delta} + 2a(1-k^2)F(x, k)\right].$$

9. $$\int \frac{(a+\operatorname{tg}^2 x)^p}{\Delta}dx = \frac{1}{(2p-1)(1-k^2)} \left[\frac{(a+\operatorname{tg}^2 x)^{p-2}\operatorname{tg} x\,\Delta}{\cos^2 x} + (2p-2)(2-3a-k^2+3k^2a)\int \frac{(a+\operatorname{tg} x)^{p-1}}{\Delta}dx + (2p-3)[1-4a+3a^2+(2a-3a^2)k^2]\int \frac{(a+\operatorname{tg}^2 x)^{p-2}}{\Delta}dx + 2(p-2)a(1-a)(1-a+k^2a)\int \frac{(a+\operatorname{tg}^2 x)^{p-3}}{\Delta}dx\right].$$

10. $$\int \frac{a+\operatorname{tg}^2 x}{\Delta}dx = \frac{\Delta}{1-k^2}\operatorname{tg} x + aF(x, k) - \frac{1}{1-k^2}E(x, k).$$

11. $$\int \frac{dx}{(a+\operatorname{tg}^2 x)\Delta} = \frac{1}{a-1}F(x, k) + \frac{1}{a(1-a)}\Pi\left(x, \frac{a-1}{a}, k\right).$$

1.5.36. Интегралы вида $\int R\left(\sin x, \cos x, \sqrt{1-p^2\sin^2 x}\right)dx$

Условие: $p^2 > 1$.

Обозначение: $\varphi = \arcsin(p \sin x)$.

1. $$\int \sqrt{1-p^2\sin^2 x}\,dx = pE\left(\varphi, \frac{1}{p}\right) - \frac{p^2-1}{p}F\left(\varphi, \frac{1}{p}\right).$$

2. $$\int \frac{dx}{\sqrt{1-p^2\sin^2 x}} = \frac{1}{p}F\left(\varphi, \frac{1}{p}\right).$$

3. $\displaystyle\int \frac{dx}{(1-r^2\sin^2 x)\sqrt{1-p^2\sin^2 x}} = \frac{1}{p}\Pi\left(\varphi,\ \frac{r^2}{p^2},\ \frac{1}{p}\right)$

Для вычисления интегралов вида

$$\int R\left(\sin x,\ \cos x,\ \sqrt{1-p^2\sin^2 x}\right)dx$$

при $p^2>1$ можно пользоваться формулами пп. 1.5.22—1.5.30, произведя в них следующие замены [5]:

1) $F(x,\ k)\to\dfrac{1}{p}F\left(\varphi,\dfrac{1}{p}\right)$;

2) $E(x,\ k)\to pE\left(\varphi,\ \dfrac{1}{p}\right)-\dfrac{p^2-1}{p}F\left(\varphi,\ \dfrac{1}{p}\right)$

3) $k\to p$.

1.5.37. Интегралы вида $\int R\left(\sin x,\ \cos x,\ \sqrt{1+p^2\sin^2 x}\right)dx$.

Обозначение: $\varphi=\arcsin\dfrac{\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}}$.

1. $\displaystyle\int\sqrt{1+p^2\sin^2 x}\,dx=\sqrt{1+p^2}\,E\left(\varphi,\ \frac{p}{\sqrt{1+p^2}}\right)-p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}$.

2. $\displaystyle\int\frac{\sqrt{1+p^2\sin^2 x}\,dx}{1+(p^2-r^2p^2-r^2)\sin^2 x}=\frac{1}{\sqrt{1+p^2}}\Pi\left(\varphi,\ r^2,\ \frac{p}{\sqrt{1+p^2}}\right)$.

3. $\displaystyle\int\frac{dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{\sqrt{1+p^2}}F\left(\varphi,\ \frac{p}{\sqrt{1+p^2}}\right)$.

4. $\displaystyle\int\frac{\sin x\,dx}{\sqrt{1+p^2\sin^2 x}}=-\frac{1}{p}\arcsin\left(\frac{p\cos x}{\sqrt{1+p^2}}\right)$.

5. $\displaystyle\int\frac{\cos x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{p}\ln\left(p\sin x+\sqrt{1+p^2\sin^2 x}\right)$.

6. $\displaystyle\int\frac{dx}{\sin x\sqrt{1+p^2\sin^2 x}}=\frac{1}{2}\ln\frac{\sqrt{1+p^2\sin^2 x}-\cos x}{\sqrt{1+p^2\sin^2 x}+\cos x}$.

7. $\displaystyle\int\frac{dx}{\cos x\sqrt{1+p^2\sin^2 x}}=\frac{1}{2\sqrt{1+p^2}}\ln\frac{\sqrt{1+p^2\sin^2 x}+\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}-\sqrt{1+p^2}\sin x}$.

8. $\displaystyle\int\frac{\operatorname{tg} x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{2\sqrt{1+p^2}}\ln\frac{\sqrt{1+p^2\sin^2 x}+\sqrt{1+p^2}}{\sqrt{1+p^2\sin^2 x}-\sqrt{1+p^2}}$.

9. $\displaystyle\int\frac{\operatorname{ctg} x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{2}\ln\frac{1-\sqrt{1+p^2\sin^2 x}}{1+\sqrt{1+p^2\sin^2 x}}$.

Для вычисления интегралов вида

$$\int R\left(\sin x,\ \cos x,\ \sqrt{1+p^2\sin^2 x}\right)dx$$

можно также пользоваться формулами пп. 1.5.22—1.5.30, произведя в них следующие замены [5]:

1) $k^2\to -p^2$;

2) $F(x,\ k)\to\dfrac{1}{\sqrt{1+p^2}}F\left(\varphi,\ \dfrac{p}{\sqrt{1+p^2}}\right)$;

3) $E(x,\ k) \to \sqrt{1+p^2}\, E\left(\varphi,\ -\frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}}$;

4) $\frac{1}{k} \ln(k \cos x + \Delta) \to \frac{1}{p} \arcsin \frac{p \cos x}{\sqrt{1+p^2}}$;

5) $\frac{1}{k} \arcsin(k \sin x) \to \frac{1}{p} \ln\left(p \sin x + \sqrt{1+p^2 \sin^2 x}\right)$.

1.5.38. Интегралы вида $\int R\left(\sin x,\ \cos x,\ \sqrt{a^2 \sin^2 x - 1}\right) dx$.
Условие: $a^2 > 1$.
Обозначение: $\varphi = \arcsin \frac{a \cos x}{\sqrt{a^2-1}}$.

1. $\int \sqrt{a^2 \sin^2 x - 1}\, dx = \frac{1}{a} F\left(\varphi,\ \frac{\sqrt{a^2-1}}{a}\right) - aE\left(\varphi,\ \frac{\sqrt{a^2-1}}{a}\right)$.

2. $\int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{1}{a} F\left(\varphi;\ \frac{\sqrt{a^2-1}}{a}\right)$.

3. $\int \frac{\sin x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{\varphi}{a}$.

4. $\int \frac{\cos x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln\left(a \sin x + \sqrt{a^2 \sin^2 x - 1}\right)$.

5. $\int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}}$.

6. $\int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} \sin x - \sqrt{a^2 \sin^2 x - 1}}$.

7. $\int \frac{dx}{(1 - r^2 \sin^2 x)\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a(r^2-1)} \Pi\left(\varphi,\ \frac{r^2(a^2-1)}{a^2(r^2-1)},\ \frac{\sqrt{a^2-1}}{a}\right)$
[$r^2 > 1$

8. $\int \frac{\operatorname{tg} x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2 \sin^2 x - 1}}$.

9. $\int \frac{\operatorname{ctg} x\, dx}{\sqrt{a^2 \sin^2 x - 1}} = -\arcsin \frac{1}{a \sin x}$.

Для вычисления интегралов вида

$$\int R\left(\sin x,\ \cos x,\ \sqrt{a^2 \sin^2 x - 1}\right) dx \qquad (a^2 > 1)$$

можно также воспользоваться формулами пп. 1.5.22—1.5.30. Для этого надо произвести в них следующие замены [5]:

1) $F(x,\ k) \to -\frac{1}{ia} F\left(\varphi,\ \frac{\sqrt{a^2-1}}{a}\right)$;

2) $E(x,\ k) \to \frac{i}{a} F\left(\varphi,\ \frac{\sqrt{a^2-1}}{a}\right) - iaE\left(\varphi,\ \frac{\sqrt{a^2-1}}{a}\right)$;

3) $\frac{1}{k} \ln(k \cos x + \Delta) \to -\frac{\varphi}{ia}$;

4) $\frac{1}{k}\arcsin(k\sin x)\to\frac{1}{ia}\ln\left(a\sin x+\sqrt{a^2\sin^2 x-1}\right)$;

5) $\frac{1}{2}\ln\frac{\Delta-\cos x}{\Delta+\cos x}\to-\frac{1}{i}\operatorname{arctg}\frac{\cos x}{\sqrt{a^2\sin^2 x-1}}$;

6) $\frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\sin x}{\Delta-\sqrt{1-k^2}\sin x}\to$

$$\to\frac{1}{2i\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}\sin x+\sqrt{a^2\sin^2 x-1}}{\sqrt{a^2-1}\sin x-\sqrt{a^2\sin^2 x-1}};$$

7) $\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}\to-\frac{1}{i}\arcsin\frac{1}{a\sin x}$;

8) $\frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}\to\frac{1}{2i\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}+\sqrt{a^2\sin^2 x-1}}{\sqrt{a^2-1}-\sqrt{a^2\sin^2 x-1}}$;

9) Оставшиеся $\Delta\to i\sqrt{a^2\sin^2 x-1}$;

10) $k\to a$;

11) умножить полученные равенства на i.

1.5.39. Интегралы вида $\int f(\operatorname{tg}x)\,dx$, $\int f(\operatorname{ctg}x)\,dx$.

Обозначение: $t(x)=\begin{Bmatrix}\operatorname{tg}x\\ \operatorname{ctg}x\end{Bmatrix}$.

1. $\int R\left(\sin^2 x,\ \cos^2 x,\ \operatorname{tg}x,\ \operatorname{ctg}x,\ \sqrt{a^2\pm b^2\operatorname{tg}^2 x}\right)dx=$

$$=\int R\left(\frac{1}{1+\tau^2},\ \frac{\tau^2}{1+\tau^2},\ \tau,\ \frac{1}{\tau},\sqrt{a^2\pm b^2\tau^2}\right)\frac{d\tau}{1+\tau^2}\qquad[\tau=\operatorname{tg}x].$$

2. $\int R\left(\sin^2 x,\ \cos^2 x,\ \operatorname{tg}x,\ \operatorname{ctg}x,\ \sqrt{a^2\pm b^2\operatorname{ctg}^2 x}\right)dx=$

$$=\int R\left(\frac{\tau^2}{1+\tau^2},\ \frac{1}{1+\tau^2},\ \frac{1}{\tau},\ \tau,\ \sqrt{a^2\pm b^2\tau^2}\right)\frac{d\tau}{1+\tau^2}\qquad[\tau=\operatorname{ctg}x].$$

3. $\int [a^2+b^2t^2(x)]^{n+1/2}\,t(x)\,dx=$

$$=\pm\sqrt{a^2+b^2t^2(x)}\sum_{k=0}^{n}(a^2-b^2)^k\frac{[a^2+b^2t^2(x)]^{n-k}}{2n-2k+1}+$$

$$+\begin{cases}\pm\frac{1}{2}(a^2-b^2)^{n+1/2}\ln\frac{\sqrt{a^2-b^2}-\sqrt{a^2+b^2t^2(x)}}{\sqrt{a^2-b^2}+\sqrt{a^2+b^2t^2(x)}} & [a^2>b^2],\\ (-1)^{n+1}(b^2-a^2)^{n+1/2}\begin{Bmatrix}\operatorname{arctg}\sqrt{\frac{a^2+b^2t^2(x)}{b^2-a^2}}\\ \operatorname{arcctg}\sqrt{\frac{a^2+b^2t^2(x)}{b^2-a^2}}\end{Bmatrix} & [a^2<b^2].\end{cases}$$

4. $$\int [a^2-b^2t^2(x)]^{n+1/2}\, t(x)\, dx = \pm\sqrt{a^2-b^2t^2(x)}\sum_{k=0}^{n}(a^2+b^2)^k \frac{[a^2-b^2t^2(x)]^{n-k}}{2n-2k+1} \pm \pm\frac{1}{2}(a^2+b^2)^{n+1/2}\ln\frac{\sqrt{a^2+b^2}-\sqrt{a^2-b^2t^2(x)}}{\sqrt{a^2+b^2}+\sqrt{a^2-b^2t^2(x)}}.$$

5. $$\int\frac{t(x)\,dx}{[a^2+b^2t^2(x)]^{n+1/2}} = \pm\frac{1}{\sqrt{a^2+b^2t^2(x)}}\sum_{k=0}^{n-1}\frac{1}{(2k+1)(a^2-b^2)^{n-k}[a^2+b^2t^2(x)]^k} + \begin{cases} \pm\dfrac{1}{2(a^2-b^2)^{n+1/2}}\ln\dfrac{\sqrt{a^2-b^2}-\sqrt{a^2+b^2t^2(x)}}{\sqrt{a^2-b^2}+\sqrt{a^2+b^2t^2(x)}} & [a^2>b^2], \\ \dfrac{(-1)^n}{(b^2-a^2)^{n+1/2}}\begin{Bmatrix}\operatorname{arctg}\sqrt{\dfrac{a^2+b^2t^2(x)}{b^2-a^2}}\\ \operatorname{arcctg}\sqrt{\dfrac{a^2+b^2t^2(x)}{b^2-a^2}}\end{Bmatrix} & [a^2<b^2]. \end{cases}$$

6. $$\int\frac{t(x)\,dx}{[a^2-b^2t^2(x)]^{n+1/2}} = \pm\frac{1}{\sqrt{a^2-b^2t^2(x)}}\sum_{k=0}^{n-1}\frac{1}{(2k+1)(a^2+b^2)^{n-k}[a^2-b^2t^2(x)]^k} \pm \pm\frac{1}{2(a^2+b^2)^{n+1/2}}\ln\frac{\sqrt{a^2+b^2}-\sqrt{a^2-b^2t^2(x)}}{\sqrt{a^2+b^2}+\sqrt{a^2-b^2t^2(x)}}.$$

7. $$\int\frac{dx}{\sqrt{a^2+b^2t^2(x)}} = \frac{1}{2\sqrt{b^2-a^2}}\ln\frac{\sqrt{b^2-a^2}\,t(x)\pm\sqrt{a^2+b^2t^2(x)}}{\sqrt{b^2-a^2}\,t(x)\mp\sqrt{a^2+b^2t^2(x)}} \quad [a^2<b^2].$$

8. $$= \frac{1}{\sqrt{a^2-b^2}}\operatorname{arctg}\frac{\sqrt{a^2-b^2}\,t(x)}{\sqrt{a^2+b^2t^2(x)}} \quad [a^2>b^2].$$

9. $$\int\frac{dx}{\sqrt{a^2-b^2t^2(x)}} = \frac{1}{\sqrt{a^2+b^2}}\begin{Bmatrix}\operatorname{arctg}\dfrac{\sqrt{a^2+b^2}\,t(x)}{\sqrt{a^2-b^2t^2(x)}}\\ \operatorname{arcctg}\dfrac{\sqrt{a^2+b^2}\,t(x)}{\sqrt{a^2-b^2t^2(x)}}\end{Bmatrix}.$$

10. $$\int\frac{dx}{\sqrt{b^2t^2(x)-a^2}} = \frac{1}{2\sqrt{a^2+b^2}}\ln\frac{\sqrt{a^2+b^2}\,t(x)\pm\sqrt{b^2t^2(x)-a^2}}{\sqrt{a^2+b^2}\,t(x)\mp\sqrt{b^2t^2(x)-a^2}}.$$

11. $$\int\frac{t(x)\,dx}{\sqrt{a^2+b^2t^2(x)}} = \frac{1}{\sqrt{b^2-a^2}}\begin{Bmatrix}\operatorname{arctg}\dfrac{\sqrt{a^2+b^2t^2(x)}}{\sqrt{b^2-a^2}}\\ \operatorname{arcctg}\dfrac{\sqrt{a^2+b^2t^2(x)}}{\sqrt{b^2-a^2}}\end{Bmatrix} \quad [a^2<b^2].$$

12. $$= \frac{1}{2\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2} \mp \sqrt{a^2+b^2t^2(x)}}{\sqrt{a^2-b^2} \pm \sqrt{a^2+b^2t^2(x)}} \qquad [a^2>b^2].$$

13. $$\int \frac{t(x)\,dx}{\sqrt{a^2-b^2t^2(x)}} = \frac{1}{2\sqrt{a^2+b^2}} \ln \frac{\sqrt{a^2+b^2} \mp \sqrt{a^2-b^2t^2(x)}}{\sqrt{a^2+b^2} \pm \sqrt{a^2-b^2t^2(x)}}.$$

14. $$\int \frac{t(x)\,dx}{\sqrt{b^2t^2(x)-a^2}} = \frac{1}{\sqrt{a^2+b^2}} \begin{Bmatrix} \operatorname{arctg} \dfrac{\sqrt{b^2t^2(x)-a^2}}{\sqrt{a^2+b^2}} \\ \operatorname{arcctg} \dfrac{\sqrt{b^2t^2(x)-a^2}}{\sqrt{a^2+b^2}} \end{Bmatrix}.$$

15. $$\int \frac{t(x)\,dx}{[a^2 \pm b^2t^2(x)]^{n+1/2}} =$$
$$= \frac{1}{\sqrt{a^2 \pm b^2t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1)(a^2 \mp b^2)^{n-k}(a^2 \pm b^2t^2(x))^k} +$$
$$+ \frac{1}{(a^2-b^2)^n} \int \frac{t(x)\,dx}{\sqrt{a^2 \pm b^2t^2(x)}}.$$

1.5.40. Интегралы вида $\int x^p \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q dx$.

1. $$\int x^p \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q dx = \frac{x^{p-1}}{q^2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{q-1} \times$$
$$\times \left[p \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \mp qx \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} \right] + \frac{q-1}{q} \int x^p \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{q-2} dx - \frac{p(p-1)}{q^2} \int x^{p-2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q dx.$$

2. $$\int x^m \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2n} dx = \binom{2n}{n} \frac{x^{m+1}}{2^{2n}(m+1)} +$$
$$+ \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (\mp 1)^{n+k} \binom{2n}{k} \int x^m \cos(2n-2k)x\,dx.$$

3. $$\int x^m \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (\mp 1)^{n+k} \binom{2n+1}{k} \int x^m \begin{Bmatrix} \sin(2n-2k+1)x \\ \cos(2n-2k+1)x \end{Bmatrix} dx.$$

4. $$\int x^n \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \mp \sum_{k=0}^{n} k! \binom{n}{k} x^{n-k} \begin{Bmatrix} \cos(x+k\pi/2) \\ \sin(x+k\pi/2) \end{Bmatrix}.$$

5. $$\int x^{2n} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx =$$
$$= (2n)! \left[\mp \sum_{k=0}^{n} (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right].$$

6. $$\int x^{2n+1} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx =$$
$$= (2n+1)! \left[\mp \sum_{k=0}^{n} (-1)^k \frac{x^{2n-2k+1}}{(2n-2k+1)!} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \sum_{k=0}^{n} (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right].$$

7. $$\int x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n} dx = \frac{(2n-1)!!}{(2n)!!}\frac{x^2}{2} + \frac{1}{2n+1}\sum_{k=0}^{n-1}\frac{(2n+1)(2n-1)\dots(2n-2k+1)}{2^{k+1}n(n-1)\dots(n-k)} \times \left[\frac{1}{2n-2k}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n-2k} \mp x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n-2k-1}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}\right].$$

8. $$\int x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n+1} dx = \frac{1}{n+1}\sum_{k=0}^{n}\frac{2^k(n+1)n\dots(n-k+1)}{(2n+1)(2n-1)\dots(2n-2k+1)} \times \left[\frac{1}{2n-2k+1}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n-2k+1} \mp x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2n-2k}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}\right]$$

9. $$\int x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} \mp x \begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}.$$

10. $$\int x^2 \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = 2x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} \mp (x^2-2) \begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}.$$

11. $$\int x^3 \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = (3x^2-6) \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} \mp (x^3-6x) \begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}.$$

12. $$\int x^n \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2 dx = \frac{x^{n+1}}{2(n+1)} \pm \frac{n!}{4}\left\{\sum_{k=0}^{[n/2]}\frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{[(n-1)/2]}\frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}.$$

13. $$\int x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2 dx = \frac{x^2}{4} \mp \frac{x}{4}\sin 2x \mp \frac{1}{8}\cos 2x.$$

14. $$\int x^2 \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2 dx = \frac{x^3}{6} \mp \left(\frac{x^2}{4}-\frac{1}{8}\right)\sin 2x \mp \frac{x}{4}\cos 2x.$$

15. $$\int x^3 \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2 dx = \frac{x^4}{8} \mp \left(\frac{x^3}{4}-\frac{3x}{8}\right)\sin 2x \mp \left(\frac{3x^2}{8}-\frac{3}{16}\right)\cos 2x$$

16. $$\int x^n \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^3 dx = \frac{n!}{4}\left\{\sum_{k=0}^{[n/2]}\frac{(-1)^k x^{2n-k}}{(n-2k)!}\left[\frac{1}{3^{2k+1}}\begin{Bmatrix}\cos 3x\\ \sin 3x\end{Bmatrix} \mp 3\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}\right] \mp \sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k x^{n-2k-1}}{(n-2k-1)!}\left[\frac{1}{3^{2k+2}}\begin{Bmatrix}\sin 3x\\ \cos 3x\end{Bmatrix} \mp 3\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}\right]\right\}.$$

17. $$\int x \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^3 dx = \frac{3}{4}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} \mp \frac{1}{36}\begin{Bmatrix}\sin 3x\\ \cos 3x\end{Bmatrix} \mp \frac{3}{4}x\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix} + \frac{x}{12}\begin{Bmatrix}\cos 3x\\ \sin 3x\end{Bmatrix}.$$

18. $$\int x^{\mu} \begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = \mp x^{\mu}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix} \pm \mu \int x^{\mu-1}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix} dx.$$

19. $$\int_0^x x^{\mu-1} \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} dx = \frac{1}{2} \begin{Bmatrix} i \\ 1 \end{Bmatrix} \left[(ia)^{-\mu} \gamma(\mu, iax) \mp (-ia)^{-\mu} \gamma(\mu, -iax) \right]$$
$$\left[\operatorname{Re} \mu > \begin{Bmatrix} -1 \\ 0 \end{Bmatrix};\ x > 0 \right].$$

20. $$\int_x^\infty x^{\mu-1} \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} dx = \frac{1}{2a^\mu} \begin{Bmatrix} i \\ 1 \end{Bmatrix} \left[\mp e^{i\mu\pi/2} \Gamma(\mu, -iax) + e^{-i\mu\pi/2} \Gamma(\mu, iax) \right]$$
$$[\operatorname{Re} \mu < 1;\ x > 0].$$

21. $$\int_x^\infty x^{\mu-1} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \begin{Bmatrix} \mathrm{S}(x, \mu) \\ \mathrm{C}(x, \mu) \end{Bmatrix}.$$

1.5.41. Интегралы вида $\int \frac{1}{x^p} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q dx$.

1. $$\int \frac{1}{x^p} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q dx =$$
$$= -\frac{1}{(p-1)x^{p-1}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q \mp \frac{q}{(p-1)(p-2)x^{p-2}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{q-1} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} +$$
$$+ \frac{q(q-1)}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{q-2} dx - \frac{q^2}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^q dx.$$

2. $$\int \frac{1}{x^p} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = -\frac{1}{(p-1)x^{p-1}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \pm \frac{1}{p-1} \int \frac{1}{x^{p-1}} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} dx.$$

3. $$= -\frac{1}{(p-1)x^{p-1}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \mp \frac{1}{(p-1)(p-2)x^{p-2}} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} -$$
$$- \frac{1}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx.$$

4. $$\int \frac{1}{x^{2n}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \frac{(-1)^{n+1}}{x(2n-1)!} \left[\pm \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \right.$$
$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right] \pm \frac{(-1)^{n+1}}{(2n-1)!} \begin{Bmatrix} \operatorname{ci}(x) \\ \operatorname{si}(x) \end{Bmatrix}.$$

5. $$\int \frac{1}{x^{2n+1}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx = \frac{(-1)^{n+1}}{x(2n)!} \left[\pm \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} + \right.$$
$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \right] + \frac{(-1)^n}{(2n)!} \begin{Bmatrix} \operatorname{si}(x) \\ \operatorname{ci}(x) \end{Bmatrix}.$$

6. $$\int \frac{1}{x} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2m} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (\mp 1)^{m+k} \binom{2m}{k} \operatorname{ci}[(2m-2k)x].$$

7. $$\int \frac{1}{x} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2m+1} dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (\mp 1)^{m+k} \binom{2m+1}{k} \begin{Bmatrix} \operatorname{si}(2m-2k+1)x \\ \operatorname{ci}(2m-2k+1)x \end{Bmatrix}.$$

8. $$\int \frac{1}{x^2}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2m} dx = -\binom{2m}{m}\frac{1}{2^{2m}x} \pm$$
$$\pm\frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(\mp 1)^{m+k+1}\binom{2m}{k}\left[\frac{\cos(2m-2k)x}{x}+(2m-2k)\operatorname{si}[(2m-2k)x]\right].$$

9. $$\int \frac{1}{x^2}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2m+1} dx = \pm\frac{1}{2^{2m}}\sum_{k=0}^{m}(\mp 1)^{m+k+1}\binom{2m+1}{k}\times$$
$$\times\left[\frac{1}{x}\begin{Bmatrix}\sin(2m-2k+1)x\\ \cos(2m-2k+1)x\end{Bmatrix}\mp(2m-2k+1)\begin{Bmatrix}\operatorname{ci}(2m-2k+1)x\\ \operatorname{si}(2m-2k+1)x\end{Bmatrix}\right].$$

10. $$\int \frac{1}{x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = \begin{Bmatrix}\operatorname{si}(x)\\ \operatorname{ci}(x)\end{Bmatrix}.$$
11. $$\int_x^\infty \frac{1}{x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = -\begin{Bmatrix}\operatorname{si}(x)\\ \operatorname{ci}(x)\end{Bmatrix}.$$

12. $$\int_0^x \frac{\sin x}{x}dx = \operatorname{Si}(x).$$
13. $$\int_0^x \frac{1-\cos x}{x}dx = C + \ln x - \operatorname{ci}(x).$$

14. $$\int \frac{1}{x^2}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = -\frac{1}{x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}\pm\begin{Bmatrix}\operatorname{ci}(x)\\ \operatorname{si}(x)\end{Bmatrix}.$$

15. $$\int \frac{1}{x^3}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix} dx = -\frac{1}{2x^2}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}\mp\frac{1}{2x}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}-\frac{1}{2}\begin{Bmatrix}\operatorname{si}(x)\\ \operatorname{ci}(x)\end{Bmatrix}.$$

16. $$\int \frac{1}{x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^2 dx = \frac{1}{2}\ln|x|\mp\frac{1}{2}\operatorname{ci}(2x).$$

17. $$\int \frac{1}{x}\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^3 dx = \frac{3}{4}\begin{Bmatrix}\operatorname{si}(x)\\ \operatorname{ci}(x)\end{Bmatrix}\mp\frac{1}{4}\begin{Bmatrix}\operatorname{si}(3x)\\ \operatorname{ci}(3x)\end{Bmatrix}.$$

1.5.42. Интегралы вида $\int x^p\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^q dx$.

1. $$\int x^p \operatorname{tg} x\,dx = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k}(2^{2k}-1)}{(p+2k)(2k)!}B_{2k}x^{p+2k}$$ $[p \geqslant -1;\ |x| < \pi/2]$.

2. $$\int x^p \operatorname{ctg} x\,dx = \sum_{k=0}^{\infty}(-1)^{k}\frac{2^{2k}B_{2k}}{(p+2k)(2k)!}x^{p+2k}$$ $[p \geqslant 1;\ |x| < \pi]$.

3. $$\int x^p\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{2n} dx = (-1)^n\frac{x^{p+1}}{p+1}+\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\int x^p\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}^{2k-2n}dx$$

[см. 1.5.44, 1.5.45].

4. $$\int x^p\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}^{2n+1} dx = \pm\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}\binom{n}{k}x^p\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}^{2k-2n}+$$
$$+(-1)^n\int x^p\begin{Bmatrix}\operatorname{tg} x\\ \operatorname{ctg} x\end{Bmatrix}dx\mp p\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}\binom{n}{k}\int x^{p-1}\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}^{2k-2n}dx$$

[см. 1.5.44, 1.5.45].

5. $\int x \begin{Bmatrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{Bmatrix}^2 dx = \pm x \begin{Bmatrix} \operatorname{tg} x \\ \operatorname{ctg} x \end{Bmatrix} + \ln \begin{Bmatrix} |\cos x| \\ |\sin x| \end{Bmatrix} - \frac{x^2}{2}.$

1.5.43. Интегралы вида $\int x^r \sin^p x \cos^q x\, dx$.

1. $$\int x^r \sin^p x \cos^q x\, dx = \frac{1}{(p+q)^2}\Big\{(p+q)\, x^r \sin^{p+1} x \cos^{q-1} x + r x^{r-1} \sin^p x \cos^q x - r(r-1)\int x^{r-2} \sin^p x \cos^q x\, dx - rp \int x^{r-1} \sin^{p-1} x \cos^{q-1} x\, dx + (q-1)(p+q)\int x^r \sin^p x \cos^{q-2} x\, dx\Big].$$

2. $$= \frac{1}{(p+q)^2}\Big[-(p+q)\, x^r \sin^{p-1} x \cos^{q+1} x + r x^{r-1} \sin^p x \cos^q x - r(r-1)\int x^{r-2} \sin^p x \cos^q x\, dx + rq \int x^{r-1} \sin^{p-1} x \cos^{q-1} x\, dx + (p-1)(p+q)\int x^r \sin^{p-2} x \cos^q x\, dx\Big].$$

3. $$\int x \sin^p x \cos^q x\, dx = \frac{1}{p+q}\Big[x \sin^{p+1} x \cos^{q-1} x - \int \sin^{p+1} x \cos^{q-1} x\, dx + (q-1)\int x \sin^p x \cos^{q-2} x\, dx\Big].$$

4. $$= \frac{1}{p+q}\Big[-x \sin^{p-1} x \cos^{q+1} x + \int \sin^{p-1} x \cos^{q+1} x\, dx + (p-1)\int x \sin^{p-2} x \cos^q x\, dx\Big].$$

5. $$\int x^p \frac{\sin^{2m} x}{\cos^n x}\, dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\, dx}{\cos^{n-2k} x}$$ [см. 1.5.45].

6. $$\int x^p \frac{\sin^{2m+1} x}{\cos^n x}\, dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x}\, dx$$ [см. 1.5.46].

7. $$\int x^p \frac{\sin x}{\cos^n x}\, dx = \frac{x^p}{(n-1)\cos^{n-1} x} - \frac{p}{n-1}\int \frac{x^{p-1}}{\cos^{n-1} x}\, dx$$ [см. 1.5.45].

8. $$\int x^p \frac{\cos^{2m} x}{\sin^n x}\, dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p}{\sin^{n-2k} x}\, dx$$ [см. 1.5.44].

9. $$\int x^p \frac{\cos^{2m+1} x}{\sin^n x}\, dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x}\, dx$$ [см. 1.5.46].

10. $$\int x^p \frac{\cos x}{\sin^n x}\, dx = -\frac{x^p}{(n-1)\sin^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}}{\sin^{n-1} x}\, dx$$ [см. 1.5.44].

11. $$\int \frac{x \cos x}{\sin^2 x}\, dx = -\frac{x}{\sin x} + \ln\left|\operatorname{tg} \frac{x}{2}\right|.$$

12. $$\int \frac{x \sin x}{\cos^2 x}\, dx = \frac{x}{\cos x} - \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$$

1.5.44. Интегралы вида $\int \frac{x^p\,dx}{\sin^q x}$.

1. $$\int \frac{x^p\,dx}{\sin^q x} = -\frac{x^{p-1}[p\sin x + (q-2)x\cos x]}{(q-1)(q-2)\sin^{q-1} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\sin^{q-2} x}.$$

2. $$\int \frac{x^p\,dx}{\sin x} = \frac{x^p}{p} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(p+2k)(2k)!} B_{2k} x^{p+2k} \qquad [|x|<\pi;\ p>0].$$

3. $$\int \frac{x^p\,dx}{\sin^2 x} = -x^p \operatorname{ctg} x + \frac{p}{p-1} x^{p-1} + p\sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{p+2k-1}}{(p+2k-1)(2k)!} B_{2k} \qquad [|x|<\pi;\ p>1].$$

4. $$\int \frac{x\,dx}{\sin^q x} = -\frac{x\cos x}{(q-1)\sin^{q-1} x} - \frac{1}{(q-1)(q-2)\sin^{q-2} x} + \frac{q-2}{q-1}\int \frac{x\,dx}{\sin^{q-2} x}.$$

5. $$\int \frac{x\,dx}{\sin^{2n} x} = -\frac{1}{2n}\sum_{k=1}^{n-1} \frac{2n(2n-2)\ldots(2n-2k+2)[\sin x + (2n-2k)x\cos x]}{(2n-1)(2n-3)\ldots(2n-2k+3)(2n-2k+1)(2n-2k)\sin^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!}(\ln|\sin x| - x\operatorname{ctg} x).$$

6. $$\int \frac{x\,dx}{\sin^{2n+1} x} = -\frac{1}{2n+1}\sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)[\sin x + (2n-2k-1)x\cos x]}{2n(2n-2)\ldots(2n-2k+2)(2n-2k)(2n-2k-1)\sin^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!}\int \frac{x\,dx}{\sin x}.$$

7. $$\int \frac{dx}{x^p \sin^q x} = \frac{p}{(q-1)(q-2)x^{p+1}\sin^{q-2} x} - \frac{\cos x}{(q-1)x^p \sin^{q-1} x} + \frac{q-2}{q-1}\int \frac{dx}{x^p \sin^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)}\int \frac{dx}{x^{p+2}\sin^{q-2} x}.$$

8. $$\int \frac{dx}{x^n \sin x} = -\frac{1}{nx^n} - [1+(-1)^n](-1)^{n/2}\frac{2^{n-1}-1}{n!} B_n \ln x - \sum_{\substack{k=1 \\ k\neq n/2}}^{\infty} (-1)^k \frac{2(2^{2k-1}-1)}{(2k)!(2k-n)} B_{2k} x^{2k-n} \qquad [n>1;\ |x|<\pi].$$

9. $$\int \frac{dx}{x^n \sin^2 x} = -\frac{\operatorname{ctg} x}{x^n} + \frac{n}{(n+1)x^{n+1}} - $$
$$-[1-(-1)^n](-1)^{(n+1)/2}\frac{2^n n}{(n+1)!}B_{n+1}\ln x -$$
$$-\frac{n}{x^{n+1}}\sum_{\substack{k=1\\ k\neq(n+1)/2}}^{\infty}\frac{(-1)^k(2x)^{2k}}{(2k)!\,(2k-n-1)}B_{2k} \qquad [|x|<\pi].$$

10. $$\int \frac{x\,dx}{\sin x} = x + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2(2^{2k-1}-1)}{(2k+1)!}B_{2k}x^{2k+1}.$$

11. $$\int \frac{x^2\,dx}{\sin x} = \frac{x^2}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k-1}-1}{(2k)!\,(k+1)}B_{2k}x^{2k+2}.$$

12. $$\int \frac{x\,dx}{\sin^2 x} = -x\operatorname{ctg} x + \ln|\sin x|.$$

13. $$\int \frac{x\,dx}{\sin^3 x} = -\frac{\sin x + x\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{x\,dx}{\sin x}.$$

14. $$\int \frac{x\,dx}{\sin^4 x} = -\frac{x\cos x}{3\sin^3 x} - \frac{1}{6\sin^2 x} - \frac{2}{3}x\operatorname{ctg} x + \frac{2}{3}\ln|\sin x|.$$

1.5.45. Интегралы вида $\int \frac{x^p\,dx}{\cos^q x}$.

1. $$\int \frac{x^p\,dx}{\cos^q x} = -\frac{x^{p-1}[p\cos x - (q-2)x\sin x]}{(q-1)(q-2)\cos^{q-1} x} +$$
$$+\frac{q-2}{q-1}\int \frac{x^p\,dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\cos^{q-2} x}.$$

2. $$\int \frac{x^p\,dx}{\cos x} = \sum_{k=0}^{\infty}\frac{|E_{2k}|\,x^{p+2k+1}}{(p+2k+1)\,(2k)!} \qquad [|x|<\pi/2;\ p>0].$$

3. $$\int \frac{x^p\,dx}{\cos^2 x} = x^p \operatorname{tg} x + p\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}(2^{2k}-1)\,x^{p+2k-1}}{(p+2k-1)\,(2k)!}B_{2k}$$
$$[p>1;\ |x|<\pi/2]$$

4. $$\int \frac{x\,dx}{\cos^{2n} x} =$$
$$=\frac{2n+1}{(2n+2)\,2n}\sum_{k=0}^{n-1}\frac{(2n+2)\,2n\ldots(2n-2k+2)\,[(2n-2k)\,x\sin x - \cos x]}{(2n+1)(2n-1)\ldots(2n-2k+3)(2n-2k+1)(2n-2k)\cos^{2n-2k+1} x} +$$
$$+\frac{2^{n-1}(n-1)!}{(2n-1)!!}(x\operatorname{tg} x + \ln|\cos x|).$$

5. $$\int \frac{x\,dx}{\cos^{2n+1}x} = \frac{1}{2n+1}\sum_{k=0}^{n-1}\frac{(2n+1)(2n-1)\dots(2n-2k+1)\,[(2n-2k+1)\,x\sin x-\cos x]}{2n\,(2n-2)\dots(2n-2k+2)\,(2n-2k)\,(2n-2k-1)\cos^{2n-2k}x} + \frac{(2n-1)!!}{2^n n!}\int\frac{x\,dx}{\cos x}.$$

6. $$\int\frac{dx}{x^p\cos^q x} = \frac{p}{(q-1)(q-2)\,x^{p+1}\cos^{q-2}x} + \frac{\sin x}{(q-1)\,x^p\cos^{q-1}x} + \frac{q-2}{q-1}\int\frac{dx}{x^p\cos^{q-2}x} + \frac{p\,(p+1)}{(q-1)(q-2)}\int\frac{dx}{x^{p+2}\cos^{q-2}x}.$$

7. $$\int\frac{dx}{x^n\cos x} = \frac{1}{2}\,[1-(-1)^n]\,\frac{|E_{n-1}|}{(n-1)!}\ln x + \sum_{\substack{k=0\\ k\neq (n-1)/2}}^{\infty}\frac{|E_{2k}|\,x^{2k-n+1}}{(2k-n+1)\,(2k)!}$$ $[|x|<\pi/2]$.

8. $$\int\frac{dx}{x^n\cos^2 x} = \frac{\operatorname{tg} x}{x^n} - [1-(-1)^n]\,(-1)^{(n+1)/2}\,\frac{2^n n}{(n+1)!}\,(2^{n+1}-1)\,B_{n+1}\ln x - \frac{n}{x^{n+1}}\sum_{\substack{k=1\\ k\neq (n+1)/2}}^{\infty}\frac{(-1)^k\,(2^{2k}-1)\,(2x)^{2k}}{(2k-n-1)\,(2k)!}\,B_{2k}$$ $[|x|<\pi/2]$.

9. $$\int\frac{x\,dx}{\cos x} = \sum_{k=0}^{\infty}\frac{|E_{2k}|\,x^{2k+2}}{(2k+2)\,(2k)!}.$$

10. $$\int\frac{x\,dx}{\cos^2 x} = x\operatorname{tg} x + \ln|\cos x|.$$

11. $$\int\frac{x\,dx}{\cos^3 x} = \frac{x\sin x-\cos x}{2\cos^2 x} + \frac{1}{2}\int\frac{x\,dx}{\cos x}.$$

12. $$\int\frac{x\,dx}{\cos^4 x} = \frac{x\sin x}{3\cos^3 x} - \frac{1}{6\cos^2 x} + \frac{2}{3}\,x\operatorname{tg} x + \frac{2}{3}\ln|\cos x|.$$

1.5.46. Интегралы вида $\int R\,(x^p,\ \sin x,\ \cos x,\ a+b\sin x+c\cos x)\,dx$.

1. $$\int\frac{x^p\cos x\,dx}{(a+b\sin x)^q} = -\frac{x^p}{(q-1)\,b\,(a+b\sin x)^{q-1}} + \frac{p}{(q-1)\,b}\int\frac{x^{p-1}\,dx}{(a+b\sin x)^{q-1}}.$$

2. $$\int\frac{x^p\sin x\,dx}{(a+b\cos x)^q} = \frac{x^p}{(q-1)\,b\,(a+b\cos x)^{q-1}} - \frac{p}{(q-1)\,b}\int\frac{x^{p-1}\,dx}{(a+b\cos x)^{q-1}}.$$

3. $$\int\frac{x\,dx}{1\pm\sin x} = \mp x\begin{Bmatrix}\operatorname{tg}[(\pi-2x)/4]\\ \operatorname{ctg}[(\pi-2x)/4]\end{Bmatrix} + 2\ln\left|\begin{Bmatrix}\cos[(\pi-2x)/4]\\ \sin[(\pi-2x)/4]\end{Bmatrix}\right|.$$

4. $$\int\frac{x\,dx}{1\pm\cos x} = \pm x\begin{Bmatrix}\operatorname{tg}(x/2)\\ \operatorname{ctg}(x/2)\end{Bmatrix} + 2\ln\begin{Bmatrix}\cos(x/2)\\ \sin(x/2)\end{Bmatrix}.$$

5. $$\int\frac{x\cos x\,dx}{(1\pm\sin x)^2} = \mp\frac{x}{1\pm\sin x} + \begin{Bmatrix}\operatorname{tg}[(2x-\pi)/4]\\ \operatorname{ctg}[(2x-\pi)/4]\end{Bmatrix}.$$

6. $$\int\frac{x\sin x\,dx}{(1\pm\cos x)^2} = \pm\frac{x}{1\pm\cos x} - \begin{Bmatrix}\operatorname{tg}(x/2)\\ \operatorname{ctg}(x/2)\end{Bmatrix}.$$

7. $\int \frac{x\,dx}{(\cos x + a \sin x)^2} = \frac{1}{a^2+1} \ln (\cos x + a \sin x) + x \frac{\operatorname{tg} x - a}{(a^2+1)(1+a \operatorname{tg} x)}$.

8. $\int \frac{x^2\,dx}{[(ax-b)\sin x + (a+bx)\cos x]^2} = \frac{x \sin x + \cos x}{b\,[(ax-b)\sin x + (a+bx)\cos x]}$.

9. $\int \frac{\cos^2 x\,dx}{[a \cos x + (ax+b)\sin x]^2} = \frac{\operatorname{tg} x}{a\,[a + (ax+b)\operatorname{tg} x]}$.

1.5.47. Интегралы вида $\int \frac{x \sin^m x \cos^n x}{\sqrt{(1-k^2 \sin^2 x)^r}}\,dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$.

1. $\int \frac{x \sin x \cos x}{\Delta}\,dx = -\frac{x\Delta}{k^2} + \frac{1}{k^2} E(x, k)$.

2. $\int \frac{x \sin^3 x \cos x}{\Delta}\,dx = \frac{1-k^2}{9k^4} F(x, k) + \frac{2k^2+5}{9k^4} E(x, k) - $
$- \frac{1}{9k^4}[3(3-\Delta^2)x + k^2 \sin x \cos x]\,\Delta$.

3. $\int \frac{x \sin x \cos^3 x}{\Delta}\,dx = -\frac{1-k^2}{9k^4} F(x, k) + \frac{7k^2-5}{9k^4} E(x, k) - $
$- \frac{1}{9k^4}[3(\Delta^2 - 3 + 3k^2)x - k^2 \sin x \cos x]\,\Delta$.

4. $\int \frac{x \sin x}{\Delta^3}\,dx = -\frac{x \cos x}{(1-k^2)\Delta} + \frac{1}{k(1-k^2)} \arcsin (k \sin x)$.

5. $\int \frac{x \cos x}{\Delta^3}\,dx = \frac{x \sin x}{\Delta} + \frac{1}{k} \ln (k \cos x + \Delta)$.

6. $\int \frac{x \sin x \cos x}{\Delta^3}\,dx = \frac{x}{k^2\Delta} - \frac{1}{k^2} F(x, k)$.

7. $\int \frac{x \sin^3 x \cos x}{\Delta^3}\,dx = x\frac{2-k^2\sin^2 x}{k^4\Delta} - \frac{1}{k^4}[E(x, k) + F(x, k)]$.

8. $\int \frac{x \sin x \cos^3 x}{\Delta^3}\,dx = x\frac{k^2\sin^2 x + k^2 - 2}{k^4\Delta} + \frac{1-k^2}{k^4} F(x, k) + \frac{1}{k^4} E(x, k)$.

1.5.48. Интегралы вида $\int (x+b)^{\pm n} \begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx$.

1. $\int (x+b)^n \begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx =$

$$= \frac{n!}{a}\left[\mp \sum_{k=0}^{[n/2]} (-1)^k \left(\frac{1}{a}\right)^{2k} \frac{(x+b)^{n-2k}}{(n-2k)!} \begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} + \right.$$
$$\left. + \sum_{k=0}^{[(n-1)/2]} (-1)^k \left(\frac{1}{a}\right)^{2k+1} \frac{(x+b)^{n-2k-1}}{(n-2k-1)!} \begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}\right].$$

2. $\int (x+b) \begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = \mp \frac{1}{a}(x+b)\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} + \frac{1}{a^2}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}$.

3. $$\int (x+b)^2 \begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = \mp \frac{1}{a}\left[(x+b)^2 - \frac{2b}{a^2}\right]\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} + \frac{2}{a^2}(x+b)\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}.$$

4. $$\int (x+b)^3 \begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = \mp \frac{x+b}{a}\left[(x+b)^2 - \frac{6}{a^2}\right]\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} + \frac{3}{a^2}\left[(x+b)^2 - \frac{2}{a^2}\right]\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}.$$

5. $$\int \frac{1}{(x+b)^n}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = a^{n-1}\left(\cos ab \int \frac{1}{t^n}\begin{Bmatrix}\sin t\\ \cos t\end{Bmatrix} dt \mp \mp \sin ab \int \frac{1}{t^n}\begin{Bmatrix}\cos t\\ \sin t\end{Bmatrix} dt\right) \qquad [t = a(x+b)].$$

6. $$\int_x^\infty \frac{1}{(x+b)^n}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = \sum_{k=1}^{n-1}\frac{(k-1)!}{(n-1)!}(x+b)^{-k}(-a)^{n-k-1}\begin{Bmatrix}\cos[\pi(n-k)/2-ax]\\ \sin[\pi(n-k)/2-ax]\end{Bmatrix} - \frac{(-a)^{n-1}}{(n-1)!}\left[\operatorname{ci}(ab+ax)\begin{Bmatrix}\cos(ab+\pi n/2)\\ \sin(ab+\pi n/2)\end{Bmatrix} \pm \operatorname{si}(ab+ax)\begin{Bmatrix}\sin(ab+\pi n/2)\\ \cos(ab+\pi n/2)\end{Bmatrix}\right].$$

7. $$\int \frac{1}{x+b}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = \cos ab \begin{Bmatrix}\operatorname{si}(ax+ab)\\ \operatorname{ci}(ax+ab)\end{Bmatrix} \mp \sin ab \begin{Bmatrix}\operatorname{ci}(ax+ab)\\ \operatorname{si}(ax+ab)\end{Bmatrix}.$$

8. $$\int \frac{1}{(x+b)^2}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = -\frac{1}{x+b}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} \pm a \int \frac{1}{x+b}\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} dx.$$

9. $$\int \frac{1}{(x+b)^3}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx = -\frac{1}{2(x+b)^2}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} \mp \frac{a}{2(x+b)}\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} - \frac{a^2}{2}\int \frac{1}{x+b}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} dx.$$

10. $$\int_x^\infty \frac{1}{x(x+b)}\sin ax\, dx = \frac{1}{b}\left[\cos ab \operatorname{si}(ab+ax) - \sin ab \operatorname{ci}(ab+ax) - \operatorname{si}(ax)\right].$$

1.5.49. Интегралы вида $\int e^{ax}\sin^p x \cos^q x\, dx$.

1. $$\int e^{ax}\sin^p x\cos^q x\, dx = \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^p x\cos^{q-1}x\,[a\cos x+(p+q)\sin x] - pa\int e^{ax}\sin^{p-1}x\cos^{q-1}x\, dx + (q-1)(p+q)\int e^{ax}\sin^p x\cos^{q-2}x\, dx\right\}.$$

2. $$= \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^{p-1}x\cos^q x\,[a\sin x-(p+q)\cos x] + qa\int e^{ax}\sin^{p-1}x\cos^{q-1}x\, dx + (p-1)(p+q)\int e^{ax}\sin^{p-2}x\cos^q x\, dx\right\}.$$

3. $$= \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^{p-1}x\cos^{q-1}x\,[a\sin x\cos x + q\sin^2 x - p\cos^2 x] + q(q-1)\int e^{ax}\sin^p x\cos^{q-2}x\, dx + p(p-1)\int e^{ax}\sin^{p-2}x\cos^q x\, dx\right\}.$$